# Coherency, Coherence Scales, and Spectral Shapes

A Short Tutorial for Early BSc Students

## 1 Why coherence matters

Consider a wave-like signal

$$E(t) = A(t)e^{-i\omega_0 t}, \tag{1}$$

where $\omega_0$ is the characteristic oscillation frequency. If the signal were perfectly periodic forever,

$$E(t) = E_0 e^{-i\omega_0 t}, \tag{2}$$

then knowing its phase at one moment would determine its phase at any later time. This is perfect temporal coherence.

Real astrophysical signals are not perfectly periodic. Their amplitudes and phases fluctuate. We therefore need a quantity that measures how similar the signal is to itself after a time delay $\tau$.

## 2 Temporal coherency matrix

For a field with two components,

$$\boldsymbol{E}(t) = \begin{pmatrix} E_1(t) \\ E_2(t) \end{pmatrix}, \tag{3}$$

define the temporal coherency matrix

$$\boxed{C_{ij}(\tau) = \left\langle E_i(t)E_j^*(t+\tau)\right\rangle.} \tag{4}$$

Here $i, j = 1, 2$ label the components, $\tau$ is the time lag, $^*$ denotes complex conjugation, and $\langle\cdots\rangle$ denotes an average over time or over many realizations of the same stochastic process.

At zero lag,

$$C_{ij}(0) = \left\langle E_i(t)E_j^*(t)\right\rangle, \tag{5}$$

so the signal is being compared with itself at the same time.

For a scalar signal we may drop the indices:

$$\boxed{C(\tau) = \langle E(t)E^*(t+\tau)\rangle.} \tag{6}$$

This scalar form is sufficient for deriving the basic spectral shapes.

## 3 Normalized coherence function

Define the normalized coherence function

$$\boxed{g(\tau) = \frac{C(\tau)}{C(0)}.} \tag{7}$$

Therefore $g(0) = 1$. If $|g(\tau)| \approx 1$, the signal remains strongly coherent over the time lag $\tau$. If $|g(\tau)| \ll 1$, the signal has largely lost memory of its initial phase or amplitude.

For a narrow-band oscillation centered at $\omega_0$, it is useful to write

$$C(\tau) = C_0\, g_c(\tau) e^{-i\omega_0 \tau}, \tag{8}$$

where $g_c(\tau)$ describes only the loss of coherence. Thus we separate

$$\boxed{\text{oscillation: } e^{-i\omega_0\tau} \qquad \text{coherence loss: } g_c(\tau).} \tag{9}$$

# 4 Coherence time and characteristic variation scale

The characteristic coherence time is denoted by $\tau_c$. It measures the typical time over which the signal remains correlated with itself.

For example, if

$$g_c(\tau) = e^{-|\tau|/\tau_c}, \tag{10}$$

then at $|\tau| = \tau_c$,

$$g_c(\tau_c) = e^{-1} \approx 0.37. \tag{11}$$

Thus $\tau_c$ is a natural coherence scale.

More generally, if

$$g_c(\tau) = f\left(\frac{|\tau|}{\tau_c}\right), \tag{12}$$

then $\tau_c$ is the characteristic variation time of the coherence function. Locally, one may estimate

$$\boxed{\tau_c \sim \left|\frac{g_c}{dg_c/d\tau}\right|.} \tag{13}$$

The precise numerical factor depends on the chosen profile, but the physical meaning is simple: $\tau_c$ is the time over which the coherence changes by an amount of order unity.

# 5 Spectral coherency matrix

The spectral coherency matrix is the Fourier transform of the temporal coherency matrix:

$$\boxed{J_{ij}(\omega) = \int_{-\infty}^{+\infty} C_{ij}(\tau) e^{i\omega\tau}\, d\tau.} \tag{14}$$

The observable spectral intensity is obtained from the trace,

$$\boxed{I(\omega) = \mathrm{Tr}\, J(\omega).} \tag{15}$$

For a scalar signal,

$$I(\omega) \propto \int_{-\infty}^{+\infty} C(\tau) e^{i\omega\tau}\, d\tau. \tag{16}$$

Substituting

$$C(\tau) = C_0 g_c(\tau) e^{-i\omega_0\tau} \tag{17}$$

gives

$$I(\omega) \propto \int_{-\infty}^{+\infty} g_c(\tau) e^{i(\omega-\omega_0)\tau}\, d\tau. \tag{18}$$

Define

$$\Delta\omega \equiv \omega - \omega_0. \tag{19}$$

Then

$$\boxed{I(\omega) \propto \int_{-\infty}^{+\infty} g_c(\tau) e^{i\Delta\omega\tau}\, d\tau.} \tag{20}$$

This is the central relation of the tutorial:

$$\boxed{\text{temporal coherence law} \xrightarrow{\mathcal{F}} \text{spectral line shape.}} \tag{21}$$

## 6 Perfect coherence: delta-function line

If

$$g_c(\tau) = 1, \tag{22}$$

then

$$I(\omega) \propto \int_{-\infty}^{+\infty} e^{i\Delta\omega\tau}\, d\tau = 2\pi\delta(\Delta\omega). \tag{23}$$

Therefore

$$\boxed{I(\omega) \propto \delta(\omega - \omega_0).} \tag{24}$$

A perfectly coherent oscillation produces an infinitely narrow spectral line.

## 7 Exponential coherence: Lorentzian spectrum

Assume

$$\boxed{g_c(\tau) = e^{-|\tau|/\tau_c}.} \tag{25}$$

Then

$$I(\omega) \propto \int_{-\infty}^{+\infty} e^{-|\tau|/\tau_c} e^{i\Delta\omega\tau}\, d\tau. \tag{26}$$

Because the decay factor is even,

$$I(\omega) \propto 2 \int_{0}^{\infty} e^{-\tau/\tau_c} \cos(\Delta\omega\tau)\, d\tau. \tag{27}$$

Using

$$\int_{0}^{\infty} e^{-at} \cos(bt)\, dt = \frac{a}{a^2 + b^2}, \tag{28}$$

with

$$a = \frac{1}{\tau_c}, \qquad b = \Delta\omega, \tag{29}$$

we obtain

$$I(\omega) \propto \frac{2/\tau_c}{\tau_c^{-2} + \Delta\omega^2} = \frac{2\tau_c}{1 + \Delta\omega^2\tau_c^2}. \tag{30}$$

Hence

$$\boxed{I(\omega) \propto \frac{\gamma}{(\omega - \omega_0)^2 + \gamma^2}, \qquad \gamma = \frac{1}{\tau_c}.} \tag{31}$$

This is the Lorentzian spectral profile.

At half maximum,

$$1 + \Delta\omega^2\tau_c^2 = 2, \tag{32}$$

so

$$\boxed{\Delta\omega_{\mathrm{HWHM}} = \frac{1}{\tau_c}, \qquad \Delta\omega_{\mathrm{FWHM}} = \frac{2}{\tau_c}.} \tag{33}$$

In cyclic frequency $\nu = \omega/2\pi$,

$$\boxed{\Gamma_\nu = \frac{1}{\pi\tau_c}.} \tag{34}$$

Thus the measured spectral width is directly related to the coherence time.

### Lorentzian and Cauchy

The normalized Cauchy probability distribution is

$$p(\omega) = \frac{1}{\pi} \frac{\gamma}{(\omega - \omega_0)^2 + \gamma^2}. \tag{35}$$

Therefore the Lorentzian line shape and the Cauchy distribution are mathematically the same function. The names simply come from different contexts.

## 8 Gaussian coherence: Gaussian spectrum

Now assume

$$\boxed{g_c(\tau) = \exp\left[-\frac{\tau^2}{2\tau_G^2}\right].} \tag{36}$$

Then

$$I(\omega) \propto \int_{-\infty}^{+\infty} \exp\left[-\frac{\tau^2}{2\tau_G^2}\right] e^{i\Delta\omega\tau}\, d\tau. \tag{37}$$

The Fourier transform of a Gaussian is another Gaussian:

$$\boxed{I(\omega) \propto \tau_G \exp\left[-\frac{\Delta\omega^2\tau_G^2}{2}\right].} \tag{38}$$

Thus

$$\boxed{\text{Gaussian temporal coherence} \longrightarrow \text{Gaussian spectral profile.}} \tag{39}$$

Again the characteristic width is of order

$$\Delta\omega \sim \tau_G^{-1}. \tag{40}$$

## 9 Finite coherent duration: sinc-squared spectrum

Suppose the signal is perfectly coherent, but exists only for a finite time $T$:

$$E(t) = E_0 e^{-i\omega_0 t}, \qquad -\frac{T}{2} < t < \frac{T}{2}. \tag{41}$$

Its Fourier amplitude is

$$\widetilde{E}(\omega) = E_0 \int_{-T/2}^{T/2} e^{i\Delta\omega t}\, dt = E_0 T \frac{\sin(\Delta\omega T/2)}{\Delta\omega T/2}. \tag{42}$$

Therefore

$$\boxed{I(\omega) \propto \operatorname{sinc}^2\left(\frac{\Delta\omega T}{2}\right).} \tag{43}$$

Again

$$\Delta\omega \sim \frac{1}{T}. \tag{44}$$

# 10 Voigt profile: simultaneous Lorentzian and Gaussian broadening

A very important case occurs when two statistically independent broadening mechanisms act at the same time. One produces exponential loss of coherence, while the other produces Gaussian decorrelation. The total coherence function is then the product

$$\boxed{g_V(\tau) = \exp\left(-\frac{|\tau|}{\tau_L}\right)\exp\left(-\frac{\tau^2}{2\tau_G^2}\right).} \tag{45}$$

Here $\tau_L$ is the characteristic exponential coherence time and $\tau_G$ is the characteristic Gaussian coherence time.

The spectrum is

$$I_V(\omega) \propto \int_{-\infty}^{+\infty} \exp\left(-\frac{|\tau|}{\tau_L}\right)\exp\left(-\frac{\tau^2}{2\tau_G^2}\right) e^{i\Delta\omega\tau}\, d\tau. \tag{46}$$

The Fourier transform of a product in the time domain is a convolution in the frequency domain. Therefore

$$\boxed{I_V(\Delta\omega) \propto [L * G]\,(\Delta\omega),} \tag{47}$$

where

$$L(\Delta\omega) = \frac{1}{\pi}\frac{\gamma}{\Delta\omega^2+\gamma^2}, \qquad \gamma = \frac{1}{\tau_L}, \tag{48}$$

and

$$G(\Delta\omega) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{\Delta\omega^2}{2\sigma^2}\right), \qquad \sigma = \frac{1}{\tau_G}. \tag{49}$$

The convolution is

$$\boxed{V(\Delta\omega;\sigma,\gamma) = \int_{-\infty}^{+\infty} G(\Delta\omega')L(\Delta\omega-\Delta\omega')\, d\Delta\omega'.} \tag{50}$$

This is the Voigt profile.

Substituting the explicit functions gives

$$V(\Delta\omega;\sigma,\gamma) = \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{\Delta\omega'^2}{2\sigma^2}\right)\frac{1}{\pi}\frac{\gamma}{(\Delta\omega-\Delta\omega')^2+\gamma^2}\, d\Delta\omega'. \tag{51}$$

The integral is usually written using the Faddeeva function $w(z)$:

$$\boxed{V(\Delta\omega;\sigma,\gamma) = \frac{\mathrm{Re}\ w(z)}{\sigma\sqrt{2\pi}}, \qquad z = \frac{\Delta\omega+i\gamma}{\sigma\sqrt{2}}.} \tag{52}$$

For an introductory course, however, the convolution form is usually more informative than the special-function form.

The limiting cases are immediate:

$$\sigma \to 0 \quad \Rightarrow \quad V \to L, \tag{53}$$

$$\gamma \to 0 \quad \Rightarrow \quad V \to G. \tag{54}$$

Thus

$$\boxed{\text{Voigt} = \text{Gaussian broadening} * \text{Lorentzian broadening.}} \tag{55}$$

Physically, the central part of a Voigt line may look nearly Gaussian, while its far wings remain Lorentzian-like because the Lorentzian decreases only as $|\Delta\omega|^{-2}$.

### Approximate Voigt width

If $\Gamma_L$ is the Lorentzian FWHM and $\Gamma_G$ the Gaussian FWHM, a commonly used approximation for the Voigt FWHM is

$$\boxed{\Gamma_V \approx 0.5346\,\Gamma_L + \sqrt{0.2166\,\Gamma_L^2 + \Gamma_G^2}.} \tag{56}$$

For the coherence scales used above,

$$\Gamma_L = \frac{2}{\tau_L}, \tag{57}$$

and since a Gaussian $\exp\left[-\Delta\omega^2/(2\sigma^2)\right]$ has FWHM $2\sqrt{2\ln 2}\,\sigma$,

$$\Gamma_G = \frac{2\sqrt{2\ln 2}}{\tau_G}. \tag{58}$$

## 11 One general family of coherence laws

A useful extension is the stretched exponential

$$\boxed{g_c(\tau) = \exp\left[-\left(\frac{|\tau|}{\tau_c}\right)^{\alpha}\right].} \tag{59}$$

Two special cases are

$$\alpha = 1 \quad \Rightarrow \quad \text{Lorentzian spectrum,} \tag{60}$$
$$\alpha = 2 \quad \Rightarrow \quad \text{Gaussian spectrum.} \tag{61}$$

Intermediate values $1 < \alpha < 2$ describe intermediate forms of coherence loss and produce corresponding intermediate spectral shapes.

## 12 Universal relation between temporal and spectral scales

Suppose

$$g_c(\tau) = f\left(\frac{\tau}{\tau_c}\right). \tag{62}$$

Then

$$I(\omega) \propto \int_{-\infty}^{+\infty} f\left(\frac{\tau}{\tau_c}\right) e^{i\Delta\omega\tau}\, d\tau. \tag{63}$$

Introduce $x = \tau/\tau_c$. Since $d\tau = \tau_c\, dx$,

$$I(\omega) \propto \tau_c \int_{-\infty}^{+\infty} f(x) e^{i(\Delta\omega\tau_c)x}\, dx. \tag{64}$$

Therefore the spectrum depends on the dimensionless combination

$$\Delta\omega\tau_c. \tag{65}$$

A substantial spectral change occurs when

$$\boxed{|\Delta\omega|\tau_c \sim 1.} \tag{66}$$

Hence, quite generally,

$$\boxed{\Delta\omega_c \sim \frac{1}{\tau_c}.} \tag{67}$$

Long coherence time means a narrow spectral feature; short coherence time means a broad feature.

## 13 Return to the full coherency matrix

The scalar derivation generalizes directly to the matrix form. Write

$$C_{ij}(\tau) = C_{ij}^{(0)} g_c(\tau) e^{-i\omega_0 \tau}. \tag{68}$$

Then

$$J_{ij}(\omega) = C_{ij}^{(0)} \int_{-\infty}^{+\infty} g_c(\tau) e^{i\Delta\omega\tau}\, d\tau. \tag{69}$$

Hence

$$\boxed{J_{ij}(\omega) = C_{ij}^{(0)} F(\Delta\omega),} \tag{70}$$

where $F$ is the Fourier transform of the coherence function. The intensity is

$$\boxed{I(\omega) = \operatorname{Tr} J(\omega) = \operatorname{Tr} C^{(0)}\, F(\Delta\omega).} \tag{71}$$

Thus the matrix $C_{ij}^{(0)}$ contains the relative amplitudes and cross-correlations of the components, while the function $F(\Delta\omega)$ determines the spectral line shape.

## 14 From theoretical to observed spectral profiles

The profiles derived above describe an *intrinsic* spectrum, $I_{\text{true}}(\omega)$. Real observations contain measurement noise, finite observing time, and a background signal. It is therefore useful to distinguish the physical spectral shape from observational effects.

### 14.1 Measurement errors

Suppose the spectrum is measured at discrete frequencies $\omega_k$. A simple observational model is

$$\boxed{I_k = I_{\text{true}}(\omega_k) + \epsilon_k,} \tag{72}$$

where $\epsilon_k$ is the measurement error. At an introductory level we may assume independent Gaussian errors,

$$\boxed{\epsilon_k \sim \mathcal{N}(0, \sigma_k^2).} \tag{73}$$

Thus each measured spectral point can be represented as $I_k \pm \sigma_k$ and compared with a theoretical Lorentzian, Gaussian, or Voigt profile.

For example, a Lorentzian line with a constant background may be written as

$$I(\omega) = A \frac{\gamma^2}{(\omega - \omega_0)^2 + \gamma^2} + B, \tag{74}$$

where $A$ is the peak amplitude, $\omega_0$ the central frequency, $\gamma$ the half-width at half maximum, and $B$ the background level.

The fitted parameters have observational uncertainties. Since

$$\tau_c = \frac{1}{\gamma}, \tag{75}$$

the uncertainty of the inferred coherence time follows from ordinary error propagation:

$$\delta\tau_c = \left| \frac{d\tau_c}{d\gamma} \right| \delta\gamma = \frac{\delta\gamma}{\gamma^2}. \tag{76}$$

Therefore

$$\boxed{\frac{\delta\tau_c}{\tau_c} = \frac{\delta\gamma}{\gamma}.} \tag{77}$$

An uncertainty in the measured spectral width therefore becomes directly an uncertainty in the physical coherence time.

### 14.2 Finite observing time and the spectral window

Even a noise-free observation lasts only for a finite time. Let $W(t)$ denote the observational window. The measured signal is

$$E_{\text{obs}}(t) = W(t)E(t). \tag{78}$$

For a continuous observation of duration $T_{\text{obs}}$,

$$W(t) = \begin{cases} 1, & |t| < T_{\text{obs}}/2, \\ 0, & |t| > T_{\text{obs}}/2. \end{cases} \tag{79}$$

Multiplication in the time domain becomes convolution in frequency:

$$\boxed{\widetilde{E}_{\text{obs}}(\omega) = \widetilde{W}(\omega) * \widetilde{E}(\omega).} \tag{80}$$

The window therefore limits the frequency resolution. Its characteristic scale is

$$\boxed{\Delta\omega_{\text{obs}} \sim \frac{2\pi}{T_{\text{obs}}}, \qquad \Delta\nu_{\text{obs}} \sim \frac{1}{T_{\text{obs}}}.} \tag{81}$$

A physical line can be resolved only if its width is not much smaller than the frequency resolution. In cyclic frequency this condition is approximately

$$\boxed{\Gamma_{\text{physical}} \gtrsim \frac{1}{T_{\text{obs}}}.} \tag{82}$$

If $\Gamma_{\text{physical}} \ll 1/T_{\text{obs}}$, the intrinsic line is unresolved and a reliable coherence time cannot be obtained from the linewidth alone.

### 14.3 Observational noise is not Voigt broadening

It is important to distinguish random measurement errors from physical broadening. If an intrinsic exponential coherence law produces a Lorentzian line, adding random errors to the measured spectral points does *not* convert the line into a Voigt profile.

A Voigt profile appears when two broadening mechanisms act on the signal itself, one Lorentzian and one Gaussian, so that the intrinsic spectral profile is a convolution $L * G$. Measurement noise instead produces uncertainty around the underlying profile. The distinction is

$$\boxed{\begin{array}{lcl} \text{physical decoherence} & \longrightarrow & \text{intrinsic line shape,} \\ \text{finite observation} & \longrightarrow & \text{spectral window,} \\ \text{measurement noise} & \longrightarrow & \text{uncertain spectral points.} \end{array}} \tag{83}$$

### 14.4 Power-spectrum statistics for stellar oscillations

For real stellar oscillation spectra, one further statistical point is useful. A raw periodogram power is generally not Gaussian distributed around the model spectrum. For a stochastically excited mode, each unsmoothed power value $I_k$ is approximately exponentially distributed about the theoretical mean spectrum $S_k = S(\omega_k; \boldsymbol{\theta})$:

$$\boxed{p(I_k \mid S_k) = \frac{1}{S_k} \exp\left(-\frac{I_k}{S_k}\right).} \tag{84}$$

This is equivalent to a $\chi^2$ distribution with two degrees of freedom. For independent frequency bins, the likelihood is

$$\mathcal{L}(\boldsymbol{\theta}) = \prod_k \frac{1}{S_k} \exp\left(-\frac{I_k}{S_k}\right). \tag{85}$$

Taking the logarithm gives

$$\boxed{-\ln \mathcal{L} = \sum_k \left[ \ln S_k + \frac{I_k}{S_k} \right] + \text{const.}} \tag{86}$$

For a first BSc exercise, ordinary least-squares fitting with Gaussian errors is sufficient. The exponential likelihood above can then be introduced as the more realistic method for fitting an actual stellar power spectrum.

### 14.5 Physical interpretation of an observed linewidth

The complete chain from the underlying signal to the inferred physical timescale is therefore

$$\boxed{C_{ij}(\tau) \longrightarrow J_{ij}(\omega) \longrightarrow I_{\text{true}}(\omega) \longrightarrow \text{window + noise} \longrightarrow I_{\text{obs}}(\omega) \longrightarrow \tau_c \pm \delta\tau_c.} \tag{87}$$

A measured departure from a Lorentzian profile should therefore be interpreted physically only after checking whether it can instead be produced by limited frequency resolution, observational noise, or background uncertainty.

## 15 Summary table

| Temporal behavior | Spectral shape | Characteristic width |
|---|---|---|
| $g_c(\tau) = 1$ | $\delta(\omega - \omega_0)$ | 0 |
| $e^{-\|\tau\|/\tau_c}$ | Lorentzian/Cauchy | $\sim \tau_c^{-1}$ |
| $e^{-\tau^2/(2\tau_G^2)}$ | Gaussian | $\sim \tau_G^{-1}$ |
| finite duration $T$ | sinc$^2$ | $\sim T^{-1}$ |
| $e^{-\|\tau\|/\tau_L} e^{-\tau^2/(2\tau_G^2)}$ | Voigt | Lorentz + Gaussian |

The main physical message is therefore

$$\boxed{\text{coherence law in time} \longleftrightarrow \text{spectral morphology in frequency.}} \tag{88}$$