Seminar 5: Fourier Transform and Spectral Methods

Implement the numerical tasks using a programming language of your choice (Python, MATLAB, C++, etc.). All variables are dimensionless. Unless stated otherwise, use periodic grids and store each spatial point only once. For FFT plots, label the frequency or wave-number axis clearly and state the normalization that you use.

Problem 1. Fourier Harmonics and Linearity

Consider the periodic signal

$$f(t) = 2\cos(3t) - \sin(5t) + \frac{1}{2}\cos(8t), \qquad 0 \le t < 2\pi. \tag{1}$$

Tasks:

1. Before calculating the transform, predict the nonzero frequencies in the spectrum.
2. Sample the signal using $N = 128$ equally spaced points and calculate its discrete Fourier transform.
3. Plot the amplitude spectrum and identify the peaks associated with the three Fourier harmonics.
4. Repeat the calculation for $2f(t)$. Verify that all Fourier coefficients are multiplied by two.
5. Add a constant to the original signal, $g(t) = f(t) + 1$, and identify which Fourier coefficient changes.

Questions: What does the zero-frequency coefficient represent? Why do real-valued signals contain symmetric positive and negative frequency components?

Problem 2. DFT Grid and Nyquist Wave Number

Consider a periodic spatial domain

$$0 \le x < 2\pi \tag{2}$$

with $N = 16$ equally spaced grid points,

$$x_j = j\Delta x, \qquad j = 0, \ldots, N-1. \tag{3}$$

Tasks:

1. Calculate the grid spacing
$$\Delta x = \frac{L}{N}, \qquad L = 2\pi. \tag{4}$$
2. Calculate the wave-number spacing and the Nyquist wave number,
$$\Delta k = \frac{2\pi}{L}, \qquad k_{\mathrm{N}} = \frac{\pi}{\Delta x}. \tag{5}$$
3. List the integer Fourier modes represented on this grid.
4. Sample
$$f(x) = \sin(3x) + \frac{1}{2}\cos(7x) \tag{6}$$
and calculate its DFT.
5. Plot the amplitude spectrum and verify that both modes are below the Nyquist limit.

Questions: How does increasing the domain length change Δk? How does increasing N at fixed L change k_{N}?

Problem 3. Aliasing

Use the same periodic domain $0 \leq x < 2\pi$ and the same $N = 16$ grid points as in Problem 2. Consider the unresolved mode

$$g(x) = \sin(11x). \tag{7}$$

Tasks:

1. Show that the sampled values satisfy

$$\sin(11x_j) = \sin[(11 - 16)x_j] = -\sin(5x_j). \tag{8}$$

2. Calculate the DFT of the sampled signal and identify the aliased wave number.
3. Plot the sampled values of $\sin(11x)$ and $-\sin(5x)$ on the same graph.
4. Repeat the calculation with $N = 32$. Verify that the mode $k = 11$ is then resolved without aliasing.

Questions: Why can two different continuous waves have identical sampled values? What condition must a wave number satisfy in order to be resolved?

Problem 4. Spectral Leakage and Windowing

Sample the two signals

$$f_1(t) = \sin(2\pi\, 8t), \tag{9}$$
$$f_2(t) = \sin(2\pi\, 8.5t), \tag{10}$$

on the interval $0 \le t < 1$ using $N = 128$ equally spaced points. The finite observation interval multiplies the true signal by a window $r_T(t)$,

$$f_T(t) = f(t) r_T(t), \tag{11}$$

so that multiplication in time produces convolution in frequency,

$$F_T(\omega) \propto \int_{-\infty}^{\infty} F(\omega') R_T(\omega - \omega')\, d\omega'. \tag{12}$$

The proportionality factor depends on the Fourier-transform normalization.
Tasks:

1. Calculate and plot the amplitude spectra of f_1 and f_2 using the rectangular observation window.
2. Explain why f_1 fits exactly eight periods inside the observation interval, while f_2 does not.
3. Apply the Hann window

$$w_j = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi j}{N-1}\right)\right], \qquad j = 0, \ldots, N-1, \tag{13}$$

   to f_2.
4. Plot the rectangular-window and Hann-window spectra of f_2 on the same axes. Normalize both spectra by their maximum values.

Questions: Why does spectral power spread into neighbouring frequencies for f_2? What does the Hann window improve, and what spectral resolution is lost in return?

Problem 5. Fourier Synthesis and the Gibbs Phenomenon

Consider the periodic square wave

$$s(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi, \end{cases} \qquad s(x + 2\pi) = s(x). \tag{14}$$

Its truncated Fourier series is

$$s_M(x) = \frac{4}{\pi} \sum_{m=0}^{M-1} \frac{\sin[(2m+1)x]}{2m+1}. \tag{15}$$

Tasks:

1. Plot $s_M(x)$ for $M = 1$, 3, 10, and 30.
2. Compare the approximations with the exact square wave.
3. For each value of M, find the maximum value close to the discontinuity at $x = 0$.
4. Observe how the width of the oscillatory region changes when more harmonics are included.

Questions: Does the overshoot disappear as M increases? Which part of the approximation improves when more harmonics are used? Why are discontinuities difficult for a global Fourier expansion?

Problem 6. Spectral Solution of the Diffusion Equation

Consider the diffusion equation

$$u_t = \nu u_{xx}, \qquad 0 \le x < 2\pi, \qquad \nu = 0.05, \tag{16}$$

with periodic boundary conditions and initial condition

$$u(x,0) = \sin(3x) + \frac{1}{2}\sin(8x). \tag{17}$$

For each Fourier mode,

$$\frac{d\widehat{u}_k}{dt} = -\nu k^2 \widehat{u}_k, \qquad \widehat{u}_k(t) = e^{-\nu k^2 t}\widehat{u}_k(0). \tag{18}$$

Tasks:

1. Derive the analytical solution

$$u(x,t) = e^{-9\nu t}\sin(3x) + \frac{1}{2}e^{-64\nu t}\sin(8x). \tag{19}$$

2. At $T = 1$, calculate the remaining amplitude of each mode.
3. Use $N = 64$ grid points. Calculate the FFT of the initial condition, multiply every Fourier coefficient by $e^{-\nu k^2 T}$, and apply the inverse FFT.
4. Compare the numerical and analytical solutions at $T = 1$ and report the maximum error.

Questions: Why does the $k = 8$ mode decay more rapidly than the $k = 3$ mode? Which boundary condition is built naturally into the Fourier basis? Why is the spectral update exact in time for this linear problem?

Optional Problem 7. DFT and FFT Cost

The direct DFT requires approximately N^2 operations, while an FFT requires approximately $N \log_2 N$ operations.

Tasks:

1. Estimate both operation counts for $N = 1024$.
2. Repeat for $N = 2^{20}$.
3. Calculate the approximate ratio

$$\frac{N^2}{N \log_2 N} = \frac{N}{\log_2 N}. \tag{20}$$

Questions: Why does the FFT become increasingly advantageous as N grows? Does the FFT change the mathematical definition of the DFT?

Submission Requirements

For each assigned problem, submit:

1. The program source code, when numerical calculation is required.
2. The requested plots with labelled axes.
3. Short calculations or tables where requested.
4. Brief answers to the questions.