ასტროფიზიკის და პლაზმის ფიზიკის ამოცანების მოდელირება - 2

ლექცია 4

Finite Volume

Conservation equation in the integral form

$$\frac{\partial}{\partial t}\int_{\Omega} \mathrm{U}(t,x)\, d\Omega + \int_{\Omega} \nabla \cdot \mathrm{F}\,(t,x) d\Omega = 0$$

$$\frac{\partial}{\partial t}\int_{\Omega} \mathrm{U}(t,x)\, d\Omega + \oint_{\mathrm{S}} \mathrm{F}(t,x) \cdot \mathrm{n}\, ds = 0$$

Finite Volume

Evaluating flux

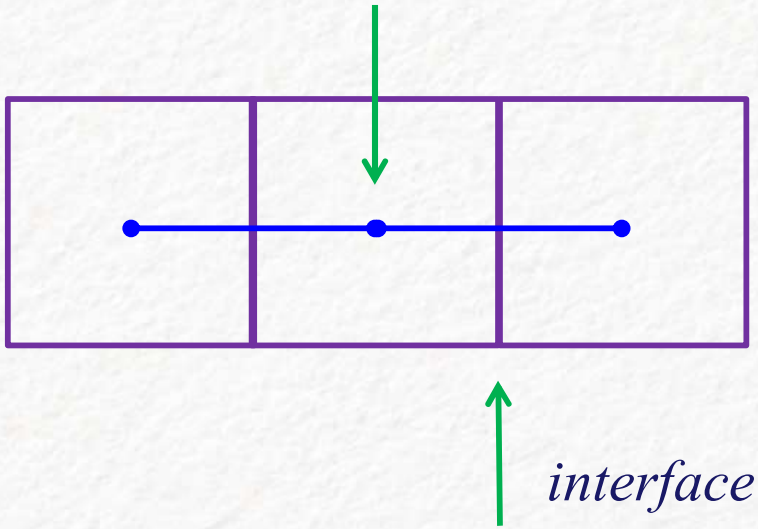
$$\oint_S \mathrm{F}(t,x)\cdot \mathrm{n}\, ds \approx \sum_{faces} F_k\, n_k$$

Numerical flux calculation problem

Cells and Faces

Grid cell centers:

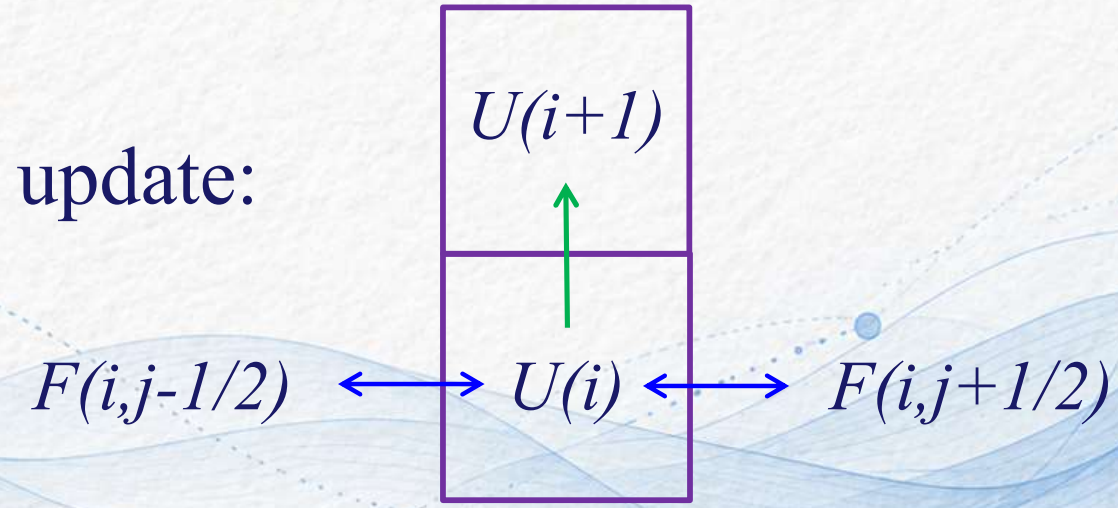
j=1, 2, 3, ..., n.



Cell interfaces:

j=1/2, 3/2, ..., n-1/2.

Finite Volume update:



Finite Volume

Time stepping algorithm

$$U(i+1,j) = U(i,j) - \frac{\Delta t}{\Delta x_j}\left(F^*(i,j+1/2) - F^*(i,j-1/2)\right)$$

flux on the cell faces

Finite Volume

Multidimensional problem:

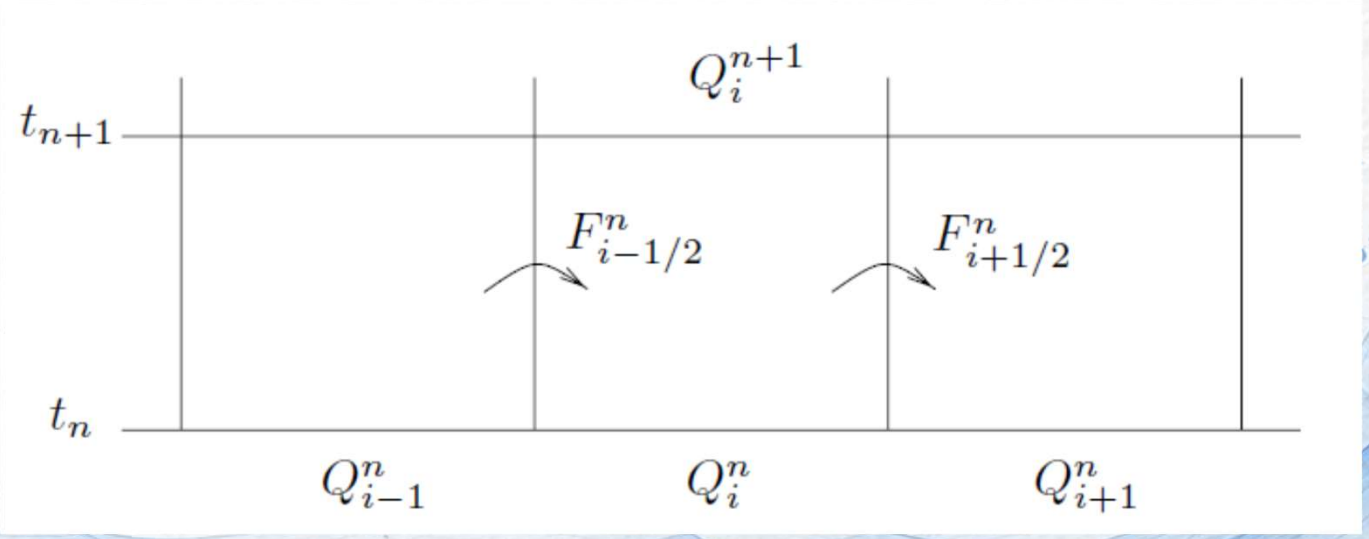
Unsplit integration

$$U(i+1,j,k) = U(i,j,k) - \frac{\Delta t}{\Delta x_j}\left(F^*(i,j+1/2,k) - F^*(i,j-1/2,k)\right) - \frac{\Delta t}{\Delta y_j}\left(G^*(i,j,k+1/2) - G^*(i,j,k-1/2)\right)$$

Finite Volume

Leveque, Sec. 4

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right),$$



Boundary Conditions

Numerical flux calculation over the domain:

$$\sum_{volume}\sum_{faces} F_k\, n_k = \sum_{boundary} F_k\, n_k$$

cancellation of fluxes on internal boundaries

Numerical Flux Function

Estimating flux on the faces

$$U(i+1,j) = U(i,j) - \frac{\Delta t}{\Delta x_j}(F(i,j+1/2) - F(i,j-1/2)) =$$

$$= U(i,j) - \frac{\Delta t}{\Delta x_j}(F(U(i,j+1/2)) - F(U(i,j-1/2))$$

U(i,j+1/2) , U(i,j-1/2) - ?

Numerical Flux Function

Estimating numerical flux:

Upwind – Zeroth approximation

Central difference – Averaging conservation variable

Central difference – Averaging fluxes

Lax-Wendroff – Finite difference analog calculation of the numerical fluxes

Upwind

$$F^*_{i+1/2} = \begin{cases} F(U_i) & \text{if } \lambda_{i+1/2} \geq 0 \\ F(U_{i+1}) & \text{if } \lambda_{i+1/2} < 0 \end{cases}$$

Finding flux direction and projecting cell centered flux value on the cell face.

"wave velocity" $\lambda = \partial F / \partial U$

Central Difference

1. Averaging U(i,j)
2. Averaging F(i,j)

$$F^*_{i+1/2} = \frac{1}{2}(F(U_i) + F(U_{i+1})) \quad \text{or} \quad F^*_{i+1/2} = F\left(\frac{U_i + U_{i+1}}{2}\right)$$

Finite Volume

Lax-Wendroff flux estimation

$$F^*_{i+1/2} = \frac{1}{2}(F(U_i) + F(U_{i+1})) - \frac{\lambda_{i+1/2}\Delta t}{2\Delta x_{i+1/2}}(F(U_{i+1}) - F(U_i))$$

Calculating numerical flux using finite difference methods on cell center to face transition.

Finite Volume Method

Discrete formulation of the conservation equation at every cell/interface

$$\frac{\partial}{\partial t}\int_{\Omega} \mathrm{U}(t,x)\, d\Omega + \int_{\Omega} \nabla \cdot \mathrm{F}\,(t,x) d\Omega =$$
$$= \frac{\partial}{\partial t} U_{cell.average}(t,x_j) + F_R(t,x_j) - F_L(t,x_j)$$
$$= 0$$

Diffusion Equations

Conservative finite-volume formulation

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[\beta(x)\frac{\partial U}{\partial x}\right]$$

$$F(t,x) = -\beta(x)\frac{\partial U}{\partial x}$$

$$\frac{d}{\mathrm{d}t}\int_{x_{j-1/2}}^{x_{j+1/2}} U(t,x)\mathrm{d}x = F(t,x_{j-1/2}) - F(t,x_{j+1/2})$$

Diffusive flux at a cell face

$$F(i,j+1/2) = -\beta_{j+1/2}\frac{U(i,j+1) - U(i,j)}{\Delta x}$$

For constant β, the equation reduces to $U_t = \beta\, U_{xx}$

Diffusion Equations

Implicit time stepping: Crank–Nicolson

$$F^*(i,j+1/2) = \frac{F(i,j+1/2) + F(i+1,j+1/2)}{2}$$

$$U(i+1,j) = U(i,j) - \frac{\Delta t}{\Delta x}[F^*(i,j+1/2) - F^*(i,j-1/2)]$$

Expanded update

$$U(i+1,j) = U(i,j) + \frac{\Delta t}{2\Delta x^2}[$$
$$\beta_{j+1/2}(U(i,j+1) - U(i,j))$$
$$-\beta_{j-1/2}(U(i,j) - U(i,j-1))$$
$$+\beta_{j+1/2}(U(i+1,j+1) - U(i+1,j))$$
$$-\beta_{j-1/2}(U(i+1,j) - U(i+1,j-1))]$$

Values at time i+1 are solved simultaneously

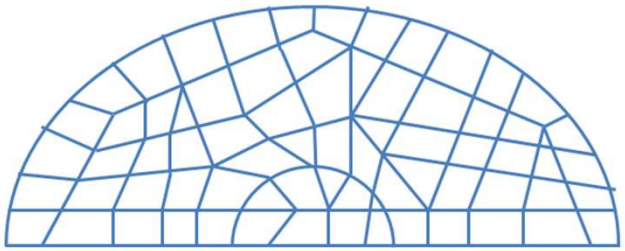
Finite Volume

Finite Difference

- Problems with conservation
- Irregular geometries

Finite Volume

+ Explicit (hyperbolic)
+ easily adaptable
+ unstructured grids
- problems towards the edges



Unstructured Grid

end

www.tevza.org/home/course/modelling-II_2026/