

Seminar 3: Explicit Schemes

Implement the numerical methods using a programming language of your choice (Python, MATLAB, C++, etc.). All variables are dimensionless. Use these exercises before the Crank–Nicolson problems. On periodic grids, store each spatial point only once. For smooth solutions, report the maximum error E_N at the final time and estimate the convergence order $p = \log_2(E_N/E_{2N})$ when a refinement study is requested.

Problem 1. Domain of Dependence and CFL

Consider the linear advection equation

$$u_t + u_x = 0, \quad 0 \leq x < 1, \quad (1)$$

with periodic boundary conditions and initial condition

$$u(x, 0) = \sin(2\pi x). \quad (2)$$

Tasks:

1. Implement the explicit upwind scheme

$$u_j^{n+1} = u_j^n - C(u_j^n - u_{j-1}^n), \quad C = \frac{\Delta t}{\Delta x}. \quad (3)$$

2. Use $N = 40, 80$, and 160 grid points, $\Delta x = 1/N$, $C = 0.8$, and final time $T = 0.6$.
3. Compare with the analytical solution

$$u(x, t) = \sin[2\pi(x - t)]. \quad (4)$$

Plot the final profiles and estimate the convergence order.

4. For a separate stability test, fix $N = 80$, add the perturbation $10^{-6}(-1)^j$ to the sampled initial values, and repeat with $C = 0.8$ and $C = 1.2$. Plot the maximum absolute grid value as a function of time.

Questions: How does the CFL condition relate the physical and numerical domains of dependence? Why can a small grid-scale perturbation reveal instability more clearly than a smooth initial profile?

Problem 2. Conservative and Nonconservative Burgers Schemes

Consider the inviscid Burgers equation

$$u_t + \left(\frac{u^2}{2} \right)_x = 0, \quad 0 \leq x \leq 1, \quad (5)$$

with initial and boundary data

$$u(x, 0) = \begin{cases} 2, & x < 0.3, \\ 1, & x \geq 0.3, \end{cases} \quad u(0, t) = 2, \quad u(1, t) = 1. \quad (6)$$

Use cell centers $x_j = (j + \frac{1}{2})\Delta x$, $j = 0, \dots, N - 1$, and the prescribed exterior states in ghost cells.

Tasks:

1. Implement the conservative upwind update

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x} \left[(u_j^n)^2 - (u_{j-1}^n)^2 \right]. \quad (7)$$

2. Implement the nonconservative update

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} u_j^n (u_j^n - u_{j-1}^n). \quad (8)$$

3. Use $N = 50, 100$, and 200 cells, $\Delta x = 1/N$, $\Delta t = 0.4\Delta x$, and final time $T = 0.2$.
4. Plot both solutions together with the exact step profile whose shock position is

$$x_s(t) = 0.3 + \frac{3}{2}t. \quad (9)$$

5. Calculate $Q(T) = \Delta x \sum_j u_j(T)$. Compare with $Q(0) = 1.3$ and the exact balance

$$Q(T) = Q(0) + \left[\frac{2^2}{2} - \frac{1^2}{2} \right] T = 1.6. \quad (10)$$

Use $x_s^{\text{eff}} = Q(T) - 1$ as an integral estimate of the shock position.

Questions: Which update satisfies the correct global balance? Does grid refinement remove the shock-position error of the nonconservative scheme? Why are backward spatial differences appropriate for the positive states used here?

Problem 3. Explicit Diffusion with a Mixed Derivative

Consider

$$u_t = \kappa(u_{xx} + 2\rho u_{xy} + u_{yy}), \quad \kappa = 0.01, \quad \rho = 0.5, \quad (11)$$

on the periodic square $0 \leq x, y < 1$, with

$$u(x, y, 0) = \cos[2\pi(x + y)]. \quad (12)$$

Tasks:

1. Derive and implement the centered mixed-derivative stencil

$$(u_{xy})_{j,k} \approx \frac{u_{j+1,k+1} - u_{j+1,k-1} - u_{j-1,k+1} + u_{j-1,k-1}}{4h^2}, \quad h = \Delta x = \Delta y. \quad (13)$$

2. Use centered differences for u_{xx} and u_{yy} , and advance all terms explicitly with forward Euler.
3. Use $N = 20, 40$, and 80 grid points in each direction, $h = 1/N$, $\Delta t = 0.1h^2/\kappa$, and final time $T = 0.5$.
4. Compare with

$$u(x, y, t) = e^{-8\pi^2\kappa(1+\rho)t} \cos[2\pi(x + y)]. \quad (14)$$

Plot the final field and estimate the convergence order.

5. At $N = 40$, repeat with $\rho = 0$ and compare the decay of the mode amplitude.

Questions: Which grid points enter the complete spatial stencil? Why does changing ρ change the decay rate? Why can an overall second-order convergence rate be observed when forward Euler is only first order in time?

Problem 4. Source Terms: Unsplit and Split Updates

Consider transport with a position-dependent loss term,

$$u_t + u_x = -xu, \quad 0 \leq x \leq 1, \quad (15)$$

with initial condition and inflow boundary condition

$$u(x, 0) = 1, \quad u(0, t) = e^{t^2/2}. \quad (16)$$

The boundary $x = 1$ is an outflow boundary: update its value using the same backward-difference formula as at interior nodes.

Tasks:

1. Implement the unsplit explicit update

$$u_j^{n+1} = u_j^n - C(u_j^n - u_{j-1}^n) - \Delta t x_j u_j^n, \quad C = \frac{\Delta t}{\Delta x}. \quad (17)$$

2. Implement the split update: first advance transport, then solve the local source equation exactly,

$$u_j^* = u_j^n - C(u_j^n - u_{j-1}^n), \quad (18)$$

$$u_j^{n+1} = e^{-x_j \Delta t} u_j^*. \quad (19)$$

3. Impose the inflow boundary value at each new time level. Use $N = 40, 80$, and 160 grid intervals, $\Delta x = 1/N$, $C = 0.8$, and final time $T = 0.4$.
4. Compare both methods with

$$u(x, t) = e^{-xt+t^2/2}. \quad (20)$$

Plot the final profiles and estimate their convergence orders.

Questions: Why do the split and unsplit updates differ at finite resolution? Does exact integration of the local source equation remove the error due to transport discretization? Is the source update implicit merely because it uses an analytical solution?

Problem 5. Dimensional Splitting

Consider

$$u_t + au_x + bu_y = 0, \quad a = 1, \quad b = \frac{1}{2}, \quad (21)$$

on the periodic square $0 \leq x, y < 1$, with

$$u(x, y, 0) = \sin(2\pi x) \sin(2\pi y). \quad (22)$$

Use equal spatial steps $h = \Delta x = \Delta y$ and define $C_x = a\Delta t/h$, $C_y = b\Delta t/h$.

Tasks:

1. Implement the unsplit upwind update

$$u_{j,k}^{n+1} = u_{j,k}^n - C_x(u_{j,k}^n - u_{j-1,k}^n) - C_y(u_{j,k}^n - u_{j,k-1}^n). \quad (23)$$

2. Implement sequential dimensional splitting, using the intermediate state in the second sweep:

$$u_{j,k}^* = u_{j,k}^n - C_x(u_{j,k}^n - u_{j-1,k}^n), \quad (24)$$

$$u_{j,k}^{n+1} = u_{j,k}^* - C_y(u_{j,k}^* - u_{j,k-1}^*). \quad (25)$$

3. Use $N = 20, 40$, and 80 grid points per direction, $h = 1/N$, $\Delta t = h/3$, and final time $T = 0.5$. Plot the final fields and estimate convergence orders using the exact solution

$$u(x, y, t) = \sin[2\pi(x - at)] \sin[2\pi(y - bt)]. \quad (26)$$

4. For a separate stability test, fix $N = 80$, set $\Delta t = 0.8h$, and add $10^{-6}(-1)^{j+k}$ to the sampled initial field. Compare the maximum absolute grid values of both methods up to $T = 0.5$.

Questions: How do the stability requirements of these two updates differ? Relate the results to $C_x + C_y$ and to the individual Courant numbers. Why is the split formula different from evaluating both directional contributions at the original state?

Problem 6. Method of Lines and Explicit Runge–Kutta

Consider

$$u_t + u_x = 0, \quad 0 \leq x < 1, \quad (27)$$

with periodic boundary conditions and $u(x, 0) = \sin(2\pi x)$. Discretize only the spatial derivative,

$$\frac{du_j}{dt} = L_j(\mathbf{u}), \quad L_j(\mathbf{u}) = -\frac{u_{j+1} - u_{j-1}}{2\Delta x}. \quad (28)$$

Tasks:

1. Integrate the resulting ODE system using forward Euler.
2. Integrate the same system using the classical fourth-order Runge–Kutta method. Recompute the spatial operator at every stage.
3. Use $N = 40, 80$, and 160 grid points, $\Delta x = 1/N$, $\Delta t = 0.8\Delta x$, and final time $T = 1$.
4. Compare the final profiles with

$$u(x, t) = \sin[2\pi(x - t)]. \quad (29)$$

For Runge–Kutta, calculate the maximum error and estimate the convergence order.

5. For a separate stability test at $N = 80$, add $10^{-6} \cos(\pi j/2)$ to the sampled initial values. For both time integrators, plot the maximum absolute grid value as a function of time.

Questions: Why can the same spatial discretization behave differently with different time integrators? Why does a fourth-order time integrator not generally give fourth-order convergence when centered second-order spatial differences are used?

Submission Requirements

For each problem, submit:

1. The program source code.
2. Plots of the requested profiles and stability diagnostics.
3. Error tables and convergence orders where requested.
4. Brief answers to the questions.