



ასტროფიზიკის და პლაზმის ფიზიკის ამოცანების მოდელირება - 2

ლექცია 2

Finite Difference (FD) Methods

Conservation Law:

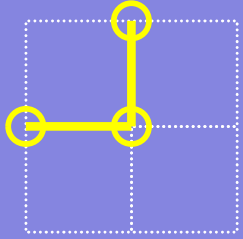
$$\frac{\partial U(t, x)}{\partial t} + \nabla \cdot J(t, x) = S(t, x)$$

$$J(t, x) = aU(t, x)$$

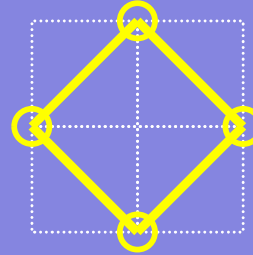
1D Linear Differential Equation:

$$\frac{\partial U(t, x)}{\partial t} + a \frac{\partial U(t, x)}{\partial x} = 0$$

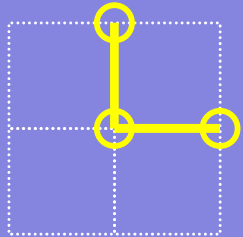
FD Methods



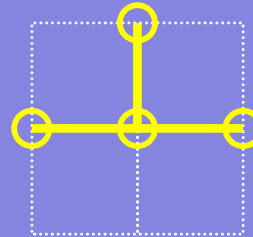
One-sided backward



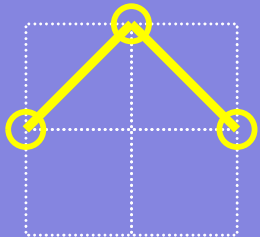
Leapfrog



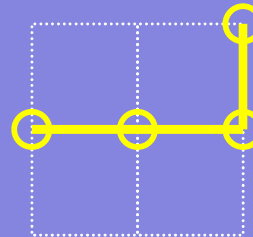
One-sided forward



Lax-Wendroff

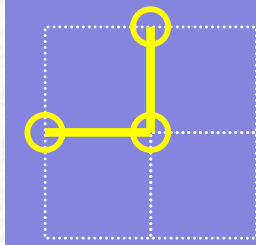


Lax-Friedrichs



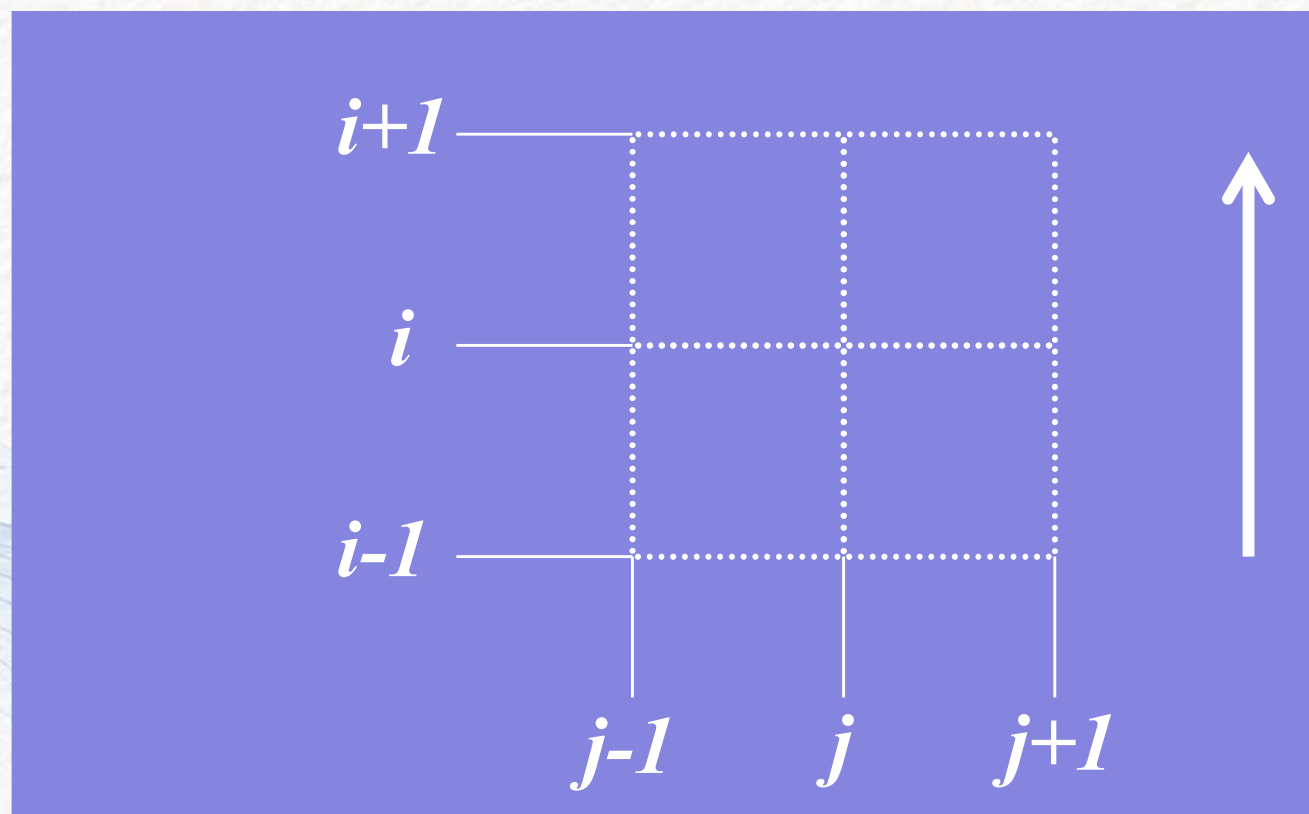
Beam-Warming

One-sided Backward



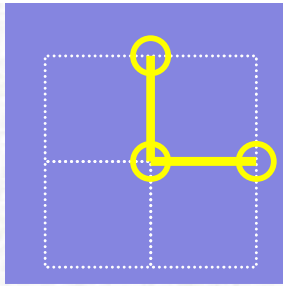
Difference equation:

$$U(i + 1, j) = U(i, j) - A \frac{\Delta t}{\Delta x} [U(i, j) - U(i, j - 1)]$$



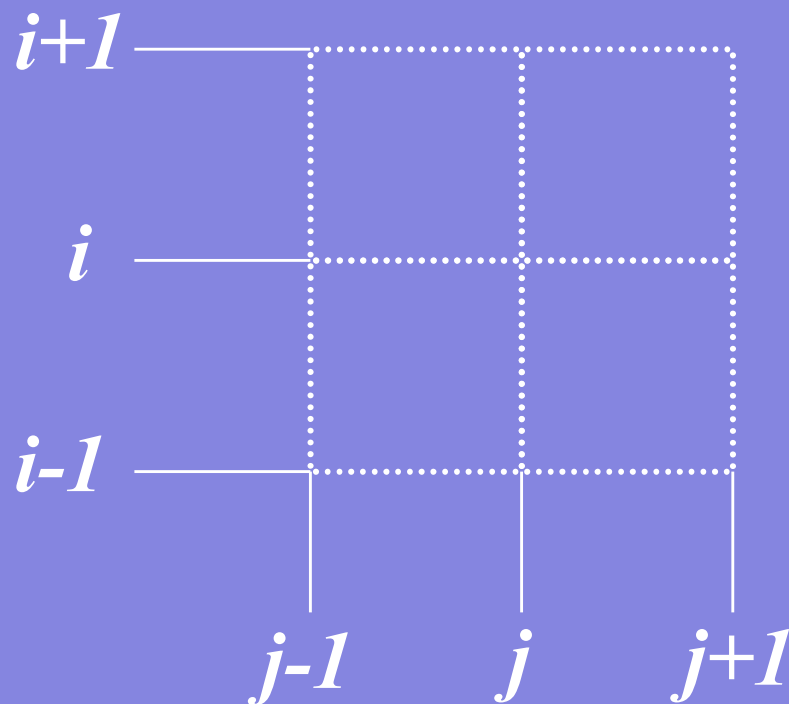
**time
stepping**

One-sided forward



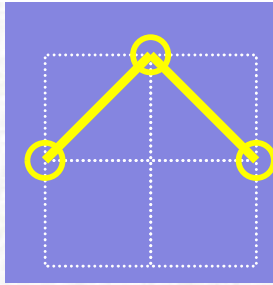
Difference equation:

$$U(i+1, j) = U(i, j) - \frac{\Delta t}{\Delta x} A[U(i, j+1) - U(i, j)]$$



time
stepping

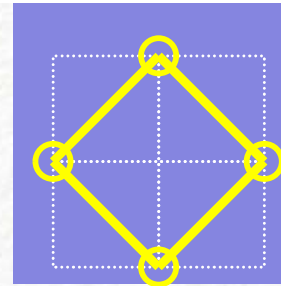
Lax-Friedrichs



Difference equation:

$$U(i + 1, j) = \frac{1}{2} [U(i, j + 1) + U(i, j - 1)] - \frac{\Delta t}{2\Delta x} A [U(i, j + 1) - U(i, j - 1)]$$

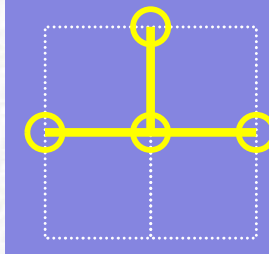
Leapfrog



Difference equation:

$$U(i + 1, j) = U(i - 1, j) - \frac{\Delta t}{\Delta x} A[U(i, j + 1) - U(i, j - 1)]$$

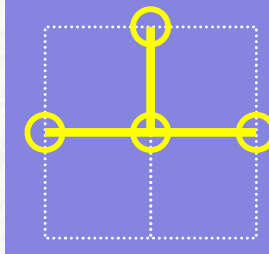
Lax-Wendroff



Difference equation:

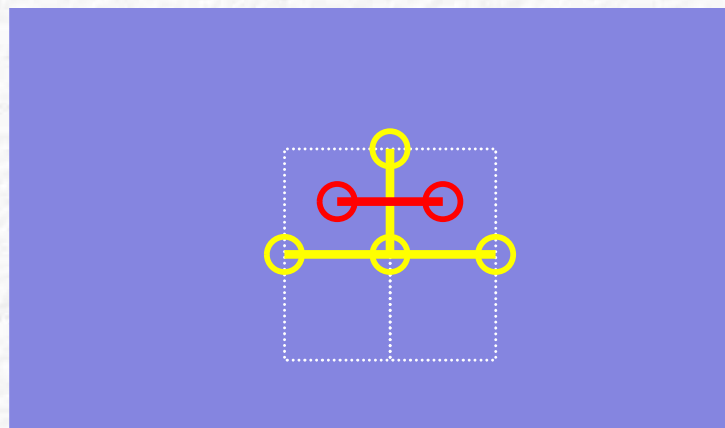
$$\begin{aligned} U(i+1, j) = & U(i, j) - \\ & - \frac{\Delta t}{2\Delta x} A [U(i, j+1) - U(i, j-1)] \\ & + \frac{\Delta t^2}{2\Delta x^2} A^2 [U(i, j+1) - 2U(i, j) + U(i, j-1)] \end{aligned}$$

Lax-Wendroff



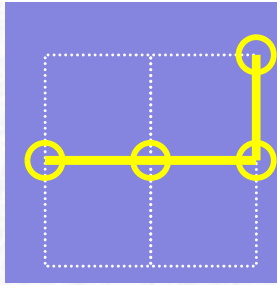
Derivation:

1. Lax-Friedrichs with half step;
2. Leapfrog half step;



derive

Beam-Warming



Difference equation:

$$\begin{aligned} U(i+1, j) = & U(i, j) - \\ & - \frac{\Delta t}{2\Delta x} A [3U(i, j) - 4U(i, j-1) + U(i, j-2)] \\ & + \frac{\Delta t^2}{2\Delta x^2} A^2 [U(i, j) - 2U(i, j-1) + U(i, j-2)] \end{aligned}$$

Comparison

One-sided schemes:

$$O(\Delta t, \Delta x)$$

Lax-Friedrichs:

$$O(\Delta t, \Delta x^2)$$

Leapfrog:

$$O(\Delta t^2, \Delta x^2)$$

Lax-Wendroff:

$$O(\Delta t^2, \Delta x^2)$$

Beam-Warming:

$$O(\Delta t^2 + \Delta x^2)$$

Finite Difference Methods: Comparison

First-order upwind

DIFFUSIVE

Damps short wavelengths.
Smooths the jump; usually no overshoot for a stable monotone update.

Lax–Friedrichs

MORE DIFFUSIVE

Strong artificial diffusion for many Courant numbers.
A sharp edge becomes broad.

Lax–Wendroff

DISPERSIVE

Better accuracy in smooth regions.
Oscillations and overshoots appear near a jump.

Beam–Warming

DISPERSIVE

Second-order upwind-biased formula.
Oscillations appear on a different side of the jump.

Leapfrog: no leading numerical diffusion in the standard smooth-mode analysis, but a parasitic computational mode can cause odd–even decoupling.

3

Time Stepping

1st order methods in time:

$$U(i+1) = F\{ U(i) \}$$

2nd order methods in time:

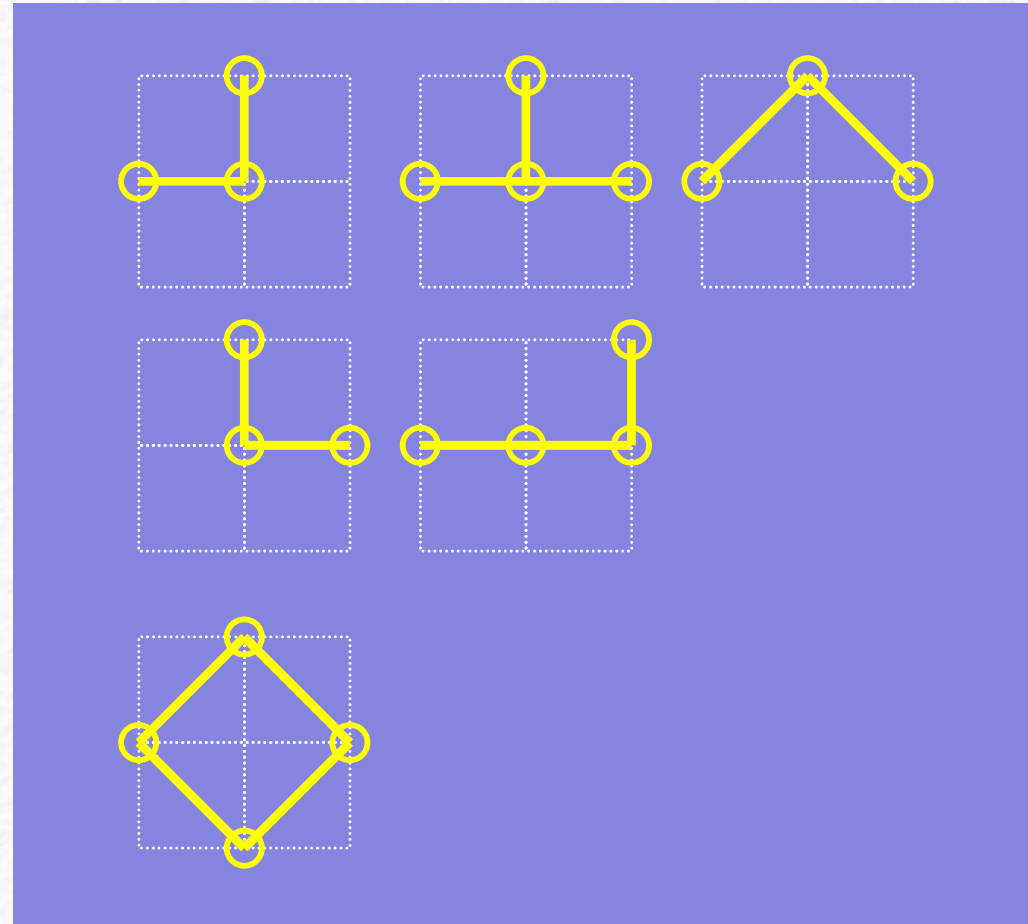
$$U(i+1) = F\{ U(i) , U(i-1) \}$$

Starting A) $U(2) = F\{ U(1) \}$

(e.g. One-Sided)

method: B) $U(3) = F\{ U(2) , U(1) \}$

(Leapfrog)



Convergence

$u(t, x)$ - Analytical solution

$U(i, j)$ - Numerical solution

Error function: $E(i, j) = U(i, j) - u(t(i), x(j))$

Numerical convergence (Norm of the Error function):

$$\|E(i, j)\| \rightarrow 0, \quad \Delta x \rightarrow 0$$

Norms

Norm for conservation laws: $\|u(x, y)\| \equiv \int_{-\infty}^{+\infty} |u(x, y)| dx$

$$\|E(i, j)\|_1 = \frac{1}{N} \sum_{j=1}^N |E(i, j)|$$

Other Norms (e.g. spectral problems)

$$\|E(i, j)\|_2 = \frac{1}{N} \sqrt{\sum_{j=1}^N |E(i, j)|^2}$$

Norms

P-Norm

$$\|E(i, j)\|_p = \frac{1}{N} \left(\sum_{j=1}^N |E(i, j)|^p \right)^{1/p}$$

Norm-2: Energy in numerical domain;
Numerical dissipation, boundary effects, etc.

Numerical Stability

Courant, Friedrichs, Levy (CFL, Courant number)

$$CFL = \max \left(\frac{|a| \Delta x}{\Delta t} \right)$$

Upwind schemes for discontinuity:

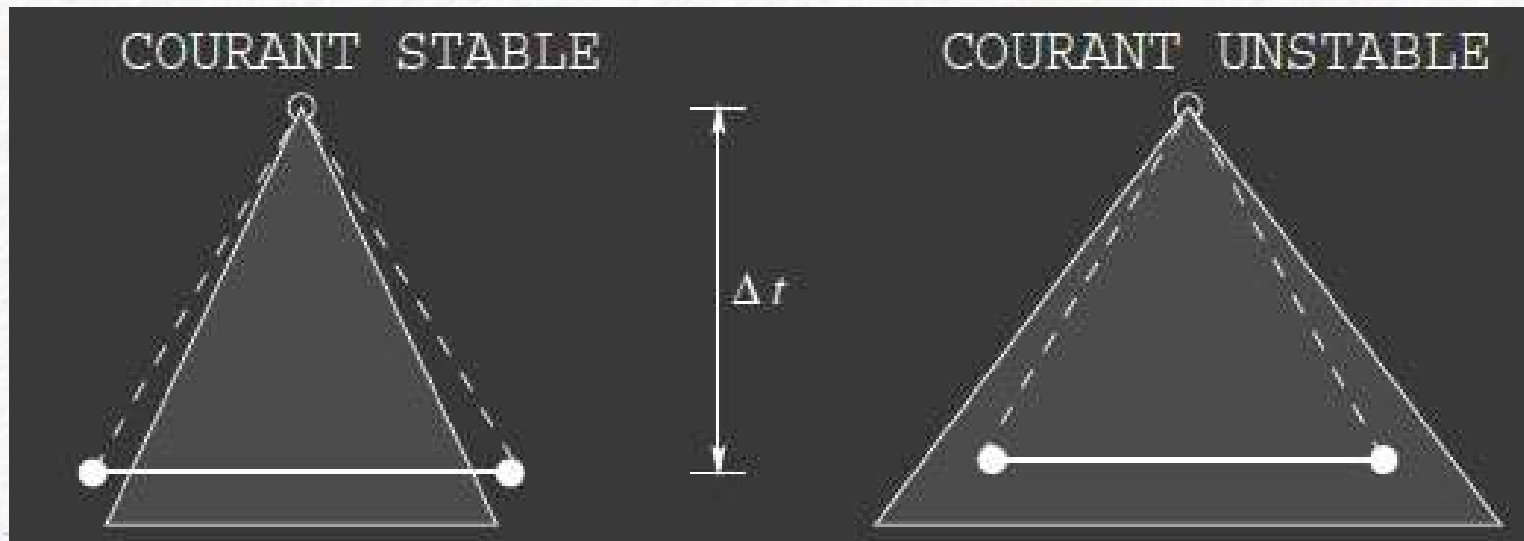
$a < 0$: One sided forward

$a > 0$: One sided backward

Numerical Stability

$$\text{CFL} < 1$$

Domain of dependence



Lax Equivalence Theorem

For a well-posed linear initial value problem, the method is convergent if and only if it is stable.

Given a properly posed initial-value problem and a finite difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence.

*Necessary and sufficient condition
for consistent linear method*

- **Well posed:** solution exists, continuous, unique;
- numerical stability (t)
- numerical convergence (xyz)

Discontinuous Solutions

Advection equation:

$$\frac{\partial U(t, x)}{\partial t} + a \frac{\partial U(t, x)}{\partial x} = 0$$

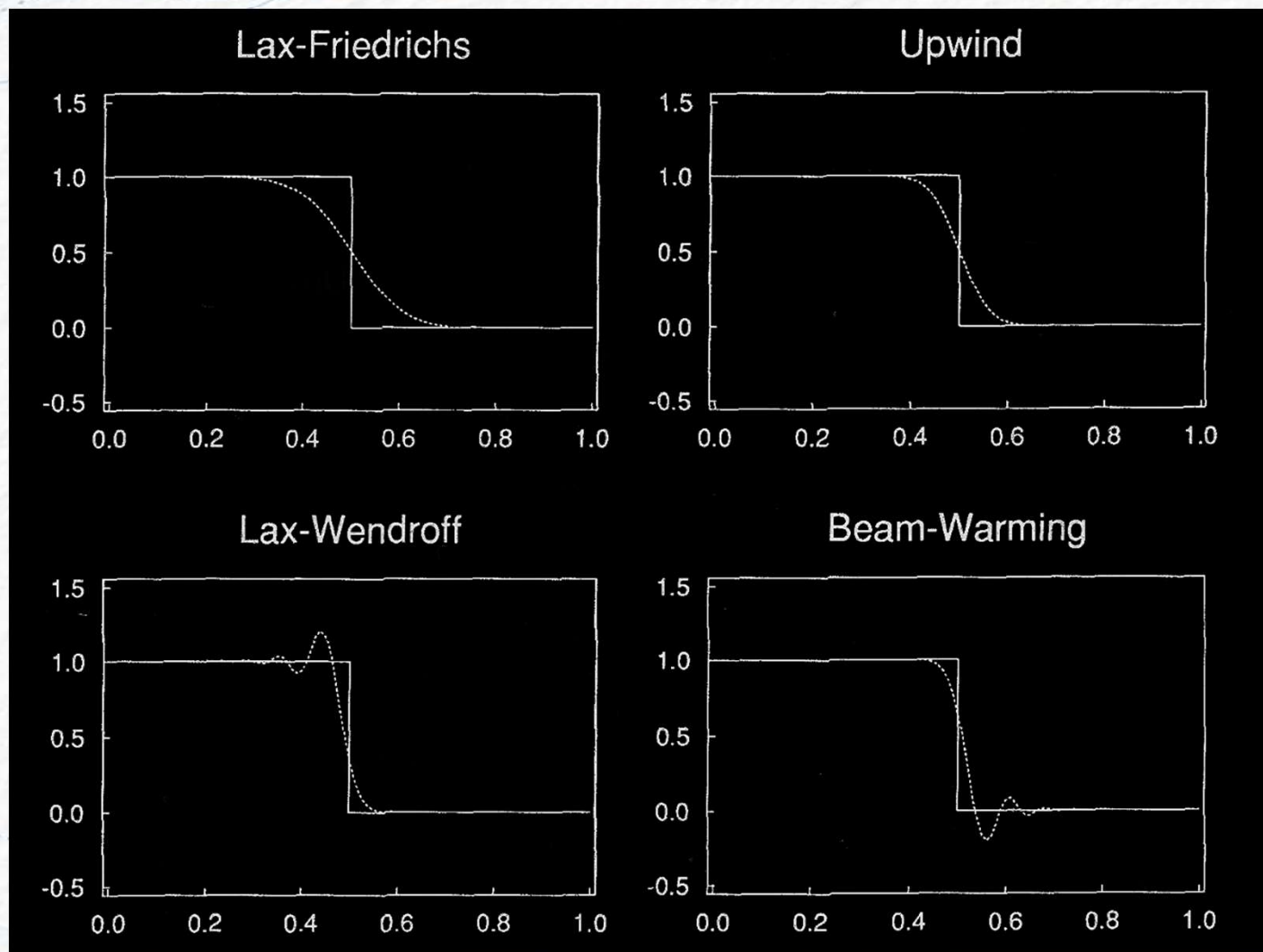
Initial condition:

$$U_0(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

Analytic solution:

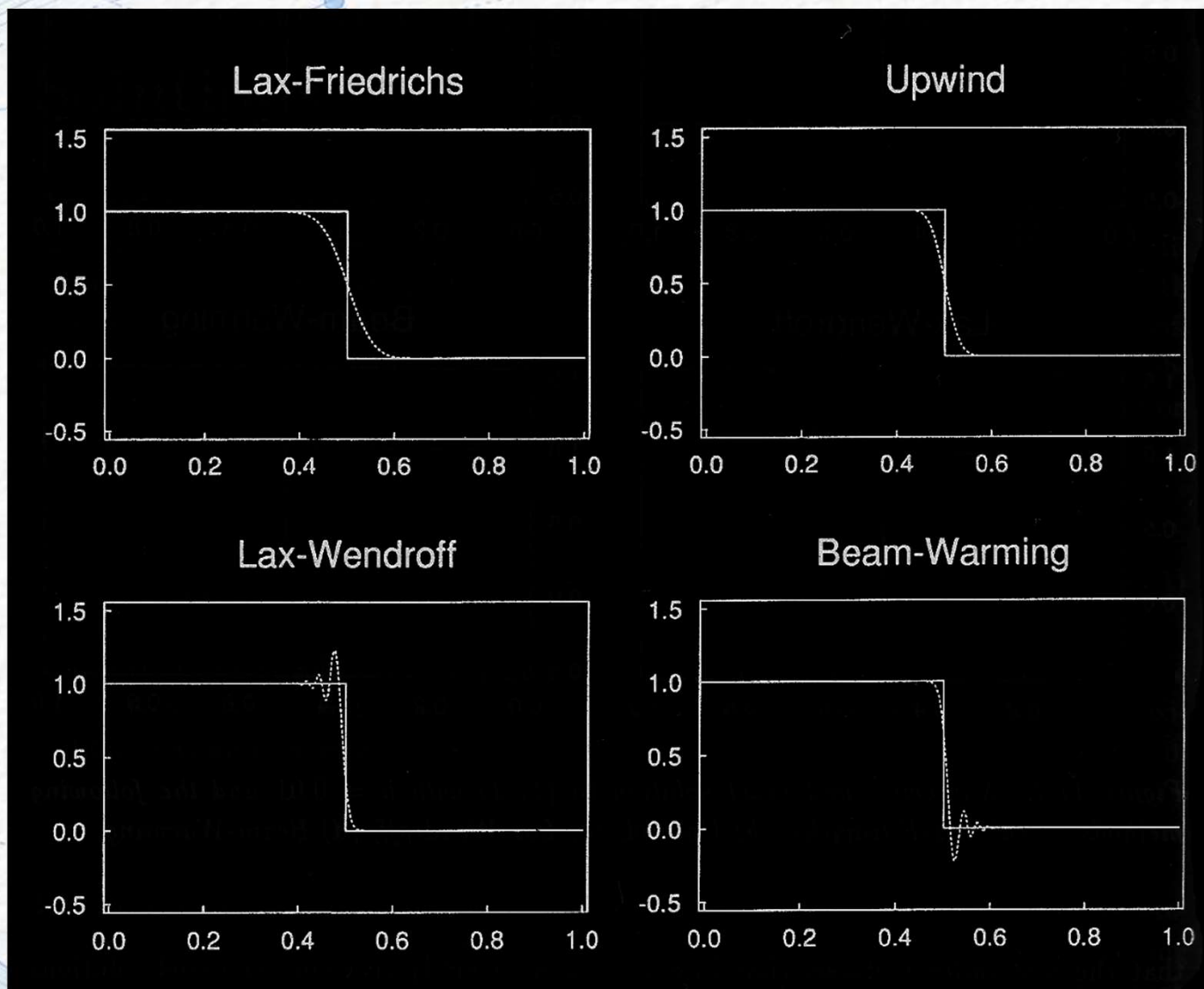
$$U(t, x) = U_0(x - at)$$

Analytic vs Numerical



$$\Delta x = 0.01$$

Analytic vs Numerical



$$\Delta x = 0.001$$

Nonlinear Equations

Linear conservation:

$$\frac{\partial U(t, x)}{\partial t} + a \frac{\partial U(t, x)}{\partial x} = 0$$

Nonlinear conservation:

$$\frac{\partial U(t, x)}{\partial t} + \frac{\partial}{\partial x} F(U) = 0$$

$$a \frac{\partial U(t, x)}{\partial x} \rightarrow \frac{\partial}{\partial x} F(U)$$

Nonlinear FD Methods

Lax-Friedrichs

linear stencil:

$$U(i+1, j) = \frac{1}{2} [U(i, j-1) + U(i, j+1)] - \frac{\Delta t}{2\Delta x} a [U(i, j+1) - U(i, j-1)]$$

nonlinear stencil:

$$U(i+1, j) = \frac{1}{2} [U(i, j-1) + U(i, j+1)] - \frac{\Delta t}{2\Delta x} [F(U(i, j+1)) - F(U(i, j-1))]$$

Nonlinear FD methods

Lax-Wendroff

$$\begin{aligned} U(i+1, j) = & U(i, j) - \\ & - \frac{\Delta t}{2\Delta x} A[U(i, j+1) - U(i, j-1)] \\ & + \frac{\Delta t^2}{2\Delta x^2} A^2[U(i, j+1) - 2U(i, j) + U(i, j-1)] \end{aligned}$$

MacCormack's two step method:

$$\begin{aligned} U'(j) = & U(i, j) - \frac{\Delta t}{\Delta x} [F(U(i, j+1)) - F(U(i, j))] \quad , \\ U(i+1, j) = & \frac{1}{2} (U(i, j) + U'(j)) - \frac{\Delta t}{2\Delta x} [F(U'(j)) - F(U'(j-1))] \end{aligned}$$

FD Methods

Methods to suppress numerical instabilities

Numerical diffusion (first order methods)

$$\frac{\partial U(t, x)}{\partial t} + a \frac{\partial U(t, x)}{\partial x} = D \frac{\partial^2 U(t, x)}{\partial x^2}$$

$$D = \frac{\Delta x^2}{2\Delta t} \left[I - \left(\frac{\Delta t}{\Delta x} a \right)^2 \right]$$

Numerical Dispersion

$$\frac{\partial U(t, x)}{\partial t} + a \frac{\partial U(t, x)}{\partial x} = \mu \frac{\partial^3 U(t, x)}{\partial x^3}$$

Lax-Wendroff (or second order methods)

$$\mu = \frac{\Delta x^2}{6} a \left[\frac{\Delta t^2}{\Delta x^2} a^2 - I \right]$$

Beam-Warming method

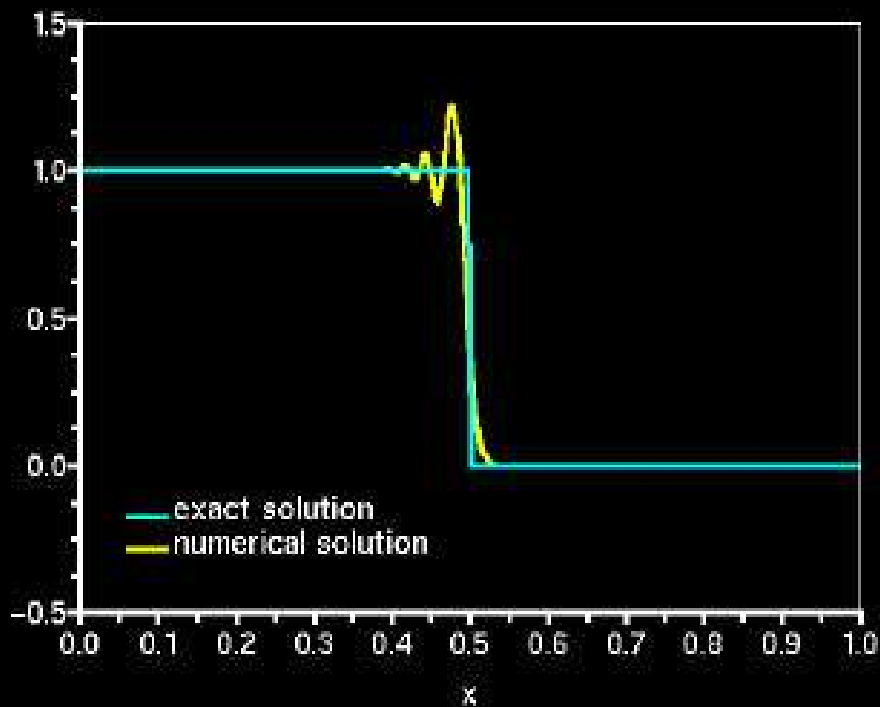
$$\mu = \frac{\Delta x^2}{6} a \left[2I - \frac{3\Delta t}{\Delta x} a + \frac{\Delta t^2}{\Delta x^2} a^2 \right]$$

Numerical Dispersion

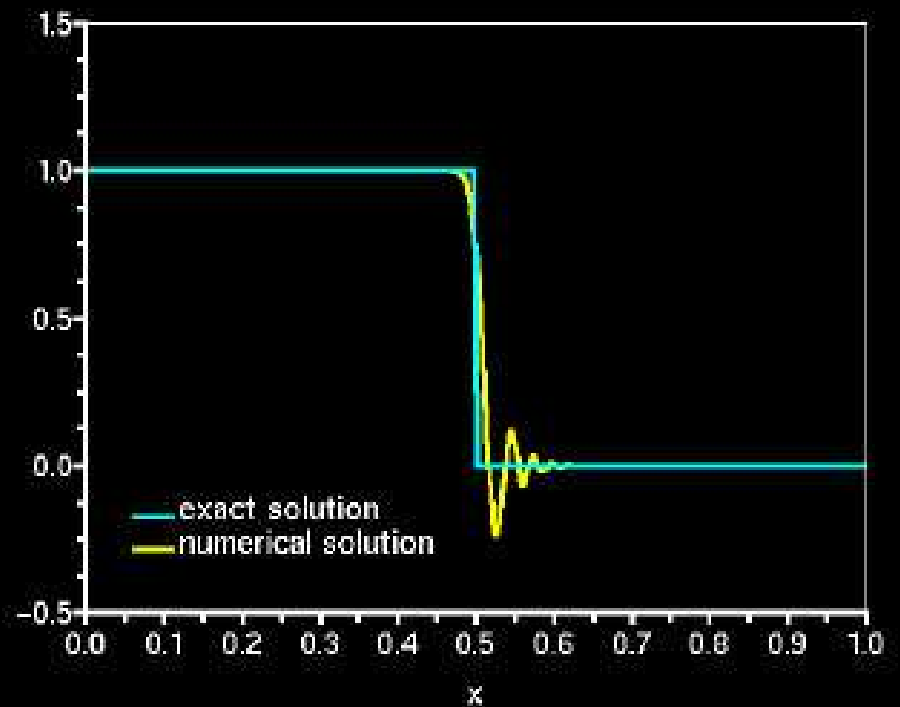
($a > 0$)

($a < 0$)

Lax-Wendroff



Beam-Warming

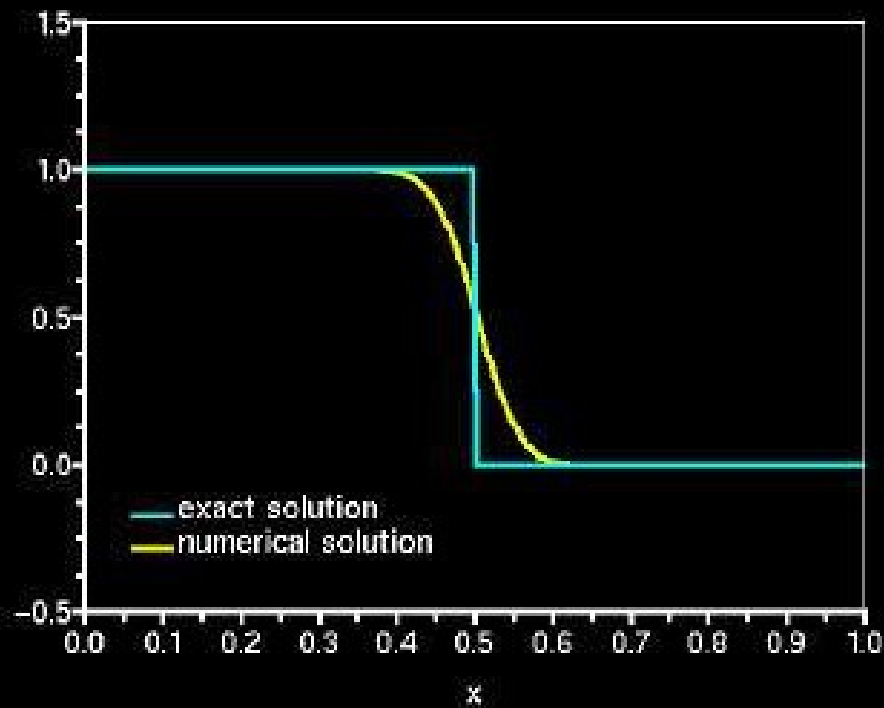


Numerical Diffusion

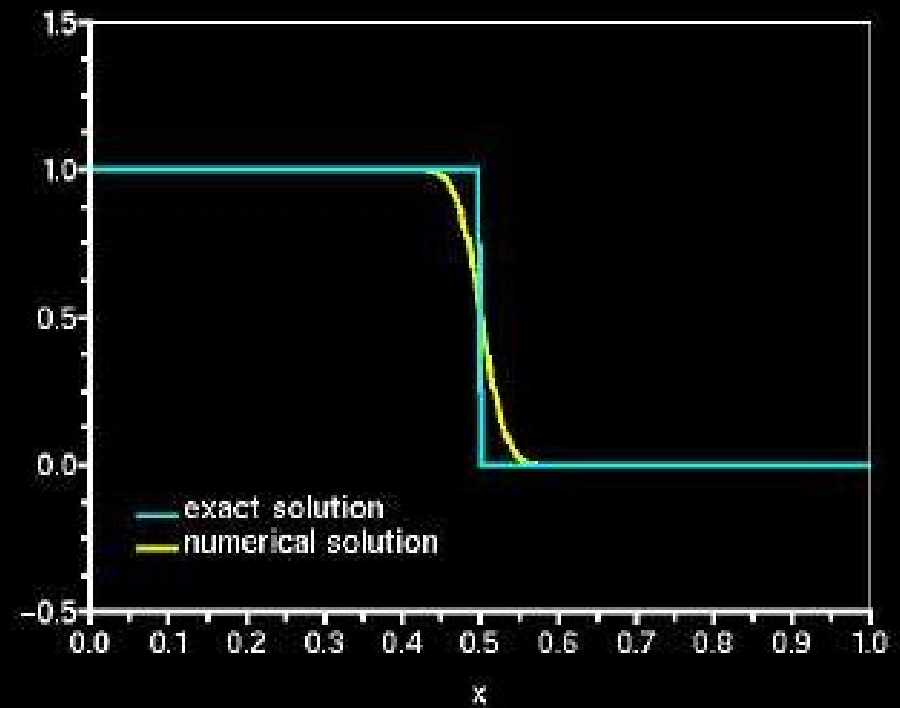
($a > 0$)

($a < 0$)

Lax-Friedrichs



Donor cell (upwind)



Dissipation vs Dispersion

Dissipation and dispersion are not the same

Modified equations and Fourier analysis explain the shapes of the numerical solutions.

NUMERICAL DIFFUSION

$$u_t + au_x = D_{\text{num}} u_{xx} + \dots$$

Even derivative $\rightarrow$ amplitude damping

Lax-Friedrichs:
$$D_{\text{num}} = (\Delta x^2 / 2\Delta t)(1 - C^2)$$

NUMERICAL DISPERSION

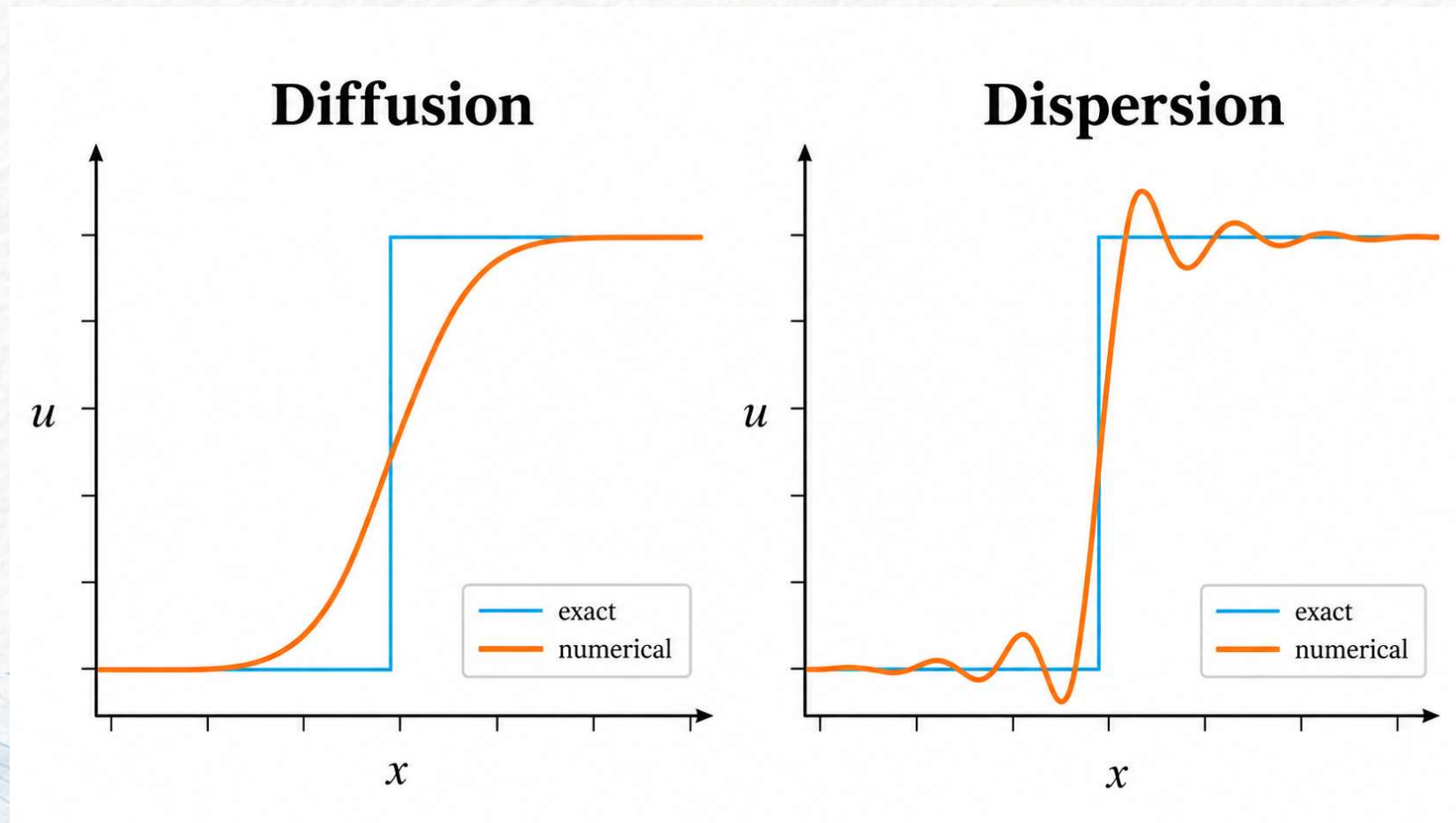
$$u_t + au_x = K_{\text{num}} u_{xxx} + \dots$$

Odd derivative $\rightarrow$ phase-speed error

Lax-Wendroff:
leading error is dispersive

- a smeared front signals numerical diffusion;
- ringing around a front signals numerical dispersion;

Diagnostics



Discontinuities

Higher formal accuracy does not guarantee a better solution near a discontinuity.

- *Lax–Friedrichs and first-order upwind tend to smear sharp gradients.*
- *Lax–Wendroff and Beam–Warming can preserve sharper transitions but introduce oscillations.*
- *Limiters can reduce these oscillations while retaining higher-order behaviour in smooth regions.*

Which numerical method should we use?

PROBLEM	EQUATION	METHOD	WATCH FOR
Advection	$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0$	Upwind (1st order) Lax-Wendroff (smooth)	CFL; diffusion / dispersion
Heat / diffusion	$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$	Centered space differences Explicit or implicit time stepping	Explicit time-step restriction
Wave	$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$	Centered space-time differences	CFL; numerical dispersion
Burgers / shocks	$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(\frac{u^2}{2} \right) = 0$	Conservative upwind flux TVD / limited high-order scheme	Oscillations; shock capturing
Poisson	$\nabla^2 u = f$	Centered differences + iterative linear solver	Grid resolution; solver convergence

Convergence

Lax-Friedrichs:

$$\|E(i, j)\| \approx C\sqrt{t \cdot \Delta x}$$

Convergence:

$$\|E(i, j)\| \rightarrow 0, \quad \Delta x \rightarrow 0$$

FD Methods

- + / Primitive
- + / Fast
- / Accuracy
- / Numerical instabilities



end

http://www.tevza.org/home/course/modelling-II_2026/