Viscorotational shear instability in astrophysical dense granular flows

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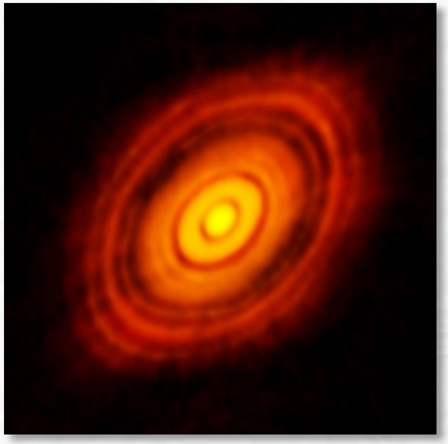
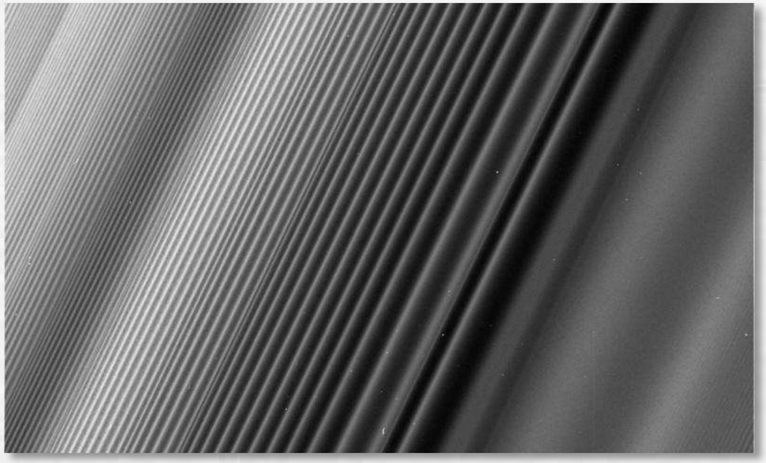
Granular Flows in Astrophysics



- **Planetary Rings**
- **Exo-Planetary Rings**
- **Debris Discs**
- **Protoplanetary Discs**

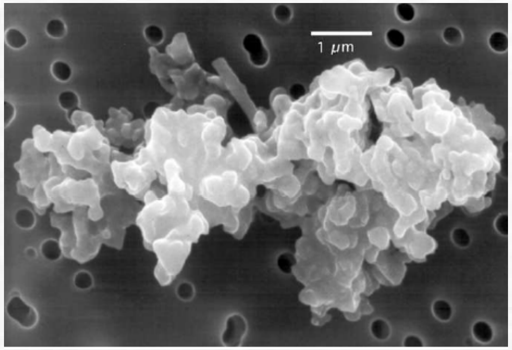
Differentially rotating “Keplerian flows”:

Particles rotate on their gravitational orbits $\Omega(r) \propto r^{-3/2}$



Granular pressure comparable with residual gravitational pressure

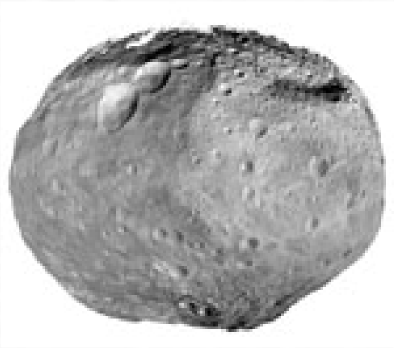
Planet Formation



Sizes of solid particles

1μ m → 10 km

dust→ planetesimal



Particle collisions: coagulation, scattering, fragmentation

Core accretion theory: ~1 meter barrier

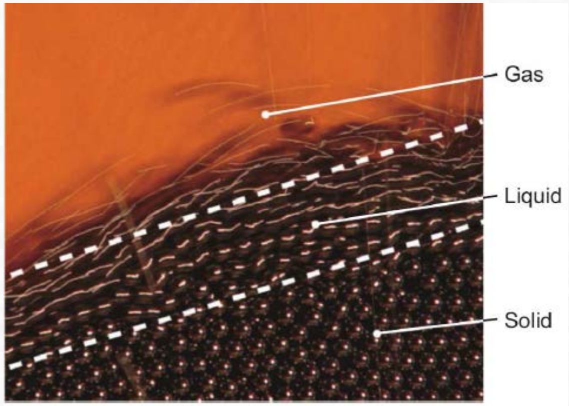
a) Fluid model: nonlinear dynamics of anticycl. vortices;

b) Fluid-Particle model: “streaming instability”;

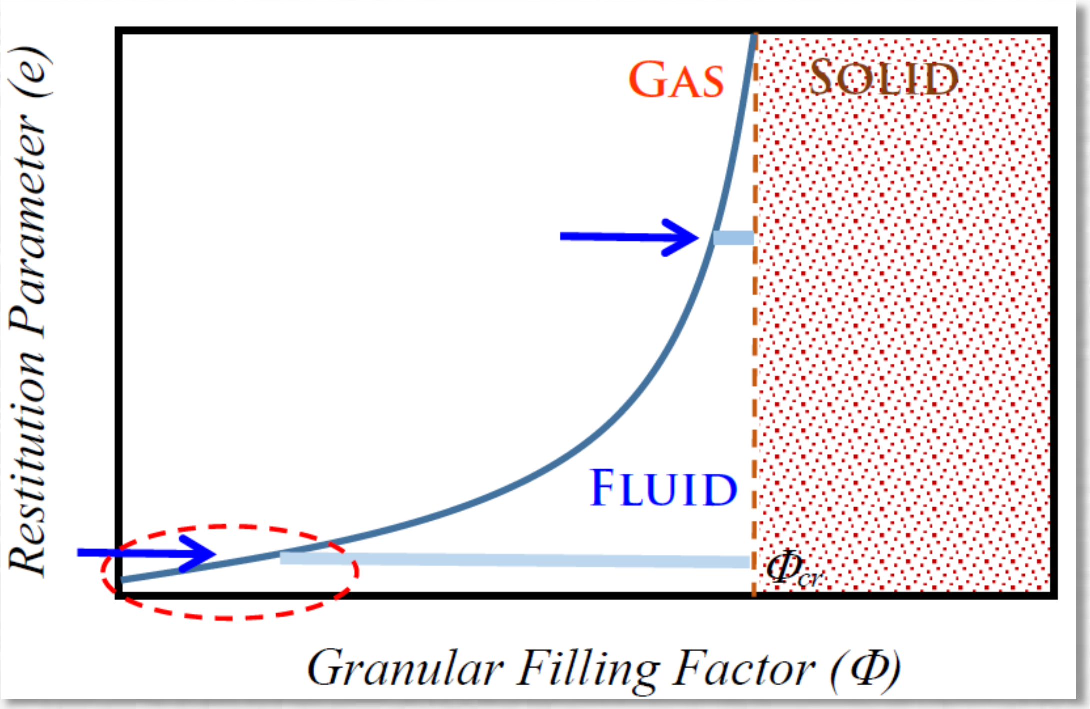
c) Particle-Particle collisions? granular fluid;

Granular Liquids

Granular flow behavior
Strongly depends on the
Particle restitution coefficient.



Granular liquid
of inelastic particles
can be observed for
a wide range of
particle filling factors



Granular Rheology
of disc flows

Viscous Stability

Turbulent (anomalous) viscosity α-model: $\nu_{eff} = \alpha hc$

Shakura Syunyaev 1973

$$\nu \propto \sigma^{\beta}$$

σ – surface density (opacity)

a) Viscous Instability: $\beta<-1$ *(incompressible)*

Lightman, Eardley 1974, Shakura Syunyaev 1976

Viscous stress is proportional to radiation pressure?

b) Viscous Overstability: $\beta>2$ *(compressible)*

Kato 1978, Blumenthal et al. 1984

Increase of viscous stress in compressed d-spiral wave;

Dense Granular Flow Model

$$\rho \left\{ \frac{\partial}{\partial t} + V_k \frac{\partial}{\partial x_k} \right\} V_i = -\frac{\partial P}{\partial x_i} + \rho \frac{\partial \Phi_0}{\partial x_i} + \frac{\partial \tau_{ik}}{\partial x_k}$$

- Incompressible flow;
- Constant Gravitational Potential;
- Granular Viscosity;
- Tensor formulation; *(Jop, Forterre & Pouliquen 2006)*

$$\frac{\partial V_k}{\partial x_k} = 0$$

$$\tau_{ij} = \eta \dot{\gamma}_{ik} \qquad \xi \equiv \sqrt{\dot{\gamma}_{ik}\dot{\gamma}_{ik}/2}$$

- Local Constitutive Law

$$\eta = \eta(P, \xi)$$

Dense Granular Flow Model

Equations of motion in cylindrical co-ordinates

$$\frac{\partial v_\phi}{\partial t} + \left(v_r \frac{\partial}{\partial r} + \frac{v_\phi}{r}\frac{\partial}{\partial \phi} + v_z \frac{\partial}{\partial z}\right) v_\phi + \frac{v_r v_\phi}{r} = -\frac{1}{\rho r}\frac{\partial P}{\partial \phi} + \frac{1}{r}\frac{\partial \Phi}{\partial \phi} +$$

$$+ \frac{\eta}{\rho}\left(\Delta v_\phi - \frac{v_\phi}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \phi}\right) + \frac{1}{\rho}\frac{\partial \eta}{\partial r}\left(\frac{1}{r}\frac{\partial v_r}{\partial \phi} + \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r}\right) +$$

$$+ \frac{2}{r}\frac{\partial \eta}{\partial \phi}\left(\frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r}\right) + \frac{1}{\rho}\frac{\partial \eta}{\partial z}\left(\frac{\partial v_\phi}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \phi}\right)$$

Linear Perturbations

Viscosity fluctuations:

$$\frac{\eta'}{\rho} = G_P \frac{P'}{\rho} + G_S \left(\frac{\partial V_x'}{\partial y} + \frac{\partial V_y'}{\partial x} \right)$$

Pressure rheology parameter:

$$G_P = \left(\frac{\partial \eta}{\partial P} \right)_\xi$$

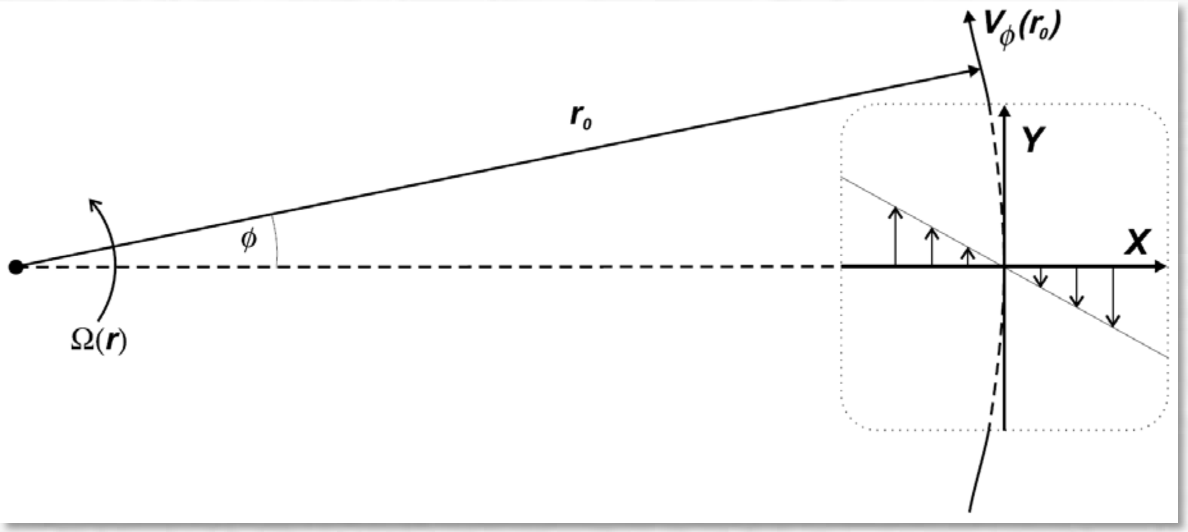
Shear rheology parameter:

$$G_S = \frac{1}{\rho} \left(\frac{\partial \eta}{\partial \xi} \right)_P$$

Linear Perturbation Analysis

Co-rotating cylindrical frame:

Shearing sheet (Kelvin modes)



Differential rotation → Planar shear flow

Shearing sheet modes

$$\psi'(t) \propto \exp\left(\mathrm{i}\mathbf{k}(t)\mathbf{r} - \mathrm{i}\omega t\right)$$

$$k_x(t) = k_0 - 2Ak_x t$$

$$\omega(t) = \omega(\mathbf{k}(t))$$

Linear Stability Analysis

Solutions of the dispersion equation:

$$\omega(\mathbf{k}) = \pm\left(\kappa^2 - W^2\right)^{1/2} + \mathrm{i}(W - \nu k^2)$$

Epicyclic frequency in Newtonian fluids:

$$\kappa_0^2 = \frac{2\Omega_{\mathrm{K}}(r)}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\Omega_{\mathrm{K}}(r)\right)$$

Rheological modification:

$$\kappa^2 = \kappa_0^2 + \Delta K^2(G_P, G_S)$$

and

$$W = \sigma_A + \sigma_P + \sigma_S$$

Linear Stability Analysis

Transient growth:

$$\sigma_A = \frac{Ak_x k_y}{k^2 - 4AG_P k_x k_y}$$

Pressure rheology effects:

$$\sigma_P = 2AG_P \frac{(\Omega k_x^2 + Bk_y^2)}{k^2 - 4AG_P k_x k_y}$$

Shear (strain) effects:

$$\sigma_S = -AG_S \frac{(k_x^2 - k_y^2)^2 + k_\perp^2 k_z^2}{k^2 - 4AG_P k_x k_y}$$

Epicyclic modification:

$$\Delta K^2(G_P, G_S) = -4G_S \frac{A^2 k_x k_y k_z^2}{k^2 - 4AG_P k_x k_y}$$

Instability

Linear instabilities:

Case 1: Pressure rheology:

$$G_P < 0 \ , \quad G_S = 0$$

"viscous instability" (cf. β<-1)

Case 2: Shear rheology:

$$G_P = 0 \ , \quad G_S > 0$$

$$W^2 > \kappa^2$$
$$W + \sqrt{W - \kappa^2} > \nu k^2$$

- Keplerian flow: $\xi(r_0) = -2A$
- Power law constitutive eq.: $\left(\frac{\ln \eta}{\ln \xi}\right)_P > 2$

Visco-rotational Shear Instability

- Vertically uniform perturbations: $k_z = 0$
- Shear rheology: $G_P = 0\ ,\quad G_S \neq 0$
- Arbitrary differential rotation: $\Omega(r) \propto r^{-q}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\ln(\mathrm{curl}\mathbf{v})_z\right) = qG_S\Omega\frac{(k_x(t)^2 - k_y^2)^2}{k(t)^2} - \nu k(t)^2$$

radial differential rotation → q

shear rheology → G_S

Angular velocity of rotation → Ω

1) *$q>0$, $Gs>0$; ($q_K=3/2$)*

2) *$q<0$, $Gs<0$;*

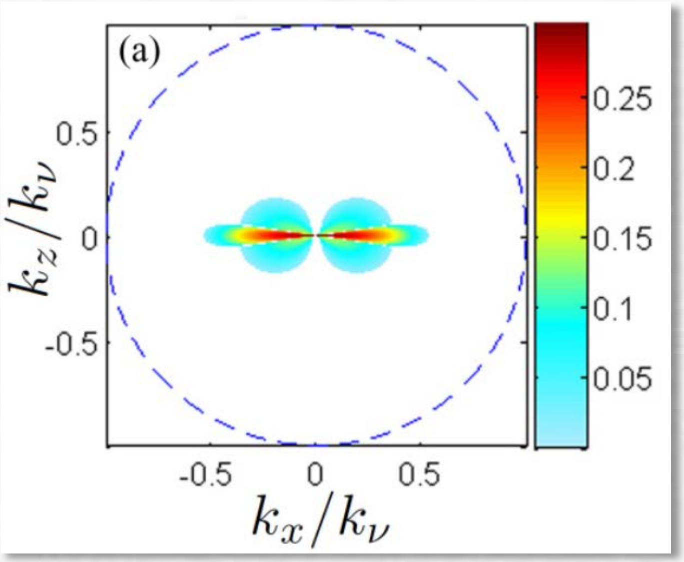
Instability Spectra

1) Pressure rheology instability:

$$G_P \neq 0\ ,\quad G_S = 0$$

Large scale

"Viscous instability"

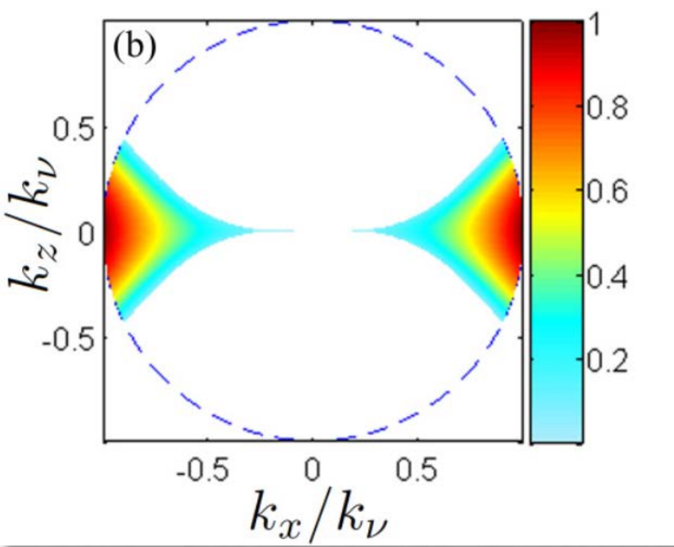


2) Shear rheology instability:

$$G_P = 0\ ,\quad G_S \neq 0$$

Small scale

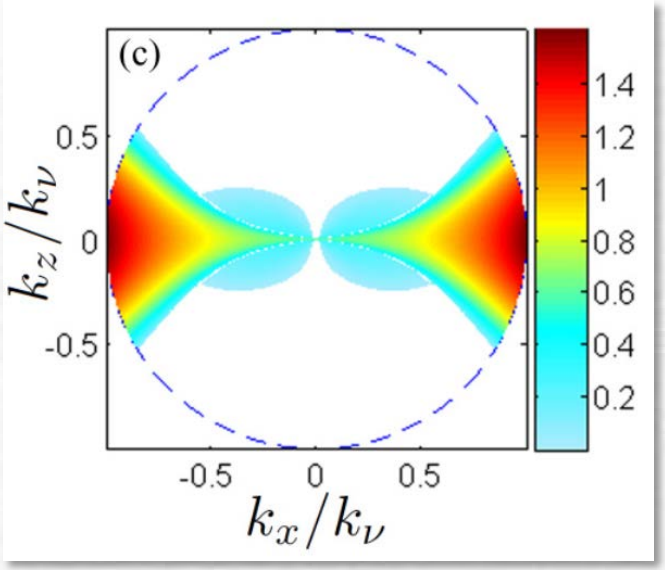
"Visco-rotational instability"



Instability Spectra

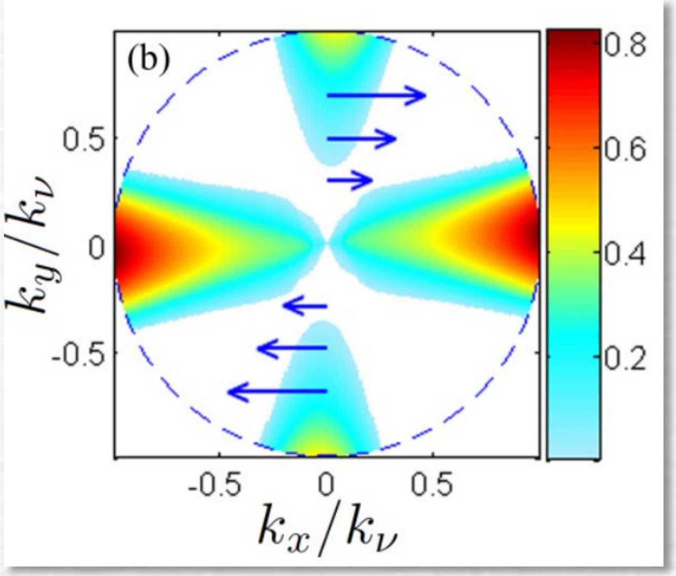
3) Instability of disk flow with pressure and shear rheologies:

$$G_P \neq 0 \ , \quad G_S \neq 0$$



4) Instability of non-axisymmetric Perturbations:

$$k_y \neq 0 \ , \quad k_x = k_x(t)$$



Summary

Shear and Pressure rheology effects on linear waves in Keplerian discs.

New type of linear instability:
"visco-rotational shear" instability

$$\left(\frac{\ln \eta}{\ln \xi}\right)_P > 2$$

- ✓ Incompressible, shear rheology,
- ✓ Epicyclic destabilization of perturbations;
- ✓ Exponential growth of the perturbation velocity curl in the shear plane;

- *Constitutive law dependence of particle sizes?*
- *Nonlinear development: numerical simulations?*

Thank You

Equilibrium Disc Flow

$$r\Omega^2 = -\frac{\partial \Phi}{\partial r},$$

$$\left(r\frac{\partial^2 \Omega}{\partial r^2} + 3\frac{\partial \Omega}{\partial r}\right)\bar{\eta} + r\frac{\partial \Omega}{\partial r}\frac{\partial \bar{\eta}}{\partial r} = 0,$$

$$\Phi(r,z) = \frac{GM}{(r^2+z^2)^{1/2}}$$

$$\Omega(r) = \Omega_0\left(\frac{r}{r_0}\right)^{-q}, \quad \Omega_0 = \left(\frac{GM}{r_0^3}\right)^{1/2}.$$

$$\frac{\partial \ln \bar{\eta}}{\partial \ln r} = q - 2.$$