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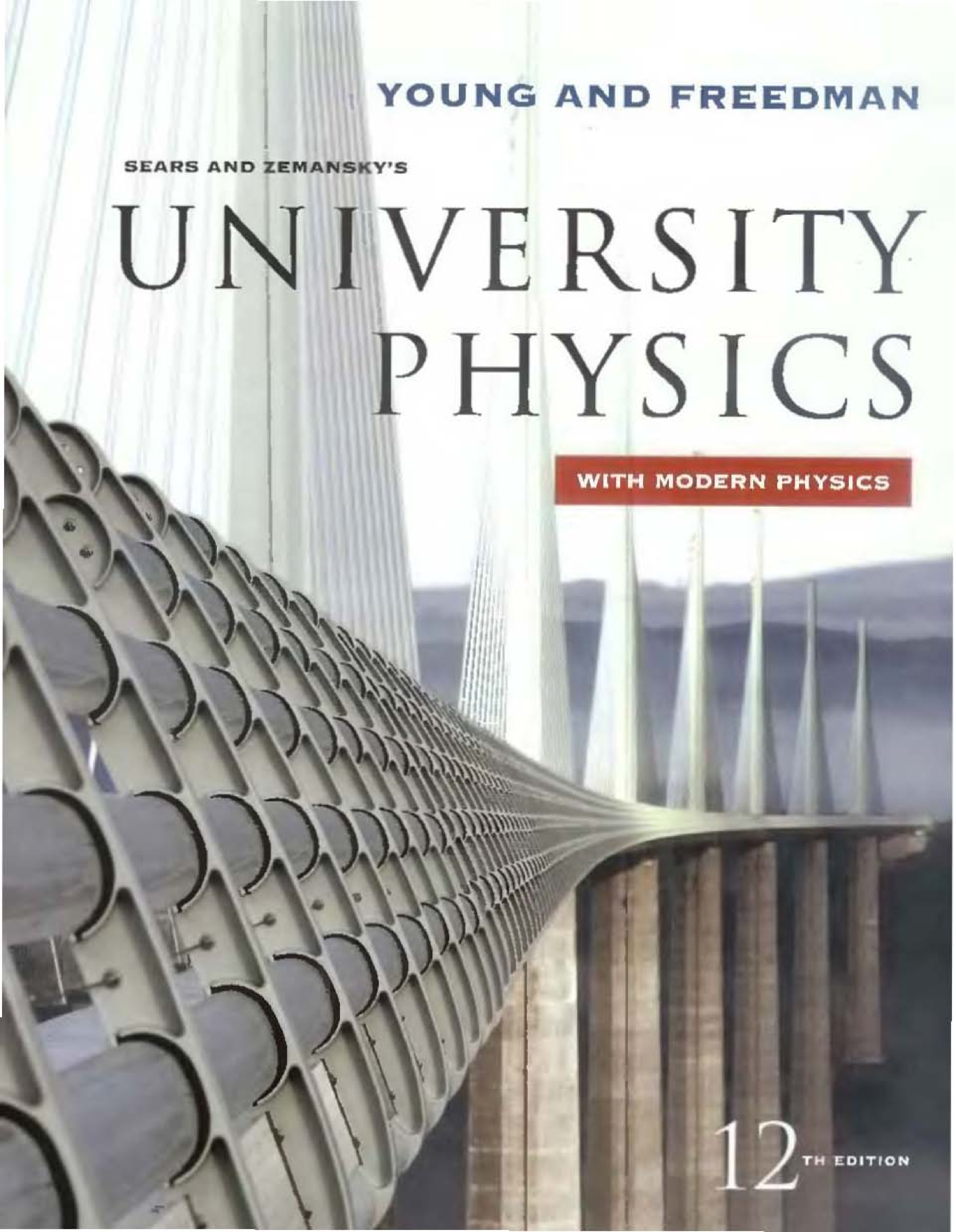
SEARS AND ZEMANSKY'S

UNIVERSITY PHYSICS

WITH MODERN PHYSICS

12TH EDITION



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12TH EDITION

WITH MODERN PHYSICS

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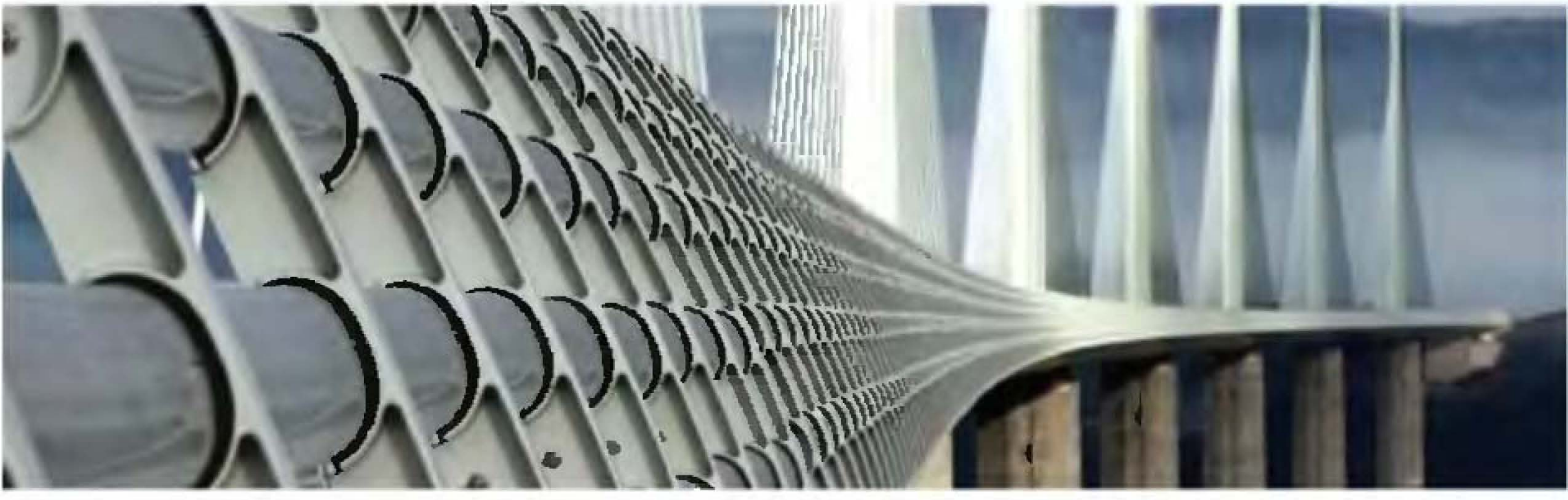

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BRIEF CONTENTS

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1.4 Predicting Motion from Equations
1.5 Problem-Solving Strategies for Kinematics
1.6 Skier Races Downhill
1.7 Balloonist Drops Lemonade
1.8 Seat Belts Save Lives
1.9 Screeching to a Halt
1.10 Pole-Vaulter Lands
1.11 Car Starts, Then Stops
1.12 Solving Two-Vehicle Problems
1.13 Car Catches Truck
1.14 Avoiding a Rear-End Collision
2.1.1 Force Magnitudes
2.1.2 Skydiver
2.1.3 Tension Change
2.1.4 Sliding on an Incline
2.1.5 Car Race
2.2 Lifting a Crate
2.3 Lowering a Crate
2.4 Rocket Blasts Off
2.5 Truck Pulls Crate
2.6 Pushing a Crate Up a Wall
2.7 Skier Goes Down a Slope
2.8 Skier and Rope Tow
2.9 Pole-Vaulter Vaults
2.10 Truck Pulls Two Crates
2.11 Modified Atwood Machine
3.1 Solving Projectile Motion Problems
3.2 Two Balls Falling
3.3 Changing the x-Velocity
3.4 Projectile x- and y-Accelerations
3.5 Initial Velocity Components
3.6 Target Practice I
3.7 Target Practice II
4.1 Magnitude of Centripetal Acceleration
4.2 Circular Motion Problem Solving
4.3 Cart Goes Over Circular Path
4.4 Ball Swings on a String
4.5 Car Circles a Track
4.6 Satellites Orbit
5.1 Work Calculations
5.2 Upward-Moving Elevator Stops
5.3 Stopping a Downward-Moving Elevator
5.4 Inverse Bungee Jumper
5.5 Spring-Launched Bowler
5.6 Skier Speed
5.7 Modified Atwood Machine
6.1 Momentum and Energy Change
6.2 Collisions and Elasticity
6.3 Momentum Conservation and Collisions
6.4 Collision Problems
6.5 Car Collision: Two Dimensions
6.6 Saving an Astronaut
6.7 Explosion Problems
6.8 Skier and Cart
6.9 Pendulum Bashes Box
6.10 Pendulum Person-Projectile Bowling
7.1 Calculating Torques
7.2 A Tilted Beam: Torques and Equilibrium
7.3 Arm Levers
7.4 Two Painters on a Beam
7.5 Lecturing from a Beam
7.6 Rotational Inertia
7.7 Rotational Kinematics
7.8 Rotoride–Dynamics Approach
7.9 Falling Ladder
7.10 Woman and Flywheel Elevator–Dynamics Approach
7.11 Race Between a Block and a Disk
7.12 Woman and Flywheel Elevator–Energy Approach
7.13 Rotoride–Energy Approach
7.14 Ball Hits Bat
8.1 Characteristics of a Gas
8.2 Maxwell-Boltzmann Distribution–Conceptual Analysis
8.3 Maxwell-Boltzmann Distribution–Quantitative Analysis
8.4 State Variables and Ideal Gas Law
8.5 Work Done By a Gas
8.6 Heat, Internal Energy, and First Law of Thermodynamics
8.7 Heat Capacity
8.8 Isochoric Process
8.9 Isobaric Process
8.10 Isothermal Process
8.11 Adiabatic Process
8.12 Cyclic Process–Strategies
8.13 Cyclic Process–Problems
8.14 Carnot Cycle
9.1 Position Graphs and Equations
9.2 Describing Vibrational Motion
9.3 Vibrational Energy
9.4 Two Ways to Weigh Young Tarzan
9.5 Ape Drops Tarzan
9.6 Releasing a Vibrating Skier I
9.7 Releasing a Vibrating Skier II
9.8 One-and Two-Spring Vibrating Systems
9.9 Vibro-Ride
9.10 Pendulum Frequency
9.11 Risky Pendulum Walk
9.12 Physical Pendulum
10.1 Properties of Mechanical Waves
10.2 Speed of Waves on a String
10.3 Speed of Sound in a Gas
10.4 Standing Waves on Strings
10.5 Tuning a Stringed Instrument: Standing Waves
10.6 String Mass and Standing Waves
10.7 Beats and Beat Frequency
10.8 Doppler Effect: Conceptual Introduction
10.9 Doppler Effect: Problems
10.10 Complex Waves: Fourier Analysis
11.1 Electric Force: Coulomb's Law
11.2 Electric Force: Superposition Principle
11.3 Electric Force Superposition Principle (Quantitative)
11.4 Electric Field: Point Charge
11.5 Electric Field Due to a Dipole
11.6 Electric Field: Problems
11.7 Electric Flux
11.8 Gauss's Law
11.9 Motion of a Charge in an Electric Field: Introduction
11.10 Motion in an Electric Field: Problems
11.11 Electric Potential: Qualitative Introduction
11.12 Electric Potential, Field, and Force
11.13 Electrical Potential Energy and Potential
12.1 DC Series Circuits (Qualitative)
12.2 DC Parallel Circuits
12.3 DC Circuit Puzzles
12.4 Using Ammeters and Voltmeters
12.5 Using Kirchhoff's Laws
12.6 Capacitance
12.7 Series and Parallel Capacitors
12.8 *RC* Circuit Time Constants
13.1 Magnetic Field of a Wire
13.2 Magnetic Field of a Loop
13.3 Magnetic Field of a Solenoid
13.4 Magnetic Force on a Particle
13.5 Magnetic Force on a Wire
13.6 Magnetic Torque on a Loop
13.7 Mass Spectrometer
13.8 Velocity Selector
13.9 Electromagnetic Induction
13.10 Motional emf
14.1 The *RL* Circuit
14.2 The *RLC* Oscillator
14.3 The Driven Oscillator
15.1 Reflection and Refraction
15.2 Total Internal Reflection
15.3 Refraction Applications
15.4 Plane Mirrors
15.5 Spherical Mirrors: Ray Diagrams
15.6 Spherical Mirror: The Mirror Equation
15.7 Spherical Mirror: Linear Magnification
15.8 Spherical Mirror: Problems
15.9 Thin-Lens Ray Diagrams
15.10 Converging Lens Problems
15.11 Diverging Lens Problems
15.12 Two-Lens Optical Systems
16.1 Two-Source Interference: Introduction
16.2 Two-Source Interference: Qualitative Questions
16.3 Two-Source Interference: Problems
16.4 The Grating: Introduction and Qualitative Questions
16.5 The Grating: Problems
16.6 Single-Slit Diffraction
16.7 Circular Hole Diffraction
16.8 Resolving Power
16.9 Polarization
17.1 Relativity of Time
17.2 Relativity of Length
17.3 Photoelectric Effect
17.4 Compton Scattering
17.5 Electron Interference
17.6 Uncertainty Principle
17.7 Wave Packets
18.1 The Bohr Model
18.2 Spectroscopy
18.3 The Laser
19.1 Particle Scattering
19.2 Nuclear Binding Energy
19.3 Fusion
19.4 Radioactivity
19.5 Particle Physics
20.1 Potential Energy Diagrams
20.2 Particle in a Box
20.3 Potential Wells
20.4 Potential Barriers

ABOUT THE AUTHORS

Hugh D. Young is Emeritus Professor of Physics at Carnegie Mellon University in Pittsburgh, PA. He attended Carnegie Mellon for both undergraduate and graduate study and earned his Ph.D. in fundamental particle theory under the direction of the late Richard Cutkosky. He joined the faculty of Carnegie Mellon in 1956 and has also spent two years as a Visiting Professor at the University of California at Berkeley.

Prof. Young's career has centered entirely around undergraduate education. He has written several undergraduate-level textbooks, and in 1973 he became a co-author with Francis Sears and Mark Zemansky for their well-known introductory texts. With their deaths, he assumed full responsibility for new editions of these books until joined by Prof. Freedman for *University Physics.*

Prof. Young is au enthusiastic skier, climber, and hiker. He also served for several years as Associate Organist at St. Paul's Cathedral in Pittsburgh, and has played numerous organ recitals in the Pittsburgh area. Prof. Young and his wife Alice usually travel extensively in the summer, especially in Europe and in the desert canyon country of southern Utah.

Roger A. Freedman is a Lecturer in Physics at the University of California, Santa Barbara. Dr. Freedman was an undergraduate at the University of California campuses in San Diego and Los Angeles, and did his doctoral research in nuclear theory at Stanford University under the direction of Professor J. Dirk Walecka. He came to UCSB in 1981 after three years teaching and doing research at the University of Washington.

At UCSB, Dr. Freedman has taught in both the Department of Physics and the College of Creative Studies, a branch of the university intended for highly gifted and motivated undergraduates. He has published research in nuclear physics, elementary particle physics, and laser physics. In recent years, he has helped to develop computer-based tools for learning introductory physics and astronomy.

When not in the classroom or slaving over a computer, Dr. Freedman can be found either flying (he holds a commercial pilot's license) or driving with his wife, Caroline, in their 1960 Nash Metropolitan convertible.

A. Lewis Ford is Professor of Physics at Texas A&M University. He received a B.A. from Rice University in 1968 and a Ph.D. in chemical physics from the University of Texas at Austin in 1972. After a one-year postdoc at Harvard University, he joined the Texas A&M physics faculty in 1973 and has been there ever since. Professor Ford's research area is theoretical atomic physics, with a specialization in atomic collisions. At Texas A&M he has taught a variety of undergraduate and graduate courses, but primarily introductory physics.

TO THE STUDENT

HOW TO SUCCEED IN PHYSICS BY REALLY TRYING

Mark Hollabaugh *Normandale Community College*

Physics encompasses the large and the small, the old and the new. From the atom to galaxies, from electrical circuitry to aerodynamics, physics is very much a part of the world around us. You probably are taking this introductory course in calculus-based physics because it is required for subsequent courses you plan to take in preparation for a career in science or engineering. Your professor wants you to learn physics and to enjoy the experience. He or she is very interested in helping you learn this fascinating subject. That is part of the reason your professor chose this textbook for your course. That is also the reason Drs. Young and Freedman asked me to write this introductory section. We want you to succeed!

The purpose of this section of *University Physics* is to give you some ideas that will assist your learning. Specific suggestions on how to use the textbook will follow a brief discussion of general study habits and strategies.

Preparation for This Course

If you had high school physics, you will probably learn concepts faster than those who have not because you will be familiar with the language of physics. If English is a second language for you, keep a glossary of new terms that you encounter and make sure you understand how they are used in physics. Likewise, if you are farther along in your mathematics courses, you will pick up the mathematical aspects of physics faster. Even if your mathematics is adequate, you may find a book such as Arnold D. Pickar's *Preparing for General Physics: Math Skill Drills and Other Useful Help (Calculus Version)* to be useful. Your professor may actually assign sections of this math review to assist your learning.

Learning to Learn

Each of us has a different learning style and a preferred means of learning. Understanding your own learning style will help you to focus on aspects of physics that may give you difficulty and to use those components of your course that will help you overcome the difficulty. Obviously you will want to spend more time on those aspects that give you the most trouble. If you learn by hearing, lectures will be very important. If you learn by explaining, then working with other students will be useful to you. If solving problems is difficult for you, spend more time learning how to solve problems. Also, it is important to understand and develop good study habits. Perhaps the most important thing you can do for yourself is to set aside adequate, regularly scheduled study time in a distraction-free environment.

Answer the following questions for yourself:

- Am I able to use fundamental mathematical concepts from algebra, geometry and trigonometry? (If not, plan a program of review with help from your professor.)
- In similar courses, what activity has given me the most trouble? (Spend more time on this.) What has been the easiest for me? (Do this first; it will help to build your confidence.)

- Do I understand the material better if I read the book before or after the lecture? (You may learn best by skimming the material, going to lecture, and then undertaking an in-depth reading.)
- Do I spend adequate time in studying physics? (A rule of thumb for a class like this is to devote, on the average, 2.5 hours out of class for each hour in class. For a course meeting 5 hours each week, that means you should spend about 10 to 15 hours per week studying physics.)
- Do I study physics every day? (Spread that 10 to 15 hours out over an entire week!) At what time of the day am I at my best for studying physics? (Pick a specific time of the day and stick to it.)
- Do I work in a quiet place where I can maintain my focus? (Distractions will break your routine and cause you to miss important points.)

Working with Others

Scientists or engineers seldom work in isolation from one another but rather work cooperatively. You will learn more physics and have more fun doing it if you work with other students. Some professors may formalize the use of cooperative learning or facilitate the formation of study groups. You may wish to form your own informal study group with members of your class who live in your neighborhood or dorm. If you have access to e-mail, use it to keep in touch with one another. Your study group is an excellent resource when reviewing for exams.

Lectures and Taking Notes

An important component of any college course is the lecture. In physics this is especially important because your professor will frequently do demonstrations of physical principles, run computer simulations, or show video clips. All of these are learning activities that will help you to understand the basic principles of physics. Don't miss lectures, and if for some reason you do, ask a friend or member of your study group to provide you with notes and let you know what happened.

Take your class notes in outline form, and fill in the details later. It can be very difficult to take word for word notes, so just write down key ideas. Your professor may use a diagram from the textbook. Leave a space in your notes and just add the diagram later. After class, edit your notes, filling in any gaps or omissions and noting things you need to study further. Make references to the textbook by page, equation number, or section number.

Make sure you ask questions in class, or see your professor during office hours. Remember the only "dumb" question is the one that is not asked. Your college may also have teaching assistants or peer tutors who are available to help you with difficulties you may have.

Examinations

Taking an examination is stressful. But if you feel adequately prepared and are well-rested, your stress will be lessened. Preparing for an exam is a continual process; it begins the moment the last exam is over. You should immediately go over the exam and understand any mistakes you made. If you worked a problem and made substantial errors, try this: Take a piece of paper and divide it down the middle with a line from top to bottom. In one column, write the proper solution to the problem. In the other column, write what you did and why, if you know, and why your solution was incorrect. If you are uncertain why you made your mistake, or how to avoid making it again, talk with your professor. Physics continually builds on fundamental ideas and it is important to correct any misunderstandings immediately. Warning: While cramming at the last minute may get you through the present exam, you will not adequately retain the concepts for use on the next exam.

TO THE INSTRUCTOR

PREFACE

This book is the product of more than half a century of leadership and innovation in physics education. When the first edition of University Physics by Francis W. Sears and Mark W. Zemansky was published in 1949, it was revolutionary among calculus-based physics textbooks in its emphasis on the fundamental principles of physics and how to apply them. The success of University Physics with generations of (several million) students and educators around the world is a testament to the merits of this approach, and to the many innovations it has introduced subsequently.

In preparing this new Twelfth Edition, we have further enhanced and developed *University Physics* to assimilate the best ideas from education research with enhanced problem-solving instruction, pioneering visual and conceptual pedagogy, the first systematically enhanced problems, and the most pedagogically proven and widely used online homework and tutorial system in the world.

New to This Edition

- **Problem solving.** The acclaimed, research-based **four-step problem-solving framework** (Identify, Set Up, Execute, and Evaluate) is now used throughout every Worked Example, chapter-specific Problem-Solving Strategy, and every Solution in the Instructor and Student Solutions Manuals. Worked Examples now incorporate black-and-white Pencil Sketches to focus students on this critical step—one that research shows students otherwise tend to skip when illustrated with highly rendered figures.

- **Instruction followed by practice.** A streamlined and systematic learning path of instruction followed by practice includes **Learning Goals** at the start of each chapter and **Visual Chapter Summaries** that consolidate each concept in words, math, and figures. Popular **Test Your Understanding** conceptual questions at the end of each section now use **multiple-choice and ranking formats** to allow students to instantly check their knowledge.

- **Instructional power of figures.** The instructional power of figures is enhanced using the research-proven technique of **"annotation"** (chalkboard-style commentary integrated into the figure to guide the student in interpreting the figure) and by **streamlined use of color and detail** (in mechanics, for example, color is used to focus the student on the object of interest while the rest of the image is in grayscale and without distracting detail).

- **Enhanced end-of-chapter problems.** Renowned for providing the most wide-ranging and best-tested problems available, the Twelfth Edition goes still further: It provides **the first library of physics problems systematically enhanced** based on student performance nationally. Using this analysis, more than 800 new problems make up the entire library of 3700.

- **MasteringPhysics™** (www.masteringphysics.com). Launched with the Eleventh Edition, MasteringPhysics is now the most widely adopted, educationally proven, and technically advanced online homework and tutorial system in the world. For the Twelfth Edition, MasteringPhysics provides a wealth of new content and technological enhancements. In addition to a library of more than 1200 tutorials and all the end-of-chapter problems, MasteringPhysics

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now also provides specific tutorials for every Problem-Solving Strategy and key Test Your Understanding questions from each chapter. Answer types include algebraic, numerical, and multiple-choice answers, as well as ranking, sorting, graph drawing, vector drawing, and ray tracing.

Key Features of *University Physics*

A Guide for the Student Many physics students experience difficulty simply because they don't know how to use their textbook. The section entitled "How to Succeed in Physics by Really Trying," which precedes this preface, is a "user's manual" to all the features of this book. This section, written by Professor Mark Hollabaugh (Normandale Community College), also gives a number of helpful study hints. *Every* student should read this section!

Chapter Organization The first section of each chapter is an *Introduction* that gives specific examples of the chapter's content and connects it with what has come before. There are also a *Chapter Opening Question* and a list of *Learning Goals* to make the reader think about the subject matter of the chapter ahead. (To find the answer to the question, look for the ? icon.) Most sections end with a *Test Your Understanding Question,* which can be conceptual or quantitative in nature. At the end of the last section of the chapter is a *Visual Chapter Summary* of the most important principles in the chapter, as well as a list of *Key Terms* with reference to the page number where each term is introduced. The answers to the Chapter Opening Question and Test Your Understanding Questions follow the Key Terms.

Questions and Problems At the end of each chapter is a collection of *Discussion Questions* that probe and extend the student's conceptual understanding. Following these are *Exercises,* which are single-concept problems keyed to specific sections of the text; *Problems,* usually requiring one or two nontrivial steps; and *Challenge Problems,* intended to challenge the strongest students. The problems include applications to such diverse fields as astrophysics, biology, and aerodynamics. Many problems have a conceptual part in which students must discuss and explain their results. The new questions, exercises, and problems for this edition were created and organized by Wayne Anderson (Sacramento City College), Laird Kramer (Florida International University), and Charlie Hibbard.

Problem-Solving Strategies and Worked Examples Throughout the book, *Problem-Solving Strategy* boxes provide students with specific tactics for solving particular types of problems. They address the needs of any students who have ever felt that they "understand the concepts but can't do the problems."

All Problem-Solving Strategy boxes follow the ISEE approach (Identify, Set Up, Execute, and Evaluate) to solving problems. This approach helps students see how to begin with a seemingly complex situation, identify the relevant physical concepts, decide what tools are needed to solve the problem, carry out the solution, and then evaluate whether the result makes sense.

Each Problem-Solving Strategy box is followed by one or more worked-out *Examples* that illustrate the strategy. Many other worked-out Examples are found in each chapter. Like the Problem-Solving Strategy boxes, all of the quantitative Examples use the ISEE approach. Several of the examples are purely qualitative and are labeled as *Conceptual Examples;* see, for instance, Conceptual Examples 6.5 (Comparing kinetic energies, p. 191), 8.1 (Momentum versus kinetic energy, p. 251) and 20.7 (A reversible adiabatic process, p. 693).

"Caution" paragraphs Two decades of physics education research have revealed a number of conceptual pitfalls that commonly plague beginning physics students. These include the ideas that force is required for motion, that electric current is "used up" as it goes around a circuit, and that the product of an object's mass and its acceleration is itself a force. The "Caution" paragraphs alert students to these and other pitfalls, and explain why the wrong way to think about a certain situation (which may have occurred to the student first) is indeed wrong. (See, for example, pp. 118, 159, and 559.)

Notation and units Students often have a hard time keeping track of which quantities are vectors and which are not. We use boldface italic symbols with an arrow on top for vector quantities, such as $\vec{\boldsymbol{v}}$, $\vec{\boldsymbol{a}}$, and $\vec{\boldsymbol{F}}$; unit vectors such as $\hat{\boldsymbol{\imath}}$, have a caret on top. Boldface $\mathbf{+}$, $\mathbf{-}$, $\boldsymbol{\times}$, and $\mathbf{=}$ signs are used in vector equations to emphasize the distinction between vector and scalar mathematical operations.

SI units are used exclusively (English unit conversions are included where appropriate). The joule is used as the standard unit of energy of all forms, including heat.

Flexibility The book is adaptable to a wide variety of course outlines. There is plenty of material for a three-semester or a five-quarter course. Most instructors will find that there is too much material for a one-year course, but it is easy to tailor the book to a variety of one-year course plans by omitting certain chapters or sections. For example, any or all of the chapters on fluid mechanics, sound and hearing, electromagnetic waves, or relativity can be omitted without loss of continuity. In any case, no instructor should feel constrained to work straight through the entire book.

Instructor Supplements

The **Instructor Solutions Manuals,** prepared by A. Lewis Ford (Texas A&M University), contain complete and detailed solutions to all end-of-chapter problems. All solutions follow consistently the same Identify/Set Up/Execute/Evaluate problem-solving framework used in the textbook. The *Instructor Solutions Manual for Volume 1* (ISBN 0-321-49968-9) covers Chapters 1–20, and the *Instructor Solutions Manual for Volumes 2 and 3* (ISBN 0-321-49210-2) covers Chapters 21–44.

The cross-platform **Media Manager CD-ROM** (ISBN 0-321-49916-6) provides a comprehensive library of more than 220 applets from ActivPhysics OnLine™ as well as all line figures from the textbook in JPEG format. In addition, all the key equations, Problem-Solving Strategies, tables, and chapter summaries are provided in editable Word format. In-class weekly multiple-choice questions for use with various Classroom Response Systems (CRS) are also provided, based on the Test Your Understanding questions in the text. The CD-ROM also provides the Instructor Solutions Manual in convenient editable Word format and as PDFs.

MasteringPhysics™ (www.masteringphysics.com) is the most advanced, educationally effective, and widely used physics homework and tutorial system in the world. It provides instructors with a library of extensively pretested end-of-chapter problems and rich, Socratic tutorials that incorporate a wide variety of answer types, wrong-answer feedback, and adaptive help (comprising hints or simpler sub-problems upon request). MasteringPhysics™ allows instructors to quickly build wide-ranging homework assignments of just the right difficulty and duration and provides them with efficient tools to analyze class trends—or the work of any student—in unprecedented detail and to compare the results either with the national average or with the performance of previous classes.

Five Easy Lessons: Strategies for Successful Physics Teaching (ISBN 0-8053-8702-1) by Randall D. Knight (California Polytechnic State University, San Luis Obispo) is packed with creative ideas on how to enhance any physics course. It is an invaluable companion for both novice and veteran physics instructors.

The **Transparency Acetates** (ISBN 0-321-50034-2) contain more than 200 key figures from *University Physics,* Twelfth Edition, in full color.

The **Printed Test Bank** (ISBN 0-321-50035-0) provides more than 2000 multiple-choice questions.

The **Computerized Test Bank** (ISBN 0-321-50126-8) includes all of the questions from the Printed Test Bank on a cross-platform CD-ROM. More than half the questions have numerical values that can be randomly assigned for each student.

Student Supplements

The **Study Guide,** by James R. Gaines, William F. Palmer, and Laird Kramer, reinforces the text's emphasis on problem-solving strategies and student misconceptions. The *Study Guide for Volume 1* (ISBN 0-321-50033-4) covers Chapters 1–20, and the *Study Guide for Volumes 2 and 3* (ISBN 0-321-50037-7) covers Chapters 21–44.

The **Student Solutions Manual, by** A. Lewis Ford (Texas A&M University), contains detailed, step-by-step solutions to more than half of the odd-numbered end-of-chapter problems from the textbook. All solutions follow consistently the same Identify/Set Up/Execute/Evaluate problem-solving framework used in the textbook. The *Student Solutions Manual for Volume 1* (ISBN 0-321-50063-6) covers Chapters 1–20, and the *Student Solutions Manual for Volumes 2 and 3* (ISBN 0-321-50038-5) covers Chapters 21–44.

MasteringPhysics™ (www.masteringphysics.com) is the most advanced, widely used, and educationally proven physics tutorial system in the world. It is the result of eight years of detailed studies of how real students work physics problems, and of precisely where they need help. Studies show that students who use MasteringPhysics™ significantly improve their scores on final exams and conceptual tests such as the Force Concept Inventory. MasteringPhysics™ achieves this by providing students with instantaneous feedback specific to their wrong answers, simpler sub-problems upon request when they get stuck, and partial credit for their method. This individualized, 24/7 tutor system is recommended by nine out of ten students to their peers as the most effective and time-efficient way to study.

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Please Tell Us What You Think!

We welcome communications from students and professors, especially concerning errors or deficiencies that you find in this edition. We have devoted a lot of time and effort to writing the best book we know how to write, and we hope it will help you to teach and learn physics. In turn, you can help us by letting us know what still needs to be improved! Please feel free to contact us either electronically or by ordinary mail. Your comments will be greatly appreciated.

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DETAILED CONTENTS

MECHANICS

WAVES/ACOUSTICS

THERMODYNAMICS

ELECTROMAGNETISM

OPTICS

MODERN PHYSICS

UNITS, PHYSICAL QUANTITIES, AND VECTORS

1

? Being able to predict the path of a hurricane is essential for minimizing the damage it does to lives and property. If a hurricane is moving at 20 km/h in a direction 53° north of east, how far north does the hurricane move in one h?

LEARNING GOALS

By studying this chapter, you will learn:

- What the fundamental quantities of mechanics are, and the units physicists use to measure them.
- How to keep track of significant figures in your calculations.
- The difference between scalars and vectors, and how to add and subtract vectors graphically.
- What the components of a vector are, and how to use them in calculations.
- What unit vectors are, and how to use them with components to describe vectors.
- Two ways of multiplying vectors.

The study of physics is important because physics is one of the most fundamental of the sciences. Scientists of all disciplines make use of the ideas of physics, including chemists who study the structure of molecules, paleontologists who try to reconstruct how dinosaurs walked, and climatologists who study how human activities affect the atmosphere and oceans. Physics is also the foundation of all engineering and technology. No engineer could design a flat-screen TV, an interplanetary spacecraft, or even a better mousetrap without first understanding the basic laws of physics.

The study of physics is also an adventure. You will find it challenging, sometimes frustrating, occasionally painful, and often richly rewarding and satisfying. It will appeal to your sense of beauty as well as to your rational intelligence. If you've ever wondered why the sky is blue, how radio waves can travel through empty space, or how a satellite stays in orbit, you can find the answers by using fundamental physics. Above all, you will come to see physics as a towering achievement of the human intellect in its quest to understand our world and ourselves.

In this opening chapter, we'll go over some important preliminaries that we'll need throughout our study. We'll discuss the nature of physical theory and the use of idealized models to represent physical systems. We'll introduce the systems of units used to describe physical quantities and discuss ways to describe the accuracy of a number. We'll look at examples of problems for which we can't (or don't want to) find a precise answer, but for which rough estimates can be useful and interesting. Finally, we'll study several aspects of vectors and vector algebra. Vectors will be needed throughout our study of physics to describe and analyze physical quantities, such as velocity and force, that have direction as well as magnitude.

1.1 The Nature of Physics

Physics is an *experimental* science. Physicists observe the phenomena of nature and try to find patterns and principles that relate these phenomena. These patterns are called physical theories or, when they are very well established and of broad use, physical laws or principles.

CAUTION **The meaning of the word "theory"** Calling an idea a theory does *not* mean that it's just a random thought or an unproven concept. Rather, a theory is an explanation of natural phenomena based on observation and accepted fundamental principles. An example is the well-established theory of biological evolution, which is the result of extensive research and observation by generations of biologists.

The development of physical theory requires creativity at every stage. The physicist has to learn to ask appropriate questions, design experiments to try to answer the questions, and draw appropriate conclusions from the results. Figure 1.1 shows two famous experimental facilities.

1.1 Two research laboratories. **(a)** According to legend, Galileo investigated falling bodies by dropping them from the Leaning Tower in Pisa, Italy, and he studied pendulum motion by observing the swinging of the chandelier in the adjacent cathedral. **(b)** The Hubble Space Telescope is the first major telescope to operate outside the earth's atmosphere. Measurements made with this telescope have helped determine the age and expansion rate of the universe.

(a)

(b)

Legend has it that Galileo Galilei (1564–1642) dropped light and heavy objects from the top of the Leaning Tower of Pisa (Fig. 1.1a) to find out whether their rates of fall were the same or different. Galileo recognized that only experimental investigation could answer this question. From examining the results of his experiments (which were actually much more sophisticated than in the legend), he made the inductive leap to the principle, or theory, that the acceleration of a falling body is independent of its weight.

The development of physical theories such as Galileo's is always a two-way process that starts and ends with observations or experiments. This development often takes an indirect path, with blind alleys, wrong guesses, and the discarding of unsuccessful theories in favor of more promising ones. Physics is not simply a collection of facts and principles; it is also the *process* by which we arrive at general principles that describe how the physical universe behaves.

No theory is ever regarded as the final or ultimate truth. The possibility always exists that new observations will require that a theory be revised or discarded. It is in the nature of physical theory that we can disprove a theory by finding behavior that is inconsistent with it, but we can never prove that a theory is always correct.

Getting back to Galileo, suppose we drop a feather and a cannonball. They certainly do *not* fall at the same rate. This does not mean that Galileo was wrong; it means that his theory was incomplete. If we drop the feather and the cannonball *in a vacuum* to eliminate the effects of the air, then they do fall at the same rate. Galileo's theory has a **range of validity:** It applies only to objects for which the force exerted by the air (due to air resistance and buoyancy) is much less than the weight. Objects like feathers or parachutes are clearly outside this range.

Every physical theory has a range of validity outside of which it is not applicable. Often a new development in physics extends a principle's range of validity. Galileo's analysis of falling bodies was greatly extended half a century later by Newton's laws of motion and law of gravitation.

1.2 Solving Physics Problems

At some point in their studies, almost all physics students find themselves thinking, "I understand the concepts, but I just can't solve the problems." But in physics, truly understanding a concept or principle is the same thing as being able to apply it to a variety of practical problems. Learning how to solve problems is absolutely essential; you don't *know* physics unless you can *do* physics.

How do you learn to solve physics problems? In every chapter of this book you will find *Problem-Solving Strategies* that offer techniques for setting up and solving problems efficiently and accurately. Following each ***Problem-Solving Strategy*** are one or more worked *Examples* that show these techniques in action.

(The *Problem-Solving Strategies* will also steer you away from some *incorrect* techniques that you may be tempted to use.) You'll also find additional examples that aren't associated with a particular *Problem-Solving Strategy.* Study these strategies and examples carefully, and work through each example for yourself on a piece of paper.

Different techniques are useful for solving different kinds of physics problems, which is why this book offers dozens of *Problem-Solving Strategies.* No matter what kind of problem you're dealing with, however, there are certain key steps that you'll always follow. (These same steps are equally useful for problems in math, engineering, chemistry, and many other fields.) In this book we've organized these steps into four stages of solving a problem.

All of the *Problem-Solving Strategies* and *Examples* in this book will follow these four steps. (In some cases we will combine the first two or three steps.) We encourage you to follow these same steps when you solve problems yourself. You may find it useful to remember the acronym ***I SEE***—short for *Identify, Set up, Execute,* and *Evaluate.*

Problem-Solving Strategy 1.1 Solving Physics Problems

IDENTIFY *the relevant concepts:* First, decide which physics ideas are relevant to the problem. Although this step doesn't involve any calculations, it's sometimes the most challenging part of solving the problem. Don't skip over this step, though; choosing the wrong approach at the beginning can make the problem more difficult than it has to be, or even lead you to an incorrect answer.

At this stage you must also identify the **target variable** of the problem—that is, is the quantity whose value you're trying to find. It could be the speed at which a projectile hits the ground, the intensity of a sound made by a siren, or the size of an image made by a lens. (Sometimes the goal will be to find a mathematical expression rather than a numerical value. Sometimes, too, the problem will have more than one target variable.) The target variable is the goal of the problem-solving process; don't lose sight of this goal as you work through the solution.

SET UP *the problem:* Based on the concepts you selected in the *Identify* step, choose the equations that you'll use to solve the problem and decide how you'll use them. If appropriate, draw a sketch of the situation described in the problem.

EXECUTE *the solution:* In this step, you "do the math." Before you launch into a flurry of calculations, make a list of all known and unknown quantities, and note which are the target variable or variables. Then solve the equations for the unknowns.

EVALUATE *your answer:* The goal of physics problem solving isn't just to get a number or a formula; it's to achieve better understanding. That means you must examine your answer to see what it's telling you. Be sure to ask yourself, "Does this answer make sense?" If your target variable was the radius of the earth and your answer is 6.38 centimeters (or if your answer is a negative number!), something went wrong in your problem-solving process. Go back and check your work, and revise your solution as necessary.

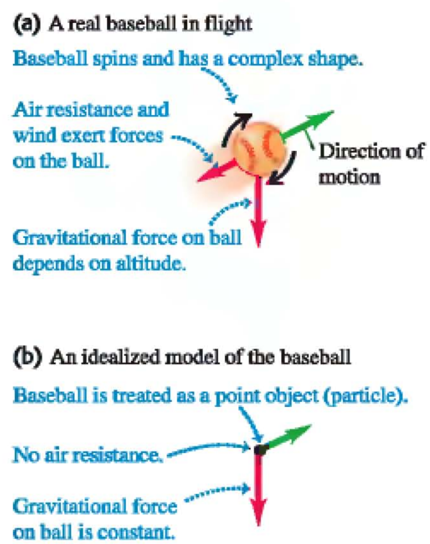
Idealized Models

In everyday conversation we use the word "model" to mean either a small-scale replica, such as a model railroad, or a person who displays articles of clothing (or the absence thereof). In physics a **model** is a simplified version of a physical system that would be too complicated to analyze in full detail.

For example, suppose we want to analyze the motion of a thrown baseball (Fig. 1.2a). How complicated is this problem? The ball is not a perfect sphere (it has raised seams), and it spins as it moves through the air. Wind and air resistance influence its motion, the ball's weight varies a little as its distance from the center of the earth changes, and so on. If we try to include all these things, the analysis gets hopelessly complicated. Instead, we invent a simplified version of the problem. We neglect the size and shape of the ball by representing it as a point object, or **particle.** We neglect air resistance by making the ball move in a vacuum, and we make the weight constant. Now we have a problem that is simple enough to deal with (Fig. 1.2b). We will analyze this model in detail in Chapter 3.

To make an idealized model, we have to overlook quite a few minor effects to concentrate on the most important features of the system. Of course, we have to be careful not to neglect too much. If we ignore the effects of gravity completely,

1.2 To simplify the analysis of **(a)** a baseball in flight, we use **(b)** an idealized model.



then our model predicts that when we throw the ball up, it will go in a straight line and disappear into space. We need to use some judgment and creativity to construct a model that simplifies a problem enough to make it manageable, yet keeps its essential features.

When we use a model to predict how a system will behave, the validity of our predictions is limited by the validity of the model. For example, Galileo's prediction about falling bodies (see Section 1.1) corresponds to an idealized model that does not include the effects of air resistance. This model works fairly well for a dropped cannonball, but not so well for a feather.

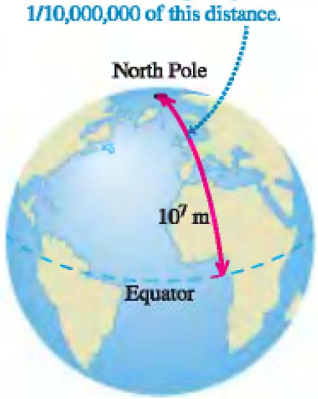
When we apply physical principles to complex systems in physical science and technology, we always use idealized models, and we have to be aware of the assumptions we are making. In fact, the principles of physics themselves are stated in terms of idealized models; we speak about point masses, rigid bodies, ideal insulators, and so on. Idealized models play a crucial role throughout this book. Watch for them in discussions of physical theories and their applications to specific problems.

1.3 Standards and Units

As we learned in Section 1.1, physics is an experimental science. Experiments require measurements, and we generally use numbers to describe the results of measurements. Any number that is used to describe a physical phenomenon quantitatively is called a **physical quantity.** For example, two physical quantities that describe you are your weight and your height. Some physical quantities are so fundamental that we can define them only by describing how to measure them. Such a definition is called an **operational definition.** Two examples are measuring a distance by using a ruler and measuring a time interval by using a stopwatch. In other cases we define a physical quantity by describing how to calculate it from other quantities that we *can* measure. Thus we might define the average speed of a moving object as the distance traveled (measured with a ruler) divided by the time of travel (measured with a stopwatch).

When we measure a quantity, we always compare it with some reference standard. When we say that a Porsche Carrera GT is 4.61 meters long, we mean that it is 4.61 times as long as a meter stick, which we define to be 1 meter long. Such a standard defines a **unit** of the quantity. The meter is a unit of distance, and the second is a unit of time. When we use a number to describe a physical quantity, we must always specify the unit that we are using; to describe a distance as simply "4.61" wouldn't mean anything.

1.3 In 1791 the distance from the North Pole to the equator was defined to be exactly 10^7 m. With the modern definition of the meter, this distance is about 0.02% more than 10^7 m.



To make accurate, reliable measurements, we need units of measurement that do not change and that can be duplicated by observers in various locations. The system of units used by scientists and engineers around the world is commonly called "the metric system," but since 1960 it has been known officially as the **International System,** or **SI** (the abbreviation for its French name, *Système International*). A list of all SI units is given in Appendix A, as are definitions of the most fundamental units.

The definitions of the basic units of the metric system have evolved over the years. When the metric system was established in 1791 by the French Academy of Sciences, the meter was defined as one ten-millionth of the distance from the North Pole to the equator (Fig. 1.3). The second was defined as the time required for a pendulum one meter long to swing from one side to the other. These definitions were cumbersome and hard to duplicate precisely, and by international agreement they have been replaced with more refined definitions.

Time

From 1889 until 1967, the unit of time was defined as a certain fraction of the mean solar day, the average time between successive arrivals of the sun at its highest point in the sky. The present standard, adopted in 1967, is much more precise. It is based on an atomic clock, which uses the energy difference between the two lowest energy states of the cesium atom. When bombarded by microwaves of precisely the proper frequency, cesium atoms undergo a transition from one of these states to the other. One **second** (abbreviated s) is defined as the time required for 9,192,631,770 cycles of this microwave radiation.

Length

In 1960 an atomic standard for the meter was also established, using the wavelength of the orange-red light emitted by atoms of krypton (^{86}Kr) in a glow discharge tube. Using this length standard, the speed of light in a vacuum was measured to be 299,792,458 m/s. In November 1983, the length standard was changed again so that the speed of light in a vacuum was *defined* to be precisely 299,792,458 m/s. The meter is defined to be consistent with this number and with the above definition of the second. Hence the new definition of the **meter** (abbreviated m) is the distance that light travels in a vacuum in 1/299,792,458 second. This provides a much more precise standard of length than the one based on a wavelength of light.

Mass

The standard of mass, the **kilogram** (abbreviated kg), is defined to be the mass of a particular cylinder of platinum–iridium alloy kept at the International Bureau of Weights and Measures at Sèvres, near Paris (Fig. 1.4). An atomic standard of mass would be more fundamental, but at present we cannot measure masses on an atomic scale with as much accuracy as on a macroscopic scale. The *gram* (which is not a fundamental unit) is 0.001 kilogram.

1.4 The metal object carefully enclosed within these nested glass containers is the international standard kilogram.

Unit Prefixes

Once we have defined the fundamental units, it is easy to introduce larger and smaller units for the same physical quantities. In the metric system these other units are related to the fundamental units (or, in the case of mass, to the gram) by multiples of 10 or $\frac{1}{10}$. Thus one kilometer (1 km) is 1000 meters, and one centimeter (1 cm) is $\frac{1}{100}$ meter. We usually express multiples of 10 or $\frac{1}{10}$ in exponential notation: $1000 = 10^3$, $\frac{1}{1000} = 10^{-3}$, and so on. With this notation, $1\ \text{km} = 10^3\ \text{m}$ and $1\ \text{cm} = 10^{-2}\ \text{m}$.

The names of the additional units are derived by adding a **prefix** to the name of the fundamental unit. For example, the prefix "kilo-," abbreviated k, always means a unit larger by a factor of 1000; thus

$$\begin{aligned} 1\ \text{kilometer} &= 1\ \text{km} = 10^3\ \text{meters} = 10^3\ \text{m} \\ 1\ \text{kilogram} &= 1\ \text{kg} = 10^3\ \text{grams} = 10^3\ \text{g} \\ 1\ \text{kilowatt} &= 1\ \text{kW} = 10^3\ \text{watts} = 10^3\ \text{W} \end{aligned}$$

A table on the inside back cover of this book lists the standard SI prefixes, with their meanings and abbreviations.

Here are several examples of the use of multiples of 10 and their prefixes with the units of length, mass, and time. Figure 1.5 shows how these prefixes help describe both large and small distances.

Length

$1\ \text{nanometer} = 1\ \text{nm} = 10^{-9}\ \text{m}$ (a few times the size of the largest atom)

$1\ \text{micrometer} = 1\ \mu\text{m} = 10^{-6}\ \text{m}$ (size of some bacteria and living cells)

$1\ \text{millimeter} = 1\ \text{mm} = 10^{-3}\ \text{m}$ (diameter of the point of a ballpoint pen)

$1\ \text{centimeter} = 1\ \text{cm} = 10^{-2}\ \text{m}$ (diameter of your little finger)

$1\ \text{kilometer} = 1\ \text{km} = 10^{3}\ \text{m}$ (a 10-minute walk)

1.5 Some typical lengths in the universe. (a) The distance to the most remote galaxies we can see is about 10^{26} m, or 10^{23} km. (b) The sun is 1.50×10^{11} m, or 1.50×10^{8} km, from earth. (c) The diameter of the earth is 1.28×10^{7} m, or 12,800 km. (d) A typical human is about 1.7 m, or 170 cm, tall. (e) Human red blood cells are about 8×10^{-6} m (0.008 mm, or 8 μm) in diameter. (f) These oxygen atoms, shown arrayed on the surface of a crystal, are about 10^{-10} m, or 10^{-4} μm, in radius. (g) Typical atomic nuclei (shown in an artist's impression) have radii of about 10^{-14} m, or 10^{-5} nm.

Mass

$$1 \text{ microgram} = 1\ \mu\text{g} = 10^{-6}\text{ g} = 10^{-9}\text{ kg (mass of a very small dust particle)}$$
$$1 \text{ milligram} = 1 \text{ mg} = 10^{-3}\text{ g} = 10^{-6}\text{ kg (mass of a grain of salt)}$$
$$1 \text{ gram} = 1 \text{ g} = 10^{-3}\text{ kg (mass of a paper clip)}$$

Time

$$1 \text{ nanosecond} = 1 \text{ ns} = 10^{-9}\text{ s (time for light to travel 0.3 m)}$$
$$1 \text{ microsecond} = 1\ \mu\text{s} = 10^{-6}\text{ s (time for an orbiting space shuttle to travel 8 mm)}$$
$$1 \text{ millisecond} = 1 \text{ ms} = 10^{-3}\text{ s (time for sound to travel 0.35 m)}$$

The British System

Finally, we mention the British system of units. These units are used only in the United States and a few other countries, and in most of these they are being replaced by SI units. British units are now officially defined in terms of SI units, as follows:

Length: 1 inch = 2.54 cm (exactly)

Force: 1 pound = 4.448221615260 newtons (exactly)

The newton, abbreviated N, is the SI unit of force. The British unit of time is the second, defined the same way as in SI. In physics, British units are used only in mechanics and thermodynamics; there is no British system of electrical units.

In this book we use SI units for all examples and problems, but we occasionally give approximate equivalents in British units. As you do problems using SI units, you may also wish to convert to the approximate British equivalents if they are more familiar to you (Fig. 1.6). But you should try to *think* in SI units as much as you can.

1.6 Many everyday items make use of both SI and British units. An example is this speedometer from a U.S.-built automobile, which shows the speed in both kilometers per hour (inner scale) and miles per hour (outer scale).

1.4 Unit Consistency and Conversions

We use equations to express relationships among physical quantities, represented by algebraic symbols. Each algebraic symbol always denotes both a number and a unit. For example, d might represent a distance of 10 m, t a time of 5 s, and v a speed of 2 m/s.

An equation must always be **dimensionally consistent.** You can't add apples and automobiles; two terms may be added or equated only if they have the same units. For example, if a body moving with constant speed v travels a distance d in a time t, these quantities are related by the equation

$$d = vt$$

If d is measured in meters, then the product vt must also be expressed in meters. Using the above numbers as an example, we may write

$$10\,\text{m} = \left(2\,\frac{\text{m}}{\cancel{\text{s}}}\right)(5\,\cancel{\text{s}})$$

Because the unit $1/\text{s}$ on the right side of the equation cancels the unit s, the product has units of meters, as it must. In calculations, units are treated just like algebraic symbols with respect to multiplication and division.

CAUTION **Always use units in calculations** When a problem requires calculations using numbers with units, *always* write the numbers with the correct units and carry the units through the calculation as in the example above. This provides a very useful check for calculations. If at some stage in a calculation you find that an equation or an expression has inconsistent units, you know you have made an error somewhere. In this book we will *always* carry units through all calculations, and we strongly urge you to follow this practice when you solve problems.

Problem-Solving Strategy 1.2 Unit Conversions

IDENTIFY *the relevant concepts:* Unit conversion is important, but it's also important to recognize when it's needed. In most cases, you're best off using the fundamental SI units (lengths in meters, masses in kilograms, and time in seconds) within a problem. If you need the answer to be in a different set of units (such as kilometers, grams, or hours), wait until the end of the problem to make the conversion. In the following examples, we'll concentrate on unit conversion alone, so we'll skip the *Identify* step.

SET UP *the problem* and **EXECUTE** *the solution:* Units are multiplied and divided just like ordinary algebraic symbols. This gives us an easy way to convert a quantity from one set of units to another. The key idea is to express the same physical quantity in two different units and form an equality.

For example, when we say that 1 min = 60 s, we don't mean that the number 1 is equal to the number 60; rather, we mean that 1 min represents the same physical time interval as 60 s. For this reason, the ratio $(1\,\text{min})/(60\,\text{s})$ equals 1, as does its reciprocal $(60\,\text{s})/(1\,\text{min})$. We may multiply a quantity by either of these factors without changing that quantity's physical meaning. For example, to find the number of seconds in 3 min, we write

$$3\,\text{min} = (3\,\cancel{\text{min}})\left(\frac{60\,\text{s}}{1\,\cancel{\text{min}}}\right) = 180\,\text{s}$$

EVALUATE *your answer:* If you do your unit conversions correctly, unwanted units will cancel, as in the example above. If instead you had multiplied 3 min by $(1\,\text{min})/(60\,\text{s})$, your result would have been $\frac{1}{20}\,\text{min}^2/\text{s}$, which is a rather odd way of measuring time. To be sure you convert units properly, you must write down the units at *all* stages of the calculation.

Finally, check whether your answer is reasonable. Is the result 3 min = 180 s reasonable? The answer is yes; the second is a smaller unit than the minute, so there are more seconds than minutes in the same time interval.

Example 1.1 Converting speed units

The official world land speed record is 1228.0 km/h, set on October 15, 1997, by Andy Green in the jet engine car *Thrust SSC*. Express this speed in meters per second.

SOLUTION

IDENTIFY AND SET UP: We want to convert the units of a speed from km/h to m/s.

EXECUTE: The prefix k means 10^3, so the speed $1228.0\,\text{km/h} = 1228.0 \times 10^3\,\text{m/h}$. We also know that there are 3600 s in 1 h. So we must combine the speed of $1228.0 \times 10^3\,\text{m/h}$ and a factor of 3600. But should we multiply or divide by this factor? If we treat the factor as a pure number without units, we're forced to guess how to proceed.

The correct approach is to carry the units with each factor. We then arrange the factor so that the hour unit cancels:

$$1228.0\,\text{km/h} = \left(1228.0 \times 10^3\,\frac{\text{m}}{\cancel{\text{h}}}\right)\left(\frac{1\,\cancel{\text{h}}}{3600\,\text{s}}\right) = 341.11\,\text{m/s}$$

If you multiplied by $(3600\,\text{s})/(1\,\text{h})$ instead of $(1\,\text{h})/(3600\,\text{s})$, the hour unit wouldn't cancel, and you would be able to easily

Continued

recognize your error. Again, the *only* way to be sure that you correctly convert units is to carry the units throughout the calculation.

EVALUATE: While you probably have a good intuition for speeds in kilometers per hour or miles per hour, speeds in meters per second are likely to be a bit more mysterious. It helps to remember that a typical walking speed is about 1 m/s: the length of an average person's stride is about one meter, and a good walking pace is about one stride per second. By comparison, a speed of 341.11 m/s is rapid indeed!

Example 1.2 Converting volume units

The world's largest cut diamond is the First Star of Africa (mounted in the British Royal Sceptre and kept in the Tower of London). Its volume is 1.84 cubic inches. What is its volume in cubic centimeters? In cubic meters?

SOLUTION

IDENTIFY AND SET UP: Here we are to convert the units of a volume from cubic inches (in.^3) to cubic centimeters (cm^3) and cubic meters (m^3).

EXECUTE: To convert cubic inches to cubic centimeters, we multiply by $[(2.54\text{ cm})/(1\text{ in.})]^3$, not just $(2.54\text{ cm})/(1\text{ in.})$. We find

$$
\begin{aligned}
1.84\text{ in.}^3 &= (1.84\text{ in.}^3)\left(\frac{2.54\text{ cm}}{1\text{ in.}}\right)^3 \\
&= (1.84)(2.54)^3\frac{\cancel{\text{in.}^3}\text{ cm}^3}{\cancel{\text{in.}^3}} = 30.2\text{ cm}^3
\end{aligned}
$$

Also, $1\text{ cm} = 10^{-2}\text{ m}$, and

$$
\begin{aligned}
30.2\text{ cm}^3 &= (30.2\text{ cm}^3)\left(\frac{10^{-2}\text{ m}}{1\text{ cm}}\right)^3 \\
&= (30.2)(10^{-2})^3\frac{\cancel{\text{cm}^3}\text{ m}^3}{\cancel{\text{cm}^3}} = 30.2\times10^{-6}\text{ m}^3 \\
&= 3.02\times10^{-5}\text{ m}^3
\end{aligned}
$$

EVALUATE: While 1 centimeter is 10^{-2} of a meter (that is, $1\text{ cm} = 10^{-2}\text{ m}$), our answer shows that a cubic centimeter (1 cm^3) is *not* 10^{-2} of a cubic meter. Rather, it is the volume of a cube whose sides are 1 cm long. So $1\text{ cm}^3 = (1\text{ cm})^3 = (10^{-2}\text{ m})^3 = (10^{-2})^3\text{ m}^3$, or $1\text{ cm}^3 = 10^{-6}\text{ m}^3$.

1.5 Uncertainty and Significant Figures

Measurements always have uncertainties. If you measure the thickness of the cover of this book using an ordinary ruler, your measurement is reliable only to the nearest millimeter, and your result will be 3 mm. It would be *wrong* to state this result as 3.00 mm; given the limitations of the measuring device, you can't tell whether the actual thickness is 3.00 mm, 2.85 mm, or 3.11 mm. But if you use a micrometer caliper, a device that measures distances reliably to the nearest 0.01 mm, the result will be 2.91 mm. The distinction between these two measurements is in their **uncertainty.** The measurement using the micrometer caliper has a smaller uncertainty; it's a more accurate measurement. The uncertainty is also called the **error** because it indicates the maximum difference there is likely to be between the measured value and the true value. The uncertainty or error of a measured value depends on the measurement technique used.

We often indicate the **accuracy** of a measured value—that is, how close it is likely to be to the true value—by writing the number, the symbol $\pm$, and a second number indicating the uncertainty of the measurement. If the diameter of a steel rod is given as 56.47 ± 0.02 mm, this means that the true value is unlikely to be less than 56.45 mm or greater than 56.49 mm. In a commonly used shorthand notation, the number 1.6454(21) means 1.6454 ± 0.0021. The numbers in parentheses show the uncertainty in the final digits of the main number.

We can also express accuracy in terms of the maximum likely **fractional error** or **percent error** (also called *fractional uncertainty* and *percent uncertainty*). A resistor labeled "47 ohms $\pm$ 10%" probably has a true resistance that differs from 47 ohms by no more than 10% of 47 ohms—that is, about 5 ohms. The resistance is probably between 42 and 52 ohms. For the diameter of the steel rod given above, the fractional error is $(0.02\text{ mm})/(56.47\text{ mm})$, or about 0.0004; the percent error is $(0.0004)(100\%)$, or about 0.04%. Even small percent errors can sometimes be very significant (Fig. 1.7).

1.7 This spectacular mishap was the result of a very small percent error—traveling a few meters too far in a journey of hundreds of thousands of meters.

In many cases the uncertainty of a number is not stated explicitly. Instead, the uncertainty is indicated by the number of meaningful digits, or **significant figures,** in the measured value. We gave the thickness of the cover of this book as 2.91 mm, which has three significant figures. By this we mean that the first two digits are known to be correct, while the third digit is uncertain. The last digit is in the hundredths place, so the uncertainty is about 0.01 mm. Two values with the *same* number of significant figures may have *different* uncertainties; a distance given as 137 km also has three significant figures, but the uncertainty is about 1 km.

When you use numbers having uncertainties to compute other numbers, the computed numbers are also uncertain. When numbers are multiplied or divided, the number of significant figures in the result can be no greater than in the factor with the fewest significant figures. For example, $3.1416 \times 2.34 \times 0.58 = 4.3$. When we add and subtract numbers, it's the location of the decimal point that matters, not the number of significant figures. For example, $123.62 + 8.9 = 132.5$. Although 123.62 has an uncertainty of about 0.01, 8.9 has an uncertainty of about 0.1. So their sum has an uncertainty of about 0.1 and should be written as 132.5, not 132.52. Table 1.1 summarizes these rules for significant figures.

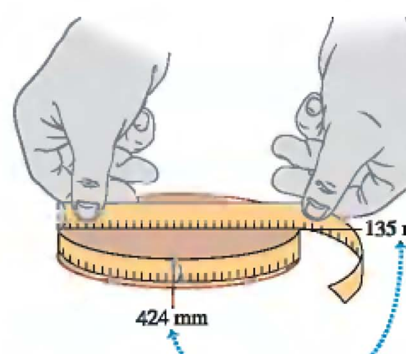
Table 1.1 Using Significant Figures

Mathematical Operation	Significant Figures in Result
Multiplication or division	No more than in the number with the fewest significant figures *Example:* $(0.745 \times 2.2)/3.885 = 0.42$ *Example:* $(1.32578 \times 10^7) \times (4.11 \times 10^{-3}) = 5.45 \times 10^4$
Addition or subtraction	Determined by the number with the largest uncertainty (i.e., the fewest digits to the right of the decimal point) *Example:* $27.153 + 138.2 - 11.74 = 153.6$

Note: In this book we will usually give numerical values with three significant figures.

As an application of these ideas, suppose you want to verify the value of π, the ratio of the circumference of a circle to its diameter. The true value of this ratio to ten digits is 3.141592654. To test this, you draw a large circle and measure its circumference and diameter to the nearest millimeter, obtaining the values 424 mm and 135 mm (Fig. 1.8). You punch these into your calculator and obtain the quotient 3.140740741. This may seem to disagree with the true value of π, but keep in mind that each of your measurements has three significant figures, so your measured value of π, equal to $(424\text{ mm})/(135\text{ mm})$, can have only three significant figures. It should be stated simply as 3.14. Within the limit of three significant figures, your value does agree with the true value.

1.8 Determining the value of π from the circumference and diameter of a circle.



In the examples and problems in this book we usually give numerical values with three significant figures, so your answers should usually have no more than three significant figures. (Many numbers in the real world have even less accuracy. An automobile speedometer, for example, usually gives only two significant figures.) Even if you do the arithmetic with a calculator that displays ten digits, it would be wrong to give a ten-digit answer because it misrepresents the accuracy of the results. Always round your final answer to keep only the correct number of significant figures or, in doubtful cases, one more at most. In Example 1.1 it would have been wrong to state the answer as 341.11111 m/s. Note that when you reduce such an answer to the appropriate number of significant figures, you must *round,* not *truncate.* Your calculator will tell you that the ratio of 525 m to 311 m is 1.688102894; to three significant figures, this is 1.69, not 1.68.

When we calculate with very large or very small numbers, we can show significant figures much more easily by using **scientific notation,** sometimes called **powers-of-10 notation.** The distance from the earth to the moon is about 384,000,000 m, but writing the number in this form doesn't indicate the number of significant figures. Instead, we move the decimal point eight places to the left (corresponding to dividing by 10^8) and multiply by 10^8; that is,

$$384{,}000{,}000\text{ m} = 3.84 \times 10^8\text{ m}$$

In this form, it is clear that we have three significant figures. The number 4.00×10^{-7} also has three significant figures, even though two of them are zeros. Note that in scientific notation the usual practice is to express the quantity as a number between 1 and 10 multiplied by the appropriate power of 10.

When an integer or a fraction occurs in a general equation, we treat that number as having no uncertainty at all. For example, in the equation $v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$, which is Eq. (2.13) in Chapter 2, the coefficient 2 is *exactly* 2. We can consider this coefficient as having an infinite number of significant figures $(2.000000 \ldots)$. The same is true of the exponent 2 in v_x^2 and v_{0x}^2.

Finally, let's note that **precision** is not the same as *accuracy*. A cheap digital watch that gives the time as 10:35:17 A.M. is very *precise* (the time is given to the second), but if the watch runs several minutes slow, then this value isn't very *accurate*. On the other hand, a grandfather clock might be very accurate (that is, display the correct time), but if the clock has no second hand, it isn't very precise. A high-quality measurement, like those used to define standards (see Section 1.3), is both precise *and* accurate.

Example 1.3 Significant figures in multiplication

The rest energy E of an object with rest mass m is given by Einstein's equation

$$E = mc^2$$

where c is the speed of light in a vacuum. Find E for an object with $m = 9.11 \times 10^{-31}$ kg (to three significant figures, the mass of an electron). The SI unit for E is the joule (J); $1\ \text{J} = 1\ \text{kg} \cdot \text{m}^2/\text{s}^2$.

SOLUTION

IDENTIFY AND SET UP: Our target variable is the energy E. We are given the equation to use and the value of the mass m; from Section 1.3 the exact value of the speed of light is $c = 299{,}792{,}458\ \text{m/s} = 2.99792458 \times 10^8\ \text{m/s}$.

EXECUTE: Substituting the values of m and c into Einstein's equation, we find

$$\begin{aligned} E &= (9.11 \times 10^{-31}\ \text{kg})(2.99792458 \times 10^8\ \text{m/s})^2 \\ &= (9.11)(2.99792458)^2(10^{-31})(10^8)^2\ \text{kg} \cdot \text{m}^2/\text{s}^2 \\ &= (81.87659678)(10^{[-31+(2\times 8)]})\ \text{kg} \cdot \text{m}^2/\text{s}^2 \\ &= 8.187659678 \times 10^{-14}\ \text{kg} \cdot \text{m}^2/\text{s}^2 \end{aligned}$$

Since the value of m was given to only three significant figures, we must round this to

$$E = 8.19 \times 10^{-14}\ \text{kg} \cdot \text{m}^2/\text{s}^2 = 8.19 \times 10^{-14}\ \text{J}$$

Most calculators use scientific notation and add exponents automatically, but you should be able to do such calculations by hand when necessary.

EVALUATE: While the rest energy contained in an electron may seem ridiculously small, on the atomic scale it is tremendous. Compare our answer to 10^{-19} J, the energy gained or lost by a single atom during a typical chemical reaction; the rest energy of an electron is about 1,000,000 times larger! (We will discuss the significance of rest energy in Chapter 37.)

Test Your Understanding of Section 1.5 The density of a material is equal to its mass divided by its volume. What is the density (in kg/m^3) of a rock of mass 1.80 kg and volume $6.0 \times 10^{-4}\ \text{m}^3$? (i) $3 \times 10^3\ \text{kg/m}^3$; (ii) $3.0 \times 10^3\ \text{kg/m}^3$; (iii) $3.00 \times 10^3\ \text{kg/m}^3$; (iv) $3.000 \times 10^3\ \text{kg/m}^3$; (v) any of these—all of these answers are mathematically equivalent.

1.6 Estimates and Orders of Magnitude

We have stressed the importance of knowing the accuracy of numbers that represent physical quantities. But even a very crude estimate of a quantity often gives us useful information. Sometimes we know how to calculate a certain quantity, but we have to guess at the data we need for the calculation. Or the calculation might be too complicated to carry out exactly, so we make some rough approximations. In either case our result is also a guess, but such a guess can be useful even if it is uncertain by a factor of two, ten, or more. Such calculations are often called **order-of-magnitude estimates.** The great Italian-American nuclear physicist Enrico Fermi (1901–1954) called them "back-of-the-envelope calculations."

Exercises 1.18 through 1.29 at the end of this chapter are of the estimating, or "order-of-magnitude," variety. Some are silly, and most require guesswork for the needed input data. Don't try to look up a lot of data; make the best guesses you can. Even when they are off by a factor of ten, the results can be useful and interesting.

Example 1.4 An order-of-magnitude estimate

You are writing an adventure novel in which the hero escapes across the border with a billion dollars' worth of gold in his suitcase. Is this possible? Would that amount of gold fit in a suitcase? Would it be too heavy to carry?

SOLUTION

IDENTIFY, SET UP, AND EXECUTE: Gold sells for around \$400 an ounce. On a particular day the price might be \$200 or \$600, but never mind. An ounce is about 30 grams. Actually, an ordinary (avoirdupois) ounce is 28.35 g; an ounce of gold is a troy ounce, which is 9.45% more. Again, never mind. Ten dollars' worth of gold has a mass somewhere around one gram, so a billion (10^9) dollars' worth of gold is a hundred million (10^8) grams, or a hundred thousand (10^5) kilograms. This corresponds to a weight in British units of around 200,000 lb, or 100 tons. Whether the precise number is closer to 50 tons or 200 tons doesn't matter. Either way, the hero is not about to carry it across the border in a suitcase.

We can also estimate the *volume* of this gold. If its density were the same as that of water $(1\ \text{g/cm}^3)$, the volume would be $10^8\ \text{cm}^3$, or $100\ \text{m}^3$. But gold is a heavy metal; we might guess its density to be 10 times that of water. Gold is actually 19.3 times as dense as water. But by guessing 10, we find a volume of $10\ \text{m}^3$. Visualize 10 cubical stacks of gold bricks, each 1 meter on a side, and ask yourself whether they would fit in a suitcase!

EVALUATE: Clearly, your novel needs rewriting. Try the calculation again with a suitcase full of five-carat (1-gram) diamonds, each worth \$100,000. Would this work?

Test Your Understanding of Section 1.6 Can you estimate the total number of teeth in all the mouths of everyone (students, staff, and faculty) on your campus? (*Hint:* How many teeth are in your mouth? Count them!)

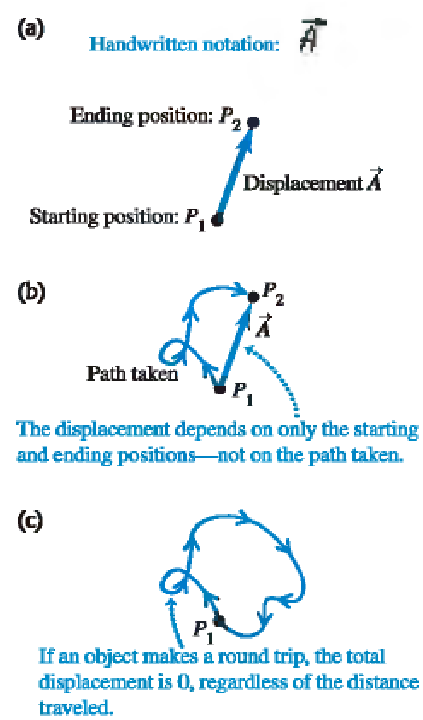
1.7 Vectors and Vector Addition

Some physical quantities, such as time, temperature, mass, and density, can be described completely by a single number with a unit. But many other important quantities in physics have a *direction* associated with them and cannot be described by a single number. A simple example is the motion of an airplane. To describe this motion completely, we must say not only how fast the plane is moving, but also in what direction. To fly from Chicago to New York, a plane has to head east, not south. The speed of the airplane combined with its direction of motion together constitute a quantity called *velocity*. Another example is *force*, which in physics means a push or pull exerted on a body. Giving a complete description of a force means describing both how hard the force pushes or pulls on the body and the direction of the push or pull.

When a physical quantity is described by a single number, we call it a **scalar quantity.** In contrast, a **vector quantity** has both a **magnitude** (the "how much" or "how big" part) and a direction in space. Calculations that combine scalar quantities use the operations of ordinary arithmetic. For example, $6\ \text{kg} + 3\ \text{kg} = 9\ \text{kg}$, or $4 \times 2\ \text{s} = 8\ \text{s}$. However, combining vectors requires a different set of operations.

To understand more about vectors and how they combine, we start with the simplest vector quantity, **displacement.** Displacement is simply a change in position of a point. (The point may represent a particle or a small body.) In Fig. 1.9a we represent the change of position from point P_1 to point P_2 by a line from P_1 to P_2, with an arrowhead at P_2 to represent the direction of motion. Displacement is a vector quantity because we must state not only how far the particle moves, but also in what direction. Walking 3 km north from your front door doesn't get you

1.9 Displacement as a vector quantity. A displacement is always a straight-line segment directed from the starting point to the ending point, even if the path is curved.

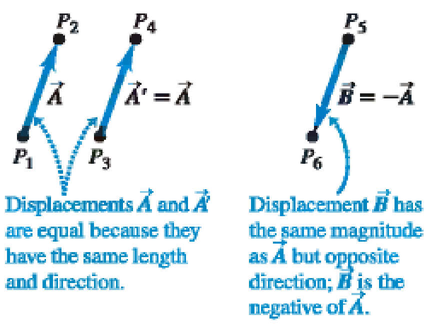


to the same place as walking 3 km southeast; these two displacements have the same magnitude, but different directions.

We usually represent a vector quantity such as displacement by a single letter, such as $\vec{A}$ in Fig. 1.9a. In this book we always print vector symbols in ***boldface italic type with an arrow above them.*** We do this to remind you that vector quantities have different properties from scalar quantities; the arrow is a reminder that vectors have direction. In handwriting, vector symbols are usually underlined or written with an arrow above them (see Fig. 1.9a). When you write a symbol for a vector, *always* write it with an arrow on top. If you don't distinguish between scalar and vector quantities in your notation, you probably won't make the distinction in your thinking either, and hopeless confusion will result.

We always *draw* a vector as a line with an arrowhead at its tip. The length of the line shows the vector's magnitude, and the direction of the line shows the vector's direction. Displacement is always a straight-line segment, directed from the starting point to the ending point, even though the actual path of the particle may be curved. In Fig. 1.9b the particle moves along the curved path shown from P_1 to P_2, but the displacement is still the vector $\vec{A}$. Note that displacement is not related directly to the total *distance* traveled. If the particle were to continue on past P_2 and then return to P_1, the displacement for the entire trip would be *zero* (Fig. 1.9c).

1.10 The meaning of vectors that have the same magnitude and the same or opposite direction.



If two vectors have the same direction, they are **parallel.** If they have the same magnitude *and* the same direction, they are *equal,* no matter where they are located in space. The vector $\vec{A}'$ from point P_3 to point P_4 in Fig. 1.10 has the same length and direction as the vector $\vec{A}$ from P_1 to P_2. These two displacements are equal, even though they start at different points. We write this as $\vec{A}' = \vec{A}$ in Fig. 1.10; the boldface equals sign emphasizes that equality of two vector quantities is not the same relationship as equality of two scalar quantities. Two vector quantities are equal only when they have the same magnitude *and* the same direction.

The vector $\vec{B}$ in Fig. 1.10, however, is not equal to $\vec{A}$ because its direction is *opposite* to that of $\vec{A}$. We define the **negative of a vector** as a vector having the same magnitude as the original vector but the *opposite* direction. The negative of vector quantity $\vec{A}$ is denoted as $-\vec{A}$, and we use a boldface minus sign to emphasize the vector nature of the quantities. If $\vec{A}$ is 87 m south, then $-\vec{A}$ is 87 m north. Thus we can write the relationship between $\vec{A}$ and $\vec{B}$ in Fig. 1.10 as $\vec{A} = -\vec{B}$ or $\vec{B} = -\vec{A}$. When two vectors $\vec{A}$ and $\vec{B}$ have opposite directions, whether their magnitudes are the same or not, we say that they are **antiparallel.**

We usually represent the *magnitude* of a vector quantity (in the case of a displacement vector, its length) by the same letter used for the vector, but in *light italic type* with *no* arrow on top, rather than boldface italic with an arrow (which is reserved for vectors). An alternative notation is the vector symbol with vertical bars on both sides:

$$(\text{Magnitude of } \vec{A}) = A = |\vec{A}| \tag{1.1}$$

By definition the magnitude of a vector quantity is a scalar quantity (a number) and is *always positive.* We also note that a vector can never be equal to a scalar because they are different kinds of quantities. The expression "$\vec{A} = 6$ m" is just as wrong as "2 oranges = 3 apples" or "6 lb = 7 km"!

When drawing diagrams with vectors, we'll generally use a scale similar to those used for maps. For example, a displacement of 5 km might be represented in a diagram by a vector 1 cm long, and a displacement of 10 km by a vector 2 cm long. In a diagram for velocity vectors, we might use a scale in which a vector that is 1 cm long represents a velocity of magnitude 5 meters per second (5 m/s). A velocity of 20 m/s would then be represented by a vector 4 cm long, with the appropriate direction.

Vector Addition

Suppose a particle undergoes a displacement $\vec{A}$ followed by a second displacement $\vec{B}$ (Fig. 1.11a). The final result is the same as if the particle had started at the same initial point and undergone a single displacement $\vec{C}$, as shown. We call displacement $\vec{C}$ the **vector sum,** or **resultant,** of displacements $\vec{A}$ and $\vec{B}$. We express this relationship symbolically as

$$\vec{C} = \vec{A} + \vec{B} \tag{1.2}$$

The boldface plus sign emphasizes that adding two vector quantities requires a geometrical process and is not the same operation as adding two scalar quantities such as $2 + 3 = 5$. In vector addition we usually place the *tail* of the *second* vector at the *head,* or tip, of the *first* vector (Fig. 1.11a).

If we make the displacements $\vec{A}$ and $\vec{B}$ in reverse order, with $\vec{B}$ first and $\vec{A}$ second, the result is the same (Fig. 1.11b). Thus

$$\vec{C} = \vec{B} + \vec{A} \quad \text{and} \quad \vec{A} + \vec{B} = \vec{B} + \vec{A} \tag{1.3}$$

This shows that the order of terms in a vector sum doesn't matter. In other words, vector addition obeys the commutative law.

Figure 1.11c shows another way to represent the vector sum: If vectors $\vec{A}$ and $\vec{B}$ are both drawn with their tails at the same point, vector $\vec{C}$ is the diagonal of a parallelogram constructed with $\vec{A}$ and $\vec{B}$ as two adjacent sides.

CAUTION **Magnitudes in vector addition** It's a common error to conclude that if $\vec{C} = \vec{A} + \vec{B}$, then the magnitude C should just equal the magnitude A plus the magnitude B. In general, this conclusion is *wrong;* for the vectors shown in Fig. 1.11, you can see that $C < A + B$. The magnitude of $\vec{A} + \vec{B}$ depends on the magnitudes of $\vec{A}$ and $\vec{B}$ *and* on the angle between $\vec{A}$ and $\vec{B}$ (see Problem 1.92). Only in the special case in which $\vec{A}$ and $\vec{B}$ are *parallel* is the magnitude of $\vec{C} = \vec{A} + \vec{B}$ equal to the sum of the magnitudes of $\vec{A}$ and $\vec{B}$ (Fig. 1.12a). By contrast, when the vectors are *antiparallel* (Fig. 1.12b) the magnitude of $\vec{C}$ equals the *difference* of the magnitudes of $\vec{A}$ and $\vec{B}$. If you're careful about distinguishing between scalar and vector quantities, you'll avoid making errors about the magnitude of a vector sum.

When we need to add more than two vectors, we may first find the vector sum of any two, add this vectorially to the third, and so on. Figure 1.13a shows three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$. In Fig. 1.13b, we first add $\vec{A}$ and $\vec{B}$ to give a vector sum $\vec{D}$; we then add vectors $\vec{C}$ and $\vec{D}$ by the same process to obtain the vector sum $\vec{R}$:

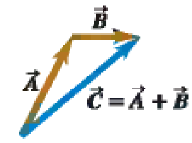
$$\vec{R} = (\vec{A} + \vec{B}) + \vec{C} = \vec{D} + \vec{C}$$

Alternatively, we can first add $\vec{B}$ and $\vec{C}$ to obtain vector $\vec{E}$ (Fig. 1.13c), and then add $\vec{A}$ and $\vec{E}$ to obtain $\vec{R}$:

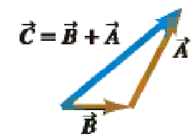
$$\vec{R} = \vec{A} + (\vec{B} + \vec{C}) = \vec{A} + \vec{E}$$

1.11 Three ways to add two vectors. As shown in (b), the order in vector addition doesn't matter; vector addition is commutative.

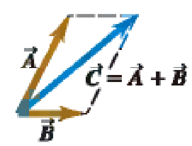
(a) We can add two vectors by placing them head to tail.



(b) Adding them in reverse order gives the same result.

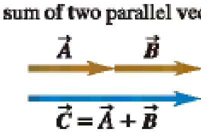


(c) We can also add them by constructing a parallelogram.

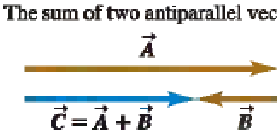


1.12 (a) Only when two vectors $\vec{A}$ and $\vec{B}$ are parallel does the magnitude of their sum equal the sum of their magnitudes: $C = A + B$. (b) When $\vec{A}$ and $\vec{B}$ are antiparallel, the magnitude of their sum equals the *difference* of their magnitudes: $C = |A - B|$.

(a) The sum of two parallel vectors

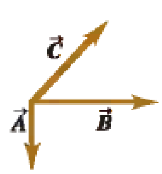


(b) The sum of two antiparallel vectors

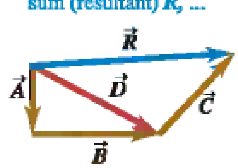


1.13 Several constructions for finding the vector sum $\vec{A} + \vec{B} + \vec{C}$.

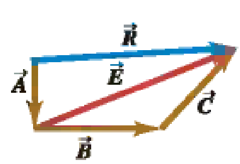
(a) To find the sum of these three vectors ...



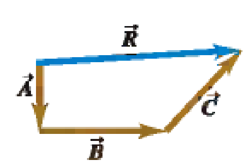
(b) we could add $\vec{A}$ and $\vec{B}$ to get $\vec{D}$ and then add $\vec{C}$ to $\vec{D}$ to get the final sum (resultant) $\vec{R}$, ...



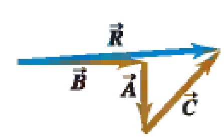
(c) or we could add $\vec{B}$ and $\vec{C}$ to get $\vec{E}$ and then add $\vec{A}$ to $\vec{E}$ to get $\vec{R}$, ...



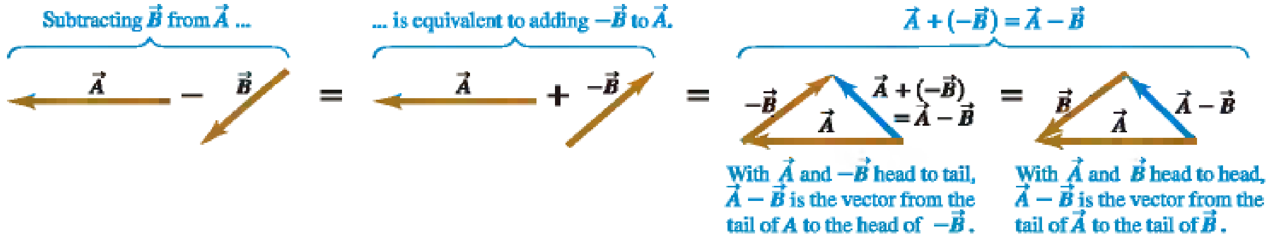
(d) or we could add $\vec{A}$, $\vec{B}$, and $\vec{C}$ to get $\vec{R}$ directly, ...



(e) or we could add $\vec{A}$, $\vec{B}$, and $\vec{C}$ in any other order and still get $\vec{R}$.



1.14 To construct the vector difference $\vec{A} - \vec{B}$, you can either place the tail of $-\vec{B}$ at the head of $\vec{A}$ or place the two vectors $\vec{A}$ and $\vec{B}$ head to head.



We don't even need to draw vectors $\vec{D}$ and $\vec{E}$; all we need to do is draw $\vec{A}$, $\vec{B}$, and $\vec{C}$ in succession, with the tail of each at the head of the one preceding it. The sum vector $\vec{R}$ extends from the tail of the first vector to the head of the last vector (Fig. 1.13d). The order makes no difference; Fig. 1.13e shows a different order, and we invite you to try others. We see that vector addition obeys the associative law.

We can *subtract* vectors as well as add them. To see how, recall that the vector $-\vec{A}$ has the same magnitude as $\vec{A}$ but the opposite direction. We define the difference $\vec{A} - \vec{B}$ of two vectors $\vec{A}$ and $\vec{B}$ to be the vector sum of $\vec{A}$ and $-\vec{B}$:

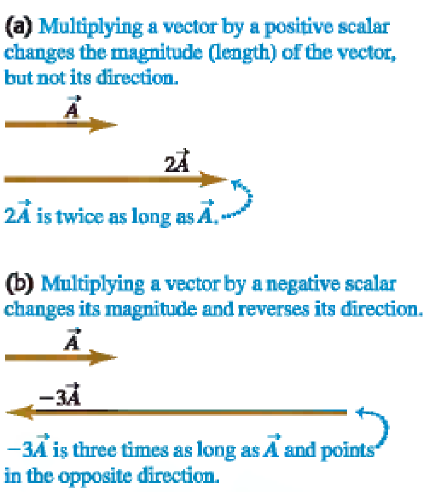
$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B}) \tag{1.4}$$

Figure 1.14 shows an example of vector subtraction.

A vector quantity such as a displacement can be multiplied by a scalar quantity (an ordinary number). The displacement $2\vec{A}$ is a displacement (vector quantity) in the same direction as the vector $\vec{A}$ but twice as long; this is the same as adding $\vec{A}$ to itself (Fig. 1.15a). In general, when a vector $\vec{A}$ is multiplied by a scalar c, the result $c\vec{A}$ has magnitude $|c|A$ (the absolute value of c multiplied by the magnitude of the vector $\vec{A}$). If c is positive, $c\vec{A}$ is in the same direction as $\vec{A}$; if c is negative, $c\vec{A}$ is in the direction opposite to $\vec{A}$. Thus $3\vec{A}$ is parallel to $\vec{A}$, while $-3\vec{A}$ is antiparallel to $\vec{A}$ (Fig. 1.15b).

The scalar quantity used to multiply a vector may also be a physical quantity having units. For example, you may be familiar with the relationship $\vec{F} = m\vec{a}$; the net force $\vec{F}$ (a vector quantity) that acts on a body is equal to the product of the body's mass m (a positive scalar quantity) and its acceleration $\vec{a}$ (a vector quantity). The direction of $\vec{F}$ is the same as that of $\vec{a}$ because m is positive, and the magnitude of $\vec{F}$ is equal to the mass m (which is positive and equals its own absolute value) multiplied by the magnitude of $\vec{a}$. The unit of force is the unit of mass multiplied by the unit of acceleration.

1.15 Multiplying a vector (a) by a positive scalar and (b) by a negative scalar.



Example 1.5 Vector addition

A cross-country skier skis 1.00 km north and then 2.00 km east on a horizontal snow field. How far and in what direction is she from the starting point?

SOLUTION

IDENTIFY: The problem involves combining displacements, so we can solve it using vector addition. The target variables are the skier's total distance and direction from her starting point. The distance is just the magnitude of her resultant displacement vector from the point of origin to where she stops, and the direction we want is the direction of the resultant displacement vector.

SET UP: Figure 1.16 is a scale diagram of the skier's displacements. We describe the direction from the starting point by the angle ϕ (the Greek letter phi). By careful measurement we find that the distance from the starting point to the ending point is about 2.2 km and that

1.16 The vector diagram, drawn to scale, for a cross-country ski trip.

ϕ is about 63°. But we can *calculate* a much more accurate result by adding the 1.00-km and 2.00-km displacement vectors.

EXECUTE: The vectors in the diagram form a right triangle; the distance from the starting point to the ending point is equal to the length of the hypotenuse. We find this length by using the Pythagorean theorem:

$$\sqrt{(1.00\ \text{km})^2 + (2.00\ \text{km})^2} = 2.24\ \text{km}$$

The angle ϕ can be found with a little simple trigonometry. If you need a review, the trigonometric functions and identities are summarized in Appendix B, along with other useful mathematical and geometrical relationships. By the definition of the tangent function,

$$\tan\phi = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{2.00\ \text{km}}{1.00\ \text{km}}$$
$$\phi = 63.4°$$

We can describe the direction as 63.4° east of north or $90° - 63.4° = 26.6°$ north of east. Take your choice!

EVALUATE: It's good practice to check the results of a vector-addition problem by making measurements on a drawing of the situation. Happily, the answers we found by calculation (2.24 km and $\phi = 63.4°$) are very close to the cruder results we found by measurement (about 2.2 km and about 63°). If they were substantially different, we would have to go back and check for errors.

Test Your Understanding of Section 1.7 Two displacement vectors, $\vec{S}$ and $\vec{T}$, have magnitudes $S = 3$ m and $T = 4$ m. Which of the following could be the magnitude of the difference vector $\vec{S} - \vec{T}$? (There may be more than one correct answer.) (i) 9 m; (ii) 7 m; (iii) 5 m; (iv) 1 m; (v) 0 m; (vi) −1 m.

1.8 Components of Vectors

In Section 1.7 we added vectors by using a scale diagram and by using properties of right triangles. Measuring a diagram offers only very limited accuracy, and calculations with right triangles work only when the two vectors are perpendicular. So we need a simple but general method for adding vectors. This is called the method of *components*.

To define what we mean by the components of a vector $\vec{A}$, we begin with a rectangular (Cartesian) coordinate system of axes (Fig. 1.17a). We then draw the vector with its tail at O, the origin of the coordinate system. We can represent any vector lying in the xy-plane as the sum of a vector parallel to the x-axis and a vector parallel to the y-axis. These two vectors are labeled $\vec{A}_x$ and $\vec{A}_y$ in Fig. 1.17a; they are called the **component vectors** of vector $\vec{A}$, and their vector sum is equal to $\vec{A}$. In symbols,

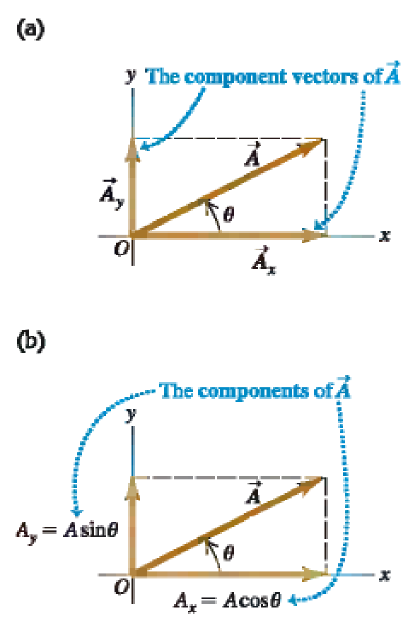
$$\vec{A} = \vec{A}_x + \vec{A}_y \tag{1.5}$$

Since each component vector lies along a coordinate-axis direction, we need only a single number to describe each one. When the component vector $\vec{A}_x$ points in the positive x-direction, we define the number A_x to be equal to the magnitude of $\vec{A}_x$. When the component vector $\vec{A}_x$ points in the negative x-direction, we define the number A_x to be equal to the negative of that magnitude (the magnitude of a vector quantity is itself never negative). We define the number A_y in the same way. The two numbers A_x and A_y are called the **components** of $\vec{A}$ (Fig. 1.17b).

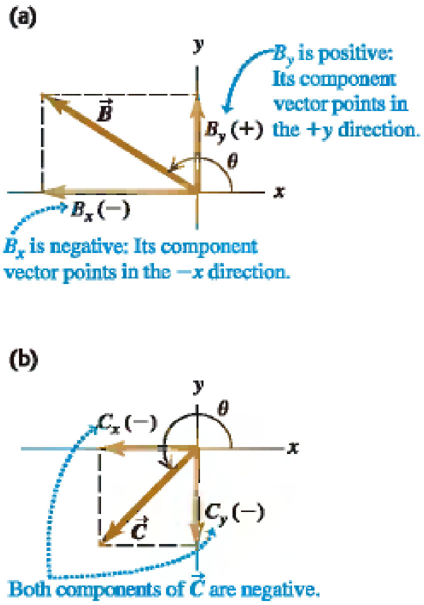
CAUTION **Components are not vectors** The components A_x and A_y of a vector $\vec{A}$ are just numbers; they are *not* vectors themselves. This is why we print the symbols for components in light italic type with *no* arrow on top instead of the boldface italic with an arrow, which is reserved for vectors.

We can calculate the components of the vector $\vec{A}$ if we know its magnitude A and its direction. We'll describe the direction of a vector by its angle relative to some reference direction. In Fig. 1.17b this reference direction is the

1.17 Representing a vector $\vec{A}$ in terms of (a) component vectors $\vec{A}_x$ and $\vec{A}_y$, and (b) components A_x and A_y (which in this case are both positive).



1.18 The components of a vector may be positive or negative numbers.



positive x-axis, and the angle between vector $\vec{A}$ and the positive x-axis is θ (the Greek letter theta). Imagine that the vector $\vec{A}$ originally lies along the $+x$-axis and that you then rotate it to its correct direction, as indicated by the arrow in Fig. 1.17b on the angle θ. If this rotation is from the $+x$-axis toward the $+y$-axis, as shown in Fig. 1.17b, then θ is *positive;* if the rotation is from the $+x$-axis toward the $-y$-axis, θ is *negative.* Thus the $+y$-axis is at an angle of 90°, the $-x$-axis at 180°, and the $-y$-axis at 270° (or $-90°$). If θ is measured in this way, then from the definition of the trigonometric functions,

$$\frac{A_x}{A} = \cos\theta \quad \text{and} \quad \frac{A_y}{A} = \sin\theta$$

$$A_x = A\cos\theta \quad \text{and} \quad A_y = A\sin\theta \tag{1.6}$$

$$(\theta \text{ measured from the } +x\text{-axis, rotating toward the } +y\text{-axis})$$

In Fig. 1.17b, A_x is positive because its direction is along the positive x-axis, and A_y is positive because its direction is along the positive y-axis. This is consistent with Eqs. (1.6); θ is in the first quadrant (between 0° and 90°), and both the cosine and the sine of an angle in this quadrant are positive. But in Fig. 1.18a the component B_x is negative; its direction is opposite to that of the positive x-axis. Again, this agrees with Eqs. (1.6); the cosine of an angle in the second quadrant is negative. The component B_y is positive ($\sin\theta$ is positive in the second quadrant). In Fig. 1.18b, both C_x and C_y are negative (both $\cos\theta$ and $\sin\theta$ are negative in the third quadrant).

CAUTION **Relating a vector's magnitude and direction to its components** Equations (1.6) are correct *only* when the angle θ is measured from the positive x-axis as described above. If the angle of the vector is given from a different reference direction or using a different sense of rotation, the relationships are different. Be careful! Example 1.6 illustrates this point.

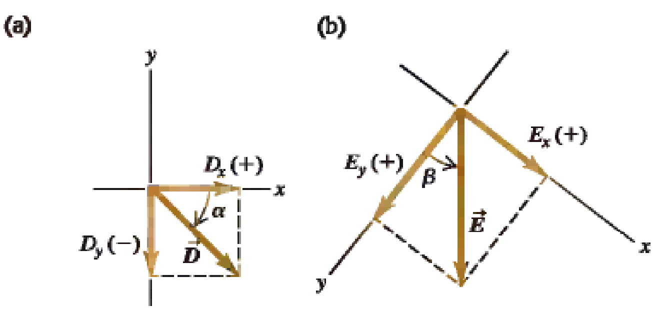
Example 1.6 Finding components

(a) What are the x- and y-components of vector $\vec{D}$ in Fig. 1.19a? The magnitude of the vector is $D = 3.00$ m and the angle $\alpha = 45°$. (b) What are the x- and y-components of vector $\vec{E}$ in Fig. 1.19b? The magnitude of the vector is $E = 4.50$ m and the angle $\beta = 37.0°$.

SOLUTION

IDENTIFY: In each case we are given the magnitude and direction of a vector, and we are asked to find its components.

1.19 Calculating the x- and y-components of vectors.



SET UP: It would seem that all we need is Eqs. (1.6). However, we need to be careful because the angles in Fig. 1.19 are *not* measured from the $+x$-axis toward the $+y$-axis.

EXECUTE: (a) The angle between $\vec{D}$ and the positive x-axis is α (the Greek letter alpha), but this angle is measured toward the *negative* y-axis. So the angle we must use in Eqs. (1.6) is $\theta = -\alpha = -45°$. We find

$$D_x = D\cos\theta = (3.00\text{ m})(\cos(-45°)) = +2.1\text{ m}$$
$$D_y = D\sin\theta = (3.00\text{ m})(\sin(-45°)) = -2.1\text{ m}$$

The vector has a positive x-component and a negative y-component, as shown in the figure. Had you been careless and substituted $+45°$ for θ in Eqs. (1.6), you would have gotten the wrong sign for D_y.

(b) The x-axis isn't horizontal in Fig. 1.19b, nor is the y-axis vertical. Don't worry, though: *Any* orientation of the x- and y-axes is permissible, just so the axes are mutually perpendicular. (In Chapter 5 we'll use axes like these to study an object sliding on an incline; one axis will lie along the incline and the other will be perpendicular to the incline.)

Here the angle β (the Greek letter beta) is the angle between $\vec{E}$ and the positive y-axis, *not* the positive x-axis, so we *cannot* use this angle in Eqs. (1.6). Instead, note that $\vec{E}$ defines the hypotenuse of a right triangle; the other two sides of the triangle are the magnitudes of E_x and E_y, the x- and y-components of $\vec{E}$. The sine of β is the opposite side (the magnitude of E_x) divided by the hypotenuse (the magnitude E), and the cosine of β is the adjacent side (the magnitude of E_y) divided by the hypotenuse (again, the magnitude E). Both components of $\vec{E}$ are positive, so

$$E_x = E\sin\beta = (4.50\text{ m})(\sin 37.0°) = +2.71\text{ m}$$
$$E_y = E\cos\beta = (4.50\text{ m})(\cos 37.0°) = +3.59\text{ m}$$

Had you used Eqs. (1.6) directly and written $E_x = E\cos 37.0°$ and $E_y = E\sin 37.0°$, your answers for E_x and E_y would have been reversed!

If you insist on using Eqs. (1.6), you must first find the angle between $\vec{E}$ and the positive x-axis, measured toward the positive y-axis; this is $\theta = 90.0° - \beta = 90.0° - 37.0° = 53.0°$. Then $E_x = E\cos\theta$ and $E_y = E\sin\theta$. You can substitute the values of E and θ into Eqs. (1.6) to show that the results for E_x and E_y are the same as those given above.

EVALUATE: Notice that the answers to part (b) have three significant figures, but the answers to part (a) have only two. Can you see why?

Doing Vector Calculations Using Components

Using components makes it relatively easy to do various calculations involving vectors. Let's look at three important examples.

1. Finding a vector's magnitude and direction from its components. We can describe a vector completely by giving either its magnitude and direction or its x- and y-components. Equations (1.6) show how to find the components if we know the magnitude and direction. We can also reverse the process: We can find the magnitude and direction if we know the components. By applying the Pythagorean theorem to Fig. 1.17b, we find that the magnitude of vector $\vec{A}$ is

$$A = \sqrt{A_x^2 + A_y^2} \qquad (1.7)$$

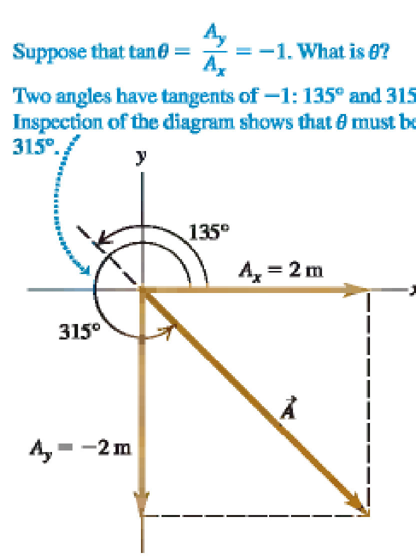
(We always take the positive root.) Equation (1.7) is valid for any choice of x-axis and y-axis, as long as they are mutually perpendicular. The expression for the vector direction comes from the definition of the tangent of an angle. If θ is measured from the positive x-axis, and a positive angle is measured toward the positive y-axis (as in Fig. 1.17b), then

$$\tan\theta = \frac{A_y}{A_x} \quad \text{and} \quad \theta = \arctan\frac{A_y}{A_x} \qquad (1.8)$$

We will always use the notation arctan for the inverse tangent function. The notation $\tan^{-1}$ is also commonly used, and your calculator may have an INV or 2ND button to be used with the TAN button.

CAUTION **Finding the direction of a vector from its components** There is one slight complication in using Eqs. (1.8) to find θ. Suppose $A_x = 2$ m and $A_y = -2$ m as in Fig. 1.20; then $\tan\theta = -1$. But there are two angles that have tangents of -1—namely, 135° and 315° (or $-45°$). In general, any two angles that differ by 180° have the same tangent. To decide which is correct, we have to look at the individual components. Because A_x is positive and A_y is negative, the angle must be in the fourth quadrant; thus $\theta = 315°$ (or $-45°$) is the correct value. Most pocket calculators give $\arctan(-1) = -45°$. In this case that is correct; but if instead we have $A_x = -2$ m and $A_y = 2$ m, then the correct angle is 135°. Similarly, when A_x and A_y are both negative, the tangent is positive, but the angle is in the third quadrant. You should *always* draw a sketch like Fig. 1.20 to check which of the two possibilities is the correct one.

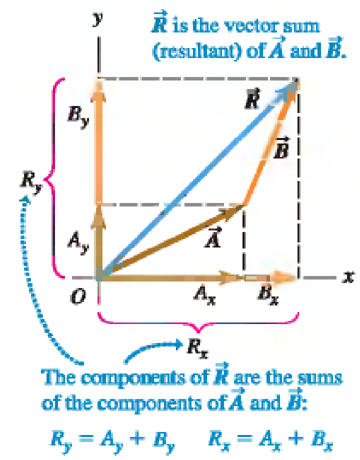
1.20 Drawing a sketch of a vector reveals the signs of its x- and y-components.



2. Multiplying a vector by a scalar. If we multiply a vector $\vec{A}$ by a scalar c, each component of the product $\vec{D} = c\vec{A}$ is just the product of c and the corresponding component of $\vec{A}$:

$$D_x = cA_x \qquad D_y = cA_y \qquad (\text{components of } \vec{D} = c\vec{A}) \qquad (1.9)$$

For example, Eq. (1.9) says that each component of the vector $2\vec{A}$ is twice as great as the corresponding component of the vector $\vec{A}$, so $2\vec{A}$ is in the same direction as $\vec{A}$ but has twice the magnitude. Each component of the vector $-3\vec{A}$ is three times as great as the corresponding component of the vector $\vec{A}$ but has the opposite sign, so $-3\vec{A}$ is in the opposite direction from $\vec{A}$ and has three times the magnitude. Hence Eqs. (1.9) are consistent with our discussion in Section 1.7 of multiplying a vector by a scalar (see Fig. 1.15).



1.21 Finding the vector sum (resultant) of $\vec{A}$ and $\vec{B}$ using components.

3. Using components to calculate the vector sum (resultant) of two or more vectors. Figure 1.21 shows two vectors $\vec{A}$ and $\vec{B}$ and their vector sum $\vec{R}$, along with the x- and y-components of all three vectors. You can see from the diagram that the x-component R_x of the vector sum is simply the sum $(A_x + B_x)$ of the x-components of the vectors being added. The same is true for the y-components. In symbols,

$$R_x = A_x + B_x \qquad R_y = A_y + B_y \qquad (\text{components of } \vec{R} = \vec{A} + \vec{B}) \tag{1.10}$$

Figure 1.21 shows this result for the case in which the components A_x, A_y, B_x, and B_y are all positive. You should draw additional diagrams to verify for yourself that Eqs. (1.10) are valid for *any* signs of the components of $\vec{A}$ and $\vec{B}$.

If we know the components of any two vectors $\vec{A}$ and $\vec{B}$, perhaps by using Eqs. (1.6), we can compute the components of the vector sum $\vec{R}$. Then if we need the magnitude and direction of $\vec{R}$, we can obtain them from Eqs. (1.7) and (1.8) with the A's replaced by R's.

We can extend this procedure to find the sum of any number of vectors. If $\vec{R}$ is the vector sum of $\vec{A}, \vec{B}, \vec{C}, \vec{D}, \vec{E}, \ldots$, the components of $\vec{R}$ are

$$\begin{aligned} R_x &= A_x + B_x + C_x + D_x + E_x + \cdots \\ R_y &= A_y + B_y + C_y + D_y + E_y + \cdots \end{aligned} \tag{1.11}$$

We have talked only about vectors that lie in the xy-plane, but the component method works just as well for vectors having any direction in space. We introduce a z-axis perpendicular to the xy-plane; then in general a vector $\vec{A}$ has components A_x, A_y, and A_z in the three coordinate directions. The magnitude A is given by

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{1.12}$$

Again, we always take the positive root. Also, Eqs. (1.11) for the components of the vector sum $\vec{R}$ have an additional member:

$$R_z = A_z + B_z + C_z + D_z + E_z + \cdots$$

Finally, while our discussion of vector addition has centered on combining *displacement* vectors, the method is applicable to all other vector quantities as well. When we study the concept of force in Chapter 4, we'll find that forces are vectors that obey the same rules of vector addition that we've used with displacement. Other vector quantities will make their appearance in later chapters.

Problem-Solving Strategy 1.3 Vector Addition

MP

IDENTIFY *the relevant concepts:* Decide what your target variable is. It may be the magnitude of the vector sum, the direction, or both.

SET UP *the problem:* Draw the individual vectors being summed and the coordinate axes being used. In your drawing, place the tail of the first vector at the origin of coordinates; place the tail of the second vector at the head of the first vector; and so on. Draw the vector sum $\vec{R}$ from the tail of the first vector to the head of the last vector. Use your drawing to make rough estimates of the magnitude and direction of $\vec{R}$; you'll use these estimates later to check your calculations.

EXECUTE *the solution* as follows:

1. Find the x- and y-components of each individual vector and record your results in a table. If a vector is described by its magnitude A and its angle θ, measured from the $+x$-axis toward the $+y$-axis, then the components are given by

$$A_x = A\cos\theta \qquad A_y = A\sin\theta$$

Some components may be positive and some may be negative, depending on how the vector is oriented (that is, what quadrant θ lies in). You can use this sign table as a check:

Quadrant	I	II	III	IV
A_x	+	−	−	+
A_y	+	+	−	−

If the angles of the vectors are given in some other way, perhaps using a different reference direction, convert them to angles measured from the $+x$-axis as described above. Be particularly careful with signs.

2. Add the individual x-components algebraically, including signs, to find R_x, the x-component of the vector sum. Do the same for the y-components to find R_y.

3. Then the magnitude R and direction θ of the vector sum are given by

$$R = \sqrt{R_x^2 + R_y^2} \qquad \theta = \arctan\frac{R_y}{R_x}$$

EVALUATE *your answer:* Check your results for the magnitude and direction of the vector sum by comparing them with the rough estimates you made from your drawing. Remember that the magnitude R is *always* positive and that θ is measured from the positive x-axis. The value of θ that you find with a calculator may be the correct one, or it may be off by 180°. You can decide by examining your drawing.

If your calculations disagree totally with the estimates from your drawing, check whether your calculator is set in "radians" or "degrees" mode. If it's in "radians" mode, entering angles in degrees will give nonsensical answers.

Example 1.7 Adding vectors with components

Three players on a reality TV show are brought to the center of a large, flat field. Each is given a meter stick, a compass, a calculator, a shovel, and (in a different order for each contestant) the following three displacements:

72.4 m, 32.0° east of north
57.3 m, 36.0° south of west
17.8 m straight south

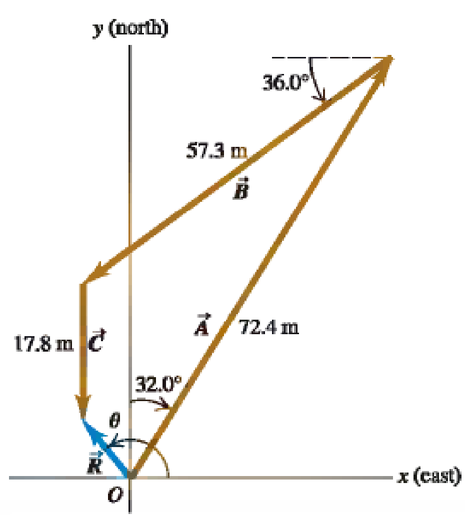
The three displacements lead to the point where the keys to a new Porsche are buried. Two players start measuring immediately, but the winner first *calculates* where to go. What does she calculate?

SOLUTION

IDENTIFY: The goal is to find the sum (resultant) of the three displacements, so this is a problem in vector addition.

SET UP: Figure 1.22 shows the situation. We have chosen the $+x$-axis as east and the $+y$-axis as north, the usual choice for maps. Let $\vec{A}$ be the first displacement, $\vec{B}$ the second, and $\vec{C}$ the third. We can estimate from the diagram that the vector sum $\vec{R}$ is about 10 m, 40° west of north.

1.22 Three successive displacements $\vec{A}$, $\vec{B}$, and $\vec{C}$ and the resultant (vector sum) displacement $\vec{R} = \vec{A} + \vec{B} + \vec{C}$.



EXECUTE: The angles of the vectors, measured from the $+x$-axis toward the $+y$-axis, are $(90.0° - 32.0°) = 58.0°$, $(180.0° + 36.0°) = 216.0°$, and 270.0°. We have to find the components of each. Because of our choice of axes, we may use Eqs. (1.6), and so the components of $\vec{A}$ are

$$A_x = A\cos\theta_A = (72.4\text{ m})(\cos 58.0°) = 38.37\text{ m}$$
$$A_y = A\sin\theta_A = (72.4\text{ m})(\sin 58.0°) = 61.40\text{ m}$$

Note that we have kept one too many significant figures in the components; we will wait until the end to round to the correct number of significant figures. The table shows the components of all the displacements, the addition of the components, and the other calculations. Always arrange your component calculations systematically like this.

Distance	Angle	x-component	y-component
$A = 72.4$ m	58.0°	38.37 m	61.40 m
$B = 57.3$ m	216.0°	−46.36 m	−33.68 m
$C = 17.8$ m	270.0°	0.00 m	−17.80 m
		$R_x = -7.99$ m	$R_y = 9.92$ m

$$R = \sqrt{(-7.99\text{ m})^2 + (9.92\text{ m})^2} = 12.7\text{ m}$$
$$\theta = \arctan\frac{9.92\text{ m}}{-7.99\text{ m}} = 129° = 39°\text{ west of north}$$

The losers try to measure three angles and three distances totaling 147.5 m, one meter at a time. The winner measured only one angle and one much shorter distance.

EVALUATE: Our calculated answers for R and θ are not too different from our estimates of 10 m and 40° west of north; that's good! Notice that $\theta = -51°$, or 51° south of east, also satisfies the equation for θ. But since the winner has made a drawing of the displacement vectors (Fig. 1.22), she knows that $\theta = 129°$ is the only correct solution for the angle.

Example 1.8 A vector in three dimensions

After an airplane takes off, it travels 10.4 km west, 8.7 km north, and 2.1 km up. How far is it from the takeoff point?

SOLUTION

Let the $+x$-axis be east, the $+y$-axis north, and the $+z$-axis up. Then $A_x = -10.4\text{ km}$, $A_y = 8.7\text{ km}$, and $A_z = 2.1\text{ km}$; Eq. (1.12) gives

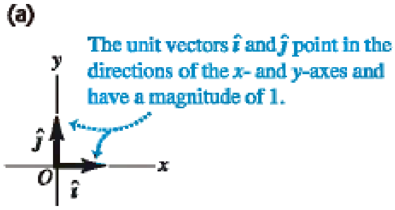
$$A = \sqrt{(-10.4\text{ km})^2 + (8.7\text{ km})^2 + (2.1\text{ km})^2} = 13.7\text{ km}$$

Test Your Understanding of Section 1.8 Two vectors $\vec{A}$ and $\vec{B}$ both lie in the xy-plane. (a) Is it possible for $\vec{A}$ to have the same magnitude as $\vec{B}$ but different components? (b) Is it possible for $\vec{A}$ to have the same components as $\vec{B}$ but a different magnitude?

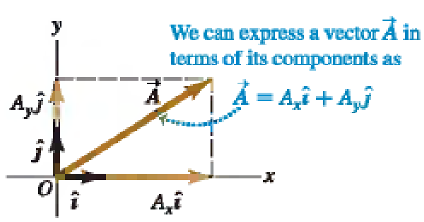
1.9 Unit Vectors

1.23 (a) The unit vectors $\hat{\imath}$ and $\hat{\jmath}$. (b) Expressing a vector $\vec{A}$ in terms of its components.

(a)



(b)



A **unit vector** is a vector that has a magnitude of 1, with no units. Its only purpose is to *point*—that is, to describe a direction in space. Unit vectors provide a convenient notation for many expressions involving components of vectors. We will always include a caret or "hat" (^) in the symbol for a unit vector to distinguish it from ordinary vectors whose magnitude may or may not be equal to 1.

In an x-y coordinate system we can define a unit vector $\hat{\imath}$ that points in the direction of the positive x-axis and a unit vector $\hat{\jmath}$ that points in the direction of the positive y-axis (Fig. 1.23a). Then we can express the relationship between component vectors and components, described at the beginning of Section 1.8, as follows:

$$\vec{A}_x = A_x\hat{\imath} \qquad \vec{A}_y = A_y\hat{\jmath} \tag{1.13}$$

Similarly, we can write a vector $\vec{A}$ in terms of its components as

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} \tag{1.14}$$

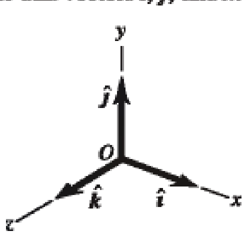
Equations (1.13) and (1.14) are vector equations; each term, such as $A_x\hat{\imath}$, is a vector quantity (Fig. 1.23b). The boldface equals and plus signs denote vector equality and addition.

When two vectors $\vec{A}$ and $\vec{B}$ are represented in terms of their components, we can express the vector sum $\vec{R}$ using unit vectors as follows:

$$\begin{aligned} \vec{A} &= A_x\hat{\imath} + A_y\hat{\jmath} \\ \vec{B} &= B_x\hat{\imath} + B_y\hat{\jmath} \\ \vec{R} &= \vec{A} + \vec{B} \\ &= (A_x\hat{\imath} + A_y\hat{\jmath}) + (B_x\hat{\imath} + B_y\hat{\jmath}) \\ &= (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} \\ &= R_x\hat{\imath} + R_y\hat{\jmath} \end{aligned} \tag{1.15}$$

Equation (1.15) restates the content of Eqs. (1.10) in the form of a single vector equation rather than two component equations.

1.24 The unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.



If the vectors do not all lie in the xy-plane, then we need a third component. We introduce a third unit vector $\hat{k}$ that points in the direction of the positive z-axis (Fig. 1.24). Then Eqs. (1.14) and (1.15) become

$$\begin{aligned} \vec{A} &= A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} \\ \vec{B} &= B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k} \end{aligned} \tag{1.16}$$

$$\vec{R} = (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + (A_z + B_z)\hat{k} = R_x\hat{\imath} + R_y\hat{\jmath} + R_z\hat{k} \tag{1.17}$$

Example 1.9 Using unit vectors

Given the two displacements

$$\vec{D} = (6\hat{\imath} + 3\hat{\jmath} - \hat{k})\text{ m} \quad \text{and} \quad \vec{E} = (4\hat{\imath} - 5\hat{\jmath} + 8\hat{k})\text{ m}$$

find the magnitude of the displacement $2\vec{D} - \vec{E}$.

SOLUTION

IDENTIFY: We are to multiply the vector $\vec{D}$ by 2 (a scalar) and then subtract the vector $\vec{E}$ from the result.

SET UP: Equation (1.9) says that to multiply $\vec{D}$ by 2, we simply multiply each of its components by 2. Then Eq. (1.17) tells us that to subtract $\vec{E}$ from $2\vec{D}$, we simply subtract the components of $\vec{E}$ from the components of $2\vec{D}$. (Recall from Section 1.7 that subtracting a vector is the same as adding the negative of that vector.) In each of these mathematical operations, the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ remain unchanged.

EXECUTE: Letting $\vec{F} = 2\vec{D} - \vec{E}$, we have

$$\begin{aligned}\vec{F} &= 2(6\hat{\imath} + 3\hat{\jmath} - \hat{k})\text{ m} - (4\hat{\imath} - 5\hat{\jmath} + 8\hat{k})\text{ m}\\ &= [(12 - 4)\hat{\imath} + (6 + 5)\hat{\jmath} + (-2 - 8)\hat{k}]\text{ m}\\ &= (8\hat{\imath} + 11\hat{\jmath} - 10\hat{k})\text{ m}\end{aligned}$$

The units of the vectors $\vec{D}$, $\vec{E}$, and $\vec{F}$ are meters, so the components of these vectors are also in meters. From Eq. (1.12),

$$\begin{aligned}F &= \sqrt{F_x^2 + F_y^2 + F_z^2}\\ &= \sqrt{(8\text{ m})^2 + (11\text{ m})^2 + (-10\text{ m})^2} = 17\text{ m}\end{aligned}$$

EVALUATE: Working with unit vectors makes vector addition and subtraction no more complicated than adding and subtracting ordinary numbers. Still, be sure to check for simple arithmetic errors.

Test Your Understanding of Section 1.9 Arrange the following vectors in order of their magnitude, with the vector of largest magnitude first. (i) $\vec{A} = (3\hat{\imath} + 5\hat{\jmath} - 2\hat{k})$ m; (ii) $\vec{B} = (-3\hat{\imath} + 5\hat{\jmath} - 2\hat{k})$ m; (iii) $\vec{C} = (3\hat{\imath} - 5\hat{\jmath} - 2\hat{k})$ m; (iv) $\vec{D} = (3\hat{\imath} + 5\hat{\jmath} + 2\hat{k})$ m.

1.10 Products of Vectors

We have seen how addition of vectors develops naturally from the problem of combining displacements, and we will use vector addition for calculating many other vector quantities later. We can also express many physical relationships concisely by using *products* of vectors. Vectors are not ordinary numbers, so ordinary multiplication is not directly applicable to vectors. We will define two different kinds of products of vectors. The first, called the *scalar product,* yields a result that is a scalar quantity. The second, the *vector product,* yields another vector.

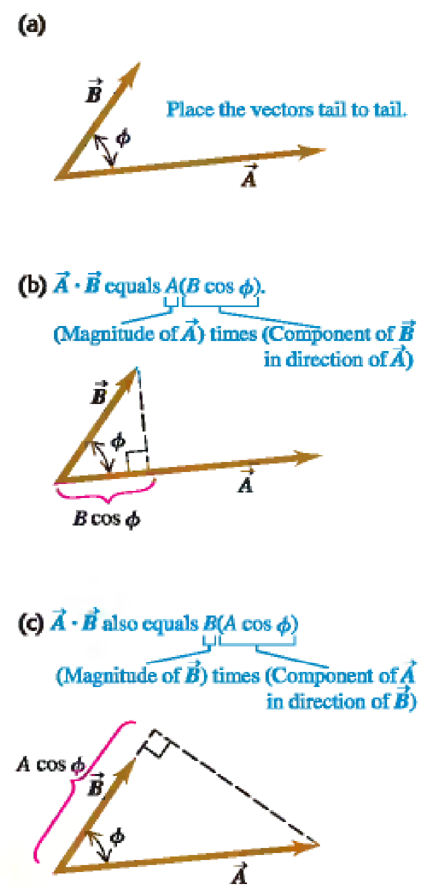
Scalar Product

The **scalar product** of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A}\cdot\vec{B}$. Because of this notation, the scalar product is also called the **dot product.** Although $\vec{A}$ and $\vec{B}$ are vectors, the quantity $\vec{A}\cdot\vec{B}$ is a scalar.

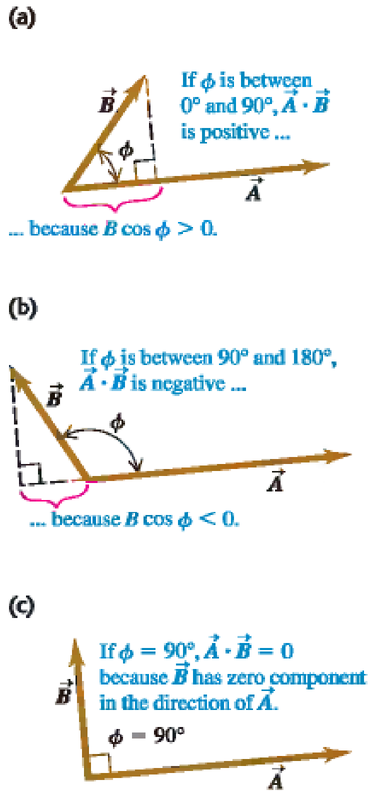
To define the scalar product $\vec{A}\cdot\vec{B}$ of two vectors $\vec{A}$ and $\vec{B}$, we draw the two vectors with their tails at the same point (Fig. 1.25a). The angle ϕ (the Greek letter phi) between their directions ranges from 0° to 180°. Figure 1.25b shows the projection of the vector $\vec{B}$ onto the direction of $\vec{A}$; this projection is the component of $\vec{B}$ in the direction of $\vec{A}$ and is equal to $B\cos\phi$. (We can take components along any direction that's convenient, not just the x- and y-axes.) We define $\vec{A}\cdot\vec{B}$ to be the magnitude of $\vec{A}$ multiplied by the component of $\vec{B}$ in the direction of $\vec{A}$. Expressed as an equation,

$$\vec{A}\cdot\vec{B} = AB\cos\phi = |\vec{A}||\vec{B}|\cos\phi \quad \text{(definition of the scalar (dot) product)} \tag{1.18}$$

1.25 Calculating the scalar product of two vectors, $\vec{A}\cdot\vec{B} = AB\cos\phi$.



1.26 The scalar product $\vec{A} \cdot \vec{B} = AB\cos\phi$ can be positive, negative, or zero, depending on the angle between $\vec{A}$ and $\vec{B}$.



Alternatively, we can define $\vec{A} \cdot \vec{B}$ to be the magnitude of $\vec{B}$ multiplied by the component of $\vec{A}$ in the direction of $\vec{B}$, as in Fig. 1.25c. Hence $\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{B} = B(A\cos\phi) = AB\cos\phi$, which is the same as Eq. (1.18).

The scalar product is a scalar quantity, not a vector, and it may be positive, negative, or zero. When ϕ is between 0° and 90°, $\cos\phi > 0$ and the scalar product is positive (Fig. 1.26a). When ϕ is between 90° and 180° so that $\cos\phi < 0$, the component of $\vec{B}$ in the direction of $\vec{A}$ is negative, and $\vec{A} \cdot \vec{B}$ is negative (Fig. 1.26b). Finally, when $\phi = 90°$, $\vec{A} \cdot \vec{B} = 0$ (Fig. 1.26c). *The scalar product of two perpendicular vectors is always zero.*

For any two vectors $\vec{A}$ and $\vec{B}$, $AB\cos\phi = BA\cos\phi$. This means that $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$. The scalar product obeys the commutative law of multiplication; the order of the two vectors does not matter.

We will use the scalar product in Chapter 6 to describe work done by a force. When a constant force $\vec{F}$ is applied to a body that undergoes a displacement $\vec{s}$, the work W (a scalar quantity) done by the force is given by

$$W = \vec{F} \cdot \vec{s}$$

The work done by the force is positive if the angle between $\vec{F}$ and $\vec{s}$ is between 0° and 90°, negative if this angle is between 90° and 180°, and zero if $\vec{F}$ and $\vec{s}$ are perpendicular. (This is another example of a term that has a special meaning in physics; in everyday language, "work" isn't something that can be positive or negative.) In later chapters we'll use the scalar product for a variety of purposes, from calculating electric potential to determining the effects that varying magnetic fields have on electric circuits.

Calculating the Scalar Product Using Components

We can calculate the scalar product $\vec{A} \cdot \vec{B}$ directly if we know the x-, y-, and z-components of $\vec{A}$ and $\vec{B}$. To see how this is done, let's first work out the scalar products of the unit vectors. This is easy, since $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ all have magnitude 1 and are perpendicular to each other. Using Eq. (1.18), we find

$$\begin{aligned} \hat{\imath} \cdot \hat{\imath} &= \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = (1)(1)\cos 0° = 1 \\ \hat{\imath} \cdot \hat{\jmath} &= \hat{\imath} \cdot \hat{k} = \hat{\jmath} \cdot \hat{k} = (1)(1)\cos 90° = 0 \end{aligned} \tag{1.19}$$

Now we express $\vec{A}$ and $\vec{B}$ in terms of their components, expand the product, and use these products of unit vectors:

$$\begin{aligned} \vec{A} \cdot \vec{B} &= (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) \cdot (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) \\ &= A_x\hat{\imath} \cdot B_x\hat{\imath} + A_x\hat{\imath} \cdot B_y\hat{\jmath} + A_x\hat{\imath} \cdot B_z\hat{k} \\ &\quad + A_y\hat{\jmath} \cdot B_x\hat{\imath} + A_y\hat{\jmath} \cdot B_y\hat{\jmath} + A_y\hat{\jmath} \cdot B_z\hat{k} \\ &\quad + A_z\hat{k} \cdot B_x\hat{\imath} + A_z\hat{k} \cdot B_y\hat{\jmath} + A_z\hat{k} \cdot B_z\hat{k} \\ &= A_xB_x\hat{\imath} \cdot \hat{\imath} + A_xB_y\hat{\imath} \cdot \hat{\jmath} + A_xB_z\hat{\imath} \cdot \hat{k} \\ &\quad + A_yB_x\hat{\jmath} \cdot \hat{\imath} + A_yB_y\hat{\jmath} \cdot \hat{\jmath} + A_yB_z\hat{\jmath} \cdot \hat{k} \\ &\quad + A_zB_x\hat{k} \cdot \hat{\imath} + A_zB_y\hat{k} \cdot \hat{\jmath} + A_zB_z\hat{k} \cdot \hat{k} \end{aligned} \tag{1.20}$$

From Eqs. (1.19) we see that six of these nine terms are zero, and the three that survive give simply

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z \quad \text{(scalar (dot) product in terms of components)} \tag{1.21}$$

Thus *the scalar product of two vectors is the sum of the products of their respective components.*

The scalar product gives a straightforward way to find the angle ϕ between any two vectors $\vec{A}$ and $\vec{B}$ whose components are known. In this case, Eq. (1.21) can be used to find the scalar product of $\vec{A}$ and $\vec{B}$. From Eq. (1.18) the scalar product is also equal to $AB\cos\phi$ The vector magnitudes A and B can be found from the vector components with Eq. (1.12), so $\cos\phi$ and hence the angle ϕ can be determined (see Example 1.11).

Example 1.10 Calculating a scalar product

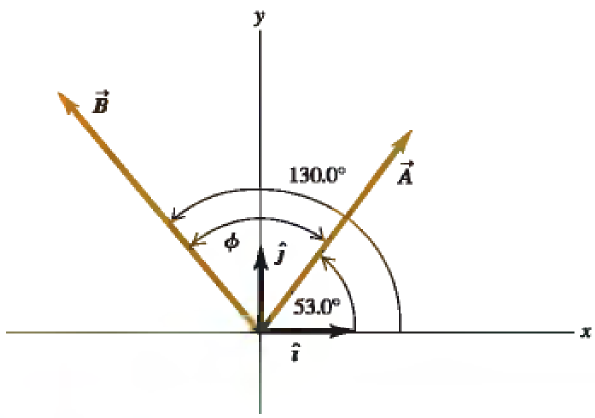
Find the scalar product $\vec{A}\cdot\vec{B}$ of the two vectors in Fig. 1.27. The magnitudes of the vectors are $A = 4.00$ and $B = 5.00$.

SOLUTION

IDENTIFY: We are given the magnitudes and directions of $\vec{A}$ and $\vec{B}$, and we wish to calculate their scalar product.

SET UP: We will calculate the scalar product in two ways: using the magnitudes of the vectors and the angle between them (Eq. 1.18), and using the components of the two vectors (Eq. 1.21).

1.27 Two vectors in two dimensions.



EXECUTE: With the first approach, the angle between the two vectors is $\phi = 130.0° - 53.0° = 77.0°$, so

$$\vec{A}\cdot\vec{B} = AB\cos\phi = (4.00)(5.00)\cos 77.0° = 4.50$$

This is positive because the angle between $\vec{A}$ and $\vec{B}$ is between 0° and 90°.

To use the second approach, we first need to find the components of the two vectors. Since the angles of $\vec{A}$ and $\vec{B}$ are given with respect to the $+x$-axis, and these angles are measured in the sense from the $+x$-axis to the $+y$-axis, we can use Eqs. (1.6):

$$\begin{aligned} A_x &= (4.00)\cos 53.0° = 2.407 \\ A_y &= (4.00)\sin 53.0° = 3.195 \\ A_z &= 0 \\ B_x &= (5.00)\cos 130.0° = -3.214 \\ B_y &= (5.00)\sin 130.0° = 3.830 \\ B_z &= 0 \end{aligned}$$

The z-components are zero because both vectors lie in the xy-plane. As in Example 1.7, we are keeping one too many significant figures in the components; we'll round to the correct number at the end. From Eq. (1.21) the scalar product is

$$\begin{aligned} \vec{A}\cdot\vec{B} &= A_xB_x + A_yB_y + A_zB_z \\ &= (2.407)(-3.214) + (3.195)(3.830) + (0)(0) = 4.50 \end{aligned}$$

EVALUATE: We get the same result for the scalar product with both methods, as we should.

Example 1.11 Finding angles with the scalar product

Find the angle between the two vectors

$$\vec{A} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k} \quad \text{and} \quad \vec{B} = -4\hat{\imath} + 2\hat{\jmath} - \hat{k}$$

SOLUTION

IDENTIFY: We are given the x-, y-, and z-components of two vectors. Our target variable is the angle ϕ between them.

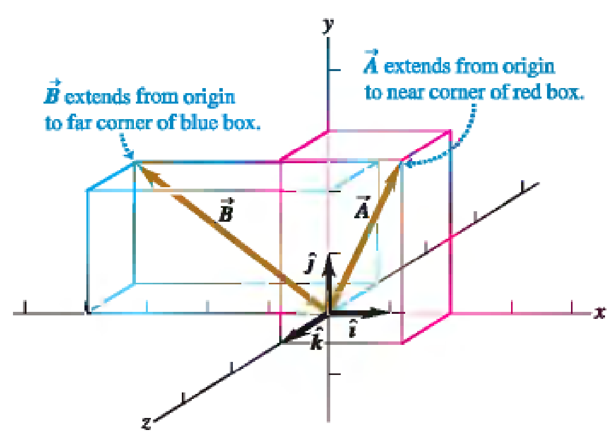
SET UP: Figure 1.28 shows the two vectors. The scalar product of two vectors $\vec{A}$ and $\vec{B}$ is related to the angle ϕ between them and to the magnitudes A and B by Eq. (1.18). The scalar product is also related to the components of the two vectors by Eq. (1.21). If we are given the components of the vectors (as we are in this example), we first determine the scalar product $\vec{A}\cdot\vec{B}$ and the values of A and B, and then determine the target variable ϕ.

EXECUTE: We set our two expressions for the scalar product, Eq. (1.18) and Eq. (1.121), equal to each other. Rearranging, we obtain

$$\cos\phi = \frac{A_xB_x + A_yB_y + A_zB_z}{AB}$$

This formula can be used to find the angle between *any* two vectors $\vec{A}$ and $\vec{B}$. For our example the components of $\vec{A}$ are $A_x = 2$,

1.28 Two vectors in three dimensions.



Continued

$A_y = 3$, and $A_z = 1$, and the components of $\vec{B}$ are $B_x = -4$, $B_y = 2$, and $B_z = -1$. Thus

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$$
$$= (2)(-4) + (3)(2) + (1)(-1) = -3$$
$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$$
$$B = \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{(-4^2) + 2^2 + (-1)^2} = \sqrt{21}$$

$$\cos\phi = \frac{A_xB_x + A_yB_y + A_zB_z}{AB} = \frac{-3}{\sqrt{14}\sqrt{21}} = -0.175$$
$$\phi = 100°$$

EVALUATE: As a check on this result, note that the scalar product $\vec{A} \cdot \vec{B}$ is negative. This means that ϕ is between 90° and 180° (see Fig. 1.26), in agreement with our answer.

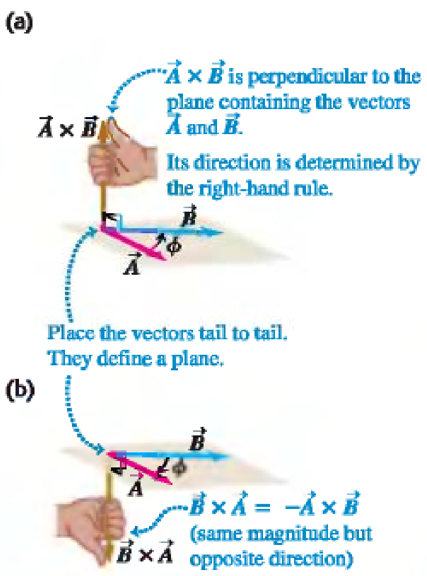
Vector Product

The **vector product** of two vectors $\vec{A}$ and $\vec{B}$, also called the **cross product,** is denoted by $\vec{A} \times \vec{B}$. As the name suggests, the vector product is itself a vector. We will use this product in Chapter 10 to describe torqne and angular momentum; in Chapters 27 and 28 we will use it extensively to describe magnetic fields and forces.

To define the vector product $\vec{A} \times \vec{B}$ of two vectors $\vec{A}$ and $\vec{B}$ we again draw the two vectors with their tails at the same point (Fig. 1.29a). The two vectors then lie in a plane. We define the vector product to be a vector quantity with a direction perpendicular to this plane (that is, perpendicular to both $\vec{A}$ and $\vec{B}$) and a magnitude equal to $AB\sin\phi$. That is, if $\vec{C} = \vec{A} \times \vec{B}$, then

$$C = AB\sin\phi \qquad \text{(magnitude of the vector (cross) product of } \vec{A} \text{ and } \vec{B}\text{)} \qquad (1.22)$$

1.29 (a) The vector product $\vec{A} \times \vec{B}$. determined by the right-hand rule. (b) $\vec{B} \times \vec{A} = -\vec{A} \times \vec{B}$; the vector product is anticommutative.



We measure the angle ϕ from $\vec{A}$ toward $\vec{B}$ and take it to be the smaller of the two possible angles, so ϕ ranges from 0° to 180°. Then $\sin\phi \geq 0$ and C in Eq. (1.22) is never negative, as must be the case for a vector magnitude. Note also that when $\vec{A}$ and $\vec{B}$ are parallel or antiparallel, $\phi = 0$ or 180° and $C = 0$. That is, *the vector product of two parallel or antiparallel vectors is always zero.* In particular, *the vector product of any vector with itself is zero.*

CAUTION **Vector product vs. scalar product** Be careful not to confuse the expression $AB\sin\phi$ for the magnitude of the vector product $\vec{A} \times \vec{B}$ with the similar expression $AB\cos\phi$ for the scalar product $\vec{A} \cdot \vec{B}$. To see the contrast between these two expressions, imagine that we vary the angle between $\vec{A}$ and $\vec{B}$ while keeping their magnitudes constant. When $\vec{A}$ and $\vec{B}$ are parallel, the magnitude of the vector product will be zero and the scalar product will be maximum. When $\vec{A}$ and $\vec{B}$ are perpendicular, the magnitude of the vector product will be maximum and the scalar product will be zero.

There are always *two* directions perpendicular to a given plane, one on each side of the plane. We choose which of these is the direction of $\vec{A} \times \vec{B}$ as follows. Imagine rotating vector $\vec{A}$ about the perpendicular line until it is aligned with $\vec{B}$, choosing the smaller of the two possible angles between $\vec{A}$ and $\vec{B}$. Curl the fingers of your right hand around the perpendicular line so that the fingertips point in the direction of rotation; your thumb will then point in the direction of $\vec{A} \times \vec{B}$. Figure 1.29a shows this **right-hand rule.**

Similarly, we determine the direction of $\vec{B} \times \vec{A}$ by rotating $\vec{B}$ into $\vec{A}$ as in Fig. 1.29b. The result is a vector that is *opposite* to the vector $\vec{A} \times \vec{B}$. The vector product is *not* commutative! In fact, for any two vectors $\vec{A}$ and $\vec{B}$,

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \qquad (1.23)$$

Just as we did for the scalar product, we can give a geometrical interpretation of the magnitude of the vector product. In Fig. 1.30a, $B\sin\phi$ is the component of vector $\vec{B}$ that is *perpendicular* to the direction of vector $\vec{A}$ From Eq. (1.22) the magnitude of $\vec{A} \times \vec{B}$ equals the magnitude of $\vec{A}$ multiplied by the component of $\vec{B}$ perpendicular to $\vec{A}$. Figure 1.30b shows that the magnitude of $\vec{A} \times \vec{B}$ also equals the magnitude of $\vec{B}$ multiplied by the component of $\vec{A}$ perpendicular to $\vec{B}$. Note that Fig. 1.30 shows the case in which ϕ is between 0° and 90°; you should draw a similar diagram for ϕ between 90° and 180° to show that the same geometrical interpretation of the magnitude of $\vec{A} \times \vec{B}$ still applies.

Calculating the Vector Product Using Components

If we know the components of $\vec{A}$ and $\vec{B}$, we can calculate the components of the vector product using a procedure similar to that for the scalar product. First we work out the multiplication table for the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, all three of which are perpendicular to each other (Fig. 1.31a). The vector product of any vector with itself is zero, so

$$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \mathbf{0}$$

The boldface zero is a reminder that each product is a zero *vector*—that is, one with all components equal to zero and an undefined direction. Using Eqs. (1.22) and (1.23) and the right-hand rule, we find

$$\begin{aligned} \hat{\imath} \times \hat{\jmath} &= -\hat{\jmath} \times \hat{\imath} = \hat{k} \\ \hat{\jmath} \times \hat{k} &= -\hat{k} \times \hat{\jmath} = \hat{\imath} \\ \hat{k} \times \hat{\imath} &= -\hat{\imath} \times \hat{k} = \hat{\jmath} \end{aligned} \quad (1.24)$$

You can verify these equations by referring to Fig. 1.31a.

Next we express $\vec{A}$ and $\vec{B}$ in terms of their components and the corresponding unit vectors, and we expand the expression for the vector product:

$$\begin{aligned} \vec{A} \times \vec{B} &= (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) \times (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) \\ &= A_x\hat{\imath} \times B_x\hat{\imath} + A_x\hat{\imath} \times B_y\hat{\jmath} + A_x\hat{\imath} \times B_z\hat{k} \\ &\quad + A_y\hat{\jmath} \times B_x\hat{\imath} + A_y\hat{\jmath} \times B_y\hat{\jmath} + A_y\hat{\jmath} \times B_z\hat{k} \\ &\quad + A_z\hat{k} \times B_x\hat{\imath} + A_z\hat{k} \times B_y\hat{\jmath} + A_z\hat{k} \times B_z\hat{k} \end{aligned} \quad (1.25)$$

We can also rewrite the individual terms in Eq. (1.25) as $A_x\hat{\imath} \times B_y\hat{\jmath} = (A_xB_y)\hat{\imath} \times \hat{\jmath}$, and so on. Evaluating these by using the multiplication table for the unit vectors in Eqs. (1.24) and then grouping the terms, we find

$$\vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\,\hat{\imath} + (A_zB_x - A_xB_z)\hat{\jmath} + (A_xB_y - A_yB_x)\hat{k} \quad (1.26)$$

Thus the components of $\vec{C} = \vec{A} \times \vec{B}$ are given by

$$C_x = A_yB_z - A_zB_y \qquad C_y = A_zB_x - A_xB_z \qquad C_z = A_xB_y - A_yB_x \quad (1.27)$$
$$\text{(components of } \vec{C} = \vec{A} \times \vec{B}\text{)}$$

The vector product can also be expressed in determinant form as

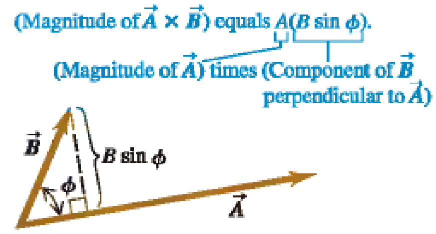
$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

If you aren't familiar with determinants, don't worry about this form.

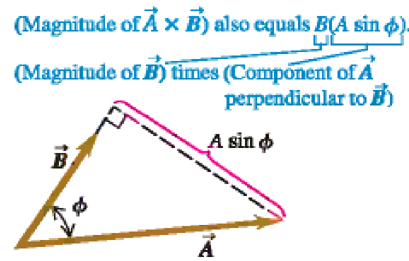
With the axis system of Fig. 1.31a, if we reverse the direction of the z-axis, we get the system shown in Fig. 1.31b. Then, as you may verify, the definition of the vector product gives $\hat{\imath} \times \hat{\jmath} = -\hat{k}$ instead of $\hat{\imath} \times \hat{\jmath} = \hat{k}$. In fact, all vector products of the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ would have signs opposite to those in Eqs. (1.24). We see that there are two kinds of coordinate systems, differing in the signs of the vector products of unit vectors. An axis system in which $\hat{\imath} \times \hat{\jmath} = \hat{k}$, as in Fig. 1.31a, is called a **right-handed system.** The usual practice is to use *only* right-handed systems, and we will follow that practice throughout this book.

1.30 Calculating the magnitude $AB\sin\phi$ of the vector product of two vectors, $\vec{A} \times \vec{B}$.

(a)

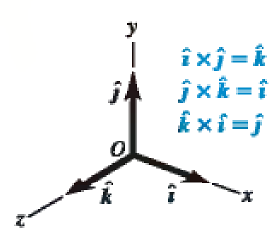


(b)

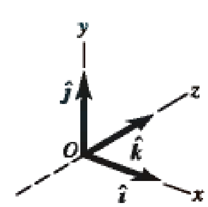


1.31 (a) We will always use a right-handed coordinate system, like this one. (b) We will never use a left handed coordinate system (in which $\hat{\imath} \times \hat{\jmath} = -\hat{k}$, and so on).

(a) A right-handed coordinate system



(b) A left-handed coordinate system; we will not use these.



Example 1.12 Calculating a vector product

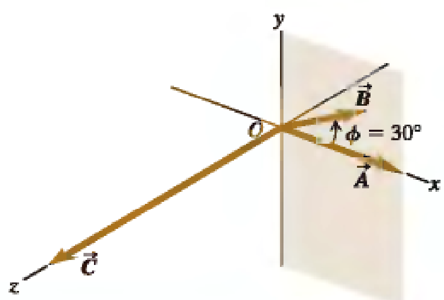
Vector $\vec{A}$ has magnitude 6 units and is in the direction of the $+x$-axis. Vector $\vec{B}$ has magnitude 4 units and lies in the xy-plane, making an angle of 30° with the $+x$-axis (Fig. 1.32). Find the vector product $\vec{A} \times \vec{B}$.

SOLUTION

IDENTIFY: We are given the magnitude and direction for each vector, and we want to find their vector product.

SET UP: We can find the vector product in one of two ways. The first way is to use Eq. (1.22) to determine the magnitude of $\vec{A} \times \vec{B}$ and then use the right-hand rule to find the direction of the vector product. The second way is to use the components of $\vec{A}$ and $\vec{B}$ to find the components of the vector product $\vec{C} = \vec{A} \times \vec{B}$ using Eqs. (1.27).

1.32 Vectors $\vec{A}$ and $\vec{B}$ and their vector product $\vec{C} = \vec{A} \times \vec{B}$. The vector $\vec{B}$ lies in the xy-plane.



EXECUTE: With the first approach, from Eq. (1.22) the magnitude of the vector product is

$$AB\sin\phi = (6)(4)(\sin 30°) = 12$$

From the right-hand rule, the direction of $\vec{A} \times \vec{B}$ is along the $+z$-axis, so we have $\vec{A} \times \vec{B} = 12\hat{k}$.

To use the second approach, we first write the components of $\vec{A}$ and $\vec{B}$:

$$A_x = 6 \qquad A_y = 0 \qquad A_z = 0$$
$$B_x = 4\cos 30° = 2\sqrt{3} \qquad B_y = 4\sin 30° = 2 \qquad B_z = 0$$

Defining $\vec{C} = \vec{A} \times \vec{B}$, we have from Eqs. (1.27) that

$$C_x = (0)(0) - (0)(2) = 0$$
$$C_y = (0)(2\sqrt{3}) - (6)(0) = 0$$
$$C_z = (6)(2) - (0)(2\sqrt{3}) = 12$$

The vector product $\vec{C}$ has only a z-component, and it lies along the $+z$-axis. The magnitude agrees with the result we obtained with the first approach, as it should.

EVALUATE: For this example the first approach was more direct because we knew the magnitudes of each vector and the angle between them, and furthermore, both vectors lay in one of the planes of the coordinate system. But often you will need to find the vector product of two vectors that are not so conveniently oriented or for which only the components are given. In such a case the second approach, using components, is more direct.

Test Your Understanding of Section 1.10 Vector $\vec{A}$ has magnitude 2 and vector $\vec{B}$ has magnitude 3. The angle ϕ between $\vec{A}$ and $\vec{B}$ is known to be either 0°, 90°, or 180°. For each of the following situations, state what the value of ϕ must be. (In each situation there may be more than one correct answer.) (a) $\vec{A} \cdot \vec{B} = 0$; (b) $\vec{A} \times \vec{B} = \mathbf{0}$; (c) $\vec{A} \cdot \vec{B} = 6$; (d) $\vec{A} \cdot \vec{B} = -6$; (e) (magnitude of $\vec{A} \times \vec{B}$) $= 6$.

CHAPTER 1 SUMMARY

Physical quantities and units: The fundamental physical quantities of mechanics are mass, length, and time. The corresponding basic SI units are the kilogram, the meter, and the second. Derived units for other physical quantities are products or quotients of the basic units. Equations must be dimensionally consistent; two terms can be added only when they have the same units. (See Examples 1.1 and 1.2.)

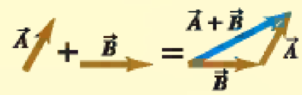
Significant figures: The accuracy of a measurement can be indicated by the number of significant figures or by a stated uncertainty. The result of a calculation usually has no more significant figures than the input data. When only crude estimates are available for input data, we can often make useful order-of-magnitude estimates. (See Examples 1.3 and 1.4.)

Significant figures in magenta

$$\pi = \frac{C}{2r} = \frac{0.424\text{ m}}{2(0.06750\text{ m})} = 3.14$$

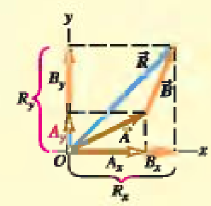
$$123.62 + 8.9 = 132.5$$

Scalars, vectors, and vector addition: Scalar quantities are numbers and combine with the usual rules of arithmetic. Vector quantities have direction as well as magnitude and combine according to the rules of vector addition. The negative of a vector has the same magnitude but points in the opposite direction. (See Example 1.5.)



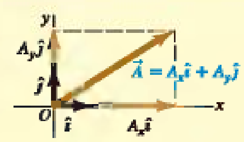
Vector components and vector addition: Vector addition can be carried out using components of vectors. The x-component of $\vec{R} = \vec{A} + \vec{B}$ is the sum of the x-components of $\vec{A}$ and $\vec{B}$, and likewise for the y- and z-components. (See Examples 1.6–1.8.)

$$R_x = A_x + B_x$$
$$R_y = A_y + B_y \qquad (1.10)$$
$$R_z = A_z + B_z$$



Unit vectors: Unit vectors describe directions in space. A unit vector has a magnitude of one, with no units. The unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, aligned with the x-, y-, and z-axes of a rectangular coordinate system, are especially useful. (See Example 1.9.)

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} \qquad (1.16)$$



Scalar product: The scalar product $C = \vec{A} \cdot \vec{B}$ of two vectors $\vec{A}$ and $\vec{B}$ is a scalar quantity. It can be expressed in terms of the magnitudes of $\vec{A}$ and $\vec{B}$ and the angle ϕ between the two vectors, or in terms of the components of $\vec{A}$ and $\vec{B}$. The scalar product is commutative; $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$. The scalar product of two perpendicular vectors is zero. (See Examples 1.10 and 1.11.)

$$\vec{A} \cdot \vec{B} = AB\cos\phi = |\vec{A}||\vec{B}|\cos\phi \qquad (1.18)$$

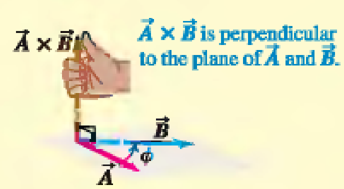
$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z \qquad (1.21)$$

Scalar product $\vec{A} \cdot \vec{B} = AB\cos\phi$

Vector product: The vector product $\vec{C} = \vec{A} \times \vec{B}$ of two vectors $\vec{A}$ and $\vec{B}$ is another vector $\vec{C}$. The magnitude of $\vec{A} \times \vec{B}$ depends on the magnitudes of $\vec{A}$ and $\vec{B}$ and the angle ϕ between the two vectors. The direction of $\vec{A} \times \vec{B}$ is perpendicular to the plane of the two vectors being multiplied, as given by the right-hand rule. The components of $\vec{C} = \vec{A} \times \vec{B}$ can be expressed in terms of the components of $\vec{A}$ and $\vec{B}$. The vector product is not commutative; $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$. The vector product of two parallel or antiparallel vectors is zero. (See Example 1.12.)

$$C = AB\sin\phi \qquad (1.22)$$

$$C_x = A_yB_z - A_zB_y$$
$$C_y = A_zB_x - A_xB_z \qquad (1.27)$$
$$C_z = A_xB_y - A_yB_x$$



(Magnitude of $\vec{A} \times \vec{B}$) $= AB\sin\phi$

Key Terms

range of validity, *2*
target variable, *3*
model, *3*
particle, *3*
physical quantity, *4*
operational definition, *4*
unit, *4*
International System (SI), *4*
second, *5*
meter, *5*
kilogram, *5*
prefix, *5*
dimensionally consistent, *7*
uncertainty (error), *8*
accuracy, *8*
fractional error, *8*
percent error, *8*
significant figures, *9*
scientific (powers-of-10) notation, *9*
precision, *10*
order-of-magnitude estimates, *11*
scalar quantity, *11*
vector quantity, *11*
magnitude of a vector, *11*
displacement, *11*
parallel vectors, *12*
negative of a vector, *12*
antiparallel vectors, *12*
vector sum (resultant), *13*
component vectors, *15*
components, *15*
unit vector, *20*
scalar (dot) product, *21*
vector (cross) product, *24*
right-hand rule, *24*
right-handed system, *25*

Answer to Chapter Opening Question ?

Take the $+x$-axis to point east and the $+y$-axis to point north. Then what we are trying to find is the y-component of the velocity vector, which has magnitude $v = 20\text{ km/h}$ and is at an angle $\theta = 53°$ measured from the $+x$-axis toward the $+y$-axis. From Eqs. (1.6) we have $v_y = v\sin\theta = (20\text{ km/h})\sin 53° = 16\text{ km/h}$. So the hurricane moves 16 km north in 1 h.

Answers to Test Your Understanding Questions

1.5 Answer: (ii) Density $= (1.80\text{ kg})/(6.0 \times 10^{-4}\text{ m}^3) = 3.0 \times 10^3\text{ kg/m}^3$. When we multiply or divide, the number with the fewest significant figures controls the number of significant figures in the result.

1.6 The answer depends on how many students are enrolled at your campus.

1.7 Answers: (ii), (iii), and (iv) The vector $-\vec{T}$ has the same magnitude as the vector $\vec{T}$, so $\vec{S} - \vec{T} = \vec{S} + (-\vec{T})$ is the *sum* of one vector of magnitude 3 m and one of magnitude 4 m. This sum has magnitude 7 m if $\vec{S}$ and $-\vec{T}$ are parallel and magnitude 1 m if $\vec{S}$ and $-\vec{T}$ are antiparallel. The magnitude of $\vec{S} - \vec{T}$ is 5 m if $\vec{S}$ and $-\vec{T}$ are perpendicular, so that the vectors $\vec{S}$, $\vec{T}$, and $\vec{S} - \vec{T}$ form a 3-4-5 right triangle. Answer (i) is impossible because the magnitude of the sum of two vectors cannot be greater than the sum of the magnitudes; answer (v) is impossible because the sum of two vectors can be zero only if the two vectors are antiparallel and have the same magnitude; and answer (vi) is impossible because the magnitude of a vector cannot be negative.

1.8 Answers: (a) yes, (b) no Vectors $\vec{A}$ and $\vec{B}$ can have the same magnitude but different components if they point in different directions. If they have the same components, however, they are the same vector $(\vec{A} = \vec{B})$ and so must have the same magnitude.

1.9 Answer: all have the same magnitude The four vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$ all point in different directions, but all have the same magnitude:

$$A = B = C = D = \sqrt{(\pm 3\text{ m})^2 + (\pm 5\text{ m})^2 + (\pm 2\text{ m})^2}$$
$$= \sqrt{9\text{ m}^2 + 25\text{ m}^2 + 4\text{ m}^2} = \sqrt{38\text{ m}^2} = 6.2\text{ m}$$

1.10 Answers: (a) $\phi = 90°$, (b) $\phi = 0°$ or $\phi = 180°$, (c) $\phi = 0°$, (d) $\phi = 180°$, (e) $\phi = 90°$ (a) The scalar product is zero only if $\vec{A}$ and $\vec{B}$ are perpendicular. (b) The vector product is zero only if $\vec{A}$ and $\vec{B}$ are either parallel or antiparallel. (c) The scalar product is equal to the product of the magnitudes $(\vec{A}\cdot\vec{B} = AB)$ only if $\vec{A}$ and $\vec{B}$ are parallel. (d) The scalar product is equal to the negative of the product of the magnitudes $(\vec{A}\cdot\vec{B} = -AB)$ only if $\vec{A}$ and $\vec{B}$ are antiparallel. (e) The magnitude of the vector product is equal to the product of the magnitudes $[(\text{magnitude of }\vec{A} \times \vec{B}) = AB]$ only if $\vec{A}$ and $\vec{B}$ are perpendicular.

PROBLEMS

For instructor-assigned homework, go to **www.masteringphysics.com**

Discussion Questions

Q1.1. How many correct experiments do we need to disprove a theory? How many to prove a theory? Explain.

Q1.2. A guidebook describes the rate of climb of a mountain trail as 120 meters per kilometer. How can you express this as a number with no units?

Q1.3. Suppose you are asked to compute the tangent of 5.00 meters. Is this possible? Why or why not?

Q1.4. A highway contractor stated that in building a bridge deck he poured 250 yards of concrete. What do you think he meant?

Q1.5. What is your height in centimeters? What is your weight in newtons?

Q1.6. The U.S. National Institute of Science and Technology (NIST) maintains several accurate copies of the international standard kilogram. Even after careful cleaning, these national standard kilograms are gaining mass at an average rate of about $1\ \mu\text{g/y}$ $(1\text{ y} = 1\text{ year})$ when compared every ten years or so to the standard international kilogram. Does this apparent change have any importance? Explain.

Q1.7. What physical phenomena (other than a pendulum or cesium clock) could you use to define a time standard?

Q1.8. Describe how you could measure the thickness of a sheet of paper with an ordinary ruler.

Q1.9. The quantity $\pi = 3.14159\ldots$ is a number with no dimensions, since it is a ratio of two lengths. Describe two or three other geometrical or physical quantities that are dimensionless.

Q1.10. What are the units of volume? Suppose another student tells you that a cylinder of radius r and height h has volume given by $\pi r^3 h$. Explain why this cannot be right.

Q1.11. Three archers each fire four arrows at a target. Joe's four arrows hit at points 10 cm above, 10 cm below, 10 cm to the left, and 10 cm to the right of the center of the target. All four of Moe's arrows hit within 1 cm of a point 20 cm from the center, and Flo's four arrows all hit within 1 cm of the center. The contest judge says that one of the archers is precise but not accurate, another archer is accurate but not precise, and the third archer is both accurate and precise. Which description goes with which archer? Explain your reasoning.

Q1.12. A circular racetrack has a radius of 500 m. What is the displacement of a bicyclist when she travels around the track from the north side to the south side? When she makes one complete circle around the track? Explain your reasoning.

Q1.13. Can you find two vectors with different lengths that have a vector sum of zero? What length restrictions are required for three vectors to have a vector sum of zero? Explain your reasoning.

Q1.14. One sometimes speaks of the "direction of time," evolving from past to future. Does this mean that time is a vector quantity? Explain your reasoning.

Q1.15. Air traffic controllers give instructions to airline pilots telling them in which direction they are to fly. These instructions are called "vectors." If these are the only instructions given, is the name "vector" used correctly? Why or why not?

Q1.16. Can you find a vector quantity that has a magnitude of zero but components that are different from zero? Explain. Can the magnitude of a vector be less than the magnitude of any of its components? Explain.

Q1.17. (a) Does it make sense to say that a vector is *negative?* Why? (b) Does it make sense to say that one vector is the negative of another? Why? Does your answer here contradict what you said in part (a)?

Q1.18. If $\vec{C}$ is the vector sum of $\vec{A}$ and $\vec{B}$, $\vec{C} = \vec{A} + \vec{B}$, what must be true if $C = A + B$? What must be true if $C = 0$?

Q1.19. If $\vec{A}$ and $\vec{B}$ are nonzero vectors, is it possible for $\vec{A} \cdot \vec{B}$ and $\vec{A} \times \vec{B}$ *both* to be zero? Explain.

Q1.20. What does $\vec{A} \cdot \vec{A}$, the scalar product of a vector with itself, give? What about $\vec{A} \times \vec{A}$, the vector product of a vector with itself?

Q1.21. Let $\vec{A}$ represent any nonzero vector. Why is $\vec{A}/A$ a unit vector and what is its direction? If θ is the angle that $\vec{A}$ makes with the $+x$-axis, explain why $(\vec{A}/A) \cdot \hat{\imath}$ is called the *direction cosine* for that axis.

Q1.22. Which of the following are legitimate mathematical operations: (a) $\vec{A} \cdot (\vec{B} - \vec{C})$; (b) $(\vec{A} - \vec{B}) \times \vec{C}$; (c) $\vec{A} \cdot (\vec{B} \times \vec{C})$; (d) $\vec{A} \times (\vec{B} \times \vec{C})$; (e) $\vec{A} \times (\vec{B} \cdot \vec{C})$? In each case, give the reason for your answer.

Q1.23. Consider the two repeated vector products $\vec{A} \times (\vec{B} \times \vec{C})$ and $(\vec{A} \times \vec{B}) \times \vec{C}$. Give an example that illustrates the general rule that these two vector products do not have the same magnitude or direction. Can you choose the vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ such that these two vector products *are* equal? If so, give an example.

Q1.24. Show that, no matter what $\vec{A}$ and $\vec{B}$ are, $\vec{A} \cdot (\vec{A} \times \vec{B}) = 0$. (*Hint:* Do not look for an elaborate mathematical proof. Rather look at the definition of the direction of the cross product.)

Q1.25. (a) If $\vec{A} \cdot \vec{B} = 0$, does it necessary follow that $A = 0$ or $B = 0$? Explain. (b) If $\vec{A} \times \vec{B} = \mathbf{0}$, does it necessary follow that $A = 0$ or $B = 0$? Explain.

Q1.26. If $\vec{A} = \mathbf{0}$ for a vector in the xy plane, does it follow that $A_x = -A_y$? What *can* you say about A_x and A_y?

Exercises

Section 1.3 Standards and Units
Section 1.4 Unit Consistency and Conversions

1.1. Starting with the definition 1 in. = 2.54 cm, find the number of (a) kilometers in 1.00 mile and (b) feet in 1.00 km.

1.2. According to the label on a bottle of salad dressing, the volume of the contents is 0.473 liter (L). Using only the conversions $1\ \text{L} = 1000\ \text{cm}^3$ and 1 in. = 2.54 cm, express this volume in cubic inches.

1.3. How many nanoseconds does it take light to travel 1.00 ft in vacuum? (This result is a useful quantity to remember.)

1.4. The density of lead is $11.3\ \text{g/cm}^3$. What is this value in kilograms per cubic meter?

1.5. The most powerful engine available for the classic 1963 Chevrolet Corvette Sting Ray developed 360 horsepower and had a displacement of 327 cubic inches. Express this displacement in liters (L) by using only the conversions $1\ \text{L} = 1000\ \text{cm}^3$ and 1 in. = 2.54 cm.

1.6. A square field measuring 100.0 m by 100.0 m has an area of 1.00 hectare. An acre has an area of $43{,}600\ \text{ft}^2$. If a country lot has an area of 12.0 acres, what is the area in hectares?

1.7. How many years older will you be 1.00 billion seconds from now? (Assume a 365-day year.)

1.8. While driving in an exotic foreign land you see a speed limit sign on a highway that reads 180,000 furlongs per fortnight. How many miles per hour is this? (One furlong is $\frac{1}{8}$ mile, and a fortnight is 14 days. A furlong originally referred to the length of a plowed furrow.)

1.9. A certain fuel-efficient hybrid car gets gasoline mileage of 55.0 mpg (miles per gallon). (a) If you are driving this car in Europe and want to compare its mileage with that of other European cars, express this mileage in km/L (L = liter). Use the conversion factors in Appendix E. (b) If this car's gas tank holds 45 L, how many tanks of gas will you use to drive 1500 km?

1.10. The following conversions occur frequently in physics and are very useful. (a) Use 1 mi = 5280 ft and 1 h = 3600 s to convert 60 mph to units of ft/s. (b) The acceleration of a freely falling object is $32\ \text{ft/s}^2$. Use 1 ft = 30.48 cm to express this acceleration in units of m/s^2. (c) The density of water is $1.0\ \text{g/cm}^3$. Convert this density to units of kg/m^3.

1.11. Neptunium. In the fall of 2002, a group of scientists at Los Alamos National Laboratory determined that the critical mass of neptunium-237 is about 60 kg. The critical mass of a fissionable material is the minimum amount that must be brought together to start a chain reaction. This element has a density of $19.5\ \text{g/cm}^3$. What would be the radius of a sphere of this material that has a critical mass?

Section 1.5 Uncertainty and Significant Figures

1.12. A useful and easy-to-remember approximate value for the number of seconds in a year is $\pi \times 10^7$. Determine the percent error in this approximate value. (There are 365.24 days in one year.)

1.13. Figure 1.7 shows the result of unacceptable error in the stopping position of a train. (a) If a train travels 890 km from Berlin to Paris and then overshoots the end of the track by 10 m, what is the percent error in the total distance covered? (b) Is it correct to write the total distance covered by the train as 890,010 m? Explain.

1.14. With a wooden ruler you measure the length of a rectangular piece of sheet metal to be 12 mm. You use micrometer calipers to measure the width of the rectangle and obtain the value 5.98 mm. Give your answers to the following questions to the correct number of significant figures. (a) What is the area of the rectangle? (b) What is the ratio of the rectangle's width to its length? (c) What is the perimeter of the rectangle? (d) What is the difference between the length and width? (e) What is the ratio of the length to the width?

1.15. Estimate the percent error in measuring (a) a distance of about 75 cm with a meter stick; (b) a mass of about 12 g with a chemical balance; (c) a time interval of about 6 min with a stopwatch.

1.16. A rectangular piece of aluminum is 5.10 ± 0.01 cm long and 1.90 ± 0.01 cm wide. (a) Find the area of the rectangle and the uncertainty in the area. (b) Verify that the fractional uncertainty in the area is equal to the sum of the fractional uncertainties in the length and in the width. (This is a general result; see Challenge Problem 1.98.)

1.17. As you eat your way through a bag of chocolate chip cookies, you observe that each cookie is a circular disk with a diameter of 8.50 ± 0.02 cm and a thickness of 0.050 ± 0.005 cm. (a) Find the average volume of a cookie and the uncertainty in the volume. (b) Find the ratio of the diameter to the thickness and the uncertainty in this ratio.

Section 1.6 Estimates and Orders of Magnitude

1.18. How many gallons of gasoline are used in the United States in one day? Assume two cars for every three people, that each car is driven an average of 10,000 mi per year, and that the average car gets 20 miles per gallon.

1.19. A rather ordinary middle-aged man is in the hospital for a routine check-up. The nurse writes the quantity 200 on his medical chart but forgets to include the units. Which of the following quantities could the 200 plausibly represent? (a) his mass in kilograms; (b) his height in meters; (c) his height in centimeters; (d) his height in millimeters; (e) his age in months.

1.20. How many kernels of corn does it take to fill a 2-L soft drink bottle?

1.21. How many words are there in this book?

1.22. Four astronauts are in a spherical space station. (a) If, as is typical, each of them breathes about 500 cm^3 of air with each breath, approximately what volume of air (in cubic meters) do these astronauts breathe in a year? (b) What would the diameter (in meters) of the space station have to be to contain all this air?

1.23. How many times does a typical person blink her eyes in a lifetime?

1.24. How many times does a human heart beat during a lifetime? How many gallons of blood does it pump? (Estimate that the heart pumps 50 cm^3 of blood with each beat.)

1.25. In Wagner's opera *Das Rheingold,* the goddess Freia is ransomed for a pile of gold just tall enough and wide enough to hide her from sight. Estimate the monetary value of this pile. The density of gold is 19.3 g/cm^3, and its value is about \$10 per gram (although this varies).

1.26. You are using water to dilute small amounts of chemicals in the laboratory, drop by drop. How many drops of water are in a 1.0 L bottle? (*Hint:* Start by estimating the diameter of a drop of water.)

1.27. How many pizzas are consumed each academic year by students at your school?

1.28. How many dollar bills would you have to stack to reach the moon? Would that be cheaper than building and launching a spacecraft? (*Hint:* Start by folding a dollar bill to see how many thicknesses make 1.0 mm.)

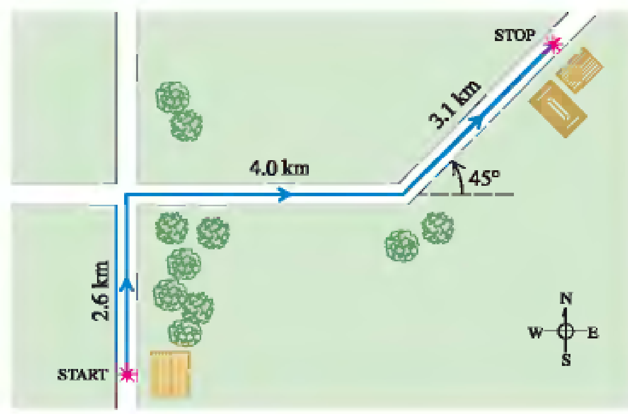
1.29. How much would it cost to paper the entire United States (including Alaska and Hawaii) with dollar bills? What would be the cost to each person in the United States?

Section 1.7 Vectors and Vector Addition

1.30. Hearing rattles from a snake, you make two rapid displacements of magnitude 1.8 m and 2.4 m. In sketches (roughly to scale), show how your two displacements might add up to give a resultant of magnitude (a) 4.2 m; (b) 0.6 m; (c) 3.0 m.

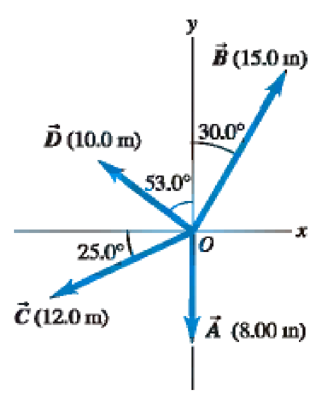
1.31. A postal employee drives a delivery truck along the route shown in Fig. 1.33. Determine the magnitude and direction of the resultant displacement by drawing a scale diagram. (See also Exercise 1.38 for a different approach to this same problem.)

Figure 1.33 Exercises 1.31 and 1.38.



1.32. For the vectors $\vec{A}$ and $\vec{B}$ in Fig. 1.34, use a scale drawing to find the magnitude and direction of (a) the vector sum $\vec{A} + \vec{B}$ and (b) the vector difference $\vec{A} - \vec{B}$. Use your answers to find the magnitude and direction of (c) $-\vec{A} - \vec{B}$ and (d) $\vec{B} - \vec{A}$. (See also Exercise 1.39 for a different approach to this problem.)

Figure 1.34 Exercises 1.32, 1.35, 1.39, 1.47, 1.53, and 1.57, and Problem 1.72.



1.33. A spelunker is surveying a cave. She follows a passage 180 m straight west, then 210 m in a direction 45° east of south, and then 280 m at 30° east of north. After a fourth unmeasured displacement, she finds herself back where she started. Use a scale drawing to determine the magnitude and direction of the fourth displacement. (See also Problem 1.73 for a different approach to this problem.)

Section 1.8 Components of Vectors

1.34. Use a scale drawing to find the x- and y-components of the following vectors. For each vector the numbers given are the magnitude of the vector and the angle, measured in the sense from the $+x$-axis toward the $+y$-axis, that it makes with the $+x$-axis: (a) magnitude 9.30 m, angle 60.0°; (b) magnitude 22.0 km, angle 135°; (c) magnitude 6.35 cm, angle 307°.

1.35. Compute the x- and y-components of the vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$ in Fig. 1.34.

1.36. Let the angle θ be the angle that the vector $\vec{A}$ makes with the $+x$-axis, measured counterclockwise from that axis. Find the angle θ for a vector that has the following components: (a) $A_x = 2.00$ m, $A_y = -1.00$ m; (b) $A_x = 2.00$ m, $A_y = 1.00$ m; (c) $A_x = -2.00$ m, $A_y = 1.00$ m; (d) $A_x = -2.00$ m, $A_y = -1.00$ m.

1.37. A rocket fires two engines simultaneously. One produces a thrust of 725 N directly forward, while the other gives a 513-N thrust at 32.4° above the forward direction. Find the magnitude and direction (relative to the forward direction) of the resultant force that these engines exert on the rocket.

1.38. A postal employee drives a delivery truck over the route shown in Fig. 1.33. Use the method of components to determine the magnitude and direction of her resultant displacement. In a vector-addition diagram (roughly to scale), show that the resultant displacement found from your diagram is in qualitative agreement with the result you obtained using the method of components.

1.39. For the vectors $\vec{A}$ and $\vec{B}$ in Fig. 1.34, use the method of components to find the magnitude and direction of (a) the vector sum $\vec{A} + \vec{B}$; (b) the vector sum $\vec{B} + \vec{A}$; (c) the vector difference $\vec{A} - \vec{B}$; (d) the vector difference $\vec{B} - \vec{A}$.

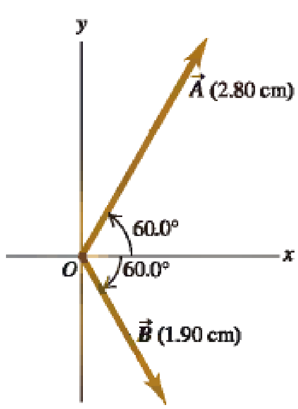
1.40. Find the magnitude and direction of the vector represented by the following pairs of components: (a) $A_x = -8.60$ cm, $A_y = 5.20$ cm; (b) $A_x = -9.70$ m, $A_y = -2.45$ m; (c) $A_x = 7.75$ km, $A_y = -2.70$ km.

1.41. A disoriented physics professor drives 3.25 km north, then 4.75 km west, and then 1.50 km south. Find the magnitude and direction of the resultant displacement, using the method of components. In a vector addition diagram (roughly to scale), show that the resultant displacement found from your diagram is in qualitative agreement with the result you obtained using the method of components.

1.42. Vector $\vec{A}$ has components $A_x = 1.30$ cm, $A_y = 2.25$ cm; vector $\vec{B}$ has components $B_x = 4.10$ cm, $B_y = -3.75$ cm. Find (a) the components of the vector sum $\vec{A} + \vec{B}$; (b) the magnitude and direction of $\vec{A} + \vec{B}$; (c) the components of the vector difference $\vec{B} - \vec{A}$; (d) the magnitude and direction of $\vec{B} - \vec{A}$.

1.43. Vector $\vec{A}$ is 2.80 cm long and is 60.0° above the x-axis in the first quadrant. Vector $\vec{B}$ is 1.90 cm long and is 60.0° below the x-axis in the fourth quadrant (Fig. 1.35). Use components to find the magnitude and direction of (a) $\vec{A} + \vec{B}$; (b) $\vec{A} - \vec{B}$; (c) $\vec{B} - \vec{A}$. In each case, sketch the vector addition or subtraction and show that your numerical answers are in qualitative agreement with your sketch.

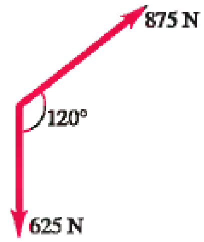
Figure 1.35 Exercises 1.43 and 1.59.



1.44. A river flows from south to north at 5.0 km/h. On this river, a boat is heading east to west perpendicular to the current at 7.0 km/h. As viewed by an eagle hovering at rest over the shore, how fast and in what direction is this boat traveling?

1.45. Use vector components to find the magnitude and direction of the vector needed to balance the two vectors shown in Figure 1.36. Let the 625-N vector be along the $-y$-axis and let the $+x$-axis be perpendicular to it toward the right.

Figure 1.36 Exercise 1.45.



1.46. Two ropes in a vertical plane exert equal magnitude forces on a hanging weight but pull with an angle of 86.0° between them. What pull does each one exert if their resultant pull is 372 N directly upward?

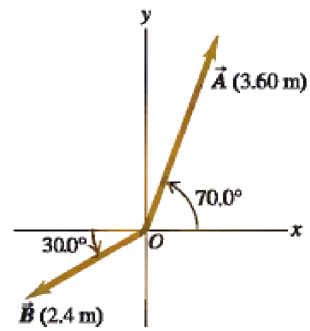
Section 1.9 Unit Vectors

1.47. Write each vector in Fig. 1.34 in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

1.48. In each case, find the x- and y-components of vector $\vec{A}$: (a) $\vec{A} = 5.0\hat{\imath} - 6.3\hat{\jmath}$; (b) $\vec{A} = 11.2\hat{\jmath} - 9.91\hat{\imath}$; (c) $\vec{A} = -15.0\hat{\imath} + 22.4\hat{\jmath}$; (d) $\vec{A} = 5.0\hat{B}$, where $\hat{B} = 4\hat{\imath} - 6\hat{\jmath}$.

1.49. (a) Write each vector in Fig. 1.37 in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$. (b) Use unit vectors to express the vector $\vec{C}$, where $\vec{C} = 3.00\vec{A} - 4.00\vec{B}$. (c) Find the magnitude and direction of $\vec{C}$.

Figure 1.37 Exercise 1.49 and Problem 1.86.



1.50. Given two vectors $\vec{A} = 4.00\hat{\imath} + 3.00\hat{\jmath}$ and $\vec{B} = 5.00\hat{\imath} - 2.00\hat{\jmath}$, (a) find the magnitude of each vector; (b) write an expression for the vector difference $\vec{A} - \vec{B}$ using unit vectors; (c) find the magnitude and direction of the vector difference $\vec{A} - \vec{B}$. (d) In a vector diagram show $\vec{A}$, $\vec{B}$, and $\vec{A} - \vec{B}$, and also show that your diagram agrees qualitatively with your answer in part (c).

1.51. (a) Is the vector $(\hat{\imath} + \hat{\jmath} + \hat{k})$ a unit vector? Justify your answer. (b) Can a unit vector have any components with magnitude greater than unity? Can it have any negative components? In each case justify your answer. (c) If $\vec{A} = a(3.0\hat{\imath} + 4.0\hat{\jmath})$, where a is a constant, determine the value of a that makes $\vec{A}$ a unit vector.

Section 1.10 Products of Vectors

1.52. (a) Use vector components to prove that two vectors commute for both addition and the scalar product. (b) Prove that two vectors *anticommute* for the vector product; that is, prove that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.

1.53. For the vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ in Fig. 1.34, find the scalar products (a) $\vec{A} \cdot \vec{B}$; (b) $\vec{B} \cdot \vec{C}$; (c) $\vec{A} \cdot \vec{C}$.

1.54. (a) Find the scalar product of the two vectors $\vec{A}$ and $\vec{B}$ given in Exercise 1.50. (b) Find the angle between these two vectors.

1.55. Find the angle between each of the following pairs of vectors:

(a) $\vec{A} = -2.00\hat{\imath} + 6.00\hat{\jmath}$ and $\vec{B} = 2.00\hat{\imath} - 3.00\hat{\jmath}$
(b) $\vec{A} = 3.00\hat{\imath} + 5.00\hat{\jmath}$ and $\vec{B} = 10.00\hat{\imath} + 6.00\hat{\jmath}$
(c) $\vec{A} = -4.00\hat{\imath} + 2.00\hat{\jmath}$ and $\vec{B} = 7.00\hat{\imath} + 14.00\hat{\jmath}$

1.56. By making simple sketches of the appropriate vector products, show that (a) $\vec{A} \cdot \vec{B}$ can be interpreted as the product of the magnitude of $\vec{A}$ times the component of $\vec{B}$ along $\vec{A}$, or the magnitude of $\vec{B}$ times the component of $\vec{A}$ along $\vec{B}$; (b) $|\vec{A} \times \vec{B}|$ can be interpreted as the product of the magnitude of $\vec{A}$ times the component of $\vec{B}$ perpendicular to $\vec{A}$, or the magnitude of $\vec{B}$ times the component of $\vec{A}$ perpendicular to $\vec{B}$.

1.57. For the vectors $\vec{A}$ and $\hat{D}$ in Fig. 1.34, (a) find the magnitude and direction of the vector product $\vec{A} \times \hat{D}$; (b) find the magnitude and direction of $\hat{D} \times \vec{A}$.

1.58. Find the vector product $\vec{A} \times \vec{B}$ (expressed in unit vectors) of the two vectors given in Exercise 1.50. What is the magnitude of the vector product?

1.58. For the two vectors in Fig. 1.35, (a) find the magnitude and direction of the vector product $\vec{A} \times \vec{B}$; (b) find the magnitude and direction of $\vec{B} \times \vec{A}$.

Problems

1.60. An acre, a unit of land measurement still in wide use, has a length of one furlong $\left(\frac{1}{8}\text{ mi}\right)$ and a width one-tenth of its length. (a) How many acres are in a square mile? (b) How many square feet are in an acre? See Appendix E. (c) An acre-foot is the volume of water that would cover 1 acre of flat land to a depth of 1 foot. How many gallons are in 1 acre-foot?

1.61. An Earthlike Planet. In January 2006, astronomers reported the discovery of a planet comparable in size to the earth orbiting another star and having a mass of about 5.5 times the earth's mass. It is believed to consist of a mixture of rock and ice, similar to Neptune. If this planet has the same density as Neptune $(1.76\ \text{g/cm}^3)$, what is its radius expressed (a) in kilometers and (b) as a multiple of earth's radius? Consult Appendix F for astronomical data.

1.62. The Hydrogen Maser. You can use the radio waves generated by a hydrogen maser as a standard of frequency. The frequency of these waves is 1,420,405,751.786 hertz. (A hertz is another name for one cycle per second.) A clock controlled by a hydrogen maser is off by only 1 s in 100,000 years. For the following questions, use only three significant figures. (The large number of significant figures given for the frequency simply illustrates the remarkable accuracy to which it has been measured.) (a) What is the time for one cycle of the radio wave? (b) How many cycles occur in 1 h? (c) How many cycles would have occurred during the age of the earth, which is estimated to be 4.6×10^9 years? (d) By how many seconds would a hydrogen maser clock be off after a time interval equal to the age of the earth?

1.63. Estimate the number of atoms in your body. (*Hint:* Based on what you know about biology and chemistry, what are the most common types of atom in your body? What is the mass of each type of atom? Appendix D gives the atomic masses for different elements, measured in atomic mass units; you can find the value of an atomic mass unit, or 1 u, in Appendix F.)

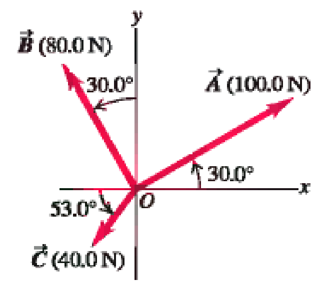
1.64. Biological tissues are typically made up of 98% water. Given that the density of water is $1.0 \times 10^3\ \text{kg/m}^3$, estimate the mass of (a) the heart of an adult human; (b) a cell with a diameter of $0.5\ \mu\text{m}$; (c) a honey bee.

1.65. Iron has a property such that a 1.00-m^3 volume has a mass of 7.86×10^3 kg (density equals $7.86 \times 10^3\ \text{kg/m}^3$). You want to manufacture iron into cubes and spheres. Find (a) the length of the side of a cube of iron that has a mass of 200.0 g and (b) the radius of a solid sphere of iron that has a mass of 200.0 g.

1.66. Stars in the Universe Astronomers frequently say that there are more stars in the universe than there are grains of sand on all the beaches on the earth. (a) Given that a typical grain of sand is about 0.2 mm in diameter, estimate the number of grains of sand on all the earth's beaches, and hence the approximate number of stars in the universe. It would be helpful to consult an atlas and do some measuring. (b) Given that a typical galaxy contains about 100 billion stars and there are more than 100 billion galaxies in the known universe, estimate the number of stars in the universe and compare this number with your result from part (a).

1.67. Physicists, mathematicians, and others often deal with large numbers. The number 10^{100} has been given the whimsical name *googol* by mathematicians. Let us compare some large numbers in physics with the googol. (*Note:* This problem requires numerical values that you can find in the appendices of the book, with which you should become familiar.) (a) Approximately how many atoms make up our planet? For simplicity, assume the average atomic mass of the atoms is 14 g/mol. Avogadro's number gives the number of atoms in a mole. (b) Approximately how many neutrons are in a neutron star? Neutron stars are composed almost entirely of neutrons and have approximately twice the mass of the sun. (c) In the leading theory of the origin of the universe, the entire universe that we can now observe occupied, at a very early time, a sphere whose radius was approximately equal to the present distance of the earth to the sun. At that time the universe had a density (mass divided by volume) of $10^{15}\ \text{g/cm}^3$. Assuming that one-third of the particles were protons, one-third of the particles were neutrons, and the remaining one-third were electrons, how many particles then made up the universe?

Figure 1.36 Problem 1.68.



1.60. Three horizontal ropes pull on a large stone stuck in the ground, producing the vector forces $\vec{A}$, $\vec{B}$, and $\vec{C}$ shown in Fig. 1.38. Find the magnitude and direction of a fourth force on the stone that will make the vector sum of the four forces zero.

1.69. Two workers pull horizontally on a heavy box, but one pulls twice as hard as the other. The larger pull is directed at 25.0° west of north, and the resultant of these two pulls is 350.0 N directly northward. Use vector components to find the magnitude of each of these pulls and the direction of the smaller pull.

1.70. Emergency Landing. A plane leaves the airport in Galisteo and flies 170 km at 68° east of north and then changes direction to fly 230 km at 48° south of east, after which it makes an immediate emergency landing in a pasture. When the airport sends out a rescue crew, in which direction and how far should this crew fly to go directly to this plane?

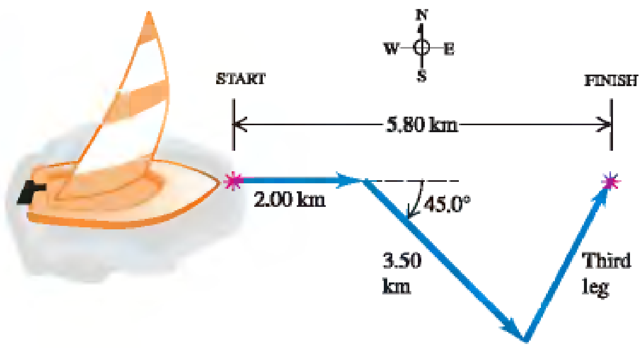
1.71. You are to program a robotic arm on an assembly line to move in the *xy*-plane. Its first displacement is $\vec{A}$; its second displacement is $\vec{B}$, of magnitude 6.40 cm and direction 63.0° measured in the sense from the $+x$-axis toward the $-y$-axis. The resultant $\vec{C} = \vec{A} + \vec{B}$ of the two displacements should also have a magnitude of 6.40 cm, but a direction 22.0° measured in the sense from the $+x$-axis toward the $+y$-axis. (a) Draw the vector addition diagram for these vectors, roughly to scale. (b) Find the components of $\vec{A}$. (c) Find the magnitude and direction of $\vec{A}$.

1.72. (a) Find the magnitude and direction of the vector $\vec{R}$ that is the sum of the three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ in Fig. 1.34. In a diagram, show how $\vec{R}$ is formed from these three vectors. (b) Find the magnitude and direction of the vector $\vec{S} = \vec{C} - \vec{A} - \vec{B}$. In a diagram, show how $\vec{S}$ is formed from these three vectors.

1.73. As noted in Exercise 1.33, a spelunker is surveying a cave. She follows a passage 180 m straight west, then 210 m in a direction 45° east of south, and then 280 m at 30° east of north. After a fourth unmeasured displacement she finds herself back where she started. Use the method of components to determine the magnitude and direction of the fourth displacement. Draw the vector addition diagram and show that it is in qualitative agreement with your numerical solution.

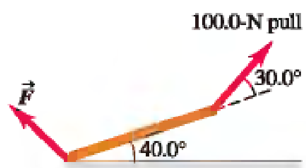
1.74. A sailor in a small sailboat encounters shifting winds. She sails 2.00 km east, then 3.50 km southeast, and then an additional distance in an unknown direction. Her final position is 5.80 km directly east of the starting point (Fig. 1.39). Find the magnitude and direction of the third leg of the journey. Draw the vector addition diagram and show that it is in qualitative agreement with your numerical solution.

Figure 1.39 Problem 1.74.



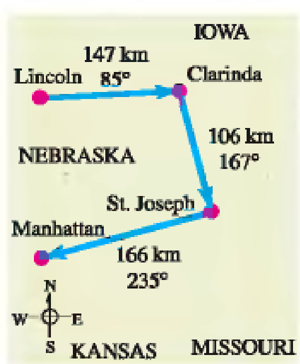
1.75. Equilibrium. We say an object is in *equilibrium* if all the forces on it balance (add up to zero). Figure 1.40 shows a beam weighing 124 N that is supported in equilibrium by a 100.0-N pull and a force $\vec{F}$ at the floor. The third force on the beam is the 124-N weight that acts vertically downward. (a) Use vector components to find the magnitude and direction of $\vec{F}$. (b) Check the reasonableness of your answer in part (a) by doing a graphical solution approximately to scale.

Figure 1.40 Problem 1.75.



1.76. On a training flight, a student pilot flies from Lincoln, Nebraska to Clarinda, Iowa, then to St. Joseph, Missouri, and then to Manhattan, Kansas (Fig. 1.41). The directions are shown relative to north: 0° is north, 90° is east, 180° is south, and 270° is west. Use the method of components to find (a) the distance she has to fly from Manhattan to get back to Lincoln, and (b) the direction (relative to north) she must fly to get there. Illustrate your solutions with a vector diagram.

Figure 1.41 Problem 1.76.



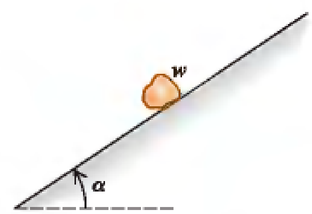
1.77. A graphic artist is creating a new logo for her company's website. In the graphics program she is using, each pixel in an image file has coordinates (x, y), where the origin $(0, 0)$ is at the upper left corner of the image, the $+x$-axis points to the right, and the $+y$-axis points down. Distances are measured in pixels. (a) The artist draws a line from the pixel location $(10, 20)$ to the location $(210, 200)$. She wishes to draw a second line that starts at $(10, 20)$, is 250 pixels long, and is at angle of 30° measured clockwise from the first line. At which pixel location should this second line end? Give your answer to the nearest pixel. (b) The artist now draws an arrow that connects the lower right end of the first line to the lower right end of the second line. Find the length and direction of this arrow. Draw a diagram showing all three lines.

1.78. Getting Back. An explorer in the dense jungles of equatorial Africa leaves his hut. He takes 40 steps northeast, then 80 steps 60° north of west, then 50 steps due south. Assume his steps all have equal length. (a) Sketch, roughly to scale, the three vectors and their resultant. (b) Save the explorer from becoming hopelessly lost in the jungle by giving him the displacement, calculated using the method of components, that will return him to his hut.

1.79. A ship leaves the island of Guam and sails 285 km at 40.0° north of west. In which direction must it now head and how far must it sail so that its resultant displacement will be 115 km directly east of Guam?

1.80. A boulder of weight w rests on a hillside that rises at a constant angle α above the horizontal, as shown in Fig. 1.42. Its weight is a force on the boulder that has direction vertically downward. (a) In terms of α and w, what is the component of the weight of the boulder in the direction parallel to the surface of the hill? (b) What is the component of the weight in the direction perpendicular to the surface of the hill? (c) An air conditioner unit is fastened to a roof that slopes upward at an angle of 35.0°. In order that the unit not slide down the roof, the component of the unit's weight parallel to the roof cannot exceed 550 N. What is the maximum allowed weight of the unit?

Figure 1.42 Problem 1.80.



1.81. Bones and Muscles. A patient in therapy has a forearm that weighs 20.5 N and that lifts a 112.0-N weight. These two forces have direction vertically downward. The only other significant forces on his forearm come from the biceps muscle (which acts perpendicularly to the forearm) and the force at the elbow. If the biceps produces a pull of 232 N when the forearm is raised 43° above the horizontal, find the magnitude and direction of the force that the elbow exerts on the forearm. (The sum of the elbow force and the biceps force must balance the weight of the arm and the weight it is carrying, so their vector sum must be 132.5 N, upward.)

1.82. You are hungry and decide to go to your favorite neighborhood fast-food restaurant. You leave your apartment and take the elevator 10 flights down (each flight is 3.0 m) and then go 15 m south to the apartment exit. You then proceed 0.2 km east, turn north, and go 0.1 km to the entrance of the restaurant. (a) Determine the displacement from your apartment to the restaurant. Use unit vector notation for your answer, being sure to make clear your choice of coordinates. (b) How far did you travel along the path you took from your apartment to the restaurant, and what is the magnitude of the displacement you calculated in part (a)?

1.83. While following a treasure map, you start at an old oak tree. You first walk 825 m directly south, then turn and walk 1.25 km at 30.0° west of north, and finally walk 1.00 km at 40.0° north of east, where you find the treasure: a biography of Isaac Newton! (a) To return to the old oak tree, in what direction should you head and how far will you walk? Use components to solve this problem.

(b) To see whether your calculation in part (a) is reasonable, check it with a graphical solution drawn roughly to scale.

1.64. You are camping with two friends, Joe and Karl. Since all three of you like your privacy, you don't pitch your tents close together. Joe's tent is 21.0 m from yours, in the direction 23.0° south of east. Karl's tent is 32.0 m from yours, in the direction 37.0° north of east. What is the distance between Karl's tent and Joe's tent?

1.85. Vectors $\vec{A}$ and $\vec{B}$ are drawn from a common point. Vector $\vec{A}$ has magnitude A and angle θ_A measured in the sense from the $+x$-axis to the $+y$-axis. The corresponding quantities for vector $\vec{B}$ are B and θ_B. Then $\vec{A} = A\cos\theta_A\hat{\imath} + A\sin\theta_A\hat{\jmath}$, $\vec{B} = B\cos\theta_B\hat{\imath} + B\sin\theta_B\hat{\jmath}$, and $\phi = |\theta_B - \theta_A|$ is the angle between $\vec{A}$ and $\vec{B}$. (a) Derive Eq. (1.18) from Eq. (1.21). (b) Derive Eq. (1.22) from Eqs. (1.27).

1.86. For the two vectors $\vec{A}$ and $\vec{B}$ in Fig. 1.37, (a) find the scalar product $\vec{A}\cdot\vec{B}$, and (b) find the magnitude and direction of the vector product $\vec{A}\times\vec{B}$.

1.67. Figure 1.11c shows a parallelogram based on the two vectors $\vec{A}$ and $\vec{B}$. (a) Show that the magnitude of the cross product of these two vectors is equal to the area of the parallelogram. (*Hint:* Area = base × height.) (b) What is the angle between the cross product and the plane of the parallelogram?

1.88. The vector $\vec{A}$ is 3.50 cm long and is directed into this page. Vector $\vec{B}$ points from the lower right corner of this page to the upper left corner of this page. Define an appropriate right-handed coordinate system and find the three components of the vector product $\vec{A}\times\vec{B}$, measured in cm^2. In a diagram, show your coordinate system and the vectors $\vec{A}$, $\vec{B}$, and $\vec{A}\times\vec{B}$.

1.89. Given two vectors $\vec{A} = -2.00\hat{\imath} + 3.00\hat{\jmath} + 4.00\hat{k}$ and $\vec{B} = 3.00\hat{\imath} + 1.00\hat{\jmath} - 3.00\hat{k}$, do the following. (a) Find the magnitude of each vector. (b) Write an expression for the vector difference $\vec{A} - \vec{B}$, using unit vectors. (c) Find the magnitude of the vector difference $\vec{A} - \vec{B}$. Is this the same as the magnitude of $\vec{B} - \vec{A}$? Explain.

1.98. Bond Angle in Methane. In the methane molecule, CH_4, each hydrogen atom is at a corner of a regular tetrahedron with the carbon atom at the center. In coordinates where one of the C—H bonds is in the direction of $\hat{\imath} + \hat{\jmath} + \hat{k}$, an adjacent C—H bond is in the $\hat{\imath} - \hat{\jmath} - \hat{k}$ direction. Calculate the angle between these two bonds.

1.91. The two vectors $\vec{A}$ and $\vec{B}$ are drawn from a common point, and $\vec{C} = \vec{A} + \vec{B}$. (a) Show that if $C^2 = A^2 + B^2$, the angle between the vectors $\vec{A}$ and $\vec{B}$ is 90°. (b) Show that if $C^2 < A^2 + B^2$, the angle between the vectors $\vec{A}$ and $\vec{B}$ is greater than 90°. (c) Show that if $C^2 > A^2 + B^2$, the angle between the vectors $\vec{A}$ and $\vec{B}$ is between 0° and 90°.

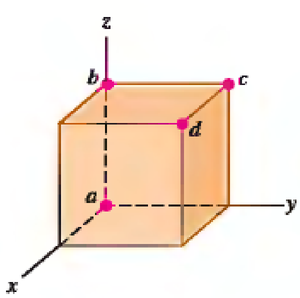
1.92. When two vectors $\vec{A}$ and $\vec{B}$ are drawn from a common point, the angle between them is ϕ. (a) Using vector techniques, show that the magnitude of their vector sum is given by

$$\sqrt{A^2 + B^2 + 2AB\cos\phi}$$

(b) If $\vec{A}$ and $\vec{B}$ have the same magnitude, for which value of ϕ will their vector sum have the same magnitude as $\vec{A}$ or $\vec{B}$?

1.93. A cube is placed so that one corner is at the origin and three edges are along the x-, y-, and z-axes of a coordinate system (Fig. 1.43). Use vectors to compute (a) the angle between the edge along the z-axis (line ab) and the diagonal from the origin to the opposite corner (line ad), and (b) the angle between line ac (the diagonal of a face) and line ad.

Figure 1.43 Problem 1.93.



1.94. Obtain a *unit vector* perpendicular to the two vectors given in Problem 1.89.

1.95. You are given vectors $\vec{A} = 5.0\hat{\imath} - 6.5\hat{\jmath}$ and $\vec{B} = -3.5\hat{\imath} + 7.0\hat{\jmath}$. A third vector $\vec{C}$ lies in the xy-plane. Vector $\vec{C}$ is perpendicular to vector $\vec{A}$, and the scalar product of $\vec{C}$ with $\vec{B}$ is 15.0. From this information, find the components of vector $\vec{C}$.

1.96. Two vectors $\vec{A}$ and $\vec{B}$ have magnitude $A = 3.00$ and $B = 3.00$. Their vector product is $\vec{A}\times\vec{B} = -5.00\hat{k} + 2.00\hat{\imath}$. What is the angle between $\vec{A}$ and $\vec{B}$?

1.97. Later in our study of physics we will encounter quantities represented by $(\vec{A}\times\vec{B})\cdot\vec{C}$. (a) Prove that for any three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, $\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$. (b) Calculate $(\vec{A}\times\vec{B})\cdot\vec{C}$ for the three vectors $\vec{A}$ with magnitude $A = 5.00$ and angle $\theta_A = 26.0°$ measured in the sense from the $+x$-axis toward the $+y$-axis, $\vec{B}$ with $B = 4.00$ and $\theta_B = 63.0°$, and $\vec{C}$ with magnitude 6.00 and in the $+z$-direction. Vectors $\vec{A}$ and $\vec{B}$ are in the xy-plane.

Challenge Problems

1.98. The length of a rectangle is given as $L \pm l$ and its width as $W \pm w$. (a) Show that the uncertainty in its area A is $a = Lw + lW$. Assume that the uncertainties l and w are small, so that the product lw is very small and you can ignore it. (b) Show that the fractional uncertainty in the area is equal to the sum of the fractional uncertainty in length and the fractional uncertainty in width. (c) A rectangular solid has dimensions $L \pm l$, $W \pm w$, and $H \pm h$. Find the fractional uncertainty in the volume, and show that it equals the sum of the fractional uncertainties in the length, width, and height.

1.99. Completed Pass. At Enormous State University (ESU), the football team records its plays using vector displacements, with the origin taken to be the position of the ball before the play starts. In a certain pass play, the receiver starts at $+1.0\hat{\imath} - 5.0\hat{\jmath}$, where the units are yards, $\hat{\imath}$ is to the right, and $\hat{\jmath}$ is downfield. Subsequent displacements of the receiver are $+9.0\hat{\imath}$ (in motion before the snap), $+11.0\hat{\jmath}$ (breaks downfield), $-6.0\hat{\imath} + 4.0\hat{\jmath}$ (zigs), and $+12.0\hat{\imath} + 18.0\hat{\jmath}$ (zags). Meanwhile, the quarterback has dropped straight back to a position $-7.0\hat{\jmath}$. How far and in which direction must the quarterback throw the ball? (Like the coach, you will be well advised to diagram the situation before solving it numerically.)

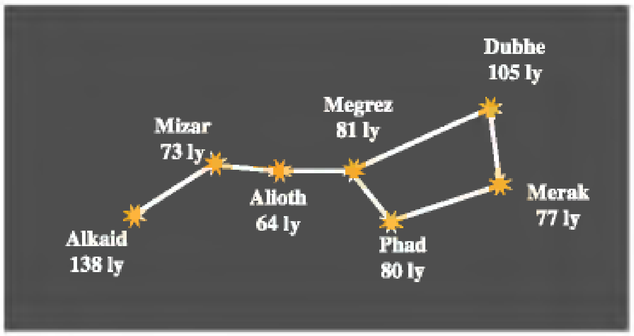
1.100. Navigating in the Solar System. The *Mars Polar Lander* spacecraft was launched on January 3, 1999. On December 3, 1999, the day that *Mars Polar Lander* touched down on the Martian surface, the positions of the earth and Mars were given by these coordinates:

	x	y	z
Earth	0.3182 AU	0.9329 AU	0.0000 AU
Mars	1.3087 AU	−0.4423 AU	−0.0414 AU

In these coordinates, the sun is at the origin and the plane of the earth's orbit is the xy-plane. The earth passes through the $+x$-axis once a year on the autumnal equinox, the first day of autumn in the northern hemisphere (on or about September 22). One AU, or *astronomical unit*, is equal to 1.496×10^8 km, the average distance from the earth to the sun. (a) In a diagram, show the positions of the sun, the earth, and Mars on December 3, 1999. (b) Find the following distances in AU on December 3, 1999: (i) from the sun to the earth; (ii) from the sun to Mars; (iii) from the earth to Mars. (c) As seen from the earth, what was the angle between the direction to the sun and the direction to Mars on December 3, 1999? (d) Explain whether Mars was visible from your location at midnight on December 3, 1999. (When it is midnight at your location, the sun is on the opposite side of the earth from you.)

1.101. Navigating in the Big Dipper. All the stars of the Big Dipper (part of the constellation Ursa Major) may appear to be the same distance from the earth, but in fact they are very far from each other. Figure 1.44 shows the distances from the earth to each of these stars. The distances are given in light years (ly), the distance that light travels in one year. One light year equals 9.461×10^{15} m. (a) Alkaid and Merak are 25.6° apart in the earth's sky. In a diagram, show the relative positions of Alkaid, Merak, and our sun. Find the distance in light years from Alkaid to Merak. (b) To an inhabitant of a planet orbiting Merak, how many degrees apart in the sky would Alkaid and our sun be?

Figure 1.44 Challenge Problem 1.101.



1.102. The vector $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, called the *position vector*, points from the origin $(0, 0, 0)$ to an arbitrary point in space with coordinates (x, y, z). Use what you know about vectors to prove the following: All points (x, y, z) that satisfy the equation $Ax + By + Cz = 0$, where A, B, and C are constants, lie in a plane that passes through the origin and that is perpendicular to the vector $A\hat{\imath} + B\hat{\jmath} + C\hat{k}$. Sketch this vector and the plane.

2 MOTION ALONG A STRAIGHT LINE

LEARNING GOALS

By studying this chapter, you will learn:

- How to describe straight-line motion in terms of average velocity, instantaneous velocity, average acceleration, and instantaneous acceleration.
- How to interpret graphs of position versus time, velocity versus time, and acceleration versus time for straight-line motion.
- How to solve problems involving straight-line motion with constant acceleration, including free-fall problems.
- How to analyze straight-line motion when the acceleration is not constant.

? A typical sprinter speeds up during the first third of a race and slows gradually over the rest of the course. Is it accurate to say that a sprinter is *accelerating* as he slows during the final two-thirds of the race?

What distance must an airliner travel down a runway before reaching takeoff speed? When you throw a baseball straight up in the air, how high does it go? When a glass slips from your hand, how much time do you have to catch it before it hits the floor? These are the kinds of questions you will learn to answer in this chapter. We are beginning our study of physics with *mechanics,* the study of the relationships among force, matter, and motion. In this chapter and the next we will study *kinematics,* the part of mechanics that enables us to describe motion. Later we will study *dynamics,* which relates motion to its causes.

In this chapter we concentrate on the simplest kind of motion: a body moving along a straight line. To describe this motion, we introduce the physical quantities *velocity* and *acceleration.* These quantities have simple definitions in physics; however, those definitions are more precise and slightly different than the ones used in everyday language. An important part of how a physicist defines velocity and acceleration is that these quantities are *vectors.* As you learned in Chapter 1, this means that they have both magnitude and direction. Our concern in this chapter is with motion along a straight line only, so we won't need the full mathematics of vectors just yet. But using vectors will be essential in Chapter 3 when we consider motion in two or three dimensions.

We'll develop simple equations to describe straight-line motion in the important special case when the acceleration is constant. An example is the motion of a freely falling body. We'll also consider situations in which the acceleration varies during the motion; in this case, it's necessary to use integration to describe the motion. (If you haven't studied integration yet, Section 2.6 is optional.)

2.1 Displacement, Time, and Average Velocity

Suppose a drag racer drives her AA-fuel dragster along a straight track (Fig. 2.1). To study the dragster's motion, we need a coordinate system. We choose the x-axis to lie along the dragster's straight-line path, with the origin O at the starting line. We also choose a point on the dragster, such as its front end, and represent the entire dragster by that point. Hence we treat the dragster as a **particle.**

A useful way to describe the motion of the particle—that is, the point that represents the dragster—is in terms of the change in the particle's coordinate x over a time interval. Suppose that 1.0 s after the start the front of the dragster is at point P_1, 19 m from the origin, and 4.0 s after the start it is at point P_2, 277 m from the origin. The *displacement* of the particle is a vector that points from P_1 to P_2 (see Section 1.7). Figure 2.1 shows that this vector points along the x-axis. The x-component of the displacement is just the change in the value of x, $(277\text{ m} - 19\text{ m}) = 258\text{ m}$, that took place during the time interval of $(4.0\text{ s} - 1.0\text{ s}) = 3.0\text{ s}$. We define the dragster's **average velocity** during this time interval as a *vector* quantity whose x-component is the change in x divided by the time interval: $(258\text{ m})/(3.0\text{ s}) = 86\text{ m/s}$.

In general, the average velocity depends on the particular time interval chosen. For a 3.0-s time interval *before* the start of the race, the average velocity would be zero because the dragster would be at rest at the starting line and would have zero displacement.

Let's generalize the concept of average velocity. At time t_1 the dragster is at point P_1, with coordinate x_1, and at time t_2 it is at point P_2, with coordinate x_2. The displacement of the dragster during the time interval from t_1 to t_2 is the vector from P_1 to P_2. The x-component of the displacement, denoted Δx, is just the change in the coordinate x:

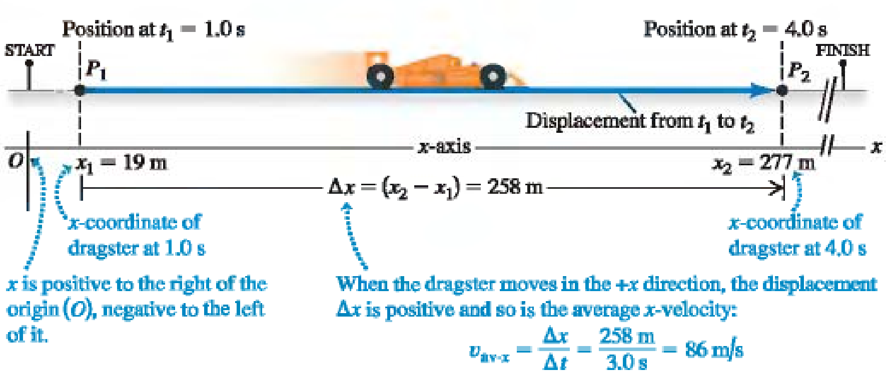
$$\Delta x = x_2 - x_1 \tag{2.1}$$

The dragster moves along the x-axis only, so the y- and z-components of the displacement are equal to zero.

CAUTION **The meaning of Δx** Note that Δx is *not* the product of Δ and x; it is a single symbol that means "the change in the quantity x." We always use the Greek capital letter Δ (delta) to represent a *change* in a quantity, equal to the *final* value of the quantity minus the *initial* value—never the reverse. Likewise, the time interval from t_1 to t_2 is Δt, the change in the quantity t: $\Delta t = t_2 - t_1$ (final time minus initial time).

The x-component of average velocity, or **average x-velocity,** is the x-component of displacement, Δx, divided by the time interval Δt during which the displacement occurs. We use the symbol $v_{\text{av-}x}$ for average x-velocity (the

2.1 Positions of a dragster at two times during its run.



subscript "av" signifies average value and the subscript x indicates that this is the x-component):

$$v_{\text{av-}x} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t} \quad \text{(average } x\text{-velocity, straight-line motion)} \qquad (2.2)$$

As an example, for the dragster $x_1 = 19$ m, $x_2 = 277$ m, $t_1 = 1.0$ s, and $t_2 = 4.0$ s, so Eq. (2.2) gives

$$v_{\text{av-}x} = \frac{277\text{ m} - 19\text{ m}}{4.0\text{ s} - 1.0\text{ s}} = \frac{258\text{ m}}{3.0\text{ s}} = 86\text{ m/s}$$

The average x-velocity of the dragster is positive. This means that during the time interval, the coordinate x increased and the dragster moved in the positive x-direction (to the right in Fig. 2.1).

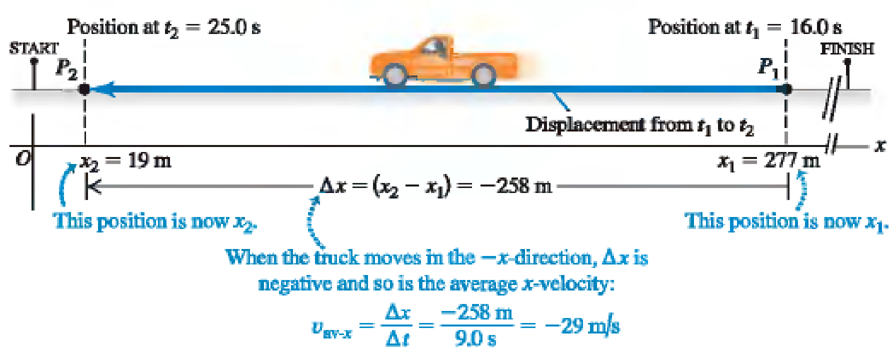
If a particle moves in the *negative* x-direction during a time interval, its average velocity for that time interval is negative. For example, suppose an official's truck moves to the left along the track (Fig. 2.2). The truck is at $x_1 = 277$ m at $t_1 = 16.0$ s and is at $x_2 = 19$ m at $t_2 = 25.0$ s. Then $\Delta x = (19\text{ m} - 277\text{ m}) = -258$ m and $\Delta t = (25.0\text{ s} - 16.0\text{ s}) = 9.0$ s. The x-component of average velocity is $v_{\text{av-}x} = \Delta x/\Delta t = (-258\text{ m})/(9.0\text{ s}) = -29$ m/s.

Here are some simple rules for the average x-velocity. **Whenever x is positive and increasing or is negative and becoming less negative, the particle is moving in the $+x$-direction and $v_{\text{av-}x}$ is positive** (Fig. 2.1). **Whenever x is positive and decreasing or is negative and becoming more negative, the particle is moving in the $-x$-direction and $v_{\text{av-}x}$ is negative** (Fig. 2.2).

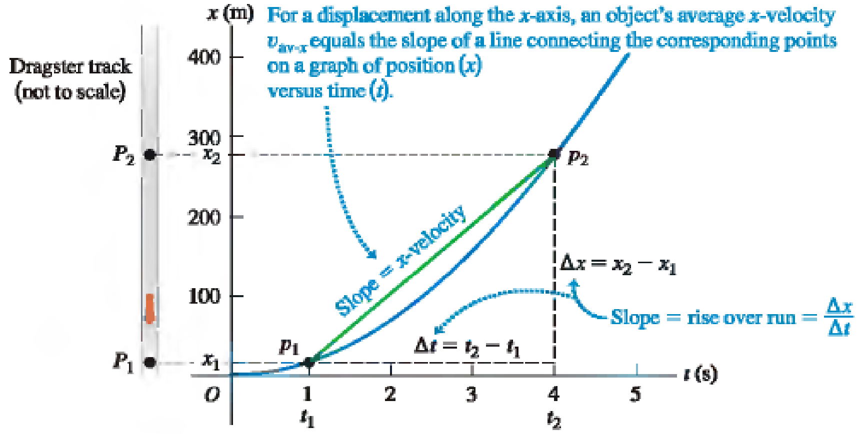
CAUTION **Choice of the positive x-direction** You might be tempted to conclude that positive average x-velocity must mean motion to the right, as in Fig. 2.1, and that negative average x-velocity must mean motion to the left, as in Fig. 2.2. But that's correct *only* if the positive x-direction is to the right, as we chose it to be in Figs. 2.1 and 2.2. Had we chosen the positive x-direction to be to the left, with the origin at the finish line, the dragster would have negative average x-velocity and the official's truck would have positive average x-velocity. In most problems the direction of the coordinate axis will be yours to choose. Once you've made your choice, you *must* take it into account when interpreting the signs of $v_{\text{av-}x}$ and other quantities that describe motion!

With straight-line motion we sometimes call Δx simply the displacement and $v_{\text{av-}x}$ simply the average velocity. But be sure to remember that these are really the x-components of vector quantities that, in this special case, have *only* x-components. In Chapter 3, displacement, velocity, and acceleration vectors will have two or three nonzero components.

Figure 2.3 is a graph of the dragster's position as a function of time—that is, an ***x-t* graph.** The curve in the figure *does not* represent the dragster's path in space; as Fig. 2.1 shows, the path is a straight line. Rather, the graph is a pictorial way to represent how the dragster's position changes with time. The points p_1



2.2 Positions of an official's truck at two times during its motion. The points P_1 and P_2 now indicate the positions of the truck, and so are the reverse of Fig. 2.1.



2.3 The position of a dragster as a function of time.

and p_2 on the graph correspond to the points P_1 and P_2 along the dragster's path. Line p_1p_2 is the hypotenuse of a right triangle with vertical side $\Delta x = x_2 - x_1$ and horizontal side $\Delta t = t_2 - t_1$. The average x-velocity $v_{av\text{-}x} = \Delta x/\Delta t$ of the dragster equals the *slope* of the line p_1p_2—that is, the ratio of the triangle's vertical side Δx to its horizontal side Δt.

The average x-velocity depends only on the total displacement $\Delta x = x_2 - x_1$ that occurs during the time interval $\Delta t = t_2 - t_1$, not on the details of what happens during the time interval. At time t_1 a motorcycle might have raced past the dragster at point P_1 in Fig. 2.1, then blown its engine and slowed down to pass through point P_2 at the same time t_2 as the dragster. Both vehicles have the same displacement during the same time interval and so have the same average x-velocity.

If distance is given in meters and time in seconds, average velocity is measured in meters per second (m/s). Other common units of velocity are kilometers per hour (km/h), feet per second (ft/s), miles per hour (mi/h), and knots (1 knot = 1 nautical mile/h = 6080 ft/h). Table 2.1 lists some typical velocity magnitudes.

Table 2.1 Typical Velocity Magnitudes

A snail's pace	10^{-3} m/s
A brisk walk	2 m/s
Fastest human	11 m/s
Running cheetah	35 m/s
Fastest car	341 m/s
Random motion of air molecules	500 m/s
Fastest airplane	1000 m/s
Orbiting communications satellite	3000 m/s
Electron orbiting in a hydrogen atom	2×10^6 m/s
Light traveling in a vacuum	3×10^8 m/s

Test Your Understanding of Section 2.1 Each of the following automobile trips takes one hour. The positive x-direction is to the east. (i) Automobile A travels 50 km due east. (ii) Automobile B travels 50 km due west. (iii) Automobile C travels 60 km due east, then turns around and travels 10 km due west. (iv) Automobile D travels 70 km due east. (v) Automobile E travels 20 km due west, then turns around and travels 20 km due east. (a) Rank the five trips in order of average x-velocity from most positive to most negative. (b) Which trips, if any, have the same average x-velocity? (c) For which trip, if any, is the average x-velocity eqnal to zero?

2.2 Instantaneous Velocity

Sometimes the average velocity is all you need to know about a particle's motion. For example, a race along a straight line is really a competition to see whose average velocity, $v_{av\text{-}x}$, has the greatest magnitude. The prize goes to the competitor who can travel the displacement Δx from the start to the finish line in the shortest time interval, Δt (Fig. 2.4).

But the average velocity of a particle during a time interval can't tell us how fast, or in what direction, the particle was moving at any given time during the interval. To do this we need to know the velocity at any specific instant of time or specific point along the path. This is called **instantaneous velocity,** and it needs to be defined carefully.

CAUTION **How long is an instant?** Note that the word "instant" has a somewhat different definition in physics than in everyday language. You might use the phrase "It lasted just an instant" to refer to something that lasted for a very short time interval. But in physics an instant has no duration at all; it refers to a single value of time.

2.4 The winner of a 50-m swimming race is the swimmer whose average velocity has the greatest magnitude—that is, the swimmer who traverses a displacement Δx of 50 m in the shortest elapsed time Δt.

2.5 Even when he's moving forward, this cyclist's instantaneous x-velocity can be negative—if he's traveling in the negative x-direction. In any problem, the choice of which direction is positive and which is negative is entirely up to you.

To find the instantaneous velocity of the dragster in Fig. 2.1 at the point P_1, we move the second point P_2 closer and closer to the first point P_1 and compute the average velocity $v_{\text{av-}x} = \Delta x/\Delta t$ over the ever-shorter displacement and time interval. Both Δx and Δt become very small, but their ratio does not necessarily become small. In the language of calculus, the limit of $\Delta x/\Delta t$ as Δt approaches zero is called the **derivative** of x with respect to t and is written dx/dt. *The instantaneous velocity is the limit of the average velocity as the time interval approaches zero; it equals the instantaneous rate of change of position with time.* We use the symbol v_x, with no "av" subscript, for the instantaneous velocity along the x-axis, or the **instantaneous x-velocity:**

$$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \qquad \text{(instantaneous } x\text{-velocity, straight-line motion)} \qquad (2.3)$$

The time interval Δt is always positive, so v_x has the same algebraic sign as Δx. A positive value of v_x means that x is increasing and the motion is in the positive x-direction; a negative value of v_x means that x is decreasing and the motion is in the negative x-direction. A body can have positive x and negative v_x, or the reverse; x tells us where the body is, while v_x tells us how it's moving (Fig. 2.5).

Instantaneous velocity, like average velocity, is a vector quantity; Eq. (2.3) defines its x-component. In straight-line motion, all other components of instantaneous velocity are zero. In this case we often call v_x simply the instantaneous velocity. (In Chapter 3 we'll deal with the general case in which the instantaneous velocity can have nonzero x-, y-, and z-components.) When we use the term "velocity," we will always mean instantaneous rather than average velocity.

The terms "velocity" and "speed" are used interchangeably in everyday language, but they have distinct definitions in physics. We use the term **speed** to denote distance traveled divided by time, on either an average or an instantaneous basis. We use the symbol v with *no* subscripts to denote instantaneous speed. Instantaneous *speed* measures how fast a particle is moving; instantaneous *velocity* measures how fast *and* in what direction it's moving. For example, a particle with instantaneous velocity $v_x = 25\text{ m/s}$ and a second particle with $v_x = -25\text{ m/s}$ are moving in opposite directions at the same instantaneous speed 25 m/s. Instantaneous speed is the magnitude of instantaneous velocity, and so instantaneous speed can never be negative.

CAUTION **Average speed and average velocity** Average speed is *not* the magnitude of average velocity. When Alexander Popov set a world record in 1994 by swimming 100.0 m in 46.74 s, his average speed was $(100.0\text{ m})/(46.74\text{ s}) = 2.139\text{ m/s}$. But because he swam two lengths in a 50-m pool, he started and ended at the same point and so had zero total displacement and zero average *velocity!* Both average speed and instantaneous speed are scalars, not vectors, because these quantities contain no information about direction.

Example 2.1 Average and instantaneous velocities

A cheetah is crouched 20 m to the east of an observer's vehicle (Fig. 2.6a). At time $t = 0$ the cheetah charges an antelope and begins to run along a straight line. During the first 2.0 s of the attack, the cheetah's coordinate x varies with time according to the equation $x = 20\text{ m} + (5.0\text{ m/s}^2)t^2$. (a) Find the displacement of the cheetah between $t_1 = 1.0\text{ s}$ and $t_2 = 2.0\text{ s}$. (b) Find the average velocity during the same time interval. (c) Find the instantaneous velocity at time $t_1 = 1.0\text{ s}$ by taking $\Delta t = 0.1\text{ s}$, then $\Delta t = 0.01\text{ s}$, then $\Delta t = 0.001\text{ s}$. (d) Derive a general expression for the instantaneous velocity as a function of time, and from it find v_x at $t = 1.0\text{ s}$ and $t = 2.0\text{ s}$.

SOLUTION

IDENTIFY: We use the definitions of displacement, average velocity, and instantaneous velocity. Using the first two of these involves algebra; the last one requires using calculus to take a derivative.

SET UP: Figure 2.6b shows our sketch of the cheetah's motion. To analyze this problem we use Eq. (2.1) for displacement, Eq. (2.2) for average velocity, and Eq. (2.3) for instantaneous velocity.

EXECUTE: (a) At time $t_1 = 1.0$ s the cheetah's position x_1 is

$$x_1 = 20\text{ m} + (5.0\text{ m/s}^2)(1.0\text{ s})^2 = 25\text{ m}$$

At time $t_2 = 2.0$ s its position x_2 is

$$x_2 = 20\text{ m} + (5.0\text{ m/s}^2)(2.0\text{ s})^2 = 40\text{ m}$$

The displacement during this interval is

$$\Delta x = x_2 - x_1 = 40\text{ m} - 25\text{ m} = 15\text{ m}$$

(b) The average x-velocity during this time interval is

$$v_{\text{av-}x} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{40\text{ m} - 25\text{ m}}{2.0\text{ s} - 1.0\text{ s}} = \frac{15\text{ m}}{1.0\text{ s}} = 15\text{ m/s}$$

(c) With $\Delta t = 0.1$ s, the time interval is from $t_1 = 1.0$ s to $t_2 = 1.1$ s. At time t_2, the position is

$$x_2 = 20\text{ m} + (5.0\text{ m/s}^2)(1.1\text{ s})^2 = 26.05\text{ m}$$

The average x-velocity during this interval is

$$v_{\text{av-}x} = \frac{26.05\text{ m} - 25\text{ m}}{1.1\text{ s} - 1.0\text{ s}} = 10.5\text{ m/s}$$

You should follow this same pattern to work out the average x-velocities for the 0.01-s and 0.001-s intervals. The results are 10.05 m/s and 10.005 m/s. As Δt gets smaller, the average x-velocity gets closer to 10.0 m/s, so we conclude that the instantaneous x-velocity at time $t = 1.0$ s is 10.0 m/s.

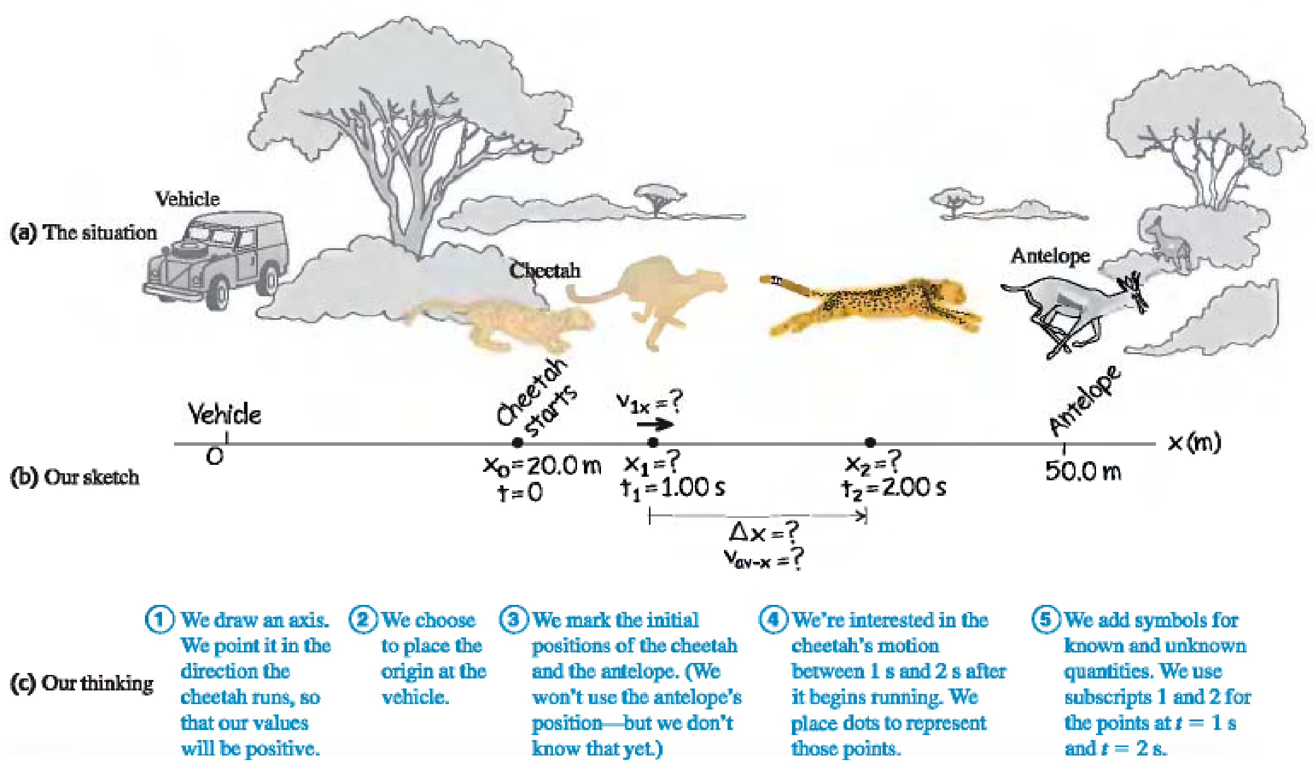
(d) To find the instantaneous x-velocity as a function of time, take the derivative of the expression for x with respect to t. The derivative of a constant is zero, and for any n the derivative of t^n is nt^{n-1}, so the derivative of t^2 is $2t$. Therefore

$$v_x = \frac{dx}{dt} = (5.0\text{ m/s}^2)(2t) = (10\text{ m/s}^2)t$$

At time $t = 1.0$ s, $v_x = 10$ m/s as we found in part (c). At time $t = 2.0$ s, $v_x = 20$ m/s.

EVALUATE: Our results show that the cheetah picked up speed from $t = 0$ (when it was at rest) to $t = 1.0$ s $(v_x = 10\text{ m/s})$ to $t = 2.0$ s $(v_x = 20\text{ m/s})$. This makes sense; the cheetah covered only 5 m during the interval $t = 0$ to $t = 1.0$ s, but covered 15 m during the interval $t = 1.0$ s to $t = 2.0$ s.

2.6 A cheetah attacking an antelope from ambush. The animals are not drawn to the same scale as the axis.



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1.1 Analyzing Motion Using Diagrams

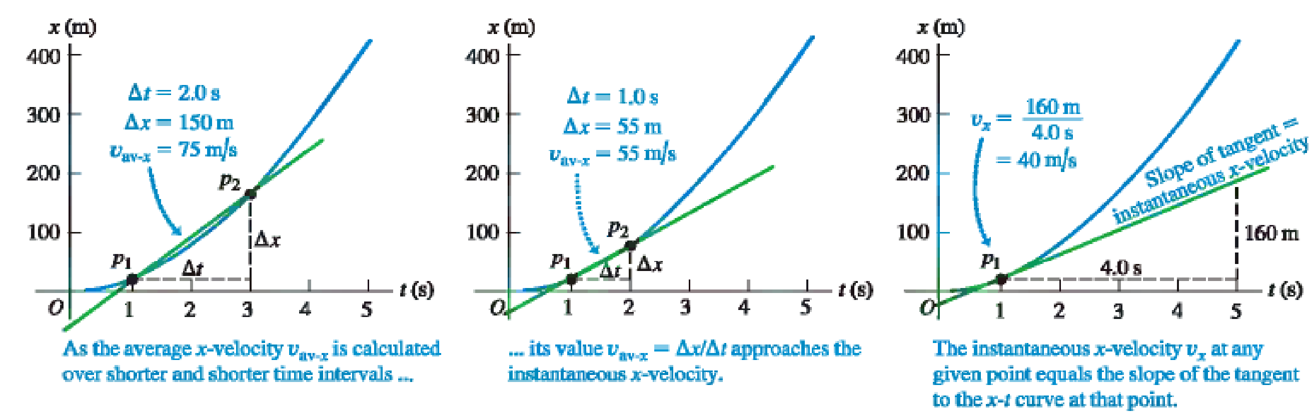
Finding Velocity on an x-t Graph

The x-velocity of a particle can also be found from the graph of its position as a function of time. Suppose we want to find the x-velocity of the dragster in Fig. 2.1 at point P_1. As point P_2 in Fig. 2.1 approaches point P_1, point p_2 in the x-t graphs of Figs. 2.7a and 2.7b approaches point p_1 and the average x-velocity is calculated over shorter time intervals Δt. In the limit that $\Delta t \to 0$, shown in Fig. 2.7c, the slope of the line p_1p_2 equals the slope of the line tangent to the curve at point p_1. Thus, *on a graph of position as a function of time for straight-line motion, the instantaneous x-velocity at any point is equal to the slope of the tangent to the curve at that point.*

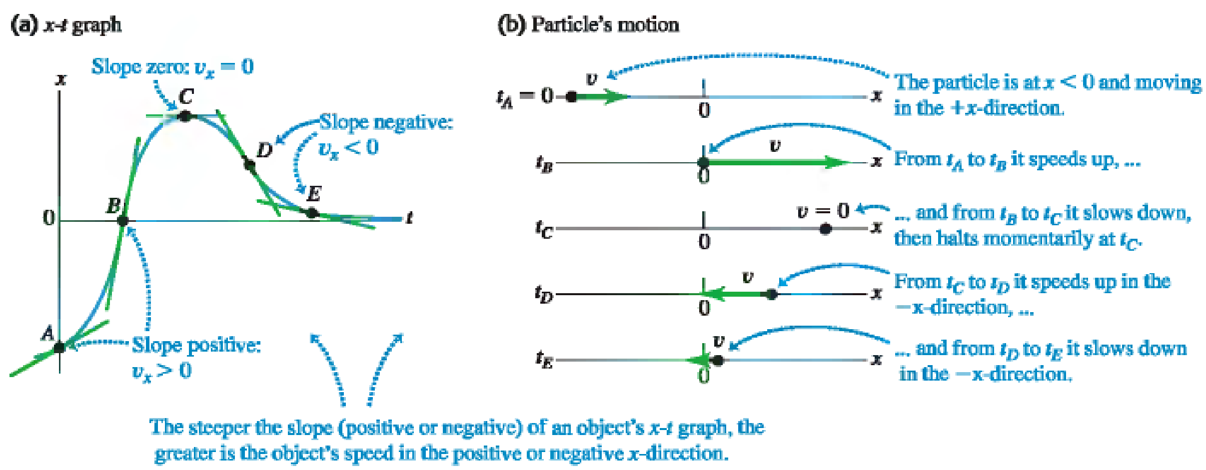
If the tangent to the x-t curve slopes upward to the right, as in Fig. 2.7c, then its slope is positive, the x-velocity is positive, and the motion is in the positive x-direction. If the tangent slopes downward to the right, the slope of the x-t graph and the x-velocity are negative, and the motion is in the negative x-direction. When the tangent is horizontal, the slope and the x-velocity are zero. Figure 2.8 illustrates these three possibilities.

Figure 2.8 actually depicts the motion of a particle in two ways: as (a) an x-t graph and (b) a **motion diagram.** A motion diagram shows the particle's posi-

2.7 Using an x-t graph to go from **(a)**, **(b)** average x-velocity to **(c)** instantaneous x-velocity v_x. In **(c)** we find the slope of the tangent to the x-t curve by dividing any vertical interval (with distance units) along the tangent by the corresponding horizontal interval (with time units).



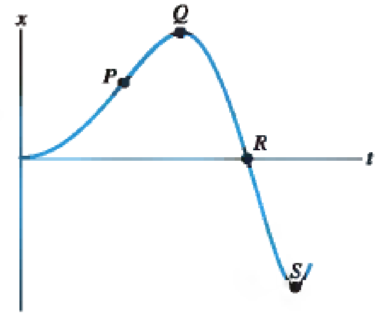
2.8 **(a)** The x-t graph of the motion of a particular particle. The slope of the tangent at any point equals the velocity at that point. **(b)** A motion diagram showing the position and velocity of the particle at each of the times labeled on the x-t graph.



tion at various times (like frames from a video of the particle's motion) as well as arrows to represent the particle's velocity at each instant. We will use both x-t graphs and motion diagrams in this chapter to help you understand motion. You will find it worth your while to draw *both* an x-t graph and a motion diagram as part of solving any problem involving motion.

Test Your Understanding of Section 2.2 Figure 2.9 is an x-t graph of the motion of a particle. (a) Rank the values of the particle's x-velocity v_x at the points P, Q, R, and S from most positive to most negative. (b) At which points is v_x positive? (c) At which points is v_x negative? (d) At which points is v_x zero? (e) Rank the values of the particle's *speed* at the points P, Q, R, and S from fastest to slowest.

2.9 An x-t graph for a particle.



2.3 Average and Instantaneous Acceleration

Just as velocity describes the rate of change of position with time, *acceleration* describes the rate of change of velocity with time. Like velocity, acceleration is a vector quantity. When the motion is along a straight line, its only nonzero component is along that line. As we'll see, acceleration in straight-line motion can refer to either speeding up or slowing down.

Average Acceleration

Let's consider again a particle moving along the x-axis. Suppose that at time t_1 the particle is at point P_1 and has x-component of (instantaneous) velocity v_{1x}, and at a later time t_2 it is at point P_2 and has x-component of velocity v_{2x}. So the x-component of velocity changes by an amount $\Delta v_x = v_{2x} - v_{1x}$ during the time interval $\Delta t = t_2 - t_1$.

We define the **average acceleration** of the particle as it moves from P_1 to P_2 to be a vector quantity whose x-component $a_{\text{av-}x}$ (called the **average x-acceleration**) equals Δv_x, the change in the x-component of velocity, divided by the time interval Δt:

$$a_{\text{av-}x} = \frac{v_{2x} - v_{1x}}{t_2 - t_1} = \frac{\Delta v_x}{\Delta t} \quad \text{(average } x\text{-acceleration, straight-line motion)} \tag{2.4}$$

For straight-line motion along the x-axis we will often call $a_{\text{av-}x}$ simply the average acceleration. (We'll encounter the other components of the average acceleration vector in Chapter 3.)

If we express velocity in meters per second and time in seconds, then average acceleration is in meters per second per second, or $(\text{m/s})/\text{s}$. This is usually written as m/s^2 and is read "meters per second squared."

CAUTION **Acceleration vs. velocity** Be very careful not to confuse acceleration with velocity! Velocity describes how a body's position changes with time; it tells us how fast and in what direction the body moves. Acceleration describes how the velocity changes with time; it tells us how the speed and direction of motion are changing. It may help to remember the phrase "acceleration is to velocity as velocity is to position." It can also help to imagine yourself riding along with the moving body. If the body accelerates forward and gains speed, you would feel pushed backward in your seat; if it accelerates backward and loses speed, you would feel pushed forward. If the velocity is constant and there's no acceleration, you would feel neither sensation. (We'll see the reason for these sensations in Chapter 4.)

Example 2.2 Average acceleration

An astronaut has left an orbiting spacecraft to test a new personal maneuvering unit. As she moves along a straight line, her partner on the spacecraft measures her velocity every 2.0 s, starting at time $t = 1.0$ s:

t	v_x	t	v_x
1.0 s	0.8 m/s	9.0 s	−0.4 m/s
3.0 s	1.2 m/s	11.0 s	−1.0 m/s
5.0 s	1.6 m/s	13.0 s	−1.6 m/s
7.0 s	1.2 m/s	15.0 s	−0.8 m/s

Find the average x-acceleration, and describe whether the speed of the astronaut increases or decreases, for each of these time intervals: (a) $t_1 = 1.0$ s to $t_2 = 3.0$ s; (b) $t_1 = 5.0$ s to $t_2 = 7.0$ s; (c) $t_1 = 9.0$ s to $t_2 = 11.0$ s; (d) $t_1 = 13.0$ s to $t_2 = 15.0$ s.

SOLUTION

IDENTIFY: We'll need the definition of average acceleration $a_{av\text{-}x}$. To find the changes in speed, we'll use the idea that speed v is the magnitude of the instantaneous velocity v_x.

SET UP: Figure 2.10 shows our graphs. We use Eq. (2.4) to find the value of $a_{av\text{-}x}$ from the change in *velocity* for each time interval.

EXECUTE: In the upper part of Fig. 2.10, we graph the x-velocity as a function of time. On this v_x-t graph, the slope of the line connecting the points at the beginning and end of each interval equals the average x-acceleration $a_{av\text{-}x} = \Delta v_x/\Delta t$ for that interval. In the lower part of Fig. 2.10, we graph the values of $a_{av\text{-}x}$. We find:

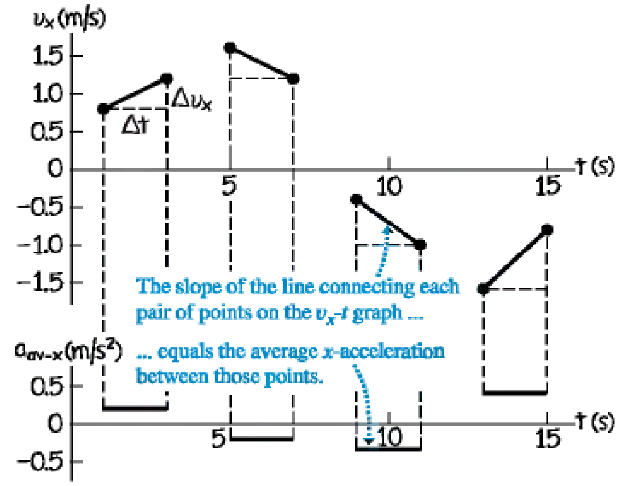
(a) $a_{av\text{-}x} = (1.2\text{ m/s} - 0.8\text{ m/s})/(3.0\text{ s} - 1.0\text{ s}) = 0.2\text{ m/s}^2$. The speed (magnitude of instantaneous x-velocity) increases from 0.8 m/s to 1.2 m/s.

(b) $a_{av\text{-}x} = (1.2\text{ m/s} - 1.6\text{ m/s})/(7.0\text{ s} - 5.0\text{ s}) = -0.2\text{ m/s}^2$. The speed decreases from 1.6 m/s to 1.2 m/s.

(c) $a_{av\text{-}x} = [-1.0\text{ m/s} - (-0.4\text{ m/s})]/(11.0\text{ s} - 9.0\text{ s}) = -0.3\text{ m/s}^2$. The speed increases from 0.4 m/s to 1.0 m/s.

(d) $a_{av\text{-}x} = [-0.8\text{ m/s} - (-1.6\text{ m/s})]/(15.0\text{ s} - 13.0\text{ s}) = 0.4\text{ m/s}^2$. The speed decreases from 1.6 m/s to 0.8 m/s.

2.10 Our graphs of x-velocity versus time (top) and average x-acceleration versus time (bottom) for the astronaut.



EVALUATE: Our results show that when the average x-acceleration has the *same* direction (same algebraic sign) as the initial velocity, as in intervals (a) and (c), the astronaut goes faster; when it has the *opposite* direction (opposite algebraic sign), as in intervals (b) and (d), she slows down. Thus positive x-acceleration means speeding up if the x-velocity is positive [interval (a)] but slowing down if the x-velocity is negative [interval (d)]. Similarly, negative x-acceleration means speeding up if the x-velocity is negative [interval (c)] but slowing down if the x-velocity is positive [interval (b)].

Instantaneous Acceleration

We can now define **instantaneous acceleration** following the same procedure that we used to define instantaneous velocity. As an example, suppose a race car driver is driving along a straightaway as shown in Fig. 2.11. To define the instantaneous acceleration at point P_1, we take the second point P_2 in Fig. 2.11 to be closer and closer to P_1 so that the average acceleration is computed over shorter and shorter time intervals. *The instantaneous acceleration is the limit of the average acceleration as the time interval approaches zero.* In the language of calculus, *instantaneous acceleration equals the instantaneous rate of change of velocity with time.* Thus

$$a_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt} \quad \text{(instantaneous } x\text{-acceleration, straight-line motion)} \tag{2.5}$$

2.11 A Grand Prix car at two points on the straightaway.

Note that a_x in Eq. (2.5) is really the x-component of the acceleration vector, or the **instantaneous x-acceleration;** in straight-line motion, all other components of this vector are zero. From now on, when we use the term "acceleration," we will always mean instantaneous acceleration, not average acceleration.

Example 2.3 Average and instantaneous accelerations

Suppose the x-velocity v_x of the car in Fig. 2.11 at any time t is given by the equation

$$v_x = 60\text{ m/s} + (0.50\text{ m/s}^3)t^2$$

(a) Find the change in x-velocity of the car in the time interval between $t_1 = 1.0$ s and $t_2 = 3.0$ s. (b) Find the average x-acceleration in this time interval. (c) Find the instantaneous x-acceleration at time $t_1 = 1.0$ s by taking Δt to be first 0.1 s, then 0.01 s, then 0.001 s. (d) Derive an expression for the instantaneous x-acceleration at any time, and use it to find the x-acceleration at $t = 1.0$ s and $t = 3.0$ s.

SOLUTION

IDENTIFY: This example is analogous to Example 2.1 in Section 2.2. (Now is a good time to review that example.) There we found the average x-velocity over shorter and shorter time intervals from the change in position, and we determined the instantaneous x-velocity by differentiating the position as a function of time. In this example, we find the *average* x-acceleration from the change in x-velocity over a time interval. Likewise, we find the *instantaneous* x-acceleration by differentiating the x-velocity as a function of time.

SET UP: We'll use Eq. (2.4) for average x-acceleration and Eq. (2.5) for instantaneous x-acceleration.

EXECUTE: (a) We first find the x-velocity at each time by substituting each value of t into the equation. At time $t_1 = 1.0$ s,

$$v_{1x} = 60\text{ m/s} + (0.50\text{ m/s}^3)(1.0\text{ s})^2 = 60.5\text{ m/s}$$

At time $t_2 = 3.0$ s,

$$v_{2x} = 60\text{ m/s} + (0.50\text{ m/s}^3)(3.0\text{ s})^2 = 64.5\text{ m/s}$$

The change in x-velocity Δv_x is

$$\Delta v_x = v_{2x} - v_{1x} = 64.5\text{ m/s} - 60.5\text{ m/s} = 4.0\text{ m/s}$$

The time interval is $\Delta t = 3.0\text{ s} - 1.0\text{ s} = 2.0$ s.

(b) The average x-acceleration during this time interval is

$$a_{\text{av-}x} = \frac{v_{2x} - v_{1x}}{t_2 - t_1} = \frac{4.0\text{ m/s}}{2.0\text{ s}} = 2.0\text{ m/s}^2$$

During the time interval from $t_1 = 1.0$ s to $t_2 = 3.0$ s, the x-velocity and average x-acceleration have the same algebraic sign (in this case, positive), and the car speeds up.

(c) When $\Delta t = 0.1$ s, $t_2 = 1.1$ s and we find

$$v_{2x} = 60\text{ m/s} + (0.50\text{ m/s}^3)(1.1\text{ s})^2 = 60.605\text{ m/s}$$

$$\Delta v_x = 0.105\text{ m/s}$$

$$a_{\text{av-}x} = \frac{\Delta v_x}{\Delta t} = \frac{0.105\text{ m/s}}{0.1\text{ s}} = 1.05\text{ m/s}^2$$

You should do these calculations for $\Delta t = 0.01$ s and $\Delta t = 0.001$ s; the results are $a_{\text{av-}x} = 1.005\text{ m/s}^2$ and $a_{\text{av-}x} = 1.0005\text{ m/s}^2$, respectively. As Δt gets smaller, the average x-acceleration gets closer to 1.0 m/s^2, so the instantaneous x-acceleration at $t = 1.0$ s is 1.0 m/s^2.

(d) The instantaneous x-acceleration is $a_x = dv_x/dt$. The derivative of a constant is zero and the derivative of t^2 is $2t$, so

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}[60\text{ m/s} + (0.50\text{ m/s}^3)t^2]$$

$$= (0.50\text{ m/s}^3)(2t) = (1.0\text{ m/s}^3)t$$

When $t = 1.0$ s,

$$a_x = (1.0\text{ m/s}^3)(1.0\text{ s}) = 1.0\text{ m/s}^2$$

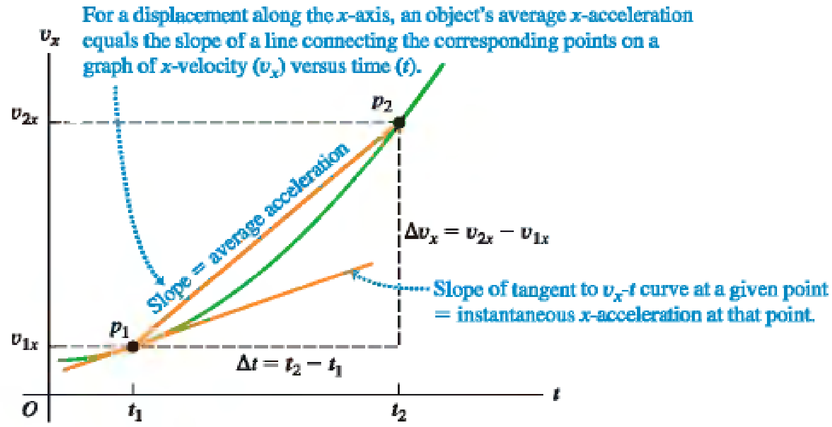
When $t = 3.0$ s,

$$a_x = (1.0\text{ m/s}^3)(3.0\text{ s}) = 3.0\text{ m/s}^2$$

EVALUATE: Note that neither of the values we found in part (d) is equal to the average x-acceleration found in part (b). That's because the car's instantaneous x-acceleration varies with time. The rate of change of acceleration with time is sometimes called the "jerk."

Finding Acceleration on a v_x-t Graph or an x-t Graph

In Section 2.2 we interpreted average and instantaneous x-velocity in terms of the slope of a graph of position versus time. In the same way, we can interpret average and instantaneous x-acceleration by using a graph with instantaneous velocity v_x on the vertical axis and time t on the horizontal axis—that is, a **v_x-t graph** (Fig. 2.12). The points on the graph labeled p_1 and p_2 correspond to points P_1 and P_2 in Fig. 2.11. The average x-acceleration $a_{\text{av-}x} = \Delta v_x/\Delta t$ during this interval is the slope of the line p_1p_2. As point P_2 in Fig. 2.11 approaches point P_1, point p_2 in the v_x-t graph of Fig. 2.12 approaches point p_1, and the slope of the line p_1p_2 approaches the slope of the line tangent to the curve at point p_1. Thus, *on a graph of x-velocity as a function of time, the instantaneous x-acceleration at any point is equal to the slope of the tangent to the curve at that point.* Tangents drawn at



2.12 A v_x-t graph of the motion in Fig. 2.11.

different points along the curve in Fig. 2.12 have different slopes, so the instantaneous x-acceleration varies with time.

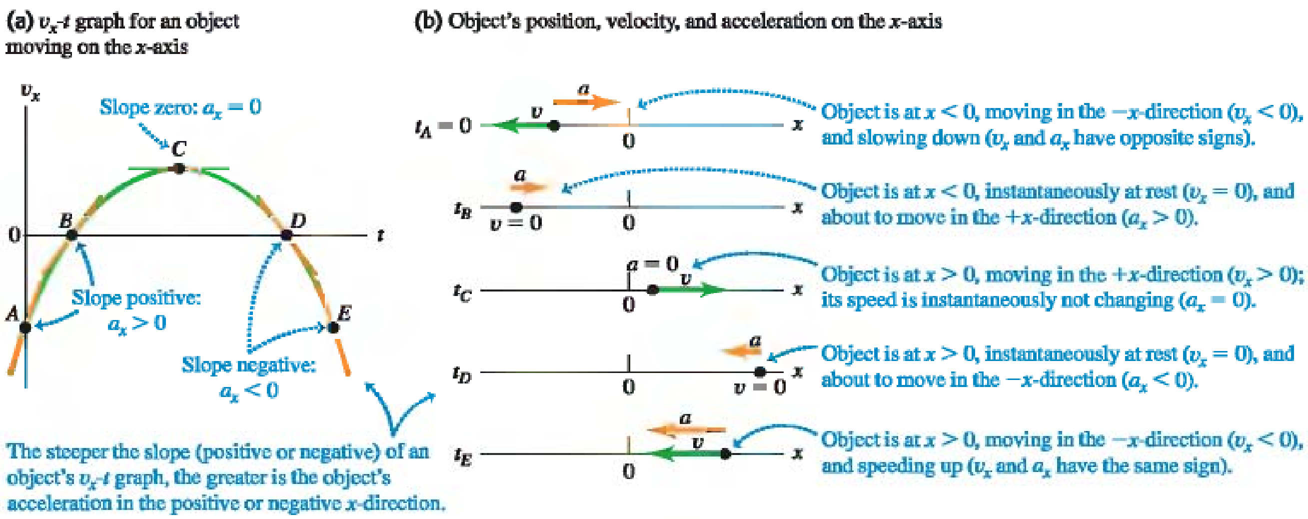
CAUTION **The signs of x-acceleration and x-velocity** By itself, the algebraic sign of the x-acceleration does *not* tell you whether a body is speeding up or slowing down. You must compare the signs of the x-velocity and the x-acceleration. When v_x and a_x have the *same* sign, the body is speeding up. If both are positive, the body is moving in the positive direction with increasing speed. If both are negative, the body is moving in the negative direction with an x-velocity that is becoming more and more negative, and again the speed is increasing. When v_x and a_x have *opposite* signs, the body is slowing down. If v_x is positive and a_x is negative, the body is moving in the positive direction with decreasing speed; if v_x is negative and a_x is positive, the body is moving in the negative direction with an x-velocity that is becoming less negative, and again the body is slowing down. Figure 2.13 illustrates some of these possibilities.

The term "deceleration" is sometimes used for a decrease in speed. Because it may mean positive or negative a_x, depending on the sign of v_x, we avoid this term.

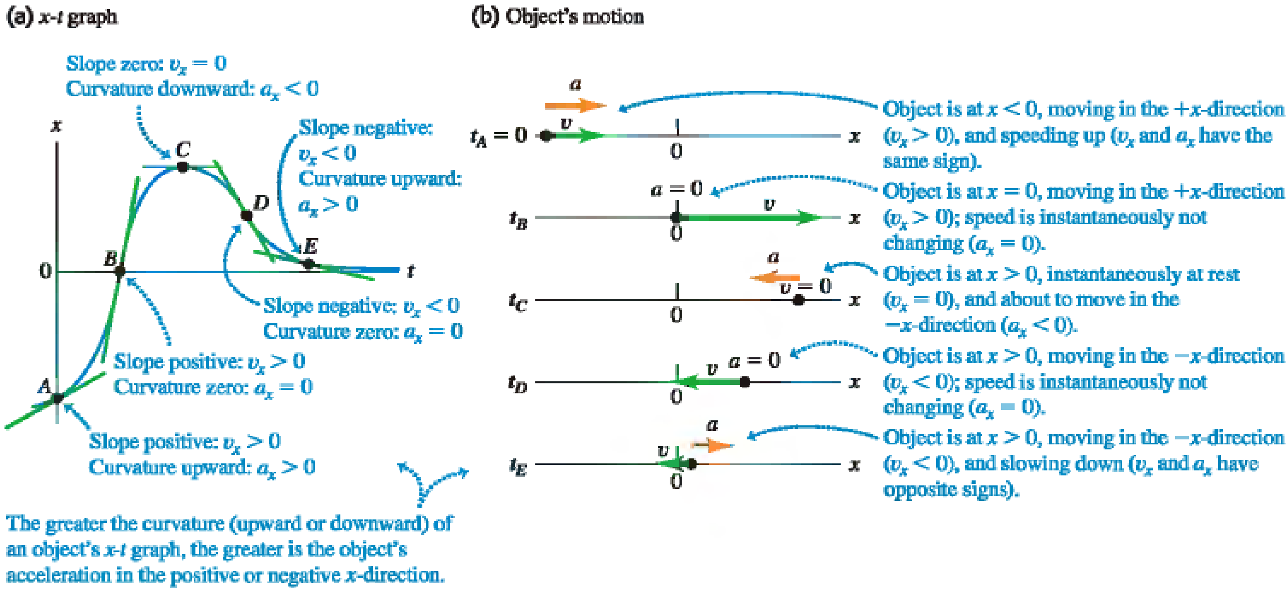
We can also learn about the acceleration of a body from a graph of its *position* versus time. Because $a_x = dv_x/dt$ and $v_x = dx/dt$, we can write

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} \qquad (2.6)$$

2.13 (a) A v_x-t graph of the motion of a different particle than that shown in Fig. 2.8. The slope of the tangent at any point equals the x-acceleration at that point. (b) A motion diagram showing the position, velocity, and acceleration of the particle at each of the times labeled on the v_x-t graph. The positions are consistent with the v_x-t graph; for instance, from t_A to t_B the velocity is negative, so at t_B the particle is at a more negative value of x than at t_A.



2.14 (a) The same x-t graph as shown in Fig. 2.8a. The x-velocity is equal to the *slope* of the graph, and the acceleration is given by the *concavity* or *curvature* of the graph. (b) A motion diagram showing the position, velocity, and acceleration of the particle at each of the times labeled on the x-t graph.



That is, a_x is the second derivative of x with respect to t. The second derivative of any function is directly related to the *concavity* or *curvature* of the graph of that function. At a point where the x-t graph is concave up (curved upward), the x-acceleration is positive and v_x is increasing; at a point where the x-t graph is concave down (curved downward), the x-acceleration is negative and v_x is decreasing. At a point where the x-t graph has no curvature, such as an inflection point, the x-acceleration is zero and the velocity is not changing. Figure 2.14 shows all three of these possibilities.

Examining the curvature of an x-t graph is an easy way to decide what the *sign* of acceleration is. This technique is less helpful for determining numerical values of acceleration because the curvature of a graph is hard to measure accurately.

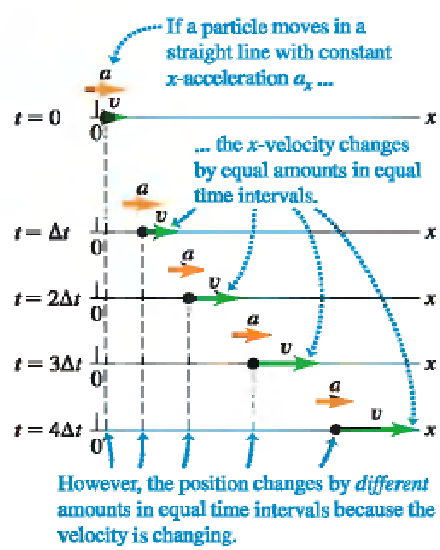
Test Your Understanding of Section 2.3 Look again at the x-t graph in Fig. 2.9 at the end of Section 2.2. (a) At which of the points P, Q, R, and S is the x-acceleration a_x positive? (b) At which points is the x-acceleration negative? (c) At which points does the x-acceleration appear to be zero? (d) At each point state whether the speed is increasing, decreasing, or not changing.

2.4 Motion with Constant Acceleration

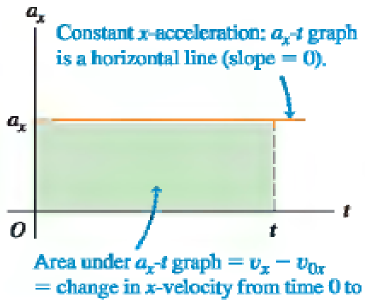
The simplest kind of accelerated motion is straight-line motion with *constant* acceleration. In this case the velocity changes at the same rate throughout the motion. This is a very special situation, yet one that occurs often in nature. A falling body has a constant acceleration if the effects of the air are not important. The same is true for a body sliding on an incline or along a rough horizontal surface. Straight-line motion with nearly constant acceleration also occurs in technology, such as an airplane being catapulted from the deck of an aircraft carrier.

Figure 2.15 is a motion diagram showing the position, velocity, and acceleration for a particle moving with constant acceleration. Figures 2.16 and 2.17 depict this same motion in the form of graphs. Since the x-acceleration is constant, the **a_x-t graph** (graph of x-acceleration versus time) in Fig. 2.16 is a horizontal line. The graph of x-velocity versus time, or v_x-t graph, has a constant *slope* because the acceleration is constant, so this graph is a straight line (Fig. 2.17).

2.15 A motion diagram for a particle moving in a straight line in the positive x-direction with constant positive x-acceleration a_x. The position, velocity, and acceleration are shown at five equally spaced times.



2.16 An acceleration-time (a_x-t) graph for straight-line motion with constant positive x-acceleration a_x.



When the x-acceleration a_x is constant, the average x-acceleration $a_{\text{av-}x}$ for any time interval is the same as a_x. This makes it easy to derive equations for the position x and the x-velocity v_x as functions of time. To find an expression for v_x, we first replace $a_{\text{av-}x}$ in Eq. (2.4) by a_x:

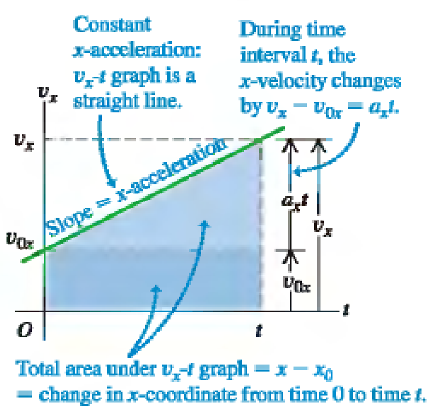
$$a_x = \frac{v_{2x} - v_{1x}}{t_2 - t_1} \qquad (2.7)$$

Now we let $t_1 = 0$ and let t_2 be any later time t. We use the symbol v_{0x} for the x-velocity at the initial time $t = 0$; the x-velocity at the later time t is v_x. Then Eq. (2.7) becomes

$$a_x = \frac{v_x - v_{0x}}{t - 0} \qquad \text{or}$$

$$v_x = v_{0x} + a_x t \qquad \text{(constant } x\text{-acceleration only)} \qquad (2.8)$$

2.17 A velocity-time (v_x-t) graph for straight-line motion with constant positive x-acceleration a_x. The initial x-velocity v_{0x} is also positive in this case.



We can interpret this equation as follows. The x-acceleration a_x is the constant rate of change of x-velocity—that is, the change in x-velocity per unit time. The term $a_x t$ is the product of the change in x-velocity per unit time, a_x, and the time interval t. Therefore it equals the *total* change in x-velocity from the initial time $t = 0$ to the later time t. The x-velocity v_x at any time t then equals the initial x-velocity v_{0x} (at $t = 0$) plus the change in x-velocity $a_x t$ (see Fig. 2.17).

Another interpretation of Eq. (2.8) is that the change in x-velocity $v_x - v_{0x}$ of the particle between $t = 0$ and any later time t equals the *area* under the a_x-t graph between those two times. In Fig. 2.16, the area under the graph of x-acceleration versus time is a rectangle of vertical side a_x and horizontal side t. The area of this rectangle is $a_x t$, which from Eq. (2.8) is indeed equal to the change in velocity $v_x - v_{0x}$. In Section 2.6 we'll show that even if the x-acceleration is not constant, the change in x-velocity during a time interval is still equal to the area under the a_x-t curve, although in that case Eq. (2.8) does not apply.

Next we'll derive an equation for the position x as a function of time when the x-acceleration is constant. To do this, we use two different expressions for the average x-velocity $v_{\text{av-}x}$ during the interval from $t = 0$ to any later time t. The first expression comes from the definition of $v_{\text{av-}x}$, Eq. (2.2), which is true whether or not the acceleration is constant. We call the position at time $t = 0$ the *initial position*, denoted by x_0. The position at the later time t is simply x. Thus for the time interval $\Delta t = t - 0$ the displacement is $\Delta x = x - x_0$, and Eq. (2.2) gives

$$v_{\text{av-}x} = \frac{x - x_0}{t} \qquad (2.9)$$

We can also get a second expression for $v_{\text{av-}x}$ that is valid only when the x-acceleration is constant, so that the v_x-t graph is a straight line (as in Fig. 2.17) and the x-velocity changes at a constant rate. In this case the average x-velocity during any time interval is simply the arithmetic average of the x-velocities at the beginning and end of the interval. For the time interval 0 to t,

$$v_{\text{av-}x} = \frac{v_{0x} + v_x}{2} \qquad \text{(constant } x\text{-acceleration only)} \qquad (2.10)$$

(This equation is *not* true if the x-acceleration varies and the v_x-t graph is a curve, as in Fig. 2.13.) We also know that with constant x-acceleration, the x-velocity v_x at any time t is given by Eq. (2.8). Substituting that expression for v_x into Eq. (2.10), we find

$$\begin{aligned} v_{\text{av-}x} &= \frac{1}{2}(v_{0x} + v_{0x} + a_x t) \\ &= v_{0x} + \frac{1}{2}a_x t \qquad \text{(constant } x\text{-acceleration only)} \qquad (2.11) \end{aligned}$$

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1.1 Analyzing Motion Using Diagrams
1.2 Analyzing Motion Using Graphs
1.3 Predicting Motion from Graphs
1.4 Predicting Motion from Equations
1.5 Problem-Solving Strategies for Kinematics
1.6 Skier Races Downhill

Finally, we set Eqs. (2.9) and (2.11) equal to each other and simplify:

$$v_{0x} + \frac{1}{2}a_x t = \frac{x - x_0}{t} \qquad \text{or}$$

$$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \qquad \text{(constant } x\text{-acceleration only)} \qquad (2.12)$$

Here's what Eq. (2.12) tells us: If at time $t = 0$ a particle is at position x_0 and has x-velocity v_{0x}, its new position x at any later time t is the sum of three terms—its initial position x_0, plus the distance $v_{0x}t$ that it would move if its x-velocity were constant, plus au additional distance $\frac{1}{2}a_x t^2$ caused by the change in x-velocity.

A graph of Eq. (2.12)—that is, an x-t graph for motion with constant x-acceleration (Fig. 2.18a)—is always a *parabola*. Figure 2.18b shows such a graph. The curve intercepts the vertical axis (x-axis) at x_0, the position at $t = 0$. The slope of the tangent at $t = 0$ equals v_{0x}, the initial x-velocity, and the slope of the tangent at any time t equals the x-velocity v_x at that time. The slope and x-velocity are continuously increasing, so the x-acceleration a_x is positive; you can also see this because the graph in Fig. 2.18b is concave up (it curves upward). If a_x is negative, the x-t graph is a parabola that is concave down (has a downward curvature).

If there is zero x-acceleration, the x-t graph is a straight line; if there is a constant x-acceleration, the additional $\frac{1}{2}a_x t^2$ term in Eq. (2.12) for x as a function of t curves the graph into a parabola (Fig. 2.19a). We can analyze the v_x-t graph in the same way. If there is zero x-acceleration this graph is a horizontal line (the x-velocity is constant); adding a constant x-acceleration gives a slope to the v_x-t graph (Fig. 2.19b).

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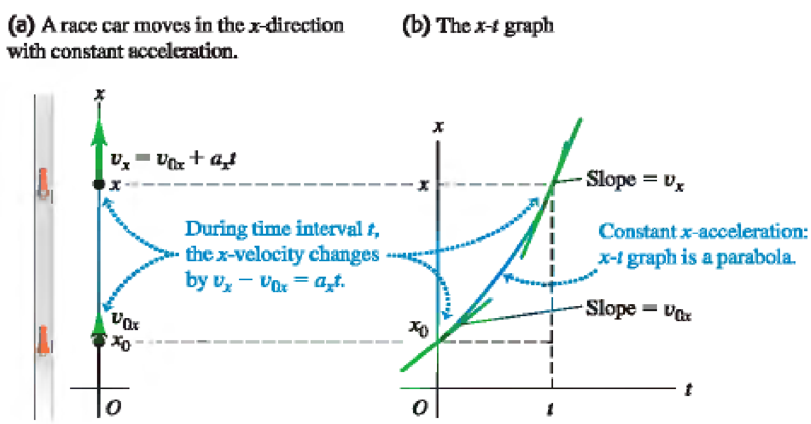
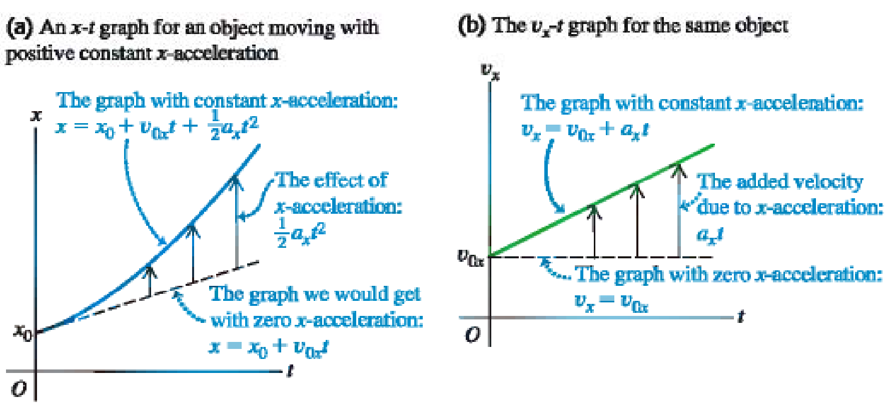
1.8 Seat Belts Save Lives
1.9 Screeching to a Halt
1.10 Car Starts, Then Stops
1.11 Solving Two-Vehicle Problems
1.12 Car Catches Truck
1.13 Avoiding a Rear-End Collision



2.18 (a) Straight-line motion with constant acceleration. (b) A position-time (x-t) graph for this motion (the same motion as is shown in Figs. 2.15, 2.16, and 2.17). For this motion the initial position x_0, the initial velocity v_{0x}, and the acceleration a_x are all positive.



2.19 (a) How a constant x-acceleration affects a body's (a) x-t graph and (b) v_x-t graph.

Just as the change in x-velocity of the particle equals the area under the a_x-t graph, the displacement—that is, the change in position—equals the area under the v_x-t graph. To be specific, the displacement $x - x_0$ of the particle between $t = 0$ and any later time t equals the area under the v_x-t graph between those two times. In Fig. 2.17 the area under the graph is divided into a dark-colored rectangle of vertical side v_{0x} and horizontal side t and a light-colored right triangle of vertical side $a_x t$ and horizontal side t. The area of the rectangle is $v_{0x}t$ and the area of the triangle is $\frac{1}{2}(a_x t)(t) = \frac{1}{2}a_x t^2$, so the total area under the v_x-t graph is

$$x - x_0 = v_{0x}t + \frac{1}{2}a_x t^2$$

in agreement with Eq. (2.12).

The displacement during a time interval can always be found from the area under the v_x-t curve. This is true even if the acceleration is *not* constant, although in that case Eq. (2.12) does not apply. (We'll show this in Section 2.6.)

We can check whether Eqs. (2.8) and (2.12) are consistent with the assumption of constant acceleration by taking the derivative of Eq. (2.12). We find

$$v_x = \frac{dx}{dt} = v_{0x} + a_x t$$

which is Eq. (2.8). Differentiating again, we find simply

$$\frac{dv_x}{dt} = a_x$$

which agrees with the definition of instantaneous x-acceleration.

It's often useful to have a relationship between position, x-velocity, and (constant) x-acceleration that does not involve the time. To obtain this, we first solve Eq. (2.8) for t, then substitute the resulting expression into Eq. (2.12), and simplify:

$$t = \frac{v_x - v_{0x}}{a_x}$$

$$x = x_0 + v_{0x}\left(\frac{v_x - v_{0x}}{a_x}\right) + \tfrac{1}{2}a_x\left(\frac{v_x - v_{0x}}{a_x}\right)^2$$

We transfer the term x_0 to the left side and multiply through by $2a_x$:

$$2a_x(x - x_0) = 2v_{0x}v_x - 2v_{0x}^{\ 2} + v_x^2 - 2v_{0x}v_x + v_{0x}^{\ 2}$$

Finally, simplifying gives us

$$v_x^2 = v_{0x}^{\ 2} + 2a_x(x - x_0) \qquad \text{(constant } x\text{-acceleration only)} \qquad (2.13)$$

We can get one more useful relationship by equating the two expressions for $v_{\text{av-}x}$, Eqs. (2.9) and (2.10), and multiplying through by t. Doing this, we obtain

$$x - x_0 = \left(\frac{v_{0x} + v_x}{2}\right)t \qquad \text{(constant } x\text{-acceleration only)} \qquad (2.14)$$

Note that Eq. (2.14) does not contain the x-acceleration a_x. This equation can be handy when a_x is constant but its value is unknown.

Equations (2.8), (2.12), (2.13), and (2.14) are the *equations of motion with constant acceleration*. By using these equations, we can solve *any* problem involving straight-line motion of a particle with constant acceleration.

For the particular case of motion with constant x-acceleration depicted in Fig. 2.15 and graphed in Figs. 2.16, 2.17, and 2.18, the values of x_0, v_{0x}, and a_x are all positive. We invite you to redraw these figures for cases in which one, two, or all three of these quantities are negative.

A special case of motion with constant x-acceleration occurs when the x-acceleration is *zero*. The x-velocity is then constant, and the equations of motion become simply

$$v_x = v_{0x} = \text{constant}$$
$$x = x_0 + v_x t$$

Problem-Solving Strategy 2.1 Motion with Constant Acceleration

IDENTIFY *the relevant concepts:* In most straight-line motion problems, you can use the constant-acceleration equations. Occasionally, however, you will encounter a situation in which the acceleration *isn't* constant. In such a case, you'll need a different approach (see Section 2.6).

SET UP *the problem* using the following steps:

1. First decide where the origin of coordinates is and which axis direction is positive. It is often easiest to place the particle at the origin at time $t = 0$; then $x_0 = 0$. It helps to make a motion diagram showing the coordinates and some later positions of the particle.
2. Remember that your choice of the positive axis direction automatically determines the positive directions for x-velocity and x-acceleration. If x is positive to the right of the origin, then v_x and a_x are also positive toward the right.
3. Restate the problem in words, and then translate it into symbols and equations. *When* does the particle arrive at a certain point (that is, what is the value of t)? *Where* is the particle when its x-velocity has a specified value (that is, what is the value of x when v_x has the specified value)? Example 2.4 asks, "Where is the motorcyclist when his velocity is 25 m/s?" In symbols, this says "What is the value of x when $v_x = 25$ m/s?"
4. Make a list of quantities such as x, x_0, v_x, v_{0x}, a_x, and t. In general, some of them will be known and some will be unknown. Write down the values of the known quantities, and decide which of the unknowns are the target variables. Be on the lookout for implicit information. For example, "A car sits at a stoplight" usually means $v_{0x} = 0$.

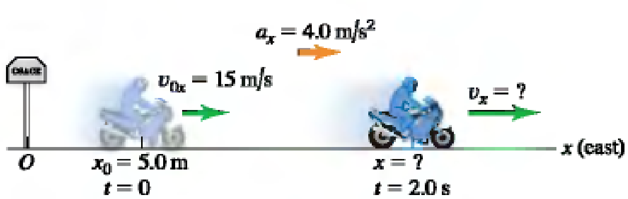
EXECUTE *the solution:* Choose an equation from Eqs. (2.8), (2.12), (2.13), and (2.14) that contains only one of the target variables. Solve this equation for the target variable, using symbols only. Then substitute the known values and compute the value of the target variable. Sometimes you will have to solve two simultaneous equations for two unknown quantities.

EVALUATE *your answer:* Take a hard look at your results to see whether they make sense. Are they within the general range of values you expected?

Example 2.4 Constant-acceleration calculations

A motorcyclist heading east through a small Iowa city accelerates after he passes the signpost marking the city limits (Fig. 2.20). His acceleration is a constant 4.0 m/s². At time $t = 0$ he is 5.0 m east of the signpost, moving east at 15 m/s. (a) Find his position and velocity at time $t = 2.0$ s. (b) Where is the motorcyclist when his velocity is 25 m/s?

2.20 A motorcyclist traveling with constant acceleration.



SOLUTION

IDENTIFY: The problem statement tells us that the acceleration is constant, so we can use the constant-acceleration equations.

SET UP: We take the signpost as the origin of coordinates ($x = 0$), and choose the positive x-axis to point east (see Fig. 2.20, which also serves as a motion diagram). At the initial time $t = 0$, the initial position is $x_0 = 5.0$ m and the initial x-velocity is $v_{0x} = 15$ m/s. The constant x-acceleration is $a_x = 4.0$ m/s². The unknown target variables in part (a) are the values of the position x and the x-velocity v_x at the later time $t = 2.0$ s; the target variable in part (b) is the value of x when $v_x = 25$ m/s.

Continued

EXECUTE: (a) We can find the position x at $t = 2.0\text{ s}$ by using Eq. (2.12), which gives x as a function of time t:

$$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$$
$$= 5.0\text{ m} + (15\text{ m/s})(2.0\text{ s}) + \frac{1}{2}(4.0\text{ m/s}^2)(2.0\text{ s})^2$$
$$= 43\text{ m}$$

We can find the x-velocity v_x at this same time by using Eq. (2.8), which gives v_x as a function of time t:

$$v_x = v_{0x} + a_x t$$
$$= 15\text{ m/s} + (4.0\text{ m/s}^2)(2.0\text{ s}) = 23\text{ m/s}$$

(b) We want to find the value of x when $v_x = 25\text{ m/s}$, but we don't know the time when the motorcycle has this x-velocity. Hence we use Eq. (2.13), which involves x, v_x, and a_x but does not involve t:

$$v_x^2 = v_{0x}{}^2 + 2a_x(x - x_0)$$

Solving for x and substituting in the known values, we find

$$x = x_0 + \frac{v_x^2 - v_{0x}{}^2}{2a_x}$$
$$= 5.0\text{ m} + \frac{(25\text{ m/s})^2 - (15\text{ m/s})^2}{2(4.0\text{ m/s}^2)}$$
$$= 55\text{ m}$$

An alternative but longer route to the same answer is to use Eq. (2.8) to first find the time when $v_x = 25\text{ m/s}$:

$$v_x = v_{0x} + a_x t \qquad \text{so}$$
$$t = \frac{v_x - v_{0x}}{a_x} = \frac{25\text{ m/s} - 15\text{ m/s}}{4.0\text{ m/s}^2} = 2.5\text{ s}$$

Given the time t, we can find x using Eq. (2.12):

$$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$$
$$= 5.0\text{ m} + (15\text{ m/s})(2.5\text{ s}) + \frac{1}{2}(4.0\text{ m/s}^2)(2.5\text{ s})^2$$
$$= 55\text{ m}$$

EVALUATE: Do these results make sense? According to our results in part (a), the motorcyclist accelerates from 15 m/s (about 34 mi/h, or 54 km/h) to 23 m/s (about 51 mi/h, or 83 km/h) in 2.0 s while traveling a distance of 38 m (about 125 ft). This is pretty brisk acceleration, but well within the capabilities of a high-performance bike.

Comparing our results in part (b) with those in part (a) tells us that the motorcycle attains an x-velocity $v_x = 25\text{ m/s}$ at a later time and after traveling a greater distance than when the motorcycle had $v_x = 23\text{ m/s}$. This makes sense, since the motorcycle has a positive x-acceleration and so its x-velocity is increasing.

Example 2.5 Two bodies with different accelerations

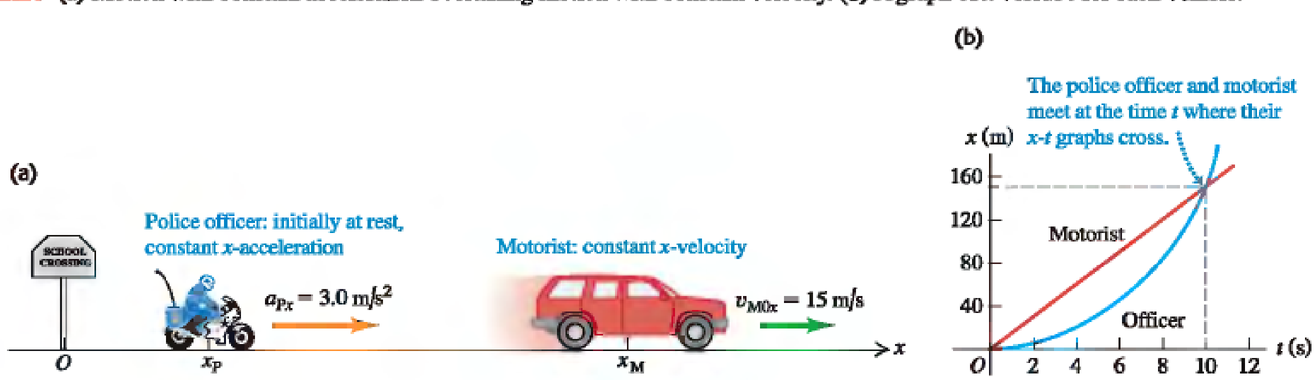
A motorist traveling with a constant speed of 15 m/s (about 34 mi/h) passes a school-crossing corner, where the speed limit is 10 m/s (about 22 mi/h). Just as the motorist passes, a police officer on a motorcycle stopped at the corner starts off in pursuit with constant acceleration of 3.0 m/s² (Fig. 2.21a). (a) How much time elapses before the officer catches up with the motorist? (b) What is the officer's speed at that point? (c) What is the total distance each vehicle has traveled at that point?

SOLUTION

IDENTIFY: The police officer and the motorist both move with constant acceleration (equal to zero for the motorist), so we can use the formulas we have developed.

SET UP: We take the origin at the corner, so $x_0 = 0$ for both, and we take the positive direction to the right. Let x_P (for police) be the officer's position and x_M (for motorist) be the motorist's position at any time. The initial x-velocities are $v_{P0x} = 0$ for the officer and $v_{M0x} = 15\text{ m/s}$ for the motorist; the constant x-accelerations are $a_{Px} = 3.0\text{ m/s}^2$ for the officer and $a_{Mx} = 0$ for the motorist. Our target variable in part (a) is the time when the officer catches the motorist—that is, when the two vehicles are at the same position. In part (b) we're looking for the officer's speed v (the magnitude of his velocity) at the time found in part (a). In part (c) we want to find the position of either vehicle at this same time. Hence we use Eq. (2.12) (which relates position and time) in

2.21 (a) Motion with constant acceleration overtaking motion with constant velocity. (b) A graph of x versus t for each vehicle.



parts (a) and (c), and Eq. (2.8) (which relates velocity and time) in part (b).

EXECUTE: (a) To find the value of the time t when the motorist and the police officer are at the same position, we apply Eq. (2.12), $x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$, to each vehicle:

$$x_M = 0 + v_{M0x}t + \frac{1}{2}(0)t^2 = v_{M0x}t$$

$$x_P = 0 + (0)t + \frac{1}{2}a_{Px}t^2 = \frac{1}{2}a_{Px}t^2$$

Since $x_M = x_P$ at time t, we set these two expressions equal to each other and solve for t:

$$v_{M0x}t = \frac{1}{2}a_{Px}t^2$$

$$t = 0 \qquad \text{or} \qquad t = \frac{2v_{M0x}}{a_{Px}} = \frac{2(15\ \text{m/s})}{3.0\ \text{m/s}^2} = 10\ \text{s}$$

There are *two* times when both the vehicles have the same x-coordinate. The first, $t = 0$, is the time when the motorist passes the parked motorcycle at the corner. The second, $t = 10$ s, is the time when the officer catches up with the motorist.

(b) We want the magnitude of the officer's x-velocity v_{Px} at the time t found in part (a). Her velocity at any time is given by Eq. (2.8):

$$v_{Px} = v_{P0x} + a_{Px}t = 0 + (3.0\ \text{m/s}^2)t$$

Using $t = 10$ s, we find $v_{Px} = 30$ m/s. When the officer overtakes the motorist, she is traveling twice as fast as the motorist is.

(c) In 10 s the distance the motorist travels is

$$x_M = v_{M0x}t = (15\ \text{m/s})(10\ \text{s}) = 150\ \text{m}$$

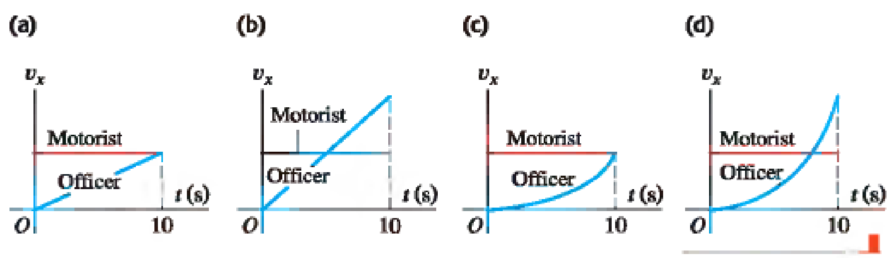
and the distance the officer travels is

$$x_P = \frac{1}{2}a_{Px}t^2 = \frac{1}{2}(3.0\ \text{m/s}^2)(10\ \text{s})^2 = 150\ \text{m}$$

This verifies that at the time the officer catches the motorist, they have gone equal distances.

EVALUATE: Figure 2.21b shows graphs of x versus t for each vehicle. We see again that there are two times when the two positions are the same (where the two graphs cross). At neither of these times do the two vehicles have the same velocity (i.e., where the two graphs cross, their slopes are different). At $t = 0$, the officer is at rest; at $t = 10$ s, the officer has twice the speed of the motorist.

Test Your Understanding of Section 2.4 Four possible v_x-t graphs are shown for the two vehicles in Example 2.5. Which graph is correct?



2.5 Freely Falling Bodies

The most familiar example of motion with (nearly) constant acceleration is a body falling under the influence of the earth's gravitational attraction. Such motion has held the attention of philosophers and scientists since ancient times. In the fourth century B.C., Aristotle thought (erroneously) that heavy bodies fall faster than light bodies, in proportion to their weight. Nineteen centuries later, Galileo (see Section 1.1) argued that a body should fall with a downward acceleration that is constant and independent of its weight.

Experiment shows that if the effects of the air can be neglected, Galileo is right; all bodies at a particular location fall with the same downward acceleration, regardless of their size or weight. If in addition the distance of the fall is small compared with the radius of the earth, and if we ignore small effects due to the earth's rotation, the acceleration is constant. The idealized motion that results under all of these assumptions is called **free fall**, although it includes rising as well as falling motion. (In Chapter 3 we will extend the discussion of free fall to include the motion of projectiles, which move both vertically and horizontally.)

Figure 2.22 is a photograph of a falling ball made with a stroboscopic light source that produces a series of short, intense flashes. As each flash occurs, an image of the ball at that instant is recorded on the photograph. There are equal

2.22 Multiflash photo of a freely falling ball.

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1.7 Balloonist Drops Lemonade
1.10 Pole-Vaulter Lands

time intervals between flashes, so the average velocity of the ball between successive flashes is proportional to the distance between corresponding images. The increasing distances between images show that the velocity is continuously changing; the ball is accelerating downward. Careful measurement shows that the velocity change is the same in each time interval, so the acceleration of the freely falling ball is constant.

The constant acceleration of a freely falling body is called the **acceleration due to gravity,** and we denote its magnitude with the letter g. We will frequently use the approximate value of g at or near the earth's surface:

$$g = 9.8\ \text{m/s}^2 = 980\ \text{cm/s}^2$$
$$= 32\ \text{ft/s}^2 \qquad \text{(approximate value near the earth's surface)}$$

The exact value varies with location, so we will often give the value of g at the earth's surface to only two significant figures. Because g is the magnitude of a vector quantity, it is always a *positive* number. On the surface of the moon, the acceleration due to gravity is caused by the attractive force of the moon rather than the earth, and $g = 1.6\ \text{m/s}^2$. Near the surface of the sun, $g = 270\ \text{m/s}^2$.

In the following examples we use the constant-acceleration equations developed in Section 2.4. You should review Problem-Solving Strategy 2.1 in that section before you study the next examples.

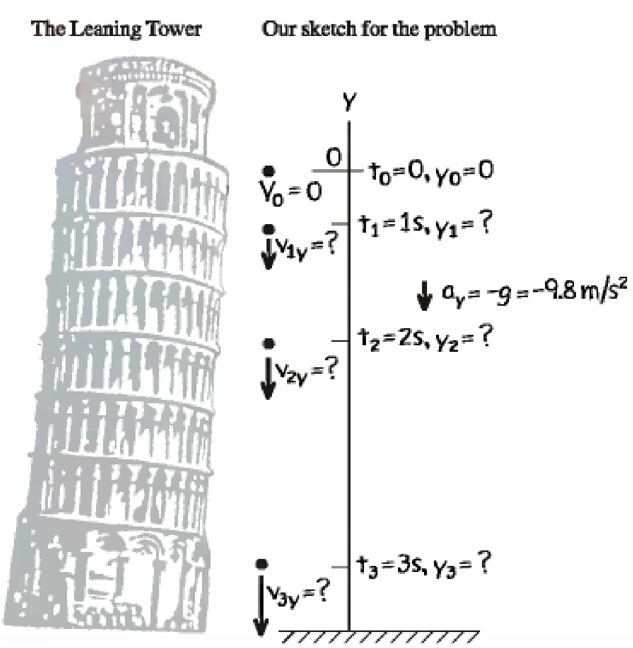
Example 2.6 A freely-falling coin

A one-euro coin is dropped from the Leaning Tower of Pisa. It starts from rest and falls freely. Compute its position and velocity after 1.0 s, 2.0 s, and 3.0 s.

SOLUTION

IDENTIFY: "Falls freely" means "has a constant acceleration due to gravity," so we can use the constant-acceleration equations to determine our target variables.

2.23 A coin freely falling from rest.



SET UP: The right side of Fig. 2.23 shows our motion diagram for the coin. The motion is vertical, so we use a vertical coordinate axis and call the coordinate y instead of x. Then we replace all the x's in the constant-acceleration equations by y's. We take the origin O at the starting point and the upward direction as positive. The initial coordinate y_0 and the initial y-velocity v_{0y} are both zero. The y-acceleration is downward, in the negative y-direction, so $a_y = -g = -9.8\ \text{m/s}^2$. (Remember that, by definition, g itself is *always* positive.) Our target variables are the values of y and v_y at the three given times. To find these, we use Eqs. (2.12) and (2.8) with x replaced by y.

EXECUTE: At a time t after the coin is dropped, its position and y-velocity are

$$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 = 0 + 0 + \frac{1}{2}(-g)t^2 = (-4.9\ \text{m/s}^2)t^2$$

$$v_y = v_{0y} + a_y t = 0 + (-g)t = (-9.8\ \text{m/s}^2)t$$

When $t = 1.0\ \text{s}$, $y = (-4.9\ \text{m/s}^2)(1.0\ \text{s})^2 = -4.9\ \text{m}$ and $v_y = (-9.8\ \text{m/s}^2)(1.0\ \text{s}) = -9.8\ \text{m/s}$; after 1 s, the coin is 4.9 m below the origin (y is negative) and has a downward velocity (v_y is negative) with magnitude 9.8 m/s.

The position and y-velocity at 2.0 s and 3.0 s are found in the same way. Can you show that $y = -19.6\ \text{m}$ and $v_y = -19.6\ \text{m/s}$ at $t = 2.0\ \text{s}$, and that $y = -44.1\ \text{m}$ and $v_y = -29.4\ \text{m/s}$ at $t = 3.0\ \text{s}$?

EVALUATE: All our answers for v_y are negative because we chose the positive y-axis to point upward. But we could just as well have chosen the positive y-axis to point downward. In that case the acceleration would have been $a_y = +g$ and all our answers for v_y would have been positive. Either choice of axis is fine; just make sure that you state your choice explicitly in your solution and confirm that the acceleration has the correct sign.

Example 2.7 Up-and-down motion in free fall

You throw a ball vertically upward from the roof of a tall building. The ball leaves your hand at a point even with the roof railing with an upward speed of 15.0 m/s; the ball is then in free fall. On its way back down, it just misses the railing. At the location of the building, $g = 9.80\ \text{m/s}^2$. Find (a) the position and velocity of the ball 1.00 s and 4.00 s after leaving your hand; (b) the velocity when the ball is 5.00 m above the railing; (c) the maximum height reached and the time at which it is reached; and (d) the acceleration of the ball when it is at its maximum height.

SOLUTION

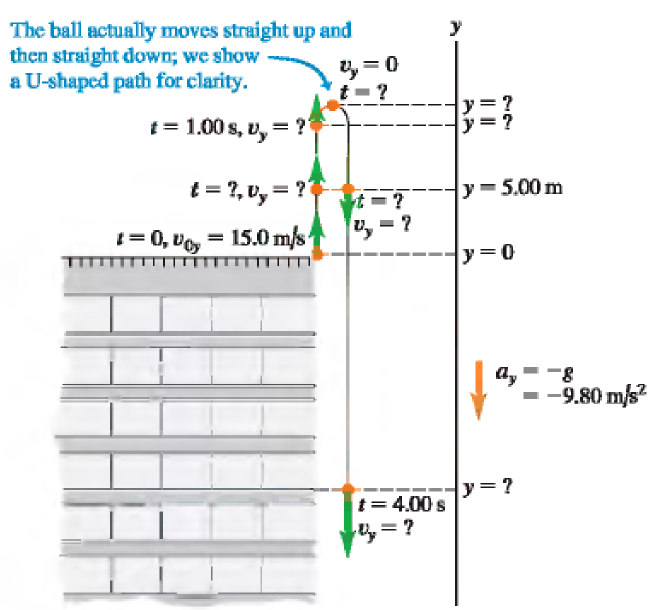
IDENTIFY: The words "free fall" in the statement of the problem mean that the acceleration is constant and due to gravity. Our target variables are position [in parts (a) and (c)], velocity [in parts (a) and (b)], and acceleration [in part (d)].

SET UP: In Fig. 2.24 (which is also a motion diagram for the ball) the downward path is displaced a little to the right of its actual position for clarity. Take the origin at the point where the ball leaves your hand, and take the positive direction to be upward. The initial position y_0 is zero, the initial y-velocity v_{0y} is $+15.0\ \text{m/s}$, and the y-acceleration is $a_y = -g = -9.80\ \text{m/s}^2$. We'll again use Eqs. (2.12) and (2.8) to find the position and velocity as functions of time. In part (b) we need to find the velocity at a certain *position* rather than at a certain *time*, so we'll use Eq. (2.13) for that part.

EXECUTE: (a) The position y and y-velocity v_y a time t after the ball leaves your hand are given by Eqs. (2.12) and (2.8) with x's replaced by y's:

$$y = y_0 = v_{0y}t + \frac{1}{2}a_yt^2 = y_0 + v_{0y}t + \frac{1}{2}(-g)t^2$$
$$= (0) + (15.0\ \text{m/s})t + \frac{1}{2}(-9.80\ \text{m/s}^2)t^2$$
$$v_y = v_{0y} + a_yt = v_{0y} + (-g)t$$
$$= 15.0\ \text{m/s} + (-9.80\ \text{m/s}^2)t$$

2.24 Position and velocity of a ball thrown vertically upward.



When $t = 1.00$ s, these equations give

$$y = +10.1\ \text{m} \qquad v_y = +5.2\ \text{m/s}$$

The ball is 10.1 m above the origin (y is positive) and moving upward (v_y is positive) with a speed of 5.2 m/s. This is less than the initial speed because the ball slows as it ascends.

When $t = 4.00$ s, the equations for y and v_y as functions of time t give

$$y = -18.4\ \text{m} \qquad v_y = -24.2\ \text{m/s}$$

The ball has passed its highest point and is 18.4 m *below* the origin (y is negative). It has a *downward* velocity (v_y is negative) with magnitude 24.2 m/s. The ball loses speed as it ascends, then gains speed as it descends; it is moving at the initial 15.0-m/s speed as it moves downward past the ball's launching point (the origin), and continues to gain speed as it descends below this point.

(b) The y-velocity v_y at any position y is given by Eq. (2.13) with x's replaced by y's:

$$v_y^2 = v_{0y}^2 + 2a_y(y - y_0) = v_{0y}^2 + 2(-g)(y - 0)$$
$$= (15.0\ \text{m/s})^2 + 2(-9.80\ \text{m/s}^2)y$$

When the ball is 5.00 m above the origin, $y = +5.00$ m, so

$$v_y^2 = (15.0\ \text{m/s})^2 + 2(-9.80\ \text{m/s}^2)(5.00\ \text{m}) = 127\ \text{m}^2/\text{s}^2$$
$$v_y = \pm 11.3\ \text{m/s}$$

We get *two* values of v_y because the ball passes through the point $y = +5.00$ m twice (see Fig. 2.24), once on the way up so v_y is positive and once on the way down so v_y is negative.

(c) Just at the instant when the ball reaches the highest point, it is momentarily at rest and $v_y = 0$. The maximum height y_1 can then be found in two ways. The first way is to use Eq. (2.13) and substitute $v_y = 0$, $y_0 = 0$, and $a_y = -g$:

$$0 = v_{0y}^2 + 2(-g)(y_1 - 0)$$
$$y_1 = \frac{v_{0y}^2}{2g} = \frac{(15.0\ \text{m/s})^2}{2(9.80\ \text{m/s}^2)} = +11.5\ \text{m}$$

The second way is find the time at which $v_y = 0$ using Eq. (2.8), $v_y = v_{0y} + a_yt$, and then substitute this value of t into Eq. (2.12) to find the position at this time. From Eq. (2.8), the time t_1 when the ball reaches the highest point is given by

$$v_y = 0 = v_{0y} + (-g)t_1$$
$$t_1 = \frac{v_{0y}}{g} = \frac{15.0\ \text{m/s}}{9.80\ \text{m/s}^2} = 1.53\ \text{s}$$

Substituting this value of t into Eq. (2.12), we find

$$y = y_0 + v_{0y}t + \frac{1}{2}a_yt^2 = (0) + (15\ \text{m/s})(1.53\ \text{s})$$
$$+ \frac{1}{2}(-9.8\ \text{m/s}^2)(1.53\ \text{s})^2 = +11.5\ \text{m}$$

Notice that the first way of finding the maximum height is easier, since it's not necessary to find the time first.

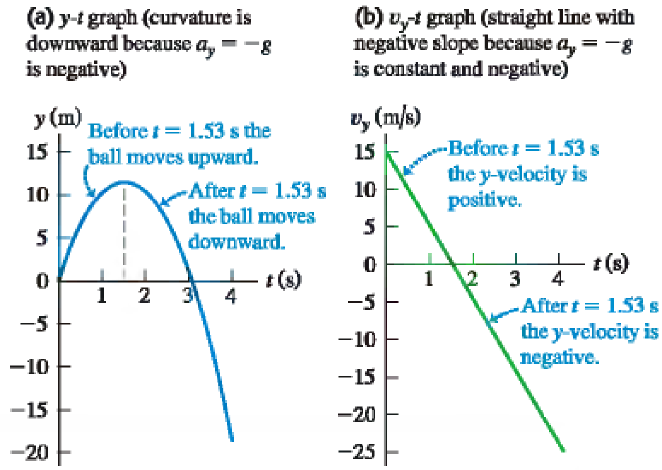
Continued

(d) **CAUTION** **A free-fall misconception** It's a common misconception that at the highest point of free-fall motion the velocity is zero *and* the acceleration is zero. If this were so, once the ball reached the highest point it would hang there suspended in midair! Remember that acceleration is the rate of change of velocity. If the acceleration were zero at the highest point, the ball's velocity would no longer change, and once the ball was instantaneously at rest, it would remain at rest forever.

At the highest point, the acceleration is still $a_y = -g = -9.80\ \text{m/s}^2$, the same value as when the ball is moving up and when it's moving down. That's because the ball's velocity is continuously changing, from positive values through zero to negative values.

EVALUATE: A useful way to check any motion problem is to draw the graphs of position and velocity versus time. Figure 2.25 shows these graphs for this problem. Since the y-acceleration is constant and negative, the y-t graph is a parabola with downward curvature and the v_y-t graph is a straight line with a negative slope.

2.25 (a) Position and (b) velocity as functions of time for a ball thrown upward with an initial speed of 15 m/s.



Example 2.8 Two solutions or one?

Find the time when the ball in Example 2.7 is 5.00 m below the roof railing.

SOLUTION

IDENTIFY: Again this a constant-acceleration problem. The target variable is the time when the ball is at a certain position.

SET UP: We again choose the y-axis as in Fig. 2.24, so y_0, v_{0y}, and $a_y = -g$ have the same values as in Example 2.7. The position y as a function of time t is again given by Eq. (2.12):

$$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 = y_0 + v_{0y}t + \frac{1}{2}(-g)t^2$$

We want to solve this for the value of t when $y = -5.00$ m. Since this equation involves t^2, it is a *quadratic* equation for t.

EXECUTE: We first rearrange the equation into the standard form of a quadratic equation for an unknown x, $Ax^2 + Bx + C = 0$:

$$\left(\frac{1}{2}g\right)t^2 + (-v_{0y})t + (y - y_0) = At^2 + Bt + C = 0$$

so $A = g/2$, $B = -v_{0y}$, and $C = y - y_0$. Using the quadratic formula (see Appendix B), we find that this equation has *two* solutions:

$$t = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{-(-v_{0y}) \pm \sqrt{(-v_{0y})^2 - 4(g/2)(y - y_0)}}{2(g/2)}$$

$$= \frac{v_{0y} \pm \sqrt{v_{0y}^2 - 2g(y - y_0)}}{g}$$

Substituting the values $y_0 = 0$, $v_{0y} = +15.0$ m/s, $g = 9.80\ \text{m/s}^2$, and $y = -5.00$ m, we find

$$t = \frac{(15.0\ \text{m/s}) \pm \sqrt{(15.0\ \text{m/s})^2 - 2(9.80\ \text{m/s}^2)(-5.00\ \text{m} - 0)}}{9.80\ \text{m/s}^2}$$

$t = +3.36$ s or $t = -0.30$ s

To decide which of these is the right answer, the key question to ask is, "Are these answers reasonable?" The second answer, $t = -0.30$ s, is simply not reasonable; it refers to a time 0.30 s *before* the ball left your hand! The correct answer is $t = +3.36$ s. The ball is 5.00 m below the railing 3.36 s *after* it leaves your hand.

EVALUATE: Where did the erroneous "solution" $t = -0.30$ s come from? Remember that the equation $y = y_0 + v_{0y}t + \frac{1}{2}(-g)t^2$ is based on the assumption that the acceleration is constant for *all* values of t, whether positive, negative, or zero. Taken at face value, this equation tells us that the ball has been moving upward in free fall ever since the dawn of time; it eventually passes your hand at $y = 0$ at the special instant we chose to call $t = 0$, then continues in free fall. But anything that this equation describes happening before $t = 0$ is pure fiction, since the ball went into free fall only after leaving your hand at $t = 0$; the "solution" $t = -0.30$ s is part of this fiction.

You should repeat these calculations to find the times when the ball is 5.00 m *above* the origin ($y = +5.00$ m). The two answers are $t = +0.38$ s and $t = +2.68$ s. These are both positive values of t, and both refer to the real motion of the ball after leaving your hand. The earlier time is when the ball passes through $y = +5.00$ m moving upward; the later time is when it passes through this point moving downward. [Compare this with part (b) of Example 2.7.]

You should also solve for the times at which $y = +15.0$ m. In this case, both solutions involve the square root of a negative number, so there are *no* real solutions. This makes sense; we found in part (c) of Example 2.7 that the ball's maximum height is only $y = +11.5$ m, so it *never* reaches $y = +15.0$ m. While a quadratic equation such as Eq. (2.12) always has two solutions, in some situations one or both of the solutions will not be physically reasonable.

Test Your Understanding of Section 2.5 If you toss a ball upward with a certain initial speed, it falls freely and reaches a maximum height h a time t after it leaves your hand. (a) If you throw the ball upward with double the initial speed, what new maximum height does the ball reach? (i) $h\sqrt{2}$; (ii) $2h$; (iii) $4h$; (iv) $8h$; (v) $16h$. (b) If you throw the ball upward with double the initial speed, how long does it take to reach its new maximum height? (i) $t/2$; (ii) $t/\sqrt{2}$; (iii) t; (iv) $t\sqrt{2}$; (v) $2t$.

2.6 *Velocity and Position by Integration

This optional section is intended for students who have already learned a little integral calculus. In Section 2.4 we analyzed the special case of straight-line motion with constant acceleration. When a_x is not constant, as is frequently the case, the equations that we derived in that section are no longer valid (Fig. 2.26). But even when a_x varies with time, we can still use the relationship $v_x = dx/dt$ to find the x-velocity v_x as a function of time if the position x is a known function of time. And we can still use $a_x = dv_x/dt$ to find the x-acceleration a_x as a function of time if the x-velocity v_x is a known function of time.

In many situations, however, position and velocity are not known as functions of time, while acceleration is. How can we find the position and velocity from the acceleration function $a_x(t)$? This problem arises in navigating an airliner between North America and Europe (Fig. 2.27). The pilots must know their position precisely at all times, but over the ocean an airliner is usually out of range of both radio navigation beacons on land and air traffic controllers' radar. To determine their position, airliners carry a device called an inertial navigation system (INS), which measures the airliner's acceleration. This is done in much the same way that you can sense changes in the velocity of a car in which you're riding, even when your eyes are closed. (In Chapter 4 we'll discuss how your body detects acceleration.) Given this information, along with the airliner's initial position (say, a particular gate at Miami International Airport) and its initial velocity (zero when parked at the gate), the INS calculates the airliner's current velocity and position at all times during the flight. (Airliners also use the Global Positioning System, or GPS, for navigation, but this supplements INS rather than replacing it.) Our goal in this section is to see how these calculations are done for the simpler case of motion in a straight line with time-varying acceleration.

We first consider a graphical approach. Figure 2.28 is a graph of x-acceleration versus time for a body whose acceleration is not constant. We can divide the time interval between times t_1 and t_2 into many smaller intervals, calling a typical one Δt. Let the average x-acceleration during Δt be $a_{\text{av-}x}$. From Eq. (2.4) the change in x-velocity Δv_x during Δt is

$$\Delta v_x = a_{\text{av-}x}\,\Delta t$$

Graphically, Δv_x equals the area of the shaded strip with height $a_{\text{av-}x}$ and width Δt—that is, the area under the curve between the left and right sides of Δt. The total change in x-velocity during any interval (say, t_1 to t_2) is the sum of the x-velocity changes Δv_x in the small subintervals. So the total x-velocity change is represented graphically by the *total* area under the a_x-t curve between the vertical lines t_1 and t_2. (In Section 2.4 we showed this for the special case in which the acceleration is constant.)

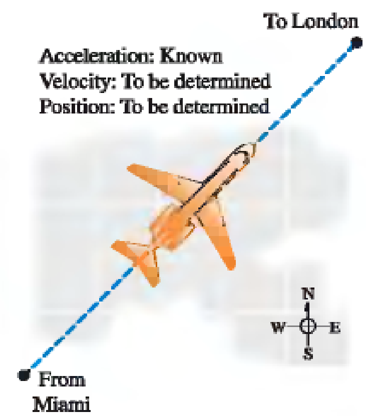
In the limit that all the Δt's become very small and their number very large, the value of $a_{\text{av-}x}$ for the interval from any time t to $t + \Delta t$ approaches the instantaneous x-acceleration a_x at time t. In this limit, the area under the a_x-t curve is the *integral* of a_x (which is in general a function of t) from t_1 to t_2. If v_{1x} is the x-velocity of the body at time t_1 and v_{2x} is the velocity at time t_2, then

$$v_{2x} - v_{1x} = \int_{v_{1x}}^{v_{2x}} dv_x = \int_{t_1}^{t_2} a_x\,dt \qquad (2.15)$$

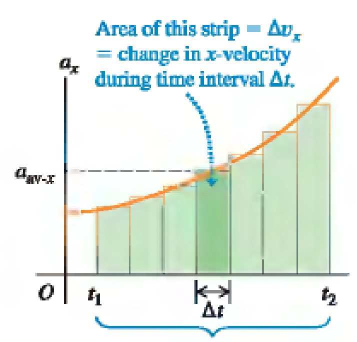
The change in the x-velocity v_x is the time integral of the x-acceleration a_x.

2.26 When you push your car's accelerator pedal to the floorboard, the resulting acceleration is *not* constant: the greater the car's speed, the more slowly it gains additional speed. A typical car takes twice as long to accelerate from 50 km/h to 100 km/h as it does to accelerate from 0 to 50 km/h.

2.27 The position and velocity of an airliner crossing the Atlantic are found by integrating its acceleration with respect to time.



2.28 An a_x-t graph for a body whose x-acceleration is not constant.



We can carry out exactly the same procedure with the curve of x-velocity versus time. If x_1 is a body's position at time t_1 and x_2 is its position at time t_2, from Eq. (2.2) the displacement Δx during a small time interval Δt is equal to $v_{\text{av-}x}\,\Delta t$, where $v_{\text{av-}x}$ is the average x-velocity during Δt. The total displacement $x_2 - x_1$ during the interval $t_2 - t_1$ is given by

$$x_2 - x_1 = \int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} v_x\,dt \tag{2.16}$$

The change in position x—that is, the displacement—is the time integral of x-velocity v_x. Graphically, the displacement between times t_1 and t_2 is the area under the v_x-t curve between those two times. [This is the same result that we obtained in Section 2.4 for the special case in which v_x is given by Eq. (2.8).]

If $t_1 = 0$ and t_2 is any later time t, and if x_0 and v_{0x} are the position and velocity, respectively, at time $t = 0$, then we can rewrite Eqs. (2.15) and (2.16) as follows:

$$v_x = v_{0x} + \int_0^t a_x\,dt \tag{2.17}$$

$$x = x_0 + \int_0^t v_x\,dt \tag{2.18}$$

Here x and v_x are the position and x-velocity at time t. If we know the x-acceleration a_x as a function of time and we know the initial velocity v_{0x}, we can use Eq. (2.17) to find the x-velocity v_x at any time; in other words, we can find v_x as a function of time. Once we know this function, and given the initial position x_0, we can use Eq. (2.18) to find the position x at any time.

Example 2.9 Motion with changing acceleration

Sally is driving along a straight highway in her classic 1965 Mustang. At time $t = 0$, when Sally is moving at 10 m/s in the positive x-direction, she passes a signpost at $x = 50$ m. Her x-acceleration is a function of time:

$$a_x = 2.0\ \text{m/s}^2 - (0.10\ \text{m/s}^3)t$$

(a) Find her x-velocity and position as functions of time. (b) When is her x-velocity greatest? (c) What is the maximum x-velocity? (d) Where is the car when it reaches the maximum x-velocity?

SOLUTION

IDENTIFY: The x-acceleration is a function of time, so we cannot use the constant-acceleration formulas of Section 2.4.

SET UP: We use Eqs. (2.17) and (2.18) to find the x-velocity and position as functions of time. Once we have those functions, we'll be able to answer a variety of questions about the motion.

EXECUTE: (a) At $t = 0$, Sally's position is $x_0 = 50$ m and her x-velocity is $v_{0x} = 10$ m/s. Since we are given the x-acceleration a_x as a function of time, we first use Eq. (2.17) to find the x-velocity v_x as a function of time t. The integral of t^n is $\int t^n\,dt = \frac{1}{n+1}t^{n+1}$ for $n \neq -1$, so

$$\begin{aligned} v_x &= 10\ \text{m/s} + \int_0^t [2.0\ \text{m/s}^2 - (0.10\ \text{m/s}^3)t]\,dt \\ &= 10\ \text{m/s} + (2.0\ \text{m/s}^2)t - \frac{1}{2}(0.10\ \text{m/s}^3)t^2 \end{aligned}$$

Then we use Eq. (2.18) to find x as a function of t:

$$\begin{aligned} x &= 50\ \text{m} + \int_0^t \left[10\ \text{m/s} + (2.0\ \text{m/s}^2)t - \frac{1}{2}(0.10\ \text{m/s}^3)t^2\right] dt \\ &= 50\ \text{m} + (10\ \text{m/s})t + \frac{1}{2}(2.0\ \text{m/s}^2)t^2 - \frac{1}{6}(0.10\ \text{m/s}^3)t^3 \end{aligned}$$

Figure 2.29 shows graphs of a_x, v_x, and x as functions of time. Note that for any time t, the slope of the v_x-t graph equals the value of a_x and the slope of the x-t graph equals the value of v_x.

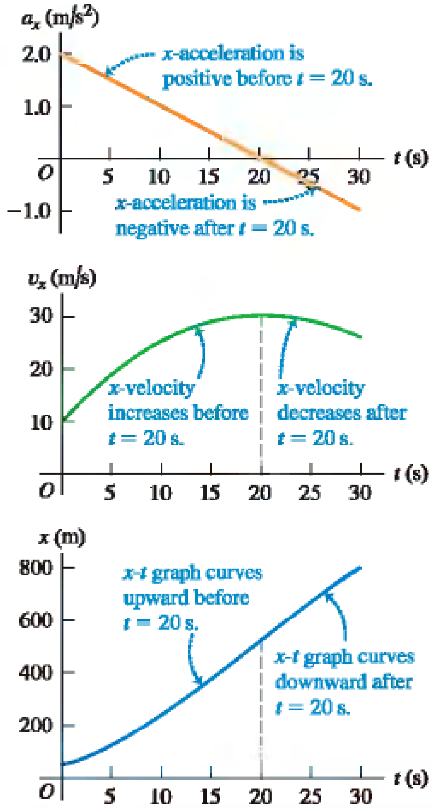
(b) The maximum value of v_x occurs when the x-velocity stops increasing and begins to decrease. At this instant, $dv_x/dt = a_x = 0$. Setting the expression for a_x equal to zero, we obtain

$$\begin{aligned} 0 &= 2.0\ \text{m/s}^2 - (0.10\ \text{m/s}^3)t \\ t &= \frac{2.0\ \text{m/s}^2}{0.10\ \text{m/s}^3} = 20\ \text{s} \end{aligned}$$

(c) We find the maximum x-velocity by substituting $t = 20$ s (when x-velocity is maximum) into the equation for v_x from part (a):

$$\begin{aligned} v_{\text{max-}x} &= 10\ \text{m/s} + (2.0\ \text{m/s}^2)(20\ \text{s}) - \frac{1}{2}(0.10\ \text{m/s}^3)(20\ \text{s})^2 \\ &= 30\ \text{m/s} \end{aligned}$$

2.29 The position, velocity, and acceleration of the car in Example 2.9 as functions of time. Can you show that if this motion continues, the car will stop at $t = 44.5$ s?



(d) The maximum value of v_x occurs at time $t = 20$ s. To obtain the position of the car at that time, we substitute $t = 20$ s into the expression for x from part (a):

$$\begin{aligned} x &= 50\text{ m} + (10\text{ m/s})(20\text{ s}) + \frac{1}{2}(2.0\text{ m/s}^2)(20\text{ s})^2 \\ &\quad - \frac{1}{6}(0.10\text{ m/s}^3)(20\text{ s})^3 \\ &= 517\text{ m} \end{aligned}$$

EVALUATE: Figure 2.29 helps us interpret our results. The top graph in this figure shows that a_x is positive between $t = 0$ and $t = 20$ s and negative after that. It is zero at $t = 20$ s, the time at which v_x is maximum (the high point in the middle graph). The car speeds up until $t = 20$ s (because v_x and a_x have the same sign) and slows down after $t = 20$ s (because v_x and a_x have opposite signs).

Since v_x is maximum at $t = 20$ s, the x-t graph (the bottom graph in Fig. 2.29) has its maximum positive slope at this time. Note that the x-t graph is concave up (curved upward) from $t = 0$ to $t = 20$ s, when a_x is positive. The graph is concave down (curved downward) after $t = 20$ s, when a_x is negative.

Example 2.10 Constant-acceleration formulas via integration

Use Eqs. (2.17) and (2.18) to find v_x and x as functions of time in the case in which the acceleration is constant.

SOLUTION

IDENTIFY: This example serves as a check on the equations we've derived in this section. If they are correct, we should end up with the same constant-acceleration equations we derived in Section 2.4 without using integration.

SET UP: We follow the same steps as in Example 2.9. The only difference is that a_x is a constant.

EXECUTE: From Eq. (2.17) the x-velocity is given by

$$v_x = v_{0x} + \int_0^t a_x\,dt = v_{0x} + a_x \int_0^t dt = v_{0x} + a_x t$$

We were able to take a_x outside the integral because it is constant. Substituting this expression for v_x into Eq. (2.18), we get

$$x = x_0 + \int_0^t v_x\,dt = x_0 + \int_0^t (v_{0x} + a_x t)\,dt$$

Since v_{0x} and a_x are constants, we can take them outside the integral:

$$x = x_0 + v_{0x}\int_0^t dt + a_x \int_0^t t\,dt = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$$

EVALUATE: Our results are the same as Eqs. (2.8) and (2.12) from Section 2.4, as they should be! Although we developed Eqs. (2.17) and (2.18) to deal with cases in which acceleration depends on time, they can be used just as well when the acceleration is constant.

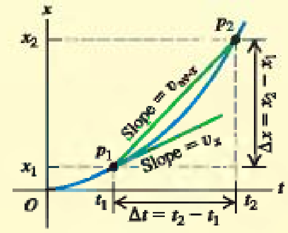
Test Your Understanding of Section 2.6 If the x-acceleration a_x is increasing with time, will the v_x-t graph be (i) a straight line, (ii) concave up (i.e., with an upward curvature), or (iii) concave down (i.e., with a downward curvature)? MP

CHAPTER 2 SUMMARY

Straight-line motion, average and instantaneous x-velocity: When a particle moves along a straight line, we describe its position with respect to an origin O by means of a coordinate such as x. The particle's average x-velocity $v_{\text{av-}x}$ during a time interval $\Delta t = t_2 - t_1$ is equal to its displacement $\Delta x = x_2 - x_1$ divided by Δt. The instantaneous x-velocity v_x at any time t is equal to the average x-velocity for the time interval from t to $t + \Delta t$ in the limit that Δt goes to zero. Equivalently, v_x is the derivative of the position function with respect to time. (See Example 2.1)

$$v_{\text{av-}x} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t} \tag{2.2}$$

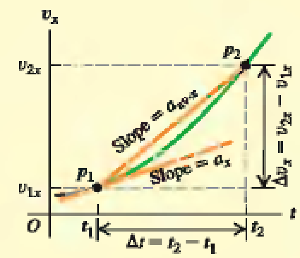
$$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \tag{2.3}$$



Average and instantaneous x-acceleration: The average x-acceleration $a_{\text{av-}x}$ during a time interval Δt is equal to the change in velocity $\Delta v_x = v_{2x} - v_{1x}$ during that time interval divided by Δt. The instantaneous x-acceleration a_x is the limit of $a_{\text{av-}x}$ as Δt goes to zero, or the derivative of v_x with respect to t. (See Examples 2.2 and 2.3.)

$$a_{\text{av-}x} = \frac{v_{2x} - v_{1x}}{t_2 - t_1} = \frac{\Delta v_x}{\Delta t} \tag{2.4}$$

$$a_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt} \tag{2.5}$$



Straight-line motion with constant acceleration: When the x-acceleration is constant, four equations relate the position x and the x-velocity v_x at any time t to the initial position x_0, the initial x-velocity v_{0x} (both measured at time $t = 0$), and the x-acceleration a_x. (See Examples 2.4 and 2.5.)

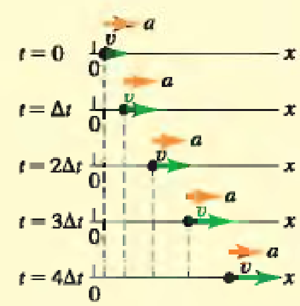
Constant x-acceleration only:

$$v_x = v_{0x} + a_x t \tag{2.8}$$

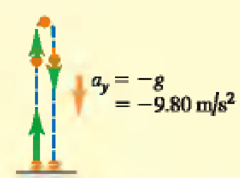
$$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \tag{2.12}$$

$$v_x^2 = v_{0x}^2 + 2a_x(x - x_0) \tag{2.13}$$

$$x - x_0 = \left(\frac{v_{0x} + v_x}{2}\right)t \tag{2.14}$$



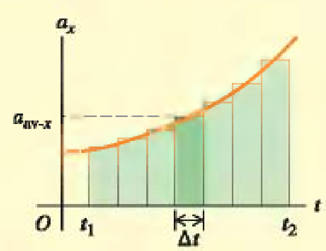
Freely falling bodies: Free fall is a case of motion with constant acceleration. The magnitude of the acceleration due to gravity is a positive quantity, g. The acceleration of a body in free fall is always downward. (See Examples 2.6–2.8.)



Straight-line motion with varying acceleration: When the acceleration is not constant but is a known function of time, we can find the velocity and position as functions of time by integrating the acceleration function. (See Examples 2.9 and 2.10.)

$$v_x = v_{0x} + \int_0^t a_x\,dt \tag{2.17}$$

$$x = x_0 + \int_0^t v_x\,dt \tag{2.18}$$



Key Terms

Answer to Chapter Opening Question ?

Yes. Acceleration refers to *any* change in velocity, including both speeding up and slowing down.

Answers to Test Your Understanding Questions

2.1 Answers to (a): (iv), (i) and (iii) (tie), (v), (ii); answer to (b): (i) and (iii); answer to (c): (v) In (a) the average x-velocity is $v_{\text{av-}x} = \Delta x/\Delta t$. For all five trips, $\Delta t = 1$ h. For the individual trips, we have (i) $\Delta x = +50$ km, $v_{\text{av-}x} = +50$ km/h; (ii) $\Delta x = -50$ km, $v_{\text{av-}x} = -50$ km/h; (iii) $\Delta x = 60 \text{ km} - 10 \text{ km} = +50$ km, $v_{\text{av-}x} = +50$ km/h; (iv) $\Delta x = +70$ km, $v_{\text{av-}x} = +70$ km/h; (v) $\Delta x = \Delta x = -20 \text{ km} + 20 \text{ km} = 0$, $v_{\text{av-}x} = 0$. In (b) both have $v_{\text{av-}x} = +50$ km/h.

2.2 Answers: (a) *P*, *Q* and *S* (tie), *R* The x-velocity is **(b)** positive when the slope of the x-t graph is positive (*P*), **(c)** negative when the slope is negative (*R*), and **(d)** zero when the slope is zero (*Q* and *S*). **(e) *R*, *P*, *Q* and *S* (tie)** The speed is greatest when the slope of the x-t graph is steepest (either positive or negative) and zero when the slope is zero.

2.3 Answers: (a) *S*, where the x-t graph is curved upward (concave up). **(b) *Q*,** where the x-t graph is curved downward (concave down). **(c) *P* and *R*,** where the x-t graph is not curved either up or down. **(d)** At *P*, $v_x > 0$ and $a_x = 0$ (speed is **not changing**); at *Q*, $v_x > 0$ and $a_x < 0$ (speed is **decreasing**); at *R*, $v_x < 0$ and $a_x = 0$ (speed is **not changing**); and at *S*, $v_x < 0$ and $a_x > 0$ (speed is **decreasing**).

2.4 Answer: (b) The officer's x-acceleration is constant, so her v_x-t graph is a straight line, and the officer's motorcycle is moving faster than the motorist's car when the two vehicles meet at $t = 10$ s.

2.5 Answers: (a) (iii) Use Eq. (2.13) with x replaced by y and $a_y = g$; $v_y^2 = v_{0y}^2 - 2g(y - y_0)$. The starting height is $y_0 = 0$ and the y-velocity at the maximum height $y = h$ is $v_y = 0$, so $0 = v_{0y}^2 - 2gh$ and $h = v_{0y}^2/2g$. If the initial y-velocity is increased by a factor of 2, the maximum height increases by a factor of $2^2 = 4$ and the ball goes to height $4h$. **(b) (v)** Use Eq. (2.8) with x replaced by y and $a_y = g$; $v_y = v_{0y} - gt$. The y-velocity at the maximum height is $v_y = 0$, so $0 = v_{0y} - gt$ and $t = v_{0y}/g$. If the initial y-velocity is increased by a factor of 2, the time to reach the maximum height increases by a factor of 2 and becomes $2t$.

2.6 Answer: (ii) The acceleration a_x is equal to the slope of the v_x-t graph. If a_x is increasing, the slope of the v_x-t graph is also increasing and the graph is concave up.

PROBLEMS

For instructor-assigned homework, go to www.masteringphysics.com

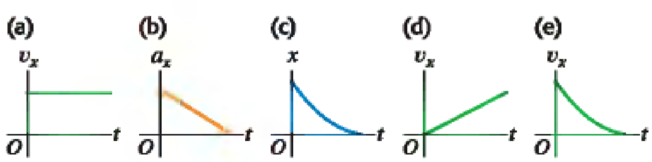
Discussion Questions

Q2.1. Does the speedometer of a car measure speed or velocity? Explain.

Q2.2. Figure 2.30 shows a series of high-speed photographs of an insect flying in a straight line from left to right (in the positive x-direction). Which of the graphs in Fig. 2.31 most plausibly depicts this insect's motion?

Figure 2.30 Question Q2.2.

Figure 2.31 Question Q2.2.



Q2.3. Can an object with constant acceleration reverse its direction of travel? Can it reverse its direction *twice?* In each case, explain your reasoning.

Q2.4. Under what conditions is average velocity equal to instantaneous velocity?

Q2.5. Is is possible for an object (a) to be slowing down while its acceleration is increasing in magnitude; (b) to be speeding up while its acceleration is decreasing? In each case, explain your reasoning.

Q2.6. Under what conditions does the magnitude of the average velocity equal the average speed?

Q2.7. When a Dodge Viper is at Elwood's Car Wash, a BMW Z3 is at Elm and Main. Later, when the Dodge reaches Elm and Main, the BMW reaches Elwood's Car Wash. How are the cars' average velocities between these two times related?

Q2.6. A driver in Massachusetts was sent to traffic court for speeding. The evidence against the driver was that a policewoman observed the driver's car alongside a second car at a certain moment, and the policewoman had already clocked the second car as going faster than the speed limit. The driver argued, "The second car was passing me. I was not speeding." The judge ruled against the driver because, in the judge's words, "If two cars were side by side, you were both speeding." If you were a lawyer representing the accused driver, how would you argue this case?

Q2.9. Can you have a zero displacement and a nonzero average velocity? A nonzero velocity? Illustrate your answers on an x-t graph.

Q2.10. Can you have zero acceleration and nonzero velocity? Explain using a v_x-t graph.

Q2.11. Can you have zero velocity and nonzero average acceleration? Zero velocity and nonzero acceleration? Explain using a v_x-t graph, and give an example of such motion.

Q2.12. An automobile is traveling west. Can it have a velocity toward the west and at the same time have an acceleration toward the east? Under what circumstances?

Q2.13. The official's truck in Fig. 2.2 is at $x_1 = 277$ m at $t_1 = 16.0$ s and is at $x_2 = 19$ m at $t_2 = 25.0$ s. (a) Sketch *two* different possible x-t graphs for the motion of the truck. (b) Does the average velocity $v_{av\text{-}x}$ during the time interval from t_1 to t_2 have the same value for both of your graphs? Why or why not?

Q2.14. Under constant acceleration the average velocity of a particle is half the sum of its initial and final velocities. Is this still true if the acceleration is *not* constant? Explain.

Q2.15. You throw a baseball straight up in the air so that it rises to a maximum height much greater than your height. Is the magnitude of the acceleration greater while it is being thrown or after it leaves your hand? Explain.

Q2.16. Prove these statements: (a) As long as you can neglect the effects of the air, if you throw anything vertically upward, it will have the same speed when it returns to the release point as when it was released. (b) The time of flight will be twice the time it takes to get to its highest point.

Q2.17. A dripping water faucet steadily releases drops 1.0 s apart. As these drops fall, will the distance between them increase, decrease, or remain the same? Prove your answer.

Q2.18. If the initial position and initial velocity of a vehicle are known and a record is kept of the acceleration at each instant, can you compute the vehicle's position after a certain time from these data? If so, explain how this might be done.

Q2.18. From the top of a tall building you throw one ball straight up with speed v_0 and one ball straight down with speed v_0. (a) Which ball has the greater speed when it reaches the ground? (b) Which ball gets to the ground first? (c) Which ball has a greater displacement when it reaches the ground? (d) Which ball has traveled the greater distance when it hits the ground?

Q2.20. A ball is dropped from rest from the top of a building of height h. At the same instant, a second ball is projected vertically upward from ground level, such that it has zero speed when it reaches the top of the building. When the two balls pass each other, which ball has the greater speed, or do they have the same speed? Explain. Where will the two balls be when they are alongside each other: at height $h/2$ above the ground, below this height, or above this height? Explain.

Exercises

Section 2.1 Displacement, Time, and Average Velocity

2.1. A rocket carrying a satellite is accelerating straight up from the earth's surface. At 1.15 s after liftoff, the rocket clears the top of its launch platform, 63 m above the ground. After an additional 4.75 s, it is 1.00 km above the ground. Calculate the magnitude of the average velocity of the rocket for (a) the 4.75-s part of its flight and (b) the first 5.90 s of its flight.

2.2. In an experiment, a shearwater (a seabird) was taken from its nest, flown 5150 km away, and released. The bird found its way back to its nest 13.5 days after release. If we place the origin in the nest and extend the $+x$-axis to the release point, what was the bird's average velocity in m/s (a) for the return flight, and (b) for the whole episode, from leaving the nest to returning?

2.3. Trip Home. You normally drive on the freeway between San Diego and Los Angeles at an average speed of 105 km/h (65 mi/h), and the trip takes 2 h and 20 min. On a Friday afternoon, however, heavy traffic slows you down and you drive the same distance at an average speed of only 70 km/h (43 mi/h). How much longer does the trip take?

2.4. From Pillar to Post. Starting from a pillar, you run 200 m east (the $+x$-direction) at an average speed of 5.0 m/s, and then run 280 m west at an average speed of 4.0 m/s to a post. Calculate (a) your average speed from pillar to post and (b) your average velocity from pillar to post.

2.5. Two runners start simultaneously from the same point on a circular 200-m track and run in *opposite* directions. One runs at a constant speed of 6.20 m/s, and the other runs at a constant speed of 5.50 m/s. When they first meet, (a) for how long a time will they have been running, and (b) how far will each one have run along the track?

2.6. Suppose the two runners in Exercise 2.5 start at the same time from the same place but run in the *same* direction. (a) When will the fast one first overtake ("lap") the slower one, and how far from the starting point will each have run? (b) When will the fast one overtake the slower one for the *second* time, and how far from the starting point will they be at that instant?

2.7. Earthquake Analysis. Earthquakes produce several types of shock waves. The most well known are the P-waves (P for *primary* or *pressure*) and the S-waves (S for *secondary* or *shear*). In the earth's crust, the P-waves travel at around 6.5 km/s, while the S-waves move at about 3.5 km/s. The actual speeds vary depending on the type of material they are going through. The time delay between the arrival of these two waves at a seismic recording station tells geologists how far away the earthquake occurred. If the time delay is 33 s, how far from the seismic station did the earthquake occur?

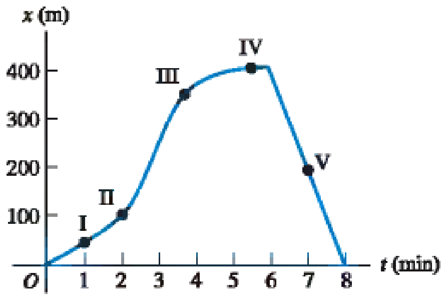
2.8. A Honda Civic travels in a straight line along a road. Its distance x from a stop sign is given as a function of time t by the equation $x(t) = \alpha t^2 - \beta t^3$, where $\alpha = 1.50\ \text{m/s}^2$ and $\beta = 0.0500\ \text{m/s}^3$. Calculate the average velocity of the car for each time interval: (a) $t = 0$ to $t = 2.00$ s; (b) $t = 0$ to $t = 4.00$ s; (c) $t = 2.00$ s to $t = 4.00$ s.

Section 2.2 Instantaneous Velocity

2.9. A car is stopped at a traffic light. It then travels along a straight road so that its distance from the light is given by $x(t) = bt^2 - ct^3$, where $b = 2.40\ \text{m/s}^2$ and $c = 0.120\ \text{m/s}^3$. (a) Calculate the average velocity of the car for the time interval $t = 0$ to $t = 10.0$ s. (b) Calculate the instantaneous velocity of the car at $t = 0$, $t = 5.0$ s, and $t = 10.0$ s. (c) How long after starting from rest is the car again at rest?

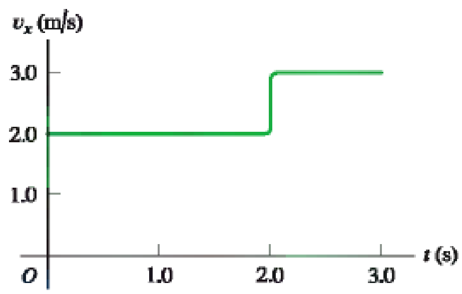
2.10. A physics professor leaves her house and walks along the sidewalk toward campus. After 5 min it starts to rain and she returns home. Her distance from her house as a function of time is shown in Fig. 2.32. At which of the labeled points is her velocity (a) zero? (b) constant and positive? (c) constant and negative? (d) increasing in magnitude? (e) decreasing in magnitude?

Figure 2.32 Exercise 2.10.



2.11. A ball moves in a straight line (the x-axis). The graph in Fig. 2.33 shows this ball's velocity as a function of time. (a) What are the ball's average speed and average velocity during the first 3.0 s? (b) Suppose that the ball moved in such a way that the graph segment after 2.0 s was $-3.0\ \text{m/s}$ instead of $+3.0\ \text{m/s}$. Find the ball's average speed and average velocity in this case.

Figure 2.33 Exercise 2.11.



Section 2.3 Average and Instantaneous Acceleration

2.12. A test driver at Incredible Motors, Inc., is testing a new model car with a speedometer calibrated to read m/s rather than mi/h. The following series of speedometer readings was obtained during a test run along a long, straight road:

Time (s)	0	2	4	6	8	10	12	14	16
Speed (m/s)	0	0	2	6	10	16	19	22	22

(a) Compute the average acceleration during each 2-s interval. Is the acceleration constant? Is it constant during any part of the test run? (b) Make a v_x-t graph of the data, using scales of 1 cm = 1 s horizontally and 1 cm = 2 m/s vertically. Draw a smooth curve through the plotted points. By measuring the slope of your curve, find the instantaneous acceleration at $t = 9$ s, 13 s, and 15 s.

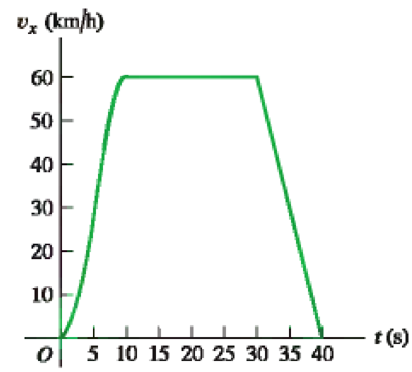
2.13. The Fastest (and Most Expensive) Car! The table shows test data for the Bugatti Veyron, the fastest car made. The car is moving in a straight line (the x-axis).

Time (s)	0	2.1	20.0	53
Speed (mi/h)	0	60	200	253

(a) Make a v_x-t graph of this car's velocity (in mi/h) as a function of time. Is its acceleration constant? (b) Calculate the car's average acceleration (in m/s^2) between (i) 0 and 2.1 s; (ii) 2.1 s and 20.0 s; (iii) 20.0 s and 53 s. Are these results consistent with your graph in part (a)? (Before you decide to buy this car, it might be helpful to know that only 300 will be built, it runs out of gas in 12 minutes at top speed, and it costs $1.25 million!)

2.14. Figure 2.34 shows the velocity of a solar-powered car as a function of time. The driver accelerates from a stop sign, cruises for 20 s at a constant speed of 60 km/h, and then brakes to come to a stop 40 s after leaving the stop sign. (a) Compute the average acceleration during the following time intervals: (i) $t = 0$ to $t = 10$ s; (ii) $t = 30$ s to $t = 40$ s; (iii) $t = 10$ s to $t = 30$ s; (iv) $t = 0$ to $t = 40$ s. (b) What is the instantaneous acceleration at $t = 20$ s and at $t = 35$ s?

Figure 2.34 Exercise 2.14.



2.15. A turtle crawls along a straight line, which we will call the x-axis with the positive direction to the right. The equation for the turtle's position as a function of time is $x(t) = 50.0\ \text{cm} + (2.00\ \text{cm/s})t - (0.0625\ \text{cm/s}^2)t^2$. (a) Find the turtle's initial velocity, initial position, and initial acceleration. (b) At what time t is the velocity of the turtle zero? (c) How long after starting does it take the turtle to return to its starting point? (d) At what times t is the turtle a distance of 10.0 cm from its starting point? What is the velocity (magnitude and direction) of the turtle at each of these times? (e) Sketch graphs of x versus t, v_x versus t, and a_x versus t, for the time interval $t = 0$ to $t = 40$ s.

2.16. An astronaut has left the International Space Station to test a new space scooter. Her partner measures the following velocity changes, each taking place in a 10-s interval. What are the magnitude, the algebraic sign, and the direction of the average acceleration in each interval? Assume that the positive direction is to the right. (a) At the beginning of the interval the astronaut is moving toward the right along the x-axis at 15.0 m/s, and at the end of the interval she is moving toward the right at 5.0 m/s. (b) At the beginning she is moving toward the left at 5.0 m/s, and at the end she is moving toward the left at 15.0 m/s. (c) At the beginning she is moving toward the right at 15.0 m/s, and at the end she is moving toward the left at 15.0 m/s.

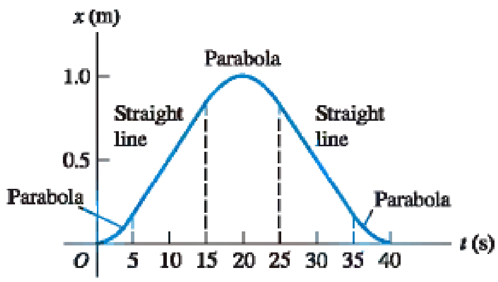
2.17. Auto Acceleration. Based on your experiences of riding in automobiles, estimate the magnitude of a car's average acceleration when it (a) accelerates onto a freeway from rest to 65 mi/h, and (b) brakes from highway speeds to a sudden stop. (c) Explain why the average acceleration in each case could be regarded as either positive or negative.

2.18. A car's velocity as a function of time is given by $v_x(t) = \alpha + \beta t^2$, where $\alpha = 3.00\ \text{m/s}$ and $\beta = 0.100\ \text{m/s}^3$. (a) Calculate the average acceleration for the time interval $t = 0$ to $t = 5.00$ s.

(b) Calculate the instantaneous acceleration for $t = 0$ and $t = 5.00$ s. (c) Draw accurate v_x-t and a_x-t graphs for the car's motion between $t = 0$ and $t = 5.00$ s.

2.19. Figure 2.35 is a graph of the coordinate of a spider crawling along the x-axis. (a) Graph its velocity and acceleration as functions of time. (b) In a motion diagram (like Fig. 2.13b and 2.14b), show the position, velocity, and acceleration of the spider at the five times $t = 2.5$ s, $t = 10$ s, $t = 20$ s, $t = 30$ s, and $t = 37.5$ s.

Figure 2.35 Exercise 2.19.



2.20. The position of the front bumper of a test car under microprocessor control is given by $x(t) = 2.17\text{ m} + (4.80\text{ m/s}^2)t^2 - (0.100\text{ m/s}^6)t^6$. (a) Find its position and acceleration at the instants when the car has zero velocity. (b) Draw x-t, v_x-t, and a_x-t graphs for the motion of the bumper between $t = 0$ and $t = 2.00$ s.

Section 2.4 Motion with Constant Acceleration

2.21. An antelope moving with constant acceleration covers the distance between two points 70.0 m apart in 7.00 s. Its speed as it passes the second point is 15.0 m/s. (a) What is its speed at the first point? (b) What is its acceleration?

2.22. The catapult of the aircraft carrier USS *Abraham Lincoln* accelerates an F/A-18 Hornet jet fighter from rest to a takeoff speed of 173 mi/h in a distance of 307 ft. Assume constant acceleration. (a) Calculate the acceleration of the fighter in m/s^2. (b) Calculate the time required for the fighter to accelerate to takeoff speed.

2.23. A Fast Pitch. The fastest measured pitched baseball left the pitcher's hand at a speed of 45.0 m/s. If the pitcher was in contact with the ball over a distance of 1.50 m and produced constant acceleration, (a) what acceleration did he give the ball, and (b) how much time did it take him to pitch it?

2.24. A Tennis Serve. In the fastest measured tennis serve, the ball left the racquet at 73.14 m/s. A served tennis ball is typically in contact with the racquet for 30.0 ms and starts from rest. Assume constant acceleration. (a) What was the ball's acceleration during this serve? (b) How far did the ball travel during the serve?

2.25. Automobile Airbags. The human body can survive an acceleration trauma incident (sudden stop) if the magnitude of the acceleration is less than 250 m/s^2. If you are in an automobile accident with an initial speed of 105 km/h (65 mi/h) and you are stopped by an airbag that inflates from the dashboard, over what distance must the airbag stop you for you to survive the crash?

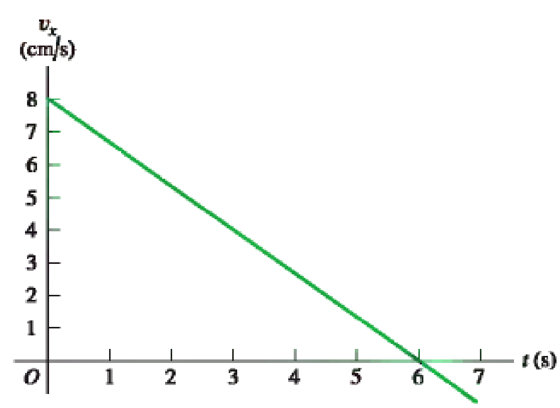
2.26. Entering the Freeway. A car sits in an entrance ramp to a freeway, waiting for a break in the traffic. The driver accelerates with constant acceleration along the ramp and onto the freeway. The car starts from rest, moves in a straight line, and has a speed of 20 m/s (45 mi/h) when it reaches the end of the 120-m-long ramp. (a) What is the acceleration of the car? (b) How much time does it take the car to travel the length of the ramp? (c) The traffic on the freeway is moving at a constant speed of 20 m/s. What distance does the traffic travel while the car is moving the length of the ramp?

2.27. Launch of the Space Shuttle. At launch the space shuttle weighs 4.5 million pounds. When it is launched from rest, it takes 8.00 s to reach 161 km/h, and at the end of the first 1.00 min its speed is 1610 km/h. (a) What is the average acceleration (in m/s^2) of the shuttle (i) during the first 8.00 s, and (ii) between 8.00 s and the end of the first 1.00 min? (b) Assuming the acceleration is constant during each time interval (but not necessarily the same in both intervals), what distance does the shuttle travel (i) during the first 8.00 s, and (ii) during the interval from 8.00 s to 1.00 min?

2.28. According to recent test data, an automobile travels 0.250 mi in 19.9 s, starting from rest. The same car, when braking from 60.0 mi/h on dry pavement, stops in 146 ft. Assume constant acceleration in each part of the motion, but not necessarily the same acceleration when slowing down as when speeding up. (a) Find the acceleration of this car when it is speeding up and when it is braking. (b) If its acceleration is constant, how fast (in mi/h) should this car be traveling after 0.250 mi of acceleration? The actual measured speed is 70.0 mi/h; what does this tell you about the motion? (c) How long does it take this car to stop while braking from 60.0 mi/h?

2.29. A cat walks in a straight line, which we shall call the x-axis with the positive direction to the right. As an observant physicist, you make measurements of this cat's motion and construct a graph of the feline's velocity as a function of time (Fig. 2.36). (a) Find the cat's velocity at $t = 4.0$ s and at $t = 7.0$ s. (b) What is the cat's acceleration at $t = 3.0$ s? At $t = 6.0$ s? At $t = 7.0$ s? (c) What distance does the cat move during the first 4.5 s? From $t = 0$ to $t = 7.5$ s? (d) Sketch clear graphs of the cat's acceleration and position as functions of time, assuming that the cat started at the origin.

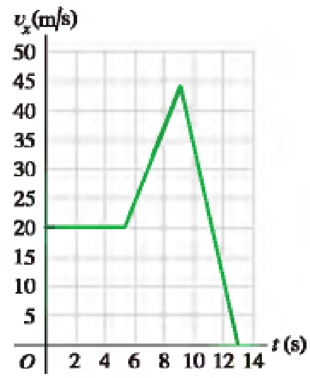
Figure 2.36 Exercise 2.29.



2.30. At $t = 0$ a car is stopped at a traffic light. When the light turns green, the car starts to speed up, and gains speed at a constant rate until it reaches a speed of 20 m/s 8 seconds after the light turns green. The car continues at a constant speed for 60 m. Then the driver sees a red light up ahead at the next intersection, and starts slowing down at a constant rate. The car stops at the red light, 180 m from where it was at $t = 0$. (a) Draw accurate x-t, v_x-t, and a_x-t graphs for the motion of the car. (b) In a motion diagram (like Figs. 2.13b and 2.14b), show the position, velocity, and acceleration of the car at 4 s after the light changes, while traveling at constant speed, and while slowing down.

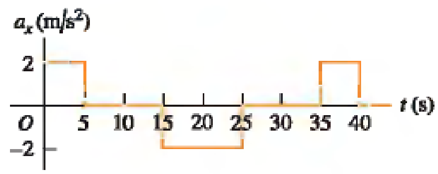
2.31. The graph in Fig. 2.37 shows the velocity of a motorcycle police officer plotted as a function of time. (a) Find the instantaneous acceleration at $t = 3$ s, at $t = 7$ s, and at $t = 11$ s. (b) How far does the officer go in the first 5 s? The first 9 s? The first 13 s?

Figure 2.37 Exercise 2.31.



2.32. Figure 2.38 is a graph of the acceleration of a model railroad locomotive moving on the x-axis. Graph its velocity and x-coordinate as functions of time if $x = 0$ and $v_x = 0$ at $t = 0$.

Figure 2.38 Exercise 2.32.

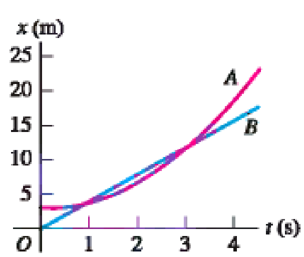


2.33. A spaceship ferrying workers to Moon Base I takes a straight-line path from the earth to the moon, a distance of 384,000 km. Suppose the spaceship starts from rest and accelerates at 20.0 m/s^2 for the first 15.0 min of the trip, and then travels at constant speed until the last 15.0 min, when it slows down at a rate of 20.0 m/s^2, just coming to rest as it reaches the moon. (a) What is the maximum speed attained? (b) What fraction of the total distance is traveled at constant speed? (c) What total time is required for the trip?

2.34. A subway train starts from rest at a station and accelerates at a rate of 1.60 m/s^2 for 14.0 s. It runs at constant speed for 70.0 s and slows down at a rate of 3.50 m/s^2 until it stops at the next station. Find the *total* distance covered.

2.35. Two cars, A and B, move along the x-axis. Figure 2.39 is a graph of the positions of A and B versus time. (a) In motion diagrams (like Figs. 2.13b and 2.14b), show the position, velocity, and acceleration of each of the two cars at $t = 0$, $t = 1$ s, and $t = 3$ s. (b) At what time(s), if any, do A and B have the same position? (c) Graph velocity versus time for both A and B. (d) At what time(s), if any, do A and B have the same velocity? (e) At what time(s), if any, does car A pass car B? (f) At what time(s), if any, does car B pass car A?

Figure 2.39 Exercise 2.35.



2.36. At the instant the traffic light turns green, a car that has been waiting at an intersection starts ahead with a constant acceleration of 3.20 m/s^2. At the same instant a truck, traveling with a constant speed of 20.0 m/s, overtakes and passes the car. (a) How far beyond its starting point does the car overtake the truck? (b) How fast is the car traveling when it overtakes the truck? (c) Sketch an x-t graph of the motion of both vehicles. Take $x = 0$ at the intersection. (d) Sketch a v_x-t graph of the motion of both vehicles.

2.37. Mars Landing. In January 2004, NASA landed exploration vehicles on Mars. Part of the descent consisted of the following stages:

Stage A: Friction with the atmosphere reduced the speed from 19,300 km/h to 1600 km/h in 4.0 min.
Stage B: A parachute then opened to slow it down to 321 km/h in 94 s.
Stage C: Retro rockets then fired to reduce its speed to zero over a distance of 75 m.

Assume that each stage followed immediately after the preceding one and that the acceleration during each stage was constant. (a) Find the rocket's acceleration (in m/s^2) during each stage. (b) What total distance (in km) did the rocket travel during stages A, B, and C?

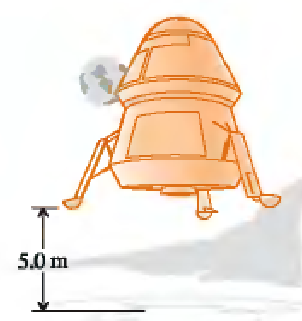
Section 2.5 Freely Falling Bodies

2.38. Raindrops. If the effects of the air acting on falling raindrops are ignored, then we can treat raindrops as freely falling objects. (a) Rain clouds are typically a few hundred meters above the ground. Estimate the speed with which raindrops would strike the ground if they were freely falling objects. Give your estimate in m/s, km/h, and mi/h. (b) Estimate (from your own personal observations of rain) the speed with which raindrops actually strike the ground. (c) Based on your answers to parts (a) and (b), is it a good approximation to neglect the effects of the air on falling raindrops? Explain.

2.39. (a) If a flea can jump straight up to a height of 0.440 m, what is its initial speed as it leaves the ground? (b) How long is it in the air?

2.40. Touchdown on the Moon. A lunar lander is making its descent to Moon Base I (Fig. 2.40). The lander descends slowly under the retro-thrust of its descent engine. The engine is cut off when the lander is 5.0 m above the surface and has a downward speed of 0.8 m/s. With the engine off, the lander is in free fall. What is the speed of the lander just before it touches the surface? The acceleration due to gravity on the moon is 1.6 m/s^2.

Figure 2.40 Exercise 2.40.



2.41. A Simple Reaction-Time Test. A meter stick is held vertically above your hand, with the lower end between your thumb and first finger. On seeing the meter stick released, you grab it with these two fingers. You can calculate your reaction time from the distance the meter stick falls, read directly from the point where your fingers grabbed it. (a) Derive a relationship for your reaction time in terms of this measured distance, d. (b) If the measured distance is 17.6 cm, what is the reaction time?

2.42. A brick is dropped (zero initial speed) from the roof of a building. The brick strikes the ground in 2.50 s. You may ignore air resistance, so the brick is in free fall. (a) How tall, in meters, is the building? (b) What is the magnitude of the brick's velocity just before it reaches the ground? (c) Sketch a_y-t, v_y-t, and y-t graphs for the motion of the brick.

2.43. Launch Failure. A 7500-kg rocket blasts off vertically from the launch pad with a constant upward acceleration of $2.25\ \text{m/s}^2$ and feels no appreciable air resistance. When it has reached a height of 525 m, its engines suddenly fail so that the only force acting on it is now gravity. (a) What is the maximum height this rocket will reach above the launch pad? (b) How much time after engine failure will elapse before the rocket comes crashing down to the launch pad, and how fast will it be moving just before it crashes? (c) Sketch a_y-t, v_y-t, and y-t graphs of the rocket's motion from the instant of blast-off to the instant just before it strikes the launch pad.

2.44. A hot-air balloonist, rising vertically with a constant velocity of magnitude 5.00 m/s, releases a sandbag at an instant when the balloon is 40.0 m above the ground (Fig. 2.41). After it is released, the sandbag is in free fall. (a) Compute the position and velocity of the sandbag at 0.250 s and 1.00 s after its release. (b) How many seconds after its release will the bag strike the ground? (c) With what magnitude of velocity does it strike the ground? (d) What is the greatest height above the ground that the sandbag reaches? (e) Sketch a_y-t, v_y-t, and y-t graphs for the motion.

Figure 2.41 Exercise 2.44.

2.45. A student throws a water balloon vertically downward from the top of a building. The balloon leaves the thrower's hand with a speed of 6.00 m/s. Air resistance may be ignored, so the water balloon is in free fall after it leaves the thrower's hand. (a) What is its speed after falling for 2.00 s? (b) How far does it fall in 2.00 s? (c) What is the magnitude of its velocity after falling 10.0 m? (d) Sketch a_y-t, v_y-t, and y-t graphs for the motion.

2.46. An egg is thrown nearly vertically upward from a point near the cornice of a tall building. It just misses the cornice on the way down and passes a point 50.0 m below its starting point 5.00 s after it leaves the thrower's hand. Air resistance may be ignored. (a) What is the initial speed of the egg? (b) How high does it rise above its starting point? (c) What is the magnitude of its velocity at the highest point? (d) What are the magnitude and direction of its acceleration at the highest point? (e) Sketch a_y-t, v_y-t, and y-t graphs for the motion of the egg.

2.47. The rocket-driven sled *Sonic Wind No. 2*, used for investigating the physiological effects of large accelerations, runs on a straight, level track 1070 m (3500 ft) long. Starting from rest, it can reach a speed of 224 m/s (500 mi/h) in 0.900 s. (a) Compute the acceleration in m/s^2, assuming that it is constant. (b) What is the ratio of this acceleration to that of a freely falling body (g)? (c) What distance is covered in 0.900 s? (d) A magazine article states that at the end of a certain run, the speed of the sled decreased from 283 m/s (632 mi/h) to zero in 1.40 s and that during this time the magnitude of the acceleration was greater than 40g. Are these figures consistent?

2.48. A large boulder is ejected vertically upward from a volcano with an initial speed of 40.0 m/s. Air resistance may be ignored. (a) At what time after being ejected is the boulder moving at 20.0 m/s upward? (b) At what time is it moving at 20.0 m/s downward? (c) When is the displacement of the boulder from its initial position zero? (d) When is the velocity of the boulder zero? (e) What are the magnitude and direction of the acceleration while the boulder is (i) moving upward? (ii) Moving downward? (iii) At the highest point? (f) Sketch a_y-t, v_y-t, and y-t graphs for the motion.

2.40. A 15-kg rock is dropped from rest on the earth and reaches the ground in 1.75 s. When it is dropped from the same height on Saturn's satellite Enceladus, it reaches the ground in 18.6 s. What is the acceleration due to gravity on Enceladus?

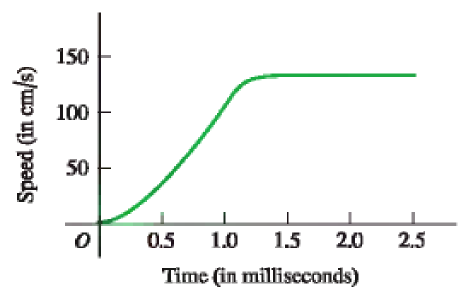
*Section 2.6 Velocity and Position by Integration

***2.50.** The acceleration of a bus is given by $a_x(t) = \alpha t$, where $\alpha = 1.2\ \text{m/s}^3$. (a) If the bus's velocity at time $t = 1.0$ s is 5.0 m/s, what is its velocity at time $t = 2.0$ s? (b) If the bus's position at time $t = 1.0$ s is 6.0 m, what is its position at time $t = 2.0$ s? (c) Sketch a_x-t, v_x-t, and x-t graphs for the motion.

***2.51.** The acceleration of a motorcycle is given by $a_x(t) = At - Bt^2$, where $A = 1.50\ \text{m/s}^3$ and $B = 0.120\ \text{m/s}^4$. The motorcycle is at rest at the origin at time $t = 0$. (a) Find its position and velocity as functions of time. (b) Calculate the maximum velocity it attains.

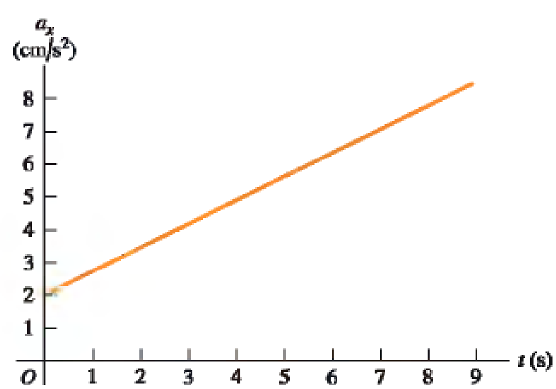
***2.52. Flying Leap of the Flea.** High-speed motion pictures (3500 frames/second) of a jumping, 210-μg flea yielded the data used to plot the graph given in Fig. 2.42. (See "The Flying Leap of the Flea" by M. Rothschild, Y. Schlein, K. Parker, C. Neville, and S. Sternberg in the November 1973 *Scientific American*.) This flea was about 2 mm long and jumped at a nearly vertical take-off angle. Use the graph to answer the questions. (a) Is the acceleration of the flea ever zero? If so, when? Justify your answer. (b) Find the maximum height the flea reached in the first 2.5 ms. (c) Find the flea's acceleration at 0.5 ms, 1.0 ms, and 1.5 ms. (d) Find the flea's height at 0.5 ms, 1.0 ms, and 1.5 ms.

Figure 2.42 Exercise 2.52.



***2.53.** The graph in Fig. 2.43 describes the acceleration as a function of time for a stone rolling down a hill starting from rest. (a) Find

Figure 2.43 Exercise 2.53



the change in the stone's velocity between $t = 2.5$ s and $t = 7.5$ s. (b) Sketch a graph of the stone's velocity as a function of time.

Problems

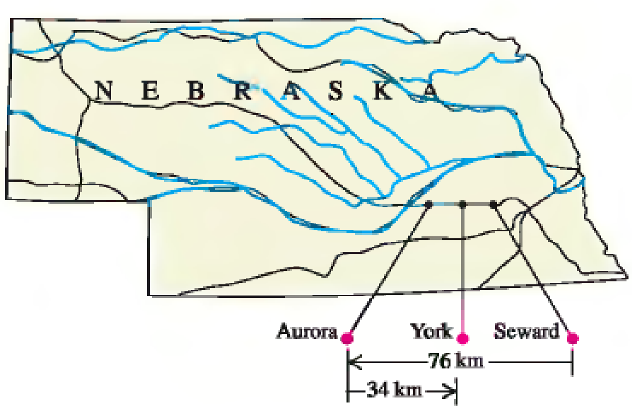
2.54. On a 20-mile bike ride, you ride the first 10 miles at an average speed of 8 mi/h. What must your average speed over the next 10 miles be to have your average speed for the total 20 miles be (a) 4 mi/h? (b) 12 mi/h? (c) Given this average speed for the first 10 miles, can you possibly attain an average speed of 16 mi/h for the total 20-mile ride? Explain.

2.55. The position of a particle between $t = 0$ and $t = 2.00$ s is given by $x(t) = (3.00\ \text{m/s}^3)t^3 - (10.0\ \text{m/s}^2)t^2 + (9.00\ \text{m/s})t$. (a) Draw the x-t, v_x-t, and a_x-t graphs of this particle. (b) At what time(s) between $t = 0$ and $t = 2.00$ s is the particle instantaneously at rest? Does your numerical result agree with the v_x-t graph in part (a)? (c) At each time calculated in part (b) is the acceleration of the particle positive or negative? Show that in each case the same answer is deduced from $a_x(t)$ and from the v_x-t graph. (d) At what time(s) between $t = 0$ and $t = 2.00$ s is the velocity of the particle instantaneously not changing? Locate this point on the v_x-t and a_x-t graphs of part (a). (e) What is the particle's greatest distance from the origin $(x = 0)$ between $t = 0$ and $t = 2.00$ s? (f) At what time(s) between $t = 0$ and $t = 2.00$ s is the particle *speeding up* at the greatest rate? At what time(s) between $t = 0$ and $t = 2.00$ s is the particle *slowing down* at the greatest rate? Locate these points on the v_x-t and a_x-t graphs of part (a).

2.56. Relay Race. In a relay race, each contestant runs 25.0 m while carrying an egg balanced on a spoon, turns around, and comes back to the starting point. Edith runs the first 25.0 m in 20.0 s. On the return trip she is more confident and takes only 15.0 s. What is the magnitude of her average velocity for (a) the first 25.0 m? (b) The return trip? (c) What is her average velocity for the entire round trip? (d) What is her average speed for the round trip?

2.57. Dan gets on Interstate Highway I–80 at Seward, Nebraska, and drives due west in a straight line and at an average velocity of magnitude 88 km/h. After traveling 76 km, he reaches the Aurora exit (Fig. 2.44). Realizing he has gone too far, he turns around and drives due east 34 km back to the York exit at an average velocity of magnitude 72 km/h. For his whole trip from Seward to the York exit, what are (a) his average speed and (b) the magnitude of his average velocity?

Figure 2.44 Problem 2.57.



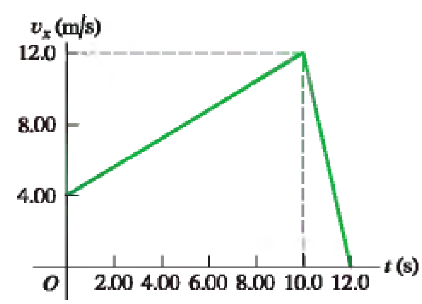
2.58. Freeway Traffic. According to a *Scientific American* article (May 1990), current freeways can sustain about 2400 vehicles per lane per hour in smooth traffic flow at 96 km/h (60 mi/h). With more vehicles the traffic flow becomes "turbulent" (stop-and-go). (a) If a vehicle is 4.6 m (15 ft) long on the average, what is the average spacing between vehicles at the above traffic density? (b) Collision-avoidance automated control systems, which operate by bouncing radar or sonar signals off surrounding vehicles and then accelerate or brake the car when necessary, could greatly reduce the required spacing between vehicles. If the average spacing is 9.2 m (two car lengths), how many vehicles per hour can a lane of traffic carry at 96 km/h?

2.59. A world-class sprinter accelerates to his maximum speed in 4.0 s. He then maintains this speed for the remainder of a 100-m race, finishing with a total time of 9.1 s. (a) What is the runner's average acceleration during the first 4.0 s? (b) What is his average acceleration during the last 5.1 s? (c) What is his average acceleration for the entire race? (d) Explain why your answer to part (c) is not the average of the answers to parts (a) and (b).

2.60. A sled starts from rest at the top of a hill and slides down with a constant acceleration. At some later time it is 14.4 m from the top; 2.00 s after that it is 25.6 m from the top, 2.00 s later 40.0 m from the top, and 2.00 s later it is 57.6 m from the top. (a) What is the magnitude of the average velocity of the sled during each of the 2.00-s intervals after passing the 14.4-m point? (b) What is the acceleration of the sled? (c) What is the speed of the sled when it passes the 14.4-m point? (d) How much time did it take to go from the top to the 14.4-m point? (e) How far did the sled go during the first second after passing the 14.4-m point?

2.61. A gazelle is running in a straight line (the x-axis). The graph in Fig. 2.45 shows this animal's velocity as a function of time. During the first 12.0 s, find (a) the total distance moved and (b) the displacement of the gazelle. (c) Sketch an a_x-t graph showing this gazelle's acceleration as a function of time for the first 12.0 s.

Figure 2.45 Problem 2.61.

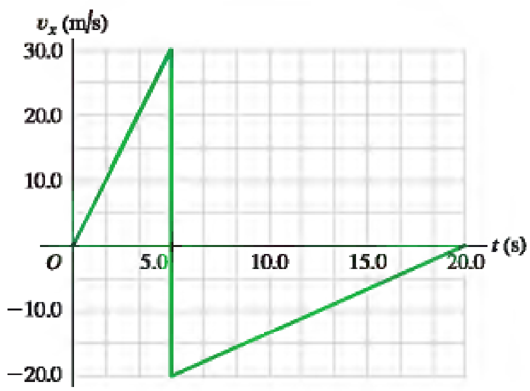


2.62. In air or vacuum light travels at a constant speed of 3.0×10^8 m/s. To answer some of these questions you may need to look up astronomical data in Appendix F. (a) One light year is defined as the distance light travels in 1 year. Use this information to determine how many meters there are in 1 light-year. (b) How far in meters does light travel in 1 nanosecond? (c) When a solar flare occurs on our sun, how soon after its occurrence can we first observe it? (d) By bouncing laser beams off a reflector placed on our moon by the Apollo astronauts, astronomers can make very accurate measurements of the earth–moon distance. How long after it is sent does it take such a laser beam (which is just a light beam) to return to earth? (e) The *Voyager* probe, which passed by Neptune in August 1989, was about 3.0 billion miles from earth at that time. Photographs and other information were sent to earth by radio waves, which travel at the speed of light. How long did it take these waves to reach earth from *Voyager*?

2.63. Use the information in Appendix F to answer the questions. (a) What is the speed of the Galapagos Islands, on the earth's equator, due to our planet's spin on its axis? (b) What is the earth's speed due to its rotation around the sun? (c) If light would bend around the curvature of the earth (which it does not), how many times would a light beam go around the equator in one second?

2.64. A rigid ball traveling in a straight line (the x-axis) hits a solid wall and suddenly rebounds during a brief instant. The v_x-t graph in Fig. 2.46 shows this ball's velocity as a function of time. During the first 20.0 s of its motion, find (a) the total distance the ball moves, and (b) its displacement. (c) Sketch a graph of a_x-t for this ball's motion. (d) Is the graph shown really vertical at 5.00 s? Explain.

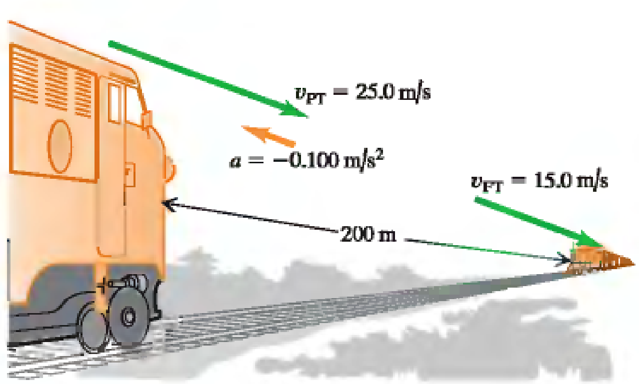
Figure 2.46 Problem 2.64.



2.65. A ball starts from rest and rolls down a hill with uniform acceleration, traveling 150 m during the second 5.0 s of its motion. How far did it roll during the first 5.0 s of motion?

2.66. Collision. The engineer of a passenger train traveling at 25.0 m/s sights a freight train whose caboose is 200 m ahead on the same track (Fig. 2.47). The freight train is traveling at 15.0 m/s in the same direction as the passenger train. The engineer of the passenger train immediately applies the brakes, causing a constant acceleration of -0.100 m/s^2, while the freight train continues with constant speed. Take $x = 0$ at the location of the front of the passenger train when the engineer applies the brakes. (a) Will the cows nearby witness a collision? (b) If so, where will it take place? (c) On a single graph, sketch the positions of the front of the passenger train and the back of the freight train.

Figure 2.47 Problem 2.66.



2.67. Large cockroaches can run as fast as 1.50 m/s in short bursts. Suppose you turn on the light in a cheap motel and see one scurrying directly away from you at a constant 1.50 m/s. If you start 0.90 m behind the cockroach with an initial speed of 0.80 m/s toward it, what minimum constant acceleration would you need to catch up with it when it has traveled 1.20 m, just short of safety under a counter?

2.68. Two cars start 200 m apart and drive toward each other at a steady 10 m/s. On the front of one of them, an energetic grasshopper jumps back and forth between the cars (he has strong legs!) with a constant horizontal velocity of 15 m/s relative to the ground. The insect jumps the instant he lands, so he spends no time resting on either car. What total distance does the grasshopper travel before the cars hit?

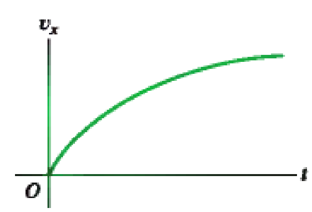
2.69. An automobile and a truck start from rest at the same instant, with the automobile initially at some distance behind the truck. The truck has a constant acceleration of 2.10 m/s^2, and the automobile an acceleration of 3.40 m/s^2. The automobile overtakes the truck after the truck has moved 40.0 m. (a) How much time does it take the automobile to overtake the truck? (b) How far was the automobile behind the truck initially? (c) What is the speed of each when they are abreast? (d) On a single graph, sketch the position of each vehicle as a function of time. Take $x = 0$ at the initial location of the truck.

2.70. Two stunt drivers drive directly toward each other. At time $t = 0$ the two cars are a distance D apart, car 1 is at rest, and car 2 is moving to the left with speed v_0. Car 1 begins to move at $t = 0$, speeding up with a constant acceleration a_x. Car 2 continues to move with a constant velocity. (a) At what time do the two cars collide? (b) Find the speed of car 1 just before it collides with car 2. (c) Sketch x-t and v_x-t graphs for car 1 and car 2. For each of the two graphs, draw the curves for both cars on the same set of axes.

2.71. A marble is released from one rim of a hemispherical bowl of diameter 50.0 cm and rolls down and up to the opposite rim in 10.0 s. Find (a) the average speed and (b) the average velocity of the marble.

2.72. You may have noticed while driving that your car's velocity does not continue to increase, even though you keep your foot on the gas pedal. This behavior is due to air resistance and friction between the moving parts of the car. Figure 2.48 shows a qualitative v_x-t graph for a typical car if it starts from rest at the origin and travels in a straight line (the x-axis). Sketch qualitative a_x-t and x-t graphs for this car.

Figure 2.48 Problem 2.72.



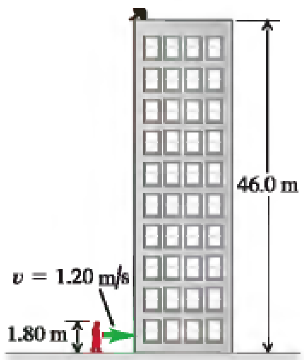
2.73. Passing. The driver of a car wishes to pass a truck that is traveling at a constant speed of 20.0 m/s (about 45 mi/h). Initially, the car is also traveling at 20.0 m/s and its front bumper is 24.0 m behind the truck's rear bumper. The car accelerates at a constant 0.600 m/s^2, then pulls back into the truck's lane when the rear of the car is 26.0 m ahead of the front of the truck. The car is 4.5 m long and the truck is 21.0 m long. (a) How much time is required for the car to pass the truck? (b) What distance does the car travel during this time? (c) What is the final speed of the car?

***2.74.** An object's velocity is measured to be $v_x(t) = \alpha - \beta t^2$, where $\alpha = 4.00$ m/s and $\beta = 2.00$ m/s^3. At $t = 0$ the object is at $x = 0$. (a) Calculate the object's position and acceleration as functions of time. (b) What is the object's maximum *positive* displacement from the origin?

***2.75.** The acceleration of a particle is given by $a_x(t) = -2.00\ \text{m/s}^2 + (3.00\ \text{m/s}^3)t$. (a) Find the initial velocity v_{0x} such that the particle will have the same x-coordinate at $t = 4.00$ s as it had at $t = 0$. (b) What will be the velocity at $t = 4.00$ s?

2.76. Egg Drop. You are on the roof of the physics building, 46.0 m above the ground (Fig. 2.49). Your physics professor, who is 1.80 m tall, is walking alongside the building at a constant speed of 1.20 m/s. If you wish to drop an egg on your professor's head, where should the professor be when you release the egg? Assume that the egg is in free fall.

Figure 2.49 Problem 2.76.



2.77. A certain volcano on earth can eject rocks vertically to a maximum height H. (a) How high (in terms of H) would these rocks go if a volcano on Mars ejected them with the same initial velocity? The acceleration due to gravity on Mars is 3.71 m/s², and you can neglect air resistance on both planets. (b) If the rocks are in the air for a time T on earth, for how long (in terms of T) will they be in the air on Mars?

2.78. An entertainer juggles balls while doing other activities. In one act, she throws a ball vertically upward, and while it is in the air, she runs to and from a table 5.50 m away at a constant speed of 2.50 m/s, returning just in time to catch the falling ball. (a) With what minimum initial speed must she throw the ball upward to accomplish this feat? (b) How high above its initial position is the ball just as she reaches the table?

2.79. Visitors at an amusement park watch divers step off a platform 21.3 m (70 ft) above a pool of water. According to the announcer, the divers enter the water at a speed of 56 mi/h (25 m/s). Air resistance may be ignored. (a) Is the announcer correct in this claim? (b) Is it possible for a diver to leap directly upward off the board so that, missing the board on the way down, she enters the water at 25.0 m/s? If so, what initial upward speed is required? Is the required initial speed physically attainable?

2.60. A flowerpot falls off a windowsill and falls past the window below. You may ignore air resistance. It takes the pot 0.420 s to pass this window, which is 1.90 m high. How far is the top of the window below the windowsill from which the flowerpot fell?

2.81. Certain rifles can fire a bullet with a speed of 965 m/s just as it leaves the muzzle (this speed is called the *muzzle velocity*). If the muzzle is 70.0 cm long and if the bullet is accelerated uniformly from rest within it, (a) what is the acceleration (in g's) of the bullet in the muzzle, and (b) for how long (in ms) is it in the muzzle? (c) If, when this rifle is fired vertically, the bullet reaches a maximum height H, what would be the maximum height (in terms of H) for a new rifle that produced half the muzzle velocity of this one?

2.62. A Multi-stage Rocket. In the first stage of a two-stage rocket, the rocket is fired from the launch pad starting from rest but with a constant acceleration of 3.50 m/s² upward. At 25.0 s after launch, the rocket fires the second stage, which suddenly boosts its speed to 132.5 m/s upward. This firing uses up all the fuel, however, so then the only force acting on the rocket is gravity. Air resistance is negligible. (a) Find the maximum height that the stage-two rocket reaches above the launch pad. (b) How much time after the stage-two firing will it take for the rocket to fall back to the launch pad? (c) How fast will the stage-two rocket be moving just as it reaches the launch pad?

2.83. Look Out Below. Sam heaves a 16-lb shot straight upward, giving it a constant upward acceleration from rest of 45.0 m/s² for 64.0 cm. He releases it 2.20 m above the ground. You may ignore air resistance. (a) What is the speed of the shot when Sam releases it? (b) How high above the ground does it go? (c) How much time does he have to get out of its way before it returns to the height of the top of his head, 1.83 m above the ground?

2.64. A physics teacher performing an outdoor demonstration suddenly falls from rest off a high cliff and simultaneously shouts "Help." When she has fallen for 3.0 s, she hears the echo of her shout from the valley floor below. The speed of sound is 340 m/s. (a) How tall is the cliff? (b) If air resistance is neglected, how fast will she be moving just before she hits the ground? (Her actual speed will be less than this, due to air resistance.)

2.65. Juggling Act. A juggler performs in a room whose ceiling is 3.0 m above the level of his hands. He throws a ball upward so that it just reaches the ceiling. (a) What is the initial velocity of the ball? (b) What is the time required for the ball to reach the ceiling? At the instant when the first ball is at the ceiling, the juggler throws a second ball upward with two-thirds the initial velocity of the first. (c) How long after the second ball is thrown did the two balls pass each other? (d) At what distance above the juggler's hand do they pass each other?

2.66. A helicopter carrying Dr. Evil takes off with a constant upward acceleration of 5.0 m/s². Secret agent Austin Powers jumps on just as the helicopter lifts off the ground. After the two men struggle for 10.0 s, Powers shuts off the engine and steps out of the helicopter. Assume that the helicopter is in free fall after its engine is shut off, and ignore the effects of air resistance. (a) What is the maximum height above ground reached by the helicopter? (b) Powers deploys a jet pack strapped on his back 7.0 s after leaving the helicopter, and then he has a constant downward acceleration with magnitude 2.0 m/s². How far is Powers above the ground when the helicopter crashes into the ground?

2.87. Building Height. Spider-Man steps from the top of a tall building. He falls freely from rest to the ground a distance of h. He falls a distance of $h/4$ in the last 1.0 s of his fall. What is the height h of the building?

2.88. Cliff Height. You are climbing in the High Sierra where you suddenly find yourself at the edge of a fog-shrouded cliff. To find the height of this cliff, you drop a rock from the top and 10.0 s later hear the sound of it hitting the ground at the foot of the cliff. (a) Ignoring air resistance, how high is the cliff if the speed of sound is 330 m/s? (b) Suppose you had ignored the time it takes the sound to reach you. In that case, would you have overestimated or underestimated the height of the cliff? Explain your reasoning.

2.69. Falling Can. A painter is standing on scaffolding that is raised at constant speed. As he travels upward, he accidentally nudges a paint can off the scaffolding and it falls 15.0 m to the ground. You are watching, and measure with your stopwatch that it takes 3.25 s for the can to reach the ground. Ignore air resistance. (a) What is the speed of the can just before it hits the ground? (b) Another painter is standing on a ledge, with his hands 4.00 m above the can when it falls off. He has lightning-fast reflexes and if the can passes in front of him, he can catch it. Does he get the chance?

2.90. Determined to test the law of gravity for himself, a student walks off a skyscraper 180 m high, stopwatch in hand, and starts his free fall (zero initial velocity). Five seconds later, Superman arrives at the scene and dives off the roof to save the student. Superman leaves the roof with an initial speed v_0 that he produces by pushing himself downward from the edge of the roof with his legs of steel. He then falls with the same acceleration as any freely falling body. (a) What must the value of v_0 be so that Superman catches the student just before they reach the ground? (b) On the same graph, sketch the positions of the student and of Superman as functions of time. Take Superman's initial speed to have the value calculated in part (a). (c) If the height of the skyscraper is less than some minimum value, even Superman can't reach the student before he hits the ground. What is this minimum height?

2.91. During launches, rockets often discard unneeded parts. A certain rocket starts from rest on the launch pad and accelerates upward at a steady $3.30\ \text{m/s}^2$. When it is 235 m above the launch pad, it discards a used fuel canister by simply disconnecting it. Once it is disconnected, the only force acting on the canister is gravity (air resistance can be ignored). (a) How high is the rocket when the canister hits the launch pad, assuming that the rocket does not change its acceleration? (b) What total distance did the canister travel between its release and its crash onto the launch pad?

2.92. A ball is thrown straight up from the ground with speed v_0. At the same instant, a second ball is dropped from rest from a height H, directly above the point where the first ball was thrown upward. There is no air resistance. (a) Find the time at which the two balls collide. (b) Find the value of H in terms of v_0 and g so that at the instant when the balls collide, the first ball is at the highest point of its motion.

2.93. Two cars, A and B, travel in a straight line. The distance of A from the starting point is given as a function of time by $x_A(t) = \alpha t + \beta t^2$, with $\alpha = 2.60\ \text{m/s}$ and $\beta = 1.20\ \text{m/s}^2$. The distance of B from the starting point is $x_B(t) = \gamma t^2 - \delta t^3$, with $\gamma = 2.80\ \text{m/s}^2$ and $\delta = 0.20\ \text{m/s}^3$. (a) Which car is ahead just after they leave the starting point? (b) At what time(s) are the cars at the same point? (c) At what time(s) is the distance from A to B neither increasing nor decreasing? (d) At what time(s) do A and B have the same acceleration?

2.94. An apple drops from the tree and falls freely. The apple is originally at rest a height H above the top of the grass of a thick lawn, which is made of blades of grass of height h. When the apple enters the grass, it slows down at a constant rate so that its speed is 0 when it reaches ground level. (a) Find the speed of the apple just before it enters the grass. (b) Find the acceleration of the apple while it is in the grass. (c) Sketch the y-t, v_y-t, and a_y-t graphs for the apple's motion.

Challenge Problems

2.95. Catching the Bus. A student is running at her top speed of $5.0\ \text{m/s}$ to catch a bus, which is stopped at the bus stop. When the student is still 40.0 m from the bus, it starts to pull away, moving with a constant acceleration of $0.170\ \text{m/s}^2$. (a) For how much time and what distance does the student have to run at $5.0\ \text{m/s}$ before she overtakes the bus? (b) When she reaches the bus, how fast is the bus traveling? (c) Sketch an x-t graph for both the student and the bus. Take $x = 0$ at the initial position of the student. (d) The equations you used in part (a) to find the time have a second solution, corresponding to a later time for which the student and bus are again at the same place if they continue their specified motions. Explain the significance of this second solution. How fast is the bus traveling at this point? (e) If the student's top speed is $3.5\ \text{m/s}$, will she catch the bus? (f) What is the *minimum* speed the student must have to just catch up with the bus? For what time and what distance does she have to run in that case?

2.90. In the vertical jump, an athlete starts from a crouch and jumps upward to reach as high as possible. Even the best athletes spend little more than 1.00 s in the air (their "hang time"). Treat the athlete as a particle and let y_{max} be his maximum height above the floor. To explain why he seems to hang in the air, calculate the ratio of the time he is above $y_{max}/2$ to the time it takes him to go from the floor to that height. You may ignore air resistance.

2.97. A ball is thrown straight up from the edge of the roof of a building. A second ball is dropped from the roof 1.00 s later. You may ignore air resistance. (a) If the height of the building is 20.0 m, what must the initial speed of the first ball be if both are to hit the ground at the same time? On the same graph, sketch the position of each ball as a function of time, measured from when the first ball is thrown. Consider the same situation, but now let the initial speed v_0 of the first ball be given and treat the height h of the building as an unknown. (b) What must the height of the building be for both balls to reach the ground at the same time (i) if v_0 is $6.0\ \text{m/s}$ and (ii) if v_0 is $9.5\ \text{m/s}$? (c) If v_0 is greater than some value v_{max}, a value of h does not exist that allows both balls to hit the ground at the same time. Solve for v_{max}. The value v_{max} has a simple physical interpretation. What is it? (d) If v_0 is less than some value v_{min}, a value of h does not exist that allows both balls to hit the ground at the same time. Solve for v_{min}. The value v_{min} also has a simple physical interpretation. What is it?

2.90. An alert hiker sees a boulder fall from the top of a distant cliff and notes that it takes 1.30 s for the boulder to fall the last third of the way to the ground. You may ignore air resistance. (a) What is the height of the cliff in meters? (b) If in part (a) you get two solutions of a quadratic equation and you use one for your answer, what does the other solution represent?

MOTION IN TWO OR THREE DIMENSIONS

3

? If a car is going around a curve at constant speed, is it accelerating? If so, in what direction is it accelerating?

LEARNING GOALS

By studying this chapter, you will learn:

- How to represent the position of a body in two or three dimensions using vectors.
- How to determine the vector velocity of a body from a knowledge of its path.
- How to find the vector acceleration of a body, and why a body can have an acceleration even if its speed is constant.
- How to interpret the components of a body's acceleration parallel to and perpendicular to its path.
- How to describe the curved path followed by a projectile.
- The key ideas behind motion in a circular path, with either constant speed or varying speed.
- How to relate the velocity of a moving body as seen from two different frames of reference.

What determines where a batted baseball lands? How do you describe the motion of a roller coaster car along a curved track or the flight of a circling hawk? If you throw a water balloon horizontally from your window, will it take the same amount of time to hit the ground as a balloon that you simply drop?

We can't answer these kinds of questions using the techniques of Chapter 2, in which particles moved only along a straight line. Instead, we need to extend our descriptions of motion to two- and three-dimensional situations. We'll still use the vector quantities displacement, velocity, and acceleration, but now these quantities will no longer lie along a single line. We'll find that several important kinds of motion take place in two dimensions only—that is, in a *plane*. These motions can be described with two components of position, velocity, and acceleration.

We also need to consider how the motion of a particle is described by different observers who are moving relative to each other. The concept of *relative velocity* will play an important role later in the book when we study collisions, when we explore electromagnetic phenomena, and when we introduce Einstein's special theory of relativity.

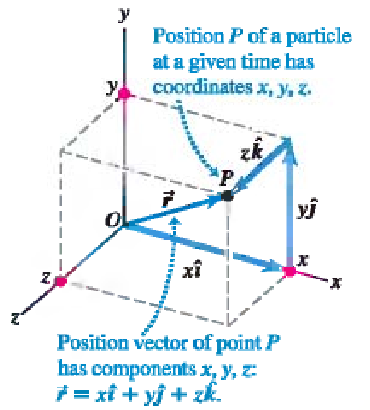
This chapter merges the vector mathematics of Chapter 1 with the kinematic language of Chapter 2. As before, we are concerned with describing motion, not with analyzing its causes. But the language you learn here will be an essential tool in later chapters when we study the relationship between force and motion.

3.1 Position and Velocity Vectors

To describe the *motion* of a particle in space, we must first be able to describe the particle's *position*. Consider a particle that is at a point P at a certain instant. The **position vector** $\vec{r}$ of the particle at this instant is a vector that goes from the origin of the coordinate system to the point P (Fig. 3.1). The Cartesian coordinates x, y, and z of point P are the x-, y-, and z-components of vector $\vec{r}$. Using the unit vectors we introduced in Section 1.9, we can write

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \qquad \text{(position vector)} \tag{3.1}$$

3.1 The position vector $\vec{r}$ from the origin to point P has components x, y, and z. The path that the particle follows through space is in general a curve (Fig. 3.2).

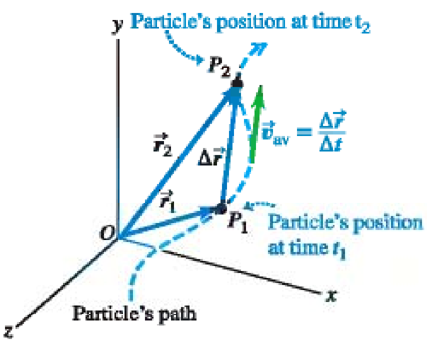


During a time interval Δt the particle moves from P_1, where its position vector is $\vec{r}_1$, to P_2, where its position vector is $\vec{r}_2$. The change in position (the displacement) during this interval is $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$. We define the **average velocity** $\vec{v}_{av}$ during this interval in the same way we did in Chapter 2 for straight-line motion, as the displacement divided by the time interval:

$$\vec{v}_{av} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \frac{\Delta\vec{r}}{\Delta t} \qquad \text{(average velocity vector)} \tag{3.2}$$

Dividing a vector by a scalar is really a special case of *multiplying* a vector by a scalar, described in Section 1.7; the average velocity $\vec{v}_{av}$ is equal to the displacement vector $\Delta\vec{r}$ multiplied by $1/\Delta t$, the reciprocal of the time interval. Note that the x-component of Eq. (3.2) is $v_{av\text{-}x} = (x_2 - x_1)/(t_2 - t_1) = \Delta x/\Delta t$. This is just Eq. (2.2), the expression for average x-velocity that we found in Section 2.1 for one-dimensional motion.

3.2 The average velocity $\vec{v}_{av}$ between points P_1 and P_2 has the same direction as the displacement $\Delta\vec{r}$.



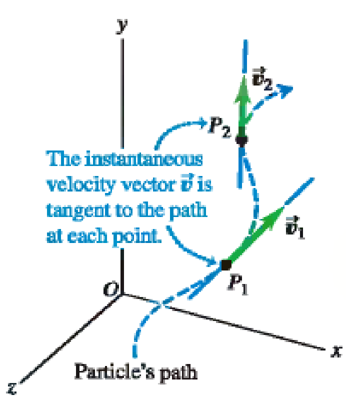
We now define **instantaneous velocity** just as we did in Chapter 2: It is the limit of the average velocity as the time interval approaches zero, and it equals the instantaneous rate of change of position with time. The key difference is that position $\vec{r}$ and instantaneous velocity $\vec{v}$ are now both vectors:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \qquad \text{(instantaneous velocity vector)} \tag{3.3}$$

The *magnitude* of the vector $\vec{v}$ at any instant is the *speed* v of the particle at that instant. The *direction* of $\vec{v}$ at any instant is the same as the direction in which the particle is moving at that instant.

Note that as $\Delta t \to 0$, points P_1 and P_2 in Fig. 3.2 move closer and closer together. In this limit, the vector $\Delta\vec{r}$ becomes tangent to the path. The direction of $\Delta\vec{r}$ in the limit is also the direction of the instantaneous velocity $\vec{v}$. This leads to an important conclusion: *At every point along the path, the instantaneous velocity vector is tangent to the path at that point* (Fig. 3.3).

3.3 The vectors $\vec{v}_1$ and $\vec{v}_2$ are the instantaneous velocities at the points P_1 and P_2 shown in Fig. 3.2.



It's often easiest to calculate the instantaneous velocity vector using components. During any displacement $\Delta\vec{r}$, the changes Δx, Δy, and Δz in the three coordinates of the particle are the *components* of $\Delta\vec{r}$. It follows that the components v_x, v_y, and v_z of the instantaneous velocity $\vec{v}$ are simply the time derivatives of the coordinates x, y, and z. That is,

$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt} \qquad v_z = \frac{dz}{dt} \qquad \text{(components of instantaneous velocity)} \tag{3.4}$$

The x-component of $\vec{v}$ is $v_x = dx/dt$, which is the same as Eq. (2.3)—the expression for instantaneous velocity for straight-line motion that we obtained in Section 2.2. Hence Eq. (3.4) is a direct extension of the idea of instantaneous velocity to motion in three dimensions.

We can also get this result by taking the derivative of Eq. (3.1). The unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are constant in magnitude and direction, so their derivatives are zero, and we find

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k} \tag{3.5}$$

This shows again that the components of $\vec{v}$ are dx/dt, dy/dt, and dz/dt.

The magnitude of the instantaneous velocity vector $\vec{v}$—that is, the speed—is given in terms of the components v_x, v_y, and v_z by the Pythagorean relation

$$|\vec{v}| = v = \sqrt{v_x^2 + v_y^2 + v_z^2} \tag{3.6}$$

Figure 3.4 shows the situation when the particle moves in the xy-plane. In this case, z and v_z are zero. Then the speed (the magnitude of $\vec{v}$) is

$$v = \sqrt{v_x^2 + v_y^2}$$

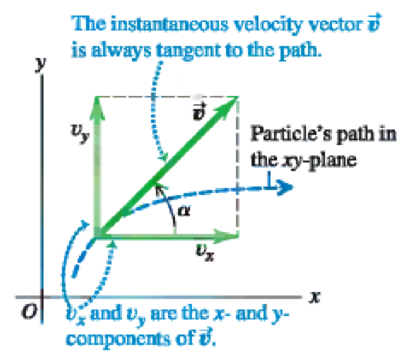
and the direction of the instantaneous velocity $\vec{v}$ is given by the angle α in the figure. We see that

$$\tan\alpha = \frac{v_y}{v_x} \tag{3.7}$$

(We always use Greek letters for angles. We use α for the direction of the instantaneous velocity vector to avoid confusion with the direction θ of the *position* vector of the particle.)

The instantaneous velocity vector is usually more interesting and useful than the average velocity vector. From now on, when we use the word "velocity," we will always mean the instantaneous velocity vector $\vec{v}$ (rather than the average velocity vector). Usually, we won't even bother to call $\vec{v}$ a vector; it's up to you to remember that velocity is a vector quantity with both magnitude and direction.

3.4 The two velocity components for motion in the xy-plane.



Example 3.1 Calculating average and instantaneous velocity

A robotic vehicle, or rover, is exploring the surface of Mars. The landing craft is the origin of coordinates, and the surrounding Martian surface lies in the xy-plane. The rover, which we represent as a point, has x- and y-coordinates that vary with time:

$$x = 2.0\text{ m} - (0.25\text{ m/s}^2)t^2$$
$$y = (1.0\text{ m/s})t + (0.025\text{ m/s}^3)t^3$$

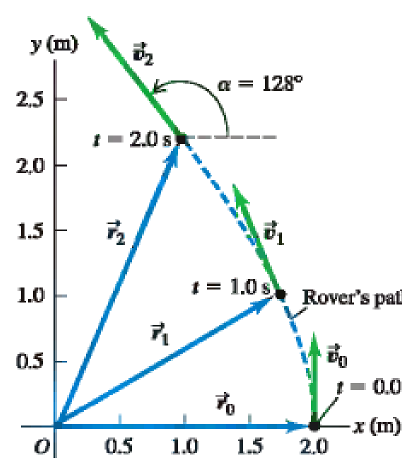
(a) Find the rover's coordinates and its distance from the lander at $t = 2.0$ s. (b) Find the rover's displacement and average velocity vectors during the interval from $t = 0.0$ s to $t = 2.0$ s. (c) Derive a general expression for the rover's instantaneous velocity vector. Express the instantaneous velocity at $t = 2.0$ s in component form and also in terms of magnitude and direction.

SOLUTION

IDENTIFY: This problem involves motion in two dimensions—that is, in a plane. Hence we must use the expressions for the displacement, average velocity, and instantaneous velocity *vectors* obtained in this section. (The simpler expressions in Sections 2.1 and 2.2 don't involve vectors; they apply only to motion along a straight line.)

SET UP: Figure 3.5 shows the rover's path. We'll use Eq. (3.1) for position $\vec{r}$, the expression $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$ for displacement, Eq. (3.2) for average velocity, and Eqs. (3.5) and (3.6) for instantaneous velocity and its direction. The target variables are stated in the problem.

3.5 At $t = 0$ the rover has position vector $\vec{r}_0$ and instantaneous velocity vector $\vec{v}_0$. Likewise, $\vec{r}_1$ and $\vec{v}_1$ are the vectors at $t = 1.0$ s; $\vec{r}_2$ and $\vec{v}_2$ are the vectors at $t = 2.0$ s.



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EXECUTE: (a) At time $t = 2.0$ s the rover's coordinates are

$$x = 2.0\text{ m} - (0.25\text{ m/s}^2)(2.0\text{ s})^2 = 1.0\text{ m}$$
$$y = (1.0\text{ m/s})(2.0\text{ s}) + (0.025\text{ m/s}^3)(2.0\text{ s})^3 = 2.2\text{ m}$$

The rover's distance from the origin at this time is

$$r = \sqrt{x^2 + y^2} = \sqrt{(1.0\text{ m})^2 + (2.2\text{ m})^2} = 2.4\text{ m}$$

(b) To find the displacement and average velocity, we express the position vector $\vec{r}$ as a function of time t. From Eq. (3.1), this is

$$\begin{aligned}\vec{r} &= x\hat{\imath} + y\hat{\jmath}\\ &= [2.0\text{ m} - (0.25\text{ m/s}^2)t^2]\hat{\imath}\\ &\quad + [(1.0\text{ m/s})t + (0.025\text{ m/s}^3)t^3]\hat{\jmath}\end{aligned}$$

At time $t = 0.0$ s the position vector $\vec{r}_0$ is

$$\vec{r}_0 = (2.0\text{ m})\hat{\imath} + (0.0\text{ m})\hat{\jmath}$$

From part (a) the position vector $\vec{r}_2$ at time $t = 2.0$ s is

$$\vec{r}_2 = (1.0\text{ m})\hat{\imath} + (2.2\text{ m})\hat{\jmath}$$

Therefore the displacement from $t = 0.0$ s to $t = 2.0$ s is

$$\begin{aligned}\Delta\vec{r} = \vec{r}_2 - \vec{r}_0 &= (1.0\text{ m})\hat{\imath} + (2.2\text{ m})\hat{\jmath} - (2.0\text{ m})\hat{\imath}\\ &= (-1.0\text{ m})\hat{\imath} + (2.2\text{ m})\hat{\jmath}\end{aligned}$$

During the time interval from $t = 0.0$ s to $t = 2.0$ s, the rover moves 1.0 m in the negative x-direction and 2.2 m in the positive y-direction. From Eq. (3.2), the average velocity during this interval is the displacement divided by the elapsed time:

$$\begin{aligned}\vec{v}_{\text{av}} = \frac{\Delta\vec{r}}{\Delta t} &= \frac{(-1.0\text{ m})\hat{\imath} + (2.2\text{ m})\hat{\jmath}}{2.0\text{ s} - 0.0\text{ s}}\\ &= (-0.50\text{ m/s})\hat{\imath} + (1.1\text{ m/s})\hat{\jmath}\end{aligned}$$

The components of this average velocity are

$$v_{\text{av-}x} = -0.50\text{ m/s} \qquad v_{\text{av-}y} = 1.1\text{ m/s}$$

(c) From Eq. (3.4), the components of instantaneous velocity are the time derivatives of the coordinates:

$$v_x = \frac{dx}{dt} = (-0.25\text{ m/s}^2)(2t)$$
$$v_y = \frac{dy}{dt} = 1.0\text{ m/s} + (0.025\text{ m/s}^3)(3t^2)$$

Then we can write the instantaneous velocity vector $\vec{v}$ as

$$\begin{aligned}\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} &= (-0.50\text{ m/s}^2)t\hat{\imath}\\ &\quad + [1.0\text{ m/s} + (0.075\text{ m/s}^3)t^2]\hat{\jmath}\end{aligned}$$

At time $t = 2.0$ s, the components of instantaneous velocity are

$$v_x = (-0.50\text{ m/s}^2)(2.0\text{ s}) = -1.0\text{ m/s}$$
$$v_y = 1.0\text{ m/s} + (0.075\text{ m/s}^3)(2.0\text{ s})^2 = 1.3\text{ m/s}$$

The magnitude of the instantaneous velocity (that is, the speed) at $t = 2.0$ s is

$$\begin{aligned}v = \sqrt{v_x^2 + v_y^2} &= \sqrt{(-1.0\text{ m/s})^2 + (1.3\text{ m/s})^2}\\ &= 1.6\text{ m/s}\end{aligned}$$

The direction of $\vec{v}$ with respect to the positive x-axis is given by the angle α, where, from Eq. (3.7),

$$\tan\alpha = \frac{v_y}{v_x} = \frac{1.3\text{ m/s}}{-1.0\text{ m/s}} = -1.3 \qquad \text{so} \qquad \alpha = 128^\circ$$

Your calculator will tell you that the inverse tangent of -1.3 is -52°. But as we learned in Section 1.8, you have to examine a sketch of a vector to decide on its direction. Figure 3.5 shows that the correct answer for α is $-52^\circ + 180^\circ = 128^\circ$.

EVALUATE: Take a moment to compare the components of *average* velocity that we found in part (b) for the interval from $t = 0.0$ s to $t = 2.0$ s $(v_{\text{av-}x} = -0.50\text{ m/s},\ v_{\text{av-}y} = 1.1\text{ m/s})$ with the components of *instantaneous* velocity at $t = 2.0$ s that we found in part (c) $(v_x = -1.0\text{ m/s},\ v_y = 1.3\text{ m/s})$. The comparison shows that, just as in one dimension, the average velocity vector $\vec{v}_{\text{av}}$ over an interval is in general *not* equal to the instantaneous velocity $\vec{v}$ at the end of the interval (see Example 2.1).

You should calculate the position vector, instantaneous velocity vector, speed, and direction of motion at $t = 0.0$ s and $t = 1.0$ s. Figure 3.5 shows the position vectors $\vec{r}$ and instantaneous velocity vectors $\vec{v}$ at $t = 0.0$ s, 1.0 s, and 2.0 s. Notice that at every point, $\vec{v}$ is tangent to the path. The magnitude of $\vec{v}$ increases as the rover moves, which shows that its speed is increasing.

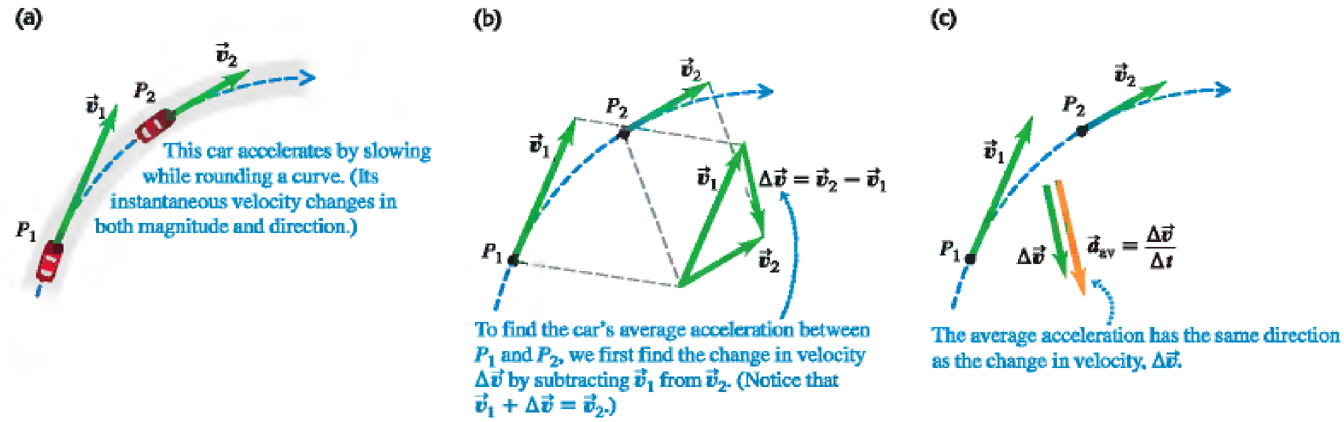
Test Your Understanding of Section 3.1 In which of these situations *would* the average velocity vector $\vec{v}_{\text{av}}$ over an interval be equal to the instantaneous velocity $\vec{v}$ at the end of the interval? (i) a body moving along a curved path at constant speed; (ii) a body moving along a curved path and speeding up; (iii) a body moving along a straight line at constant speed; (iv) a body moving along a straight line and speeding up. MP

3.2 The Acceleration Vector

Now let's consider the *acceleration* of a particle moving in space. Just as for motion in a straight line, acceleration describes how the velocity of the particle changes. But since we now treat velocity as a vector, acceleration will describe changes in the velocity magnitude (that is, the speed) *and* changes in the direction of velocity (that is, the direction in which the particle is moving).

In Fig. 3.6a, a car (treated as a particle) is moving along a curved road. The vectors $\vec{v}_1$ and $\vec{v}_2$ represent the car's instantaneous velocities at time t_1, when the

3.6 (a) A car moving along a curved road from P_1 to P_2. (b) Obtaining $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$ by vector subtraction. (c) The vector $\vec{a}_{av} = \Delta\vec{v}/\Delta t$ represents the average acceleration between P_1 and P_2.



car is at point P_1, and at time t_2, when the car is at point P_2. The two velocities may differ in both magnitude and direction. During the time interval from t_1 to t_2, the *vector change in velocity* is $\vec{v}_2 - \vec{v}_1 = \Delta\vec{v}$ (Fig. 3.6b). We define the **average acceleration** $\vec{a}_{av}$ of the car during this time interval as the velocity change divided by the time interval $t_2 - t_1 = \Delta t$:

$$\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t} \quad \text{(average acceleration vector)} \qquad (3.8)$$

Average acceleration is a *vector* quantity in the same direction as the vector $\Delta\vec{v}$ (Fig. 3.6c). Note that $\vec{v}_2$ is the vector sum of the original velocity $\vec{v}_1$ and the change $\Delta\vec{v}$ (Fig. 3.6b). The x-component of Eq. (3.8) is $a_{av\text{-}x} = (v_{2x} - v_{1x})/(t_2 - t_1) = \Delta v_x/\Delta t$, which is just Eq. (2.4) for the average acceleration in straight-line motion.

As in Chapter 2, we define the **instantaneous acceleration** $\vec{a}$ at point P_1 as the limit of the average acceleration when point P_2 approaches point P_1 and $\Delta\vec{v}$ and Δt both approach zero. The instantaneous acceleration is also equal to the instantaneous rate of change of velocity with time. Because we are not restricted to straight-line motion, instantaneous acceleration is now a vector (Fig. 3.7):

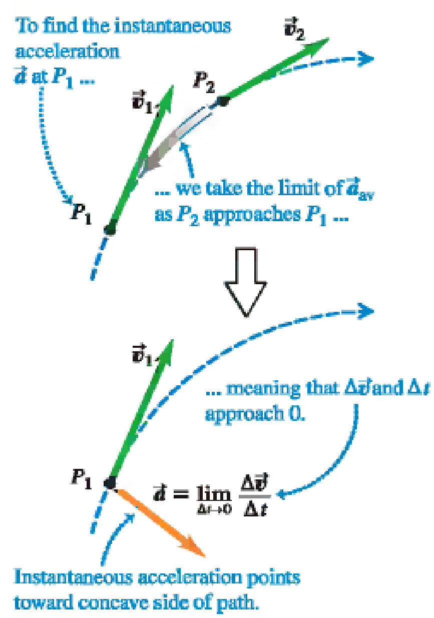
$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \quad \text{(instantaneous acceleration vector)} \qquad (3.9)$$

The velocity vector $\vec{v}$, as we have seen, is tangent to the path of the particle. But Figs. 3.6c and 3.7 show that if the path is curved, the instantaneous acceleration vector $\vec{a}$ always points toward the concave side of the path—that is, toward the inside of any turn that the particle is making.

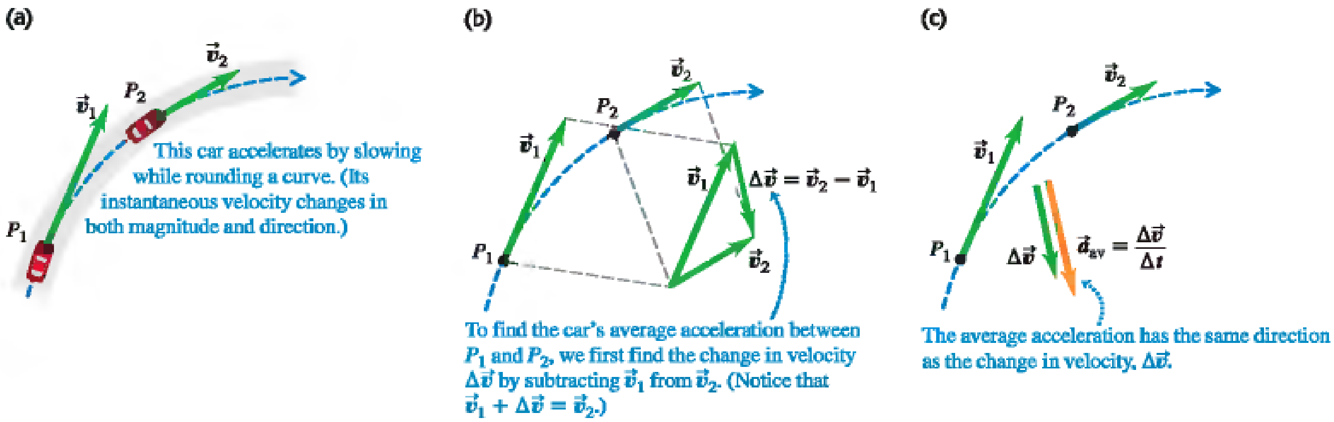
CAUTION **Any particle following a curved path is accelerating** When a particle is moving in a curved path, it always has nonzero acceleration, even when it moves with constant speed. This conclusion may seem contrary to your intuition, but it's really just contrary to the everyday use of the word "acceleration" to mean that speed is increasing. The more precise definition given in Eq. (3.9) shows that there is a nonzero acceleration whenever the velocity vector changes in any way, whether there is a change of speed, direction, or both.

? To convince yourself that a particle has a nonzero acceleration when moving on a curved path with constant speed, think of your sensations when you ride in a car. When the car accelerates, you tend to move inside the car in a

3.7 Instantaneous acceleration $\vec{a}$ at point P_1 in Fig. 3.6.



3.6 (a) A car moving along a curved road from P_1 to P_2. (b) Obtaining $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$ by vector subtraction. (c) The vector $\vec{a}_{av} = \Delta\vec{v}/\Delta t$ represents the average acceleration between P_1 and P_2.



car is at point P_1, and at time t_2, when the car is at point P_2. The two velocities may differ in both magnitude and direction. During the time interval from t_1 to t_2, the *vector change in velocity* is $\vec{v}_2 - \vec{v}_1 = \Delta\vec{v}$ (Fig. 3.6b). We define the **average acceleration** $\vec{a}_{av}$ of the car during this time interval as the velocity change divided by the time interval $t_2 - t_1 = \Delta t$:

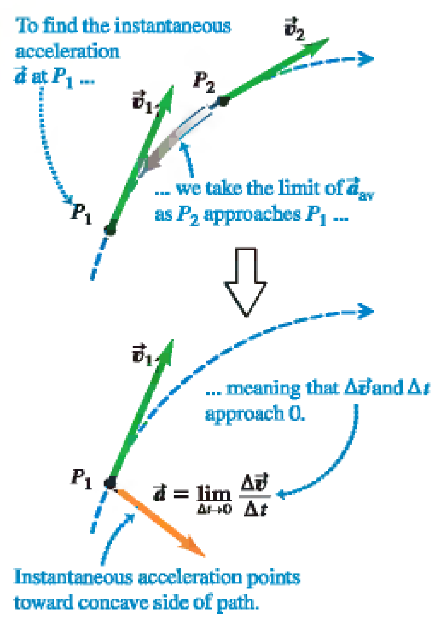
$$\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t} \qquad \text{(average acceleration vector)} \qquad (3.8)$$

Average acceleration is a *vector* quantity in the same direction as the vector $\Delta\vec{v}$ (Fig. 3.6c). Note that $\vec{v}_2$ is the vector sum of the original velocity $\vec{v}_1$ and the change $\Delta\vec{v}$ (Fig. 3.6b). The x-component of Eq. (3.8) is $a_{av\text{-}x} = (v_{2x} - v_{1x})/(t_2 - t_1) = \Delta v_x/\Delta t$, which is just Eq. (2.4) for the average acceleration in straight-line motion.

As in Chapter 2, we define the **instantaneous acceleration** $\vec{a}$ at point P_1 as the limit of the average acceleration when point P_2 approaches point P_1 and $\Delta\vec{v}$ and Δt both approach zero. The instantaneous acceleration is also equal to the instantaneous rate of change of velocity with time. Because we are not restricted to straight-line motion, instantaneous acceleration is now a vector (Fig. 3.7):

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \qquad \text{(instantaneous acceleration vector)} \qquad (3.9)$$

3.7 Instantaneous acceleration $\vec{a}$ at point P_1 in Fig. 3.6.



The velocity vector $\vec{v}$, as we have seen, is tangent to the path of the particle. But Figs. 3.6c and 3.7 show that if the path is curved, the instantaneous acceleration vector $\vec{a}$ always points toward the concave side of the path—that is, toward the inside of any turn that the particle is making.

CAUTION **Any particle following a curved path is accelerating** When a particle is moving in a curved path, it always has nonzero acceleration, even when it moves with constant speed. This conclusion may seem contrary to your intuition, but it's really just contrary to the everyday use of the word "acceleration" to mean that speed is increasing. The more precise definition given in Eq. (3.9) shows that there is a nonzero acceleration whenever the velocity vector changes in any way, whether there is a change of speed, direction, or both.

To convince yourself that a particle has a nonzero acceleration when moving on a curved path with constant speed, think of your sensations when you ride in a car. When the car accelerates, you tend to move inside the car in a ?

We can write the instantaneous acceleration vector $\vec{a}$ as

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} = (-0.50\ \mathrm{m/s^2})\hat{\imath} + (0.15\ \mathrm{m/s^3})t\hat{\jmath}$$

At time $t = 2.0$ s, the components of instantaneous acceleration are

$$a_x = -0.50\ \mathrm{m/s^2} \qquad a_y = (0.15\ \mathrm{m/s^3})(2.0\ \mathrm{s}) = 0.30\ \mathrm{m/s^2}$$

The acceleration vector at this time is

$$\vec{a} = (-0.50\ \mathrm{m/s^2})\hat{\imath} + (0.30\ \mathrm{m/s^2})\hat{\jmath}$$

The magnitude of acceleration at this time is

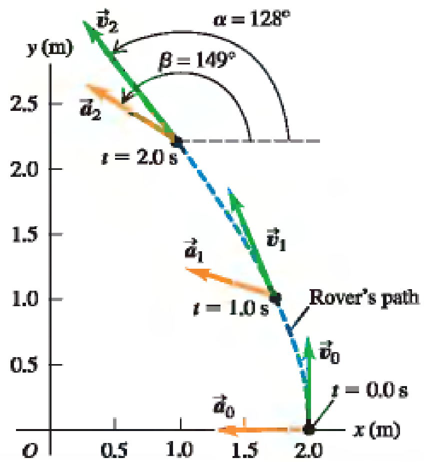
$$\begin{aligned} a &= \sqrt{a_x^2 + a_y^2} \\ &= \sqrt{(-0.50\ \mathrm{m/s^2})^2 + (0.30\ \mathrm{m/s^2})^2} = 0.58\ \mathrm{m/s^2} \end{aligned}$$

The direction of $\vec{a}$ with respect to the positive x-axis is given by the angle β, where

$$\tan\beta = \frac{a_y}{a_x} = \frac{0.30\ \mathrm{m/s^2}}{-0.50\ \mathrm{m/s^2}} = -0.60$$
$$\beta = 180° - 31° = 149°$$

EVALUATE: You should use the results of part (b) to calculate the instantaneous acceleration at $t = 0.0$ s and $t = 1.0$ s. Figure 3.9 shows the rover's path and the velocity and acceleration vectors at $t = 0.0$ s, 1.0 s, and 2.0 s. Note that $\vec{v}$ and $\vec{a}$ are *not* in the same direction at any of these times. The velocity vector $\vec{v}$ is tangent to the path at each point, and the acceleration vector $\vec{a}$ points toward the concave side of the path.

3.9 The path of the robotic rover, showing the velocity and acceleration at $t = 0.0$ s ($\vec{v}_0$ and $\vec{a}_0$), $t = 1.0$ s ($\vec{v}_1$ and $\vec{a}_1$), and $t = 2.0$ s ($\vec{v}_2$ and $\vec{a}_2$).



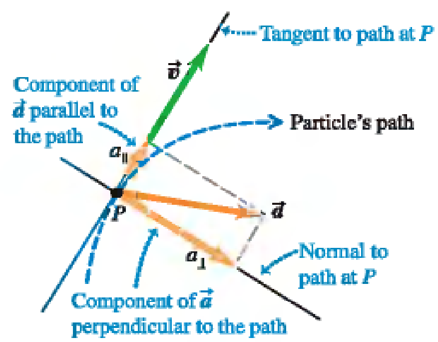
Parallel and Perpendicular Components of Acceleration

The acceleration vector $\vec{a}$ for a particle can describe changes in the particle's speed, its direction of motion, or both. It's useful to note that the component of acceleration *parallel* to a particle's path—that is, parallel to the velocity—tells us about changes in the particle's *speed,* while the acceleration component *perpendicular* to the path—and hence perpendicular to the velocity—tells us about changes in the particle's *direction of motion.* Figure 3.10 shows these components, which we label $a_\parallel$ and $a_\perp$. To see why the parallel and perpendicular components of $\vec{a}$ have these properties, let's consider two special cases.

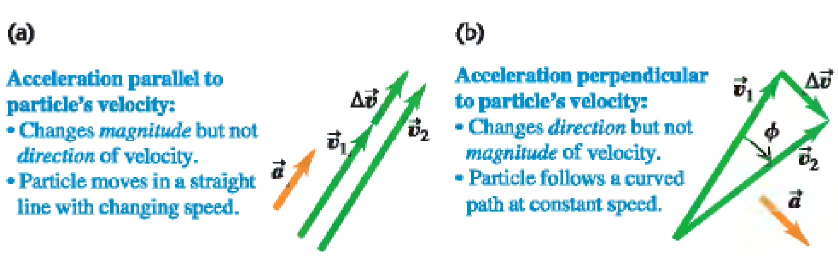
In Fig. 3.11a the acceleration vector is in the same direction as the velocity $\vec{v}_1$, so $\vec{a}$ has only a parallel component $a_\parallel$ (that is, $a_\perp = 0$). The velocity change $\Delta\vec{v}$ during a small time interval Δt is in the same direction as $\vec{a}$ and hence in the same direction as $\vec{v}_1$. The velocity $\vec{v}_2$ at the end of Δt, given by $\vec{v}_2 = \vec{v}_1 + \Delta\vec{v}$, is in the same direction as $\vec{v}_1$ but has greater magnitude. Hence during the time interval Δt the particle in Fig. 3.11a moved in a straight line with increasing speed.

In Fig. 3.11b the acceleration is *perpendicular* to the velocity, so $\vec{a}$ has only a perpendicular component $a_\perp$ (that is, $a_\parallel = 0$). In a small time interval Δt, the velocity change $\Delta\vec{v}$ is very nearly perpendicular to $\vec{v}_1$. Again $\vec{v}_2 = \vec{v}_1 + \Delta\vec{v}$, but in this case $\vec{v}_1$ and $\vec{v}_2$ have different directions. As the time interval Δt

3.10 The acceleration can be resolved into a component $a_\parallel$ parallel to the path (that is, along the tangent to the path) and a component $a_\perp$ perpendicular to the path (that is, along the normal to the path).



3.11 The effect of acceleration directed (a) parallel to and (b) perpendicular to a particle's velocity.

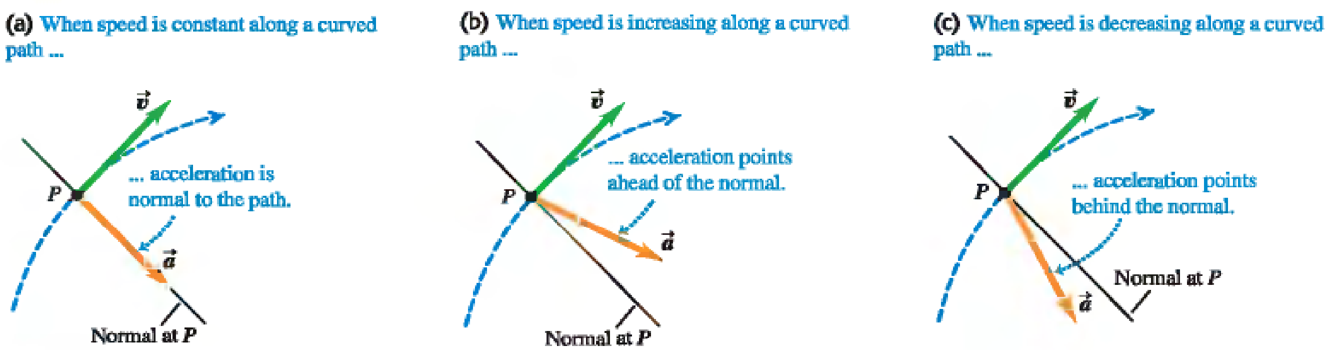


approaches zero, the angle ϕ in the figure also approaches zero, $\Delta\vec{v}$ becomes perpendicular to *both* $\vec{v}_1$ and $\vec{v}_2$, and $\vec{v}_1$ and $\vec{v}_2$ have the same magnitude. In other words, the speed of the particle stays the same, but the direction of motion changes and the path of the particle curves.

In the most general case, the acceleration $\vec{a}$ has components *both* parallel and perpendicular to the velocity $\vec{v}$, as in Fig. 3.10. Then the particle's speed will change (described by the parallel component $a_{\parallel}$) *and* its direction of motion will change (described by the perpendicular component $a_{\perp}$) so that it follows a curved path.

Figure 3.12 shows a particle moving along a curved path for three different situations: constant speed, increasing speed, and decreasing speed. If the speed is constant, $\vec{a}$ is perpendicular, or *normal,* to the path and to $\vec{v}$ and points toward the concave side of the path (Fig. 3.12a). If the speed is increasing, there is still a perpendicular component of $\vec{a}$, but there is also a parallel component having the same direction as $\vec{v}$ (Fig. 3.12b). Then $\vec{a}$ points ahead of the normal to the path. (This was the case in Example 3.2.) If the speed is decreasing, the parallel component has the direction opposite to $\vec{v}$, and $\vec{a}$ points behind the normal to the path (Fig. 3.12c). We will use these ideas again in Section 3.4 when we study the special case of motion in a circle.

3.12 Velocity and acceleration vectors for a particle moving through a point P on a curved path with (a) constant speed, (b) increasing speed, and (c) decreasing speed.



Example 3.3 Calculating parallel and perpendicular components of acceleration

For the rover of Examples 3.1 and 3.2, find the parallel and perpendicular components of the acceleration at $t = 2.0$ s.

SOLUTION

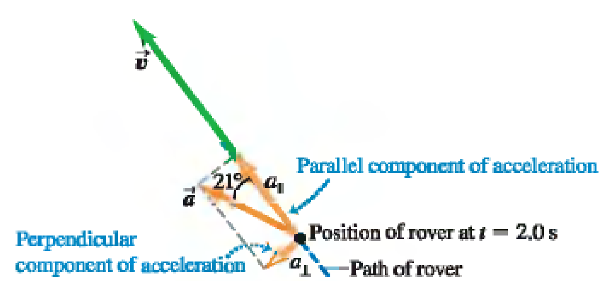
IDENTIFY: We want to find the components of the acceleration vector $\vec{a}$ that are parallel and perpendicular to the velocity vector $\vec{v}$.

SET UP: We found the directions of $\vec{a}$ and $\vec{v}$ in Examples 3.2 and 3.1, respectively. This will allow us to find the angle between the two vectors and hence the components of $\vec{a}$.

EXECUTE: In Example 3.2 we found that at $t = 2.0$ s the particle has an acceleration of magnitude $0.58\ \text{m/s}^2$ at an angle of 149° with respect to the positive x-axis. From Example 3.1, at this same time the velocity vector is at an angle of 128° with respect to the positive x-axis. So Fig. 3.9 shows that the angle between $\vec{a}$ and $\vec{v}$ is $149° - 128° = 21°$ (Fig. 3.13). The parallel and perpendicular components of acceleration are then

$$a_{\parallel} = a\cos 21° = (0.58\ \text{m/s}^2)\cos 21° = 0.54\ \text{m/s}^2$$
$$a_{\perp} = a\sin 21° = (0.58\ \text{m/s}^2)\sin 21° = 0.21\ \text{m/s}^2$$

3.13 The parallel and perpendicular components of the acceleration of the rover at $t = 2.0$ s.



EVALUATE: The parallel component $a_{\parallel}$ is in the same direction as $\vec{v}$, which means that the speed is increasing at this instant; the value of $a_{\parallel} = 0.54\ \text{m/s}^2$ means that the speed is increasing at a rate of 0.54 m/s per second. The perpendicular component $a_{\perp}$ is not zero, which means that at this instant the rover is changing direction and following a curved path; in other words, the rover is turning.

Conceptual Example 3.4 Acceleration of a skier

A skier moves along a ski-jump ramp as shown in Fig. 3.14a. The ramp is straight from point *A* to point *C* and curved from point *C* onward. The skier picks up speed as she moves downhill from point *A* to point *E*, where her speed is maximum. She slows down after passing point *E*. Draw the direction of the acceleration vector at points *B*, *D*, *E*, and *F*.

SOLUTION

Figure 3.14b shows our solution. At point *B* the skier is moving in a straight line with increasing speed, so her acceleration points downhill, in the same direction as her velocity.

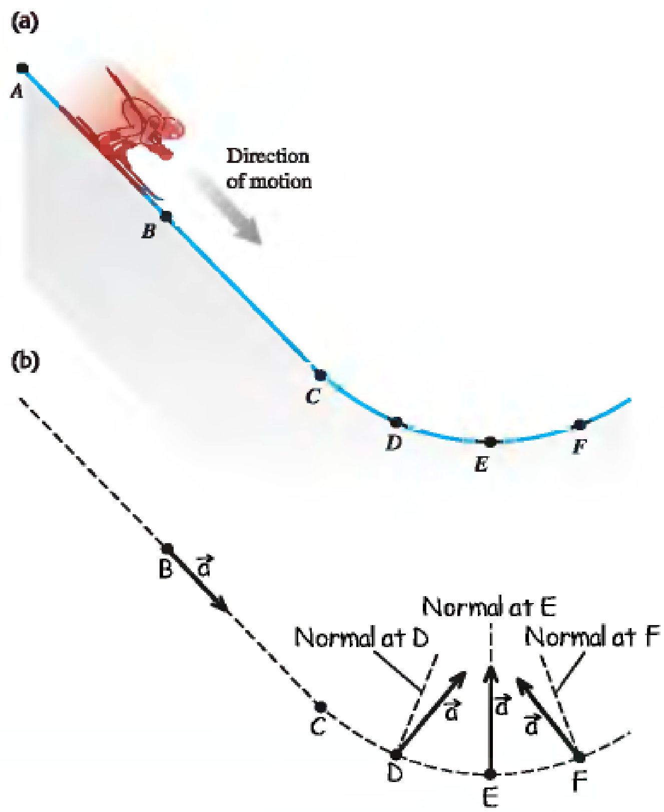
At point *D* the skier is moving along a curved path, so her acceleration has a component perpendicular to the path. There is also a component in the direction of her motion because she is still speeding up at this point. So the acceleration vector points *ahead* of the normal to her path at point *D*.

The skier's speed is instantaneously not changing at point *E*; the speed is maximum at this point, so its derivative is zero. There is no parallel component of $\vec{a}$, and the acceleration is perpendicular to her motion.

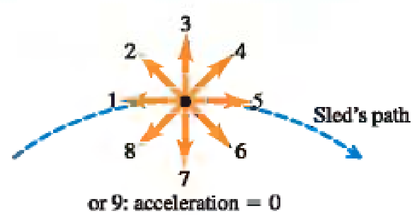
Finally, at point *F* the acceleration has a perpendicular component (because her path is curved at this point) and a parallel component *opposite* to the direction of her motion (because she's slowing down). So at this point, the acceleration vector points *behind* the normal to her path.

In the next section we'll examine the skier's acceleration after she flies off the ramp.

3.14 (a) The skier's path. (b) Our solution.



Test Your Understanding of Section 3.2 A sled travels over the crest of a snow-covered hill. The sled slows down as it climbs up one side of the hill and gains speed as it descends on the other side. Which of the vectors (1 through 9) in the figure correctly shows the direction of the sled's acceleration at the crest? (Choice 9 is that the acceleration is zero.)



3.3 Projectile Motion

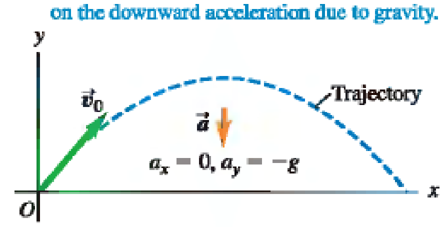
A **projectile** is any body that is given an initial velocity and then follows a path determined entirely by the effects of gravitational acceleration and air resistance. A batted baseball, a thrown football, a package dropped from an airplane, and a bullet shot from a rifle are all projectiles. The path followed by a projectile is called its **trajectory.**

To analyze this common type of motion, we'll start with an idealized model, representing the projectile as a single particle with an acceleration (due to gravity) that is constant in both magnitude and direction. We'll neglect the effects of air resistance and the curvature and rotation of the earth. Like all models, this one has limitations. Curvature of the earth has to be considered in the flight of long-range missiles, and air resistance is of crucial importance to a sky diver. Nevertheless, we can learn a lot from analysis of this simple model. For the remainder of this chapter the phrase "projectile motion" will imply that we're ignoring air resistance. In Chapter 5 we will see what happens when air resistance cannot be ignored.

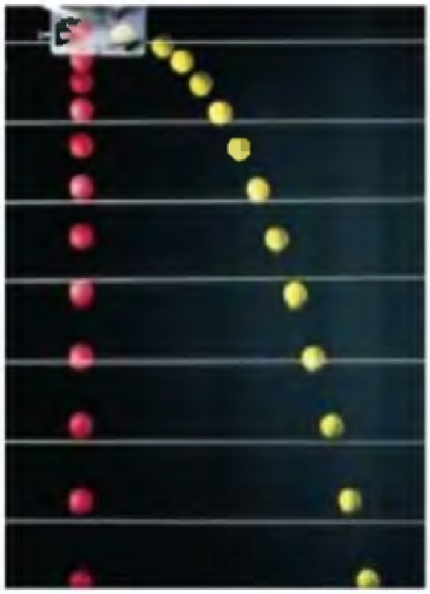
Projectile motion is always confined to a vertical plane determined by the direction of the initial velocity (Fig. 3.15). This is because the acceleration due to

3.15 The trajectory of a projectile.

- A projectile moves in a vertical plane that contains the initial velocity vector $\vec{v}_0$.
- Its trajectory depends only on $\vec{v}_0$ and on the downward acceleration due to gravity.



3.16 The red ball is dropped from rest, and the yellow ball is simultaneously projected horizontally; successive images in this stroboscopic photograph are separated by equal time intervals. At any given time, both balls have the same y-position, y-velocity, and y-acceleration, despite having different x-positions and x-velocities.



gravity is purely vertical; gravity can't move the projectile sideways. Thus projectile motion is *two-dimensional.* We will call the plane of motion the xy-coordinate plane, with the x-axis horizontal and the y-axis vertically upward.

The key to analyzing projectile motion is that we can treat the x- and y-coordinates separately. The x-component of acceleration is zero, and the y-component is constant and equal to $-g$. (By definition, g is always positive; with our choice of coordinate directions, a_y is negative.) So *we can analyze projectile motion as a combination of horizontal motion with constant velocity and vertical motion with constant acceleration.* Figure 3.16 shows two projectiles with different x-motion but identical y-motion; one is dropped from rest and the other is projected horizontally, but both projectiles fall the same distance in the same time.

We can then express all the vector relationships for the projectile's position, velocity, and acceleration by separate equations for the horizontal and vertical components. The components of $\vec{a}$ are

$$a_x = 0 \qquad a_y = -g \qquad \text{(projectile motion, no air resistance)} \tag{3.14}$$

Since the x-acceleration and y-acceleration are both constant, we can use Eqs. (2.8), (2.12), (2.13), and (2.14) directly. For example, suppose that at time $t = 0$ our particle is at the point (x_0, y_0) and that at this time its velocity components have the initial values v_{0x} and v_{0y}. The components of acceleration are $a_x = 0$, $a_y = -g$. Considering the x-motion first, we substitute 0 for a_x in Eqs. (2.8) and (2.12). We find

$$v_x = v_{0x} \tag{3.15}$$

$$x = x_0 + v_{0x}t \tag{3.16}$$

For the y-motion we substitute y for x, v_y for v_x, v_{0y} for v_{0x}, and $a_y = -g$ for a_x:

$$v_y = v_{0y} - gt \tag{3.17}$$

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2 \tag{3.18}$$

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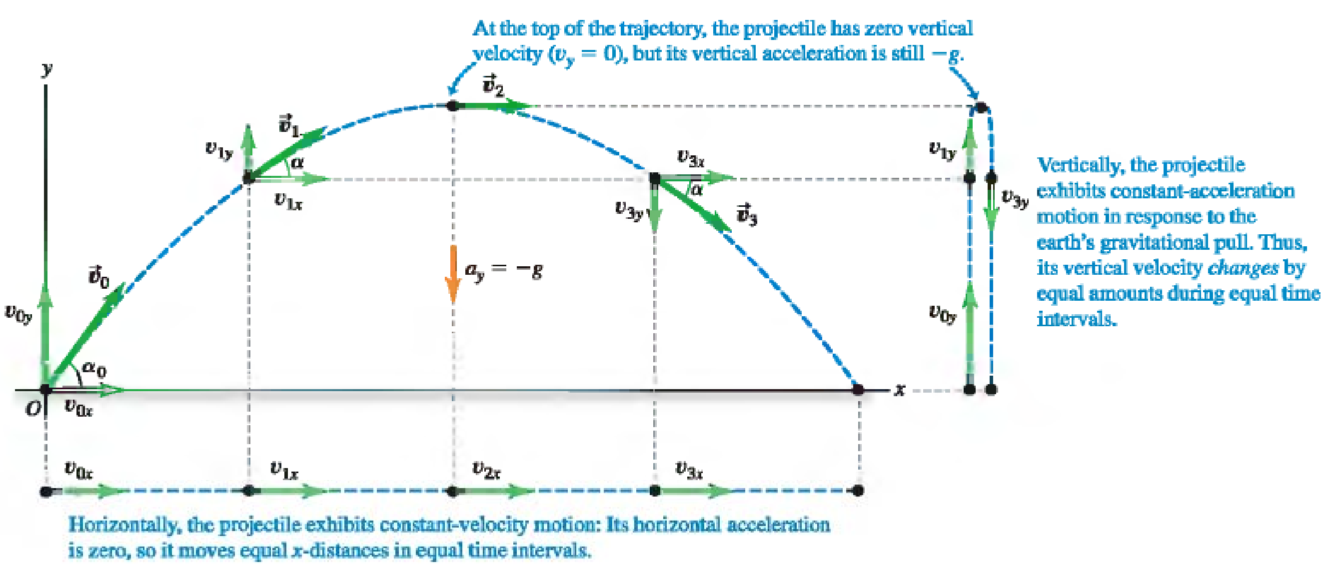
3.1 Solving Projectile Motion Problems
3.2 Two Balls Falling
3.3 Changing the x-velocity
3.4 Projectile x-y-Accelerations

It's usually simplest to take the initial position (at $t = 0$) as the origin; then $x_0 = y_0 = 0$. This might be the position of a ball at the instant it leaves the thrower's hand or the position of a bullet at the instant it leaves the gun barrel.

Figure 3.17 shows the path of a projectile that starts at (or passes through) the origin at time $t = 0$. The position, velocity, and velocity components are shown

3.17 If air resistance is negligible, the trajectory of a projectile is a combination of horizontal motion with constant velocity and vertical motion with constant acceleration.



at equal time intervals. The x-component of acceleration is zero, so v_x is constant. The y-component of acceleration is constant and not zero, so v_y changes by equal amounts in equal times, just the same as if the projectile were launched vertically with the same initial y-velocity. At the highest point in the trajectory, $v_y = 0$.

We can also represent the initial velocity $\vec{v}_0$ by its magnitude v_0 (the initial speed) and its angle α_0 with the positive x-axis (Fig. 3.18). In terms of these quantities, the components v_{0x} and v_{0y} of the initial velocity are

$$v_{0x} = v_0 \cos\alpha_0 \qquad v_{0y} = v_0 \sin\alpha_0 \tag{3.19}$$

Using these relationships in Eqs. (3.15) through (3.18) and setting $x_0 = y_0 = 0$, we find

$$x = (v_0 \cos\alpha_0)t \qquad \text{(projectile motion)} \tag{3.20}$$

$$y = (v_0 \sin\alpha_0)t - \frac{1}{2}gt^2 \qquad \text{(projectile motion)} \tag{3.21}$$

$$v_x = v_0 \cos\alpha_0 \qquad \text{(projectile motion)} \tag{3.22}$$

$$v_y = v_0 \sin\alpha_0 - gt \qquad \text{(projectile motion)} \tag{3.23}$$

These equations describe the position and velocity of the projectile in Fig. 3.17 at any time t.

We can get a lot of information from these equations. For example, at any time the distance r of the projectile from the origin (the magnitude of the position vector $\vec{r}$) is given by

$$r = \sqrt{x^2 + y^2} \tag{3.24}$$

The projectile's speed (the magnitude of its velocity) at any time is

$$v = \sqrt{v_x^2 + v_y^2} \tag{3.25}$$

The *direction* of the velocity, in terms of the angle α it makes with the positive x-direction (see Fig. 3.17), is given by

$$\tan\alpha = \frac{v_y}{v_x} \tag{3.26}$$

The velocity vector $\vec{v}$ is tangent to the trajectory at each point.

We can derive an equation for the trajectory's shape in terms of x and y by eliminating t. From Eqs. (3.20) and (3.21), which assume $x_0 = y_0 = 0$, we find $t = x/(v_0 \cos\alpha_0)$ and

$$y = (\tan\alpha_0)x - \frac{g}{2v_0^2\cos^2\alpha_0}x^2 \tag{3.27}$$

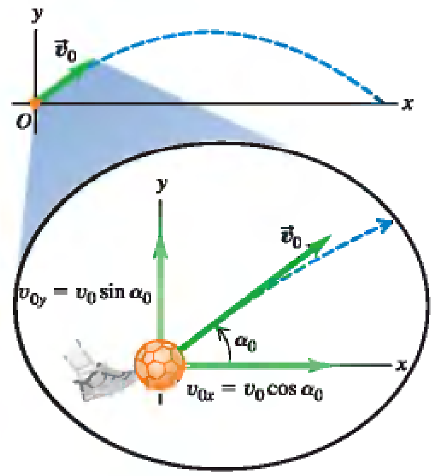
Don't worry about the details of this equation; the important point is its general form. The quantities v_0, $\tan\alpha_0$, $\cos\alpha_0$, and g are constants, so the equation has the form

$$y = bx - cx^2$$

where b and c are constants. This is the equation of a *parabola*. In projectile motion, with our simple model, the trajectory is always a parabola (Fig. 3.19).

When air resistance *isn't* always negligible and has to be included, calculating the trajectory becomes a lot more complicated; the effects of air resistance

3.18 The initial velocity components v_{0x} and v_{0y} of a projectile (such as a kicked soccer ball) are related to the initial speed v_0 and initial angle α_0.



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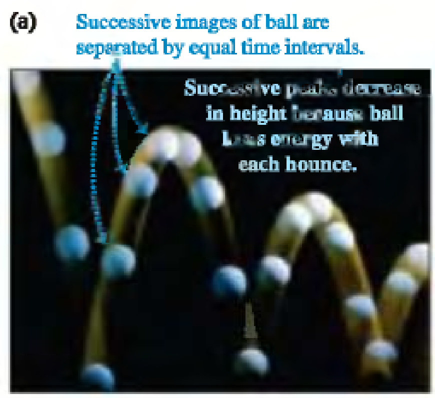
3.5 Initial Velocity Components
3.6 Target Practice I
3.7 Target Practice II

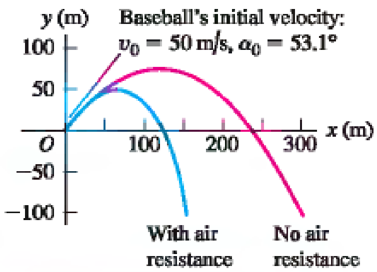
3.19 The nearly parabolic trajectories of (a) a bouncing ball and (b) blobs of molten rock ejected from a volcano.

(a)



(b)

3.20 Air resistance has a large cumulative effect on the motion of a baseball. In this simulation we allow the baseball to fall below the height from which it was thrown (for example, the baseball could have been thrown from a cliff).



depend on velocity, so the acceleration is no longer constant. Figure 3.20 shows a computer simulation of the trajectory of a baseball both without air resistance and with air resistance proportional to the square of the baseball's speed. We see that air resistance has a very large effect; the maximum height and range both decrease, and the trajectory is no longer a parabola. (If you look closely at Fig. 3.19b, you'll see that the trajectories of the volcanic blobs deviate in a similar way from a parabolic shape.)

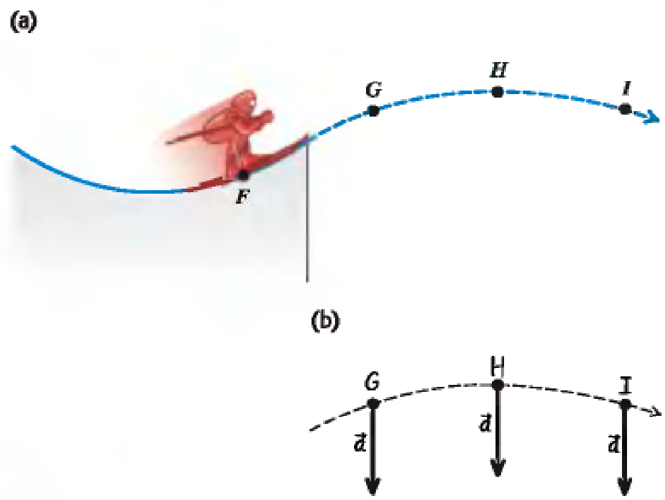
Conceptual Example 3.5 Acceleration of a skier, continued

Let's consider again the skier in Conceptual Example 3.4. What is her acceleration at points G, H, and I in Fig. 3.21a *after* she flies off the ramp? Neglect air resistance.

SOLUTION

Figure 3.21b shows our answer. The skier's acceleration changed from point to point while she was on the ramp. But as soon as she leaves the ramp, she becomes a projectile. So at points G, H, and I, and indeed at *all* points after she leaves the ramp, the skier's acceleration points vertically downward and has magnitude g. No matter how complicated the acceleration of a particle before it becomes a projectile, its acceleration as a projectile is given by $a_x = 0$, $a_y = -g$.

3.21 (a) The skier's path during the jump. (b) Our solution.



Problem Solving Strategy 3.1 Projectile Motion

NOTE: The strategies we used in Sections 2.4 and 2.5 for straight-line, constant-acceleration problems are also useful here.

IDENTIFY *the relevant concepts:* The key concept to remember is that throughout projectile motion, the acceleration is downward and has a constant magnitude g. Note that the projectile-motion equations don't apply to throwing a ball, because during the throw the ball is acted on by both the thrower's hand and gravity. These equations come into play only after the ball leaves the thrower's hand.

SET UP *the problem* using the following steps:

1. Define your coordinate system and make a sketch showing your axes. Usually it's easiest to take the x-axis as being horizontal and the y-axis as being upward and to place the origin at the initial $(t = 0)$ position where the body first becomes a projectile (such as where a ball leaves the thrower's hand). Then the components of the (constant) acceleration are $a_x = 0$, $a_y = -g$, and the initial position is $x_0 = 0$, $y_0 = 0$.

2. List the unknown and known quantities, and decide which unknowns are your target variables. For example, you might be given the initial velocity (either the components or the magnitude and direction) and asked to find the coordinates and velocity components at some later time. In any case, you'll be using Eqs. (3.20) through (3.23). (Certain other equations given in Section 3.3 may be useful as well.) Make sure that you have as many equations as there are target variables to be found.
3. State the problem in words and then translate those words into symbols. For example, *when* does the particle arrive at a certain point? (That is, at what value of t?) *Where* is the particle when its velocity has a certain value? (That is, what are the values of x and y when v_x or v_y has the specified value?) Since $v_y = 0$ at the highest point in a trajectory, the question "When does the projectile reach its highest point?" translates into "What is the value of t when $v_y = 0$?" Similarly, "When does the projectile return to its initial elevation?" translates into "What is the value of t when $y = y_0$?"

EXECUTE *the solution:* Use Eqs. (3.20) through (3.23) to find the target variables. Resist the temptation to break the trajectory into segments and analyze each segment separately. You don't have to start all over when the projectile reaches its highest point! It's almost always easier to use the same axes and time scale throughout the problem. Use the value $g = 9.8\ \text{m/s}^2$.

EVALUATE *your answer:* As always, look at your results to see whether they make sense and whether the numerical values seem reasonable.

Example 3.6 A body projected horizontally

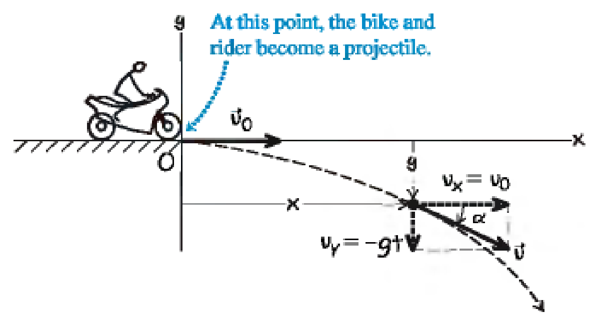
A motorcycle stunt rider rides off the edge of a cliff. Just at the edge his velocity is horizontal, with magnitude 9.0 m/s. Find the motorcycle's position, distance from the edge of the cliff, and velocity after 0.50 s.

SOLUTION

IDENTIFY: Once the rider leaves the cliff, he is in projectile motion. His velocity at the edge of the cliff is therefore his initial velocity.

SET UP: Figure 3.22 shows our sketch. We place the origin of our coordinate system at the edge of the cliff, where the motorcycle first becomes a projectile, so $x_0 = 0$ and $y_0 = 0$. The initial velocity is purely horizontal (that is, $\alpha_0 = 0$), so the initial velocity components are $v_{0x} = v_0\cos\alpha_0 = 9.0\ \text{m/s}$ and $v_{0y} = v_0\sin\alpha_0 = 0$. To find the motorcycle's position at time $t = 0.50\ \text{s}$, we use Eqs. (3.20) and (3.21), which give x and y as functions of time. We then find the distance from the origin using Eq. (3.24). Finally, we use Eqs. (3.22) and (3.23) to find the velocity components v_x and v_y at $t = 0.50\ \text{s}$.

3.22 Our sketch for this problem.



EXECUTE: Where is the motorcycle at $t = 0.50\ \text{s}$? From Eqs. (3.20) and (3.21), the x- and y-coordinates are

$$x = v_{0x}t = (9.0\ \text{m/s})(0.50\ \text{s}) = 4.5\ \text{m}$$

$$y = -\frac{1}{2}gt^2 = -\frac{1}{2}(9.8\ \text{m/s}^2)(0.50\ \text{s})^2 = -1.2\ \text{m}$$

The negative value of y shows that at this time the motorcycle is below its starting point.

What is the motorcycle's distance from the origin at this time? From Eq. (3.24),

$$r = \sqrt{x^2 + y^2} = \sqrt{(4.5\ \text{m})^2 + (-1.2\ \text{m})^2} = 4.7\ \text{m}$$

What is the velocity at time $t = 0.50\ \text{s}$? From Eqs. (3.22) and (3.23), the components of velocity at this time are

$$v_x = v_{0x} = 9.0\ \text{m/s}$$

$$v_y = -gt = (-9.8\ \text{m/s}^2)(0.50\ \text{s}) = -4.9\ \text{m/s}$$

The motorcycle has the same horizontal velocity v_x as when it left the cliff at $t = 0$, but in addition there is a downward (negative) vertical velocity v_y. If we use unit vectors, the velocity at $t = 0.50\ \text{s}$ is

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} = (9.0\ \text{m/s})\hat{\imath} + (-4.9\ \text{m/s})\hat{\jmath}$$

We can also express the velocity in terms of magnitude and direction. From Eq. (3.25), the speed (magnitude of the velocity) at this time is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(9.0\ \text{m/s})^2 + (-4.9\ \text{m/s})^2} = 10.2\ \text{m/s}$$

From Eq. (3.26), the angle α of the velocity vector is

$$\alpha = \arctan\frac{v_y}{v_x} = \arctan\left(\frac{-4.9\ \text{m/s}}{9.0\ \text{m/s}}\right) = -29^\circ$$

At this time the velocity is 29° below the horizontal.

EVALUATE: Just as shown in Fig. 3.17, the horizontal aspect of the motion is unchanged by gravity; the motorcycle continues to move horizontally at 9.0 m/s, covering 4.5 m in 0.50 s. The motorcycle initially has zero vertical velocity, so it falls vertically just like a body released from rest and descends a distance $\frac{1}{2}gt^2 = 1.2\ \text{m}$ in 0.50 s.

Example 3.7 Height and range of a projectile I: A batted baseball

A batter hits a baseball so that it leaves the bat at speed $v_0 = 37.0\ \text{m/s}$ at an angle $\alpha_0 = 53.1°$, at a location where $g = 9.80\ \text{m/s}^2$. (a) Find the position of the ball, and the magnitude and direction of its velocity, at $t = 2.00$ s. (b) Find the time when the ball reaches the highest point of its flight and find its height h at this point. (c) Find the *horizontal range R*—that is, the horizontal distance from the starting point to where the ball hits the ground.

SOLUTION

IDENTIFY: As Fig. 3.20 shows, the effects of air resistance on the motion of a baseball aren't really negligible. For the sake of simplicity, however, we'll ignore air resistance for this example and use the projectile-motion equations to describe the motion.

SET UP: Figure 3.23 shows our sketch. We use the same coordinate system as in Fig. 3.17 or 3.18 so we can use Eqs. (3.20) through (3.23) without any modifications. Our target variables are (1) the position and velocity of the ball 2.00 s after it leaves the bat, (2) the elapsed time after leaving the bat when the ball is at its maximum height—that is, when $v_y = 0$—and the y-coordinate at this time, and (3) the x-coordinate at the time when the y-coordinate is equal to the initial value y_0.

The ball leaves the bat a meter or so above ground level, but we neglect this distance and assume that it starts at ground level ($y_0 = 0$). The initial velocity of the ball has components

$$v_{0x} = v_0\cos\alpha_0 = (37.0\ \text{m/s})\cos 53.1° = 22.2\ \text{m/s}$$
$$v_{0y} = v_0\sin\alpha_0 = (37.0\ \text{m/s})\sin 53.1° = 29.6\ \text{m/s}$$

EXECUTE: (a) We want to find x, y, v_x, and v_y at time $t = 2.00$ s. From Eqs. (3.20) through (3.23),

$$x = v_{0x}t = (22.2\ \text{m/s})(2.00\ \text{s}) = 44.4\ \text{m}$$
$$y = v_{0y}t - \frac{1}{2}gt^2$$
$$= (29.6\ \text{m/s})(2.00\ \text{s}) - \frac{1}{2}(9.80\ \text{m/s}^2)(2.00\ \text{s})^2$$
$$= 39.6\ \text{m}$$
$$v_x = v_{0x} = 22.2\ \text{m/s}$$
$$v_y = v_{0y} - gt = 29.6\ \text{m/s} - (9.80\ \text{m/s}^2)(2.00\ \text{s})$$
$$= 10.0\ \text{m/s}$$

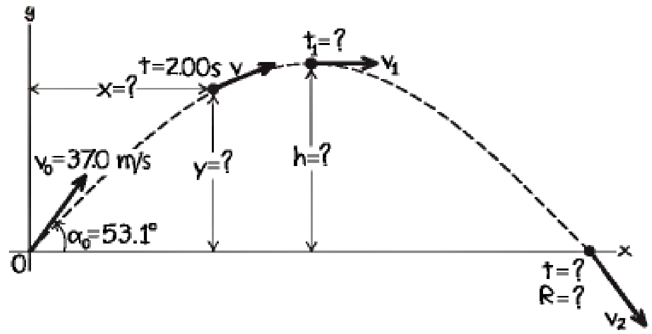
The y-component of velocity is positive, which means that the ball is still moving upward at this time (Fig. 3.23). The magnitude and direction of the velocity are found from Eqs. (3.25) and (3.26):

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(22.2\ \text{m/s})^2 + (10.0\ \text{m/s})^2}$$
$$= 24.3\ \text{m/s}$$
$$\alpha = \arctan\left(\frac{10.0\ \text{m/s}}{22.2\ \text{m/s}}\right) = \arctan 0.450 = 24.2°$$

The direction of the velocity (that is, the direction of motion) is 24.2° above the horizontal.

(b) At the highest point, the vertical velocity v_y is zero. When does this happen? Call the time t_1; then

$$v_y = v_{0y} - gt_1 = 0$$
$$t_1 = \frac{v_{0y}}{g} = \frac{29.6\ \text{m/s}}{9.80\ \text{m/s}^2} = 3.02\ \text{s}$$



3.23 Our sketch for this problem.

The height h at this time is the value of y when $t = t_1 = 3.02$ s:

$$h = v_{0y}t_1 - \frac{1}{2}gt_1^2$$
$$= (29.6\ \text{m/s})(3.02\ \text{s}) - \frac{1}{2}(9.80\ \text{m/s}^2)(3.02\ \text{s})^2$$
$$= 44.7\ \text{m}$$

(c) We'll find the horizontal range in two steps. First, *when* does the ball hit the ground? This occurs when $y = 0$. Call this time t_2; then

$$y = 0 = v_{0y}t_2 - \frac{1}{2}gt_2^2 = t_2\left(v_{0y} - \frac{1}{2}gt_2\right)$$

This is a quadratic equation for t_2. It has two roots:

$$t_2 = 0 \qquad \text{and} \qquad t_2 = \frac{2v_{0y}}{g} = \frac{2(29.6\ \text{m/s})}{9.80\ \text{m/s}^2} = 6.04\ \text{s}$$

There are two times at which $y = 0$; $t_2 = 0$ is the time the ball *leaves* the ground, and $t_2 = 2v_{0y}/g = 6.04$ s is the time of its return. This is exactly twice the time to reach the highest point that we found in part (b), $t_1 = v_{0y}/g = 3.02$ s, so the time of descent equals the time of ascent. This is *always* true if the starting and end points are at the same elevation and air resistance can be neglected.

The horizontal range R is the value of x when the ball returns to the ground—that is, at $t = 6.04$ s:

$$R = v_{0x}t_2 = (22.2\ \text{m/s})(6.04\ \text{s}) = 134\ \text{m}$$

The vertical component of velocity when the ball hits the ground is

$$v_y = v_{0y} - gt_2 = 29.6\ \text{m/s} - (9.80\ \text{m/s}^2)(6.04\ \text{s})$$
$$= -29.6\ \text{m/s}$$

That is, v_y has the same magnitude as the initial vertical velocity v_{0y} but the opposite direction (down). Since v_x is constant, the angle $\alpha = -53.1°$ (below the horizontal) at this point is the negative of the initial angle $\alpha_0 = 53.1°$.

EVALUATE: It's often useful to check results by getting them in a different way. For example, we can check our answer for the maximum height in part (b) by applying the constant-acceleration formula Eq. (2.13) to the y-motion:

$$v_y^2 = v_{0y}^2 + 2a_y(y - y_0) = v_{0y}^2 - 2g(y - y_0)$$

At the highest point, $v_y = 0$ and $y = h$. Substituting these, along with $y_0 = 0$, we find

$$0 = v_{0y}^2 - 2gh$$

$$h = \frac{v_{0y}^2}{2g} = \frac{(29.6\ \text{m/s})^2}{2(9.80\ \text{m/s}^2)} = 44.7\ \text{m}$$

which is the same height we obtained in part (b).

It's interesting to note that $h = 44.7$ m in part (b) is comparable to the 52.4-m height above the playing field of the roof of the Hubert H. Humphrey Metrodome in Minneapolis, and the horizontal range $R = 134$ m in part (c) is greater than the 99.7-m distance from home plate to the right-field fence at Safeco Field in Seattle. (The ball's height when it crosses the fence is more than enough to clear it, so this ball is a home run.)

In real life, a batted ball with the initial speed and angle we've used here won't go as high or as far as we've calculated. (If it did, home runs would be far more common and baseball would be a far less interesting game.) The reason is that air resistance, which we neglected in this example, is actually an important factor at the typical speeds of pitched and batted balls (see Fig. 3.20).

Example 3.8 Height and range of a projectile II: Maximum height, maximum range

For a projectile launched with speed v_0 at initial angle α_0 (between 0° and 90°), derive general expressions for the maximum height h and horizontal range R (Fig. 3.23). For a given v_0, what value of α_0 gives maximum height? What value gives maximum horizontal range?

SOLUTION

IDENTIFY: This is really the same exercise as parts (b) and (c) of Example 3.7. The difference is that we are looking for general expressions for h and R. We'll also be looking for the values of α_0 that give the maximum values of h and R.

SET UP: In part (b) of Example 3.7 we found that the projectile reaches the high point of its trajectory (so that $v_y = 0$) at time $t_1 = v_{0y}/g$, and in part (c) of Example 3.7 we found that the projectile returns to its starting height (so that $y = y_0$) at time $t_2 = 2v_{0y}/g$. (As we saw in Example 3.7, $t_2 = 2t_1$.) To determine the height h at the high point of the trajectory, we use Eq. (3.21) to find the y-coordinate at t_1. To determine R, we substitute t_2 into Eq. (3.20) to determine the x-coordinate at t_2. We'll express our answers in terms of the launch speed v_0 and launch angle α_0 using Eq. (3.19).

EXECUTE: From Eq. (3.19), $v_{0x} = v_0\cos\alpha_0$ and $v_{0y} = v_0\sin\alpha_0$. Hence we can write the time t_1 when $v_y = 0$ as

$$t_1 = \frac{v_{0y}}{g} = \frac{v_0\sin\alpha_0}{g}$$

Then, from Eq. (3.21), the height at this time is

$$h = (v_0\sin\alpha_0)\left(\frac{v_0\sin\alpha_0}{g}\right) - \frac{1}{2}g\left(\frac{v_0\sin\alpha_0}{g}\right)^2$$

$$= \frac{v_0^2\sin^2\alpha_0}{2g}$$

For a given launch speed v_0, the maximum value of h occurs when $\sin\alpha_0 = 1$ and $\alpha_0 = 90°$—that is, when the projectile is launched straight up. That's what we should expect. If it is launched horizontally, as in Example 3.6, $\alpha_0 = 0$ and the maximum height is zero!

The time t_2 when the projectile returns to the ground is

$$t_2 = \frac{2v_{0y}}{g} = \frac{2v_0\sin\alpha_0}{g}$$

The horizontal range R is the value of x at this time. From Eq. (3.20),

$$R = (v_0\cos\alpha_0)t_2 = (v_0\cos\alpha_0)\frac{2v_0\sin\alpha_0}{g}$$

We can now use the trigonometric identity $2\sin\alpha_0\cos\alpha_0 = \sin 2\alpha_0$ to rewrite this as

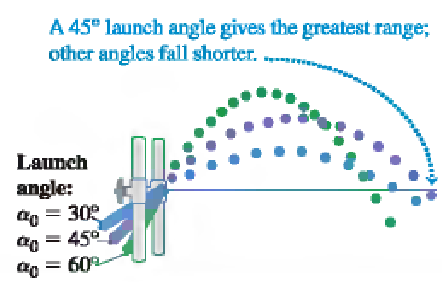
$$R = \frac{v_0^2\sin 2\alpha_0}{g}$$

The maximum value of $\sin 2\alpha_0$ is 1; this occurs when $2\alpha_0 = 90°$, or $\alpha_0 = 45°$. This angle gives the maximum range for a given initial speed.

EVALUATE: Figure 3.24 is based on a composite photograph of three trajectories of a ball projected from a spring gun at angles of 30°, 45°, and 60°. The initial speed v_0 is approximately the same in all three cases. The horizontal ranges are nearly the same for the 30° and 60° angles, and the range for 45° is greater than either. Can you prove that for a given value of v_0 the range is the same for both an initial angle α_0 and an initial angle $90° - \alpha_0$?

CAUTION **Height and range of a projectile** We don't recommend memorizing the above expressions for h and R. They are applicable only in the special circumstances we have described. In particular, the expression for the range R can be used *only* when launch and landing heights are equal. There are many end-of-chapter problems to which these equations do *not* apply.

3.24 A launch angle of 45° gives the maximum horizontal range. The range is shorter with launch angles of 30° and 60°.



Example 3.9 Different initial and final heights

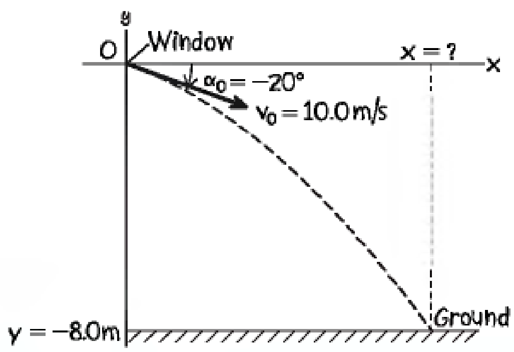
You toss a ball from your window 8.0 m above the ground. When the ball leaves your hand, it is moving at 10.0 m/s at an angle of 20° below the horizontal. How far horizontally from your window will the ball hit the ground? Ignore air resistance.

SOLUTION

IDENTIFY: As in our calculation of the horizontal range in Examples 3.7 and 3.8, we are trying to find the horizontal coordinate of a projectile when it is at a given value of y. The difference here is that this value of y is *not* equal to the initial y-coordinate.

SET UP: Once again we choose the x-axis to be horizontal and the y-axis to be upward, and we place the origin of coordinates at the point where the ball leaves your hand (Fig. 3.25). We have $v_0 = 10.0\ \text{m/s}$ and $\alpha_0 = -20°$; the angle is negative because the initial velocity is below the horizontal. Our target variable is the value of x at the point where the ball reaches the ground—that is, when $y = -8.0\ \text{m}$. Because the initial and final heights of the ball are different, we can't simply use the expression for the horizontal range found in Example 3.8. Instead, we first use Eq. (3.21) to find the time t when the ball reaches $y = -8.0\ \text{m}$ and then calculate the value of x at this time using Eq. (3.20).

3.25 Our sketch for this problem.



EXECUTE: To determine t, we rewrite Eq. (3.21) in the standard form for a quadratic equation for t:

$$\frac{1}{2}gt^2 - (v_0\sin\alpha_0)t + y = 0$$

The roots of this equation are

$$t = \frac{v_0\sin\alpha_0 \pm \sqrt{(-v_0\sin\alpha_0)^2 - 4\left(\frac{1}{2}g\right)y}}{2\left(\frac{1}{2}g\right)}$$

$$= \frac{v_0\sin\alpha_0 \pm \sqrt{v_0^2\sin^2\alpha_0 - 2gy}}{g}$$

$$= \frac{\left[\begin{array}{l}(10.0\ \text{m/s})\sin(-20°)\\ \pm\sqrt{(10.0\ \text{m/s})^2\sin^2(-20°) - 2(9.80\ \text{m/s}^2)(-8.0\ \text{m})}\end{array}\right]}{9.80\ \text{m/s}^2}$$

$$= -1.7\ \text{s} \quad \text{or} \quad 0.98\ \text{s}$$

We can discard the negative root, since it refers to a time before the ball left your hand. The positive root tells us that the ball takes 0.98 s to reach the ground. From Eq. (3.20), the ball's x-coordinate at that time is

$$x = (v_0\cos\alpha_0)t = (10.0\ \text{m/s})[\cos(-20°)](0.98\ \text{s})$$
$$= 9.2\ \text{m}$$

The ball hits the ground a horizontal distance of 9.2 m from your window.

EVALUATE: The root $t = -1.7\ \text{s}$ is an example of a "fictional" solution to a quadratic equation. We discussed these in Example 2.8 in Section 2.5; you should review that discussion.

With our choice of origin we had initial and final heights $y_0 = 0$ and $y = -8.0\ \text{m}$. Can you use Eqs. (3.16) and (3.18) to show that you get the same answers for t and x if you choose the origin to be at the point on the ground directly below where the ball leaves your hand?

Example 3.10 The zookeeper and the monkey

A monkey escapes from the zoo and climbs a tree. After failing to entice the monkey down, the zookeeper fires a tranquilizer dart directly at the monkey (Fig. 3.26). The clever monkey lets go at the same instant the dart leaves the gun barrel, intending to land on the ground and escape. Show that the dart *always* hits the monkey, regardless of the dart's muzzle velocity (provided that it gets to the monkey before he hits the ground).

SOLUTION

IDENTIFY: In this example we have *two* bodies in projectile motion: the tranquilizer dart and the monkey. The dart and the monkey have different initial positions and initial velocities, but they go into projectile motion at the same time. To show that the dart hits the monkey, we have to prove that at some time the monkey and the dart have the same x-coordinate and the same y-coordinate.

SET UP: We make the usual choice for the x- and y-directions, and place the origin of coordinates at the end of the barrel of the tranquilizer gun (Fig. 3.26). We'll first use Eq. (3.20) to find the time t when the x-coordinates x_{monkey} and x_{dart} are the same. Then we'll use Eq. (3.21) to check whether y_{monkey} and y_{dart} are also equal at this time; if they are, the dart hits the monkey.

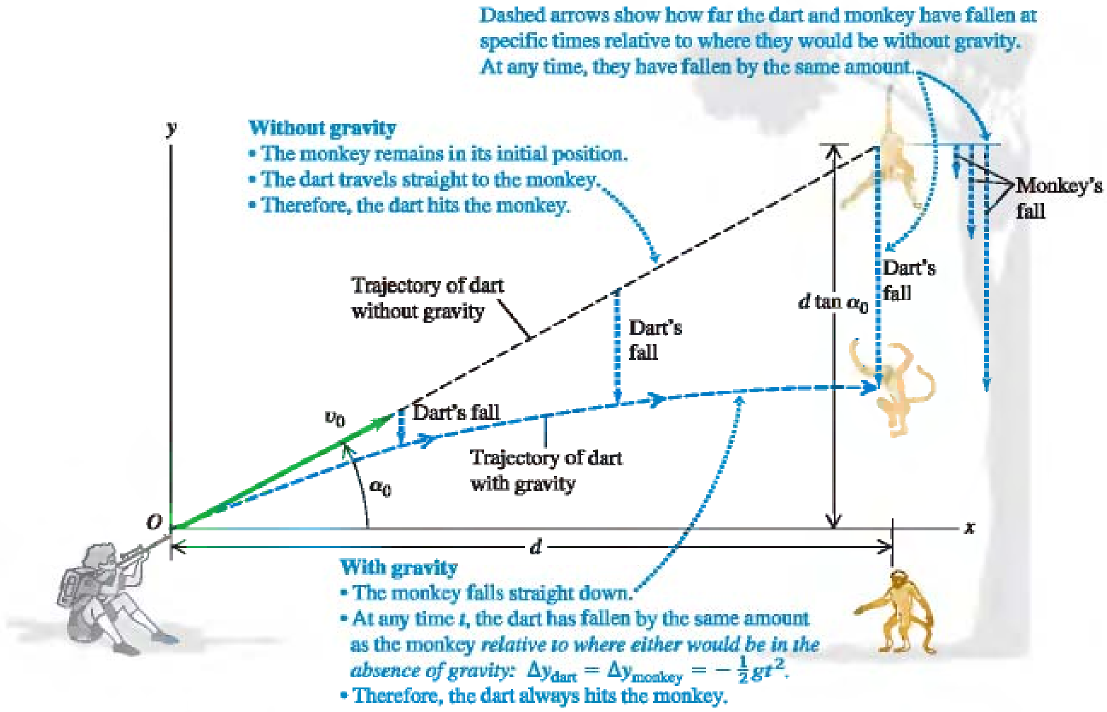
EXECUTE: The monkey drops straight down, so $x_{\text{monkey}} = d$ at *all* times. For the dart, Eq. (3.20) tells us that $x_{\text{dart}} = (v_0\cos\alpha_0)t$. When these x-coordinates are equal, $d = (v_0\cos\alpha_0)t$, or

$$t = \frac{d}{v_0\cos\alpha_0}$$

To have the dart hit the monkey, it must be true that $y_{\text{monkey}} = y_{\text{dart}}$ at this same time. The monkey is in one-dimensional free fall; his position at any time is given by Eq. (2.12), with appropriate symbol changes. Figure 3.26 shows that the monkey's initial height is $d\tan\alpha_0$ (the opposite side of a right triangle with angle α_0 and adjacent side d), and we find

$$y_{\text{monkey}} = d\tan\alpha_0 - \frac{1}{2}gt^2$$

3.26 The tranquilizer dart hits the falling monkey.



For the dart we use Eq. (3.21):

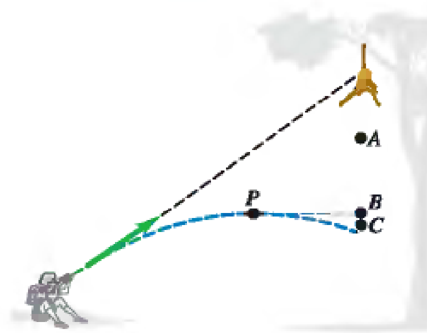
$$y_{dart} = (v_0 \sin\alpha_0)t - \frac{1}{2}gt^2$$

So we see that if $d \tan\alpha_0 = (v_0 \sin\alpha_0)t$ at the time when the two x-coordinates are equal, then $y_{monkey} = y_{dart}$, and we have a hit. To prove that this happens, we replace t with $d/(v_0 \cos\alpha_0)$, the time when $x_{monkey} = x_{dart}$. Sure enough, we find that

$$(v_0 \sin\alpha_0)t = (v_0 \sin\alpha_0)\frac{d}{v_0 \cos\alpha_0} = d\tan\alpha_0$$

EVALUATE: We have proved that at the time the x-coordinates are equal, the y-coordinates are also equal; a dart aimed at the initial position of the monkey *always* hits it, no matter what v_0 is. This result is also independent of the value of g, the acceleration due to gravity. With no gravity $(g = 0)$, the monkey would remain motionless, and the dart would travel in a straight line to hit him. With gravity, both "fall" the same distance $(\frac{1}{2}gt^2)$ below their $g = 0$ positions, and the dart still hits the monkey (Fig. 3.26).

Test Your Understanding of Section 3.3 In Example 3.10, suppose the tranquilizer dart has a relatively low muzzle velocity so that the dart reaches a maximum height at a point P before striking the monkey, as shown in the figure. When the dart is at point P, will the monkey be (i) at point A (higher than P), (ii) at point B (at the same height as P), or (iii) at point C (lower than P)? Ignore air resistance.

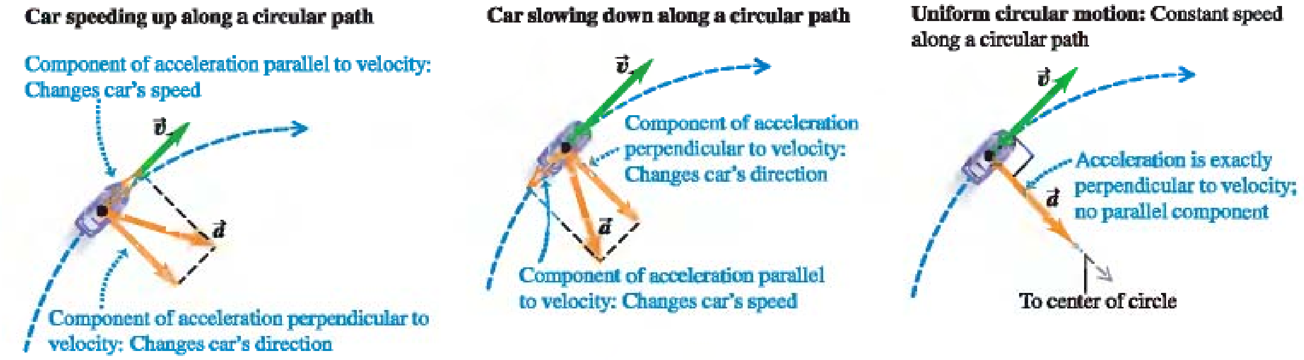


3.4 Motion in a Circle

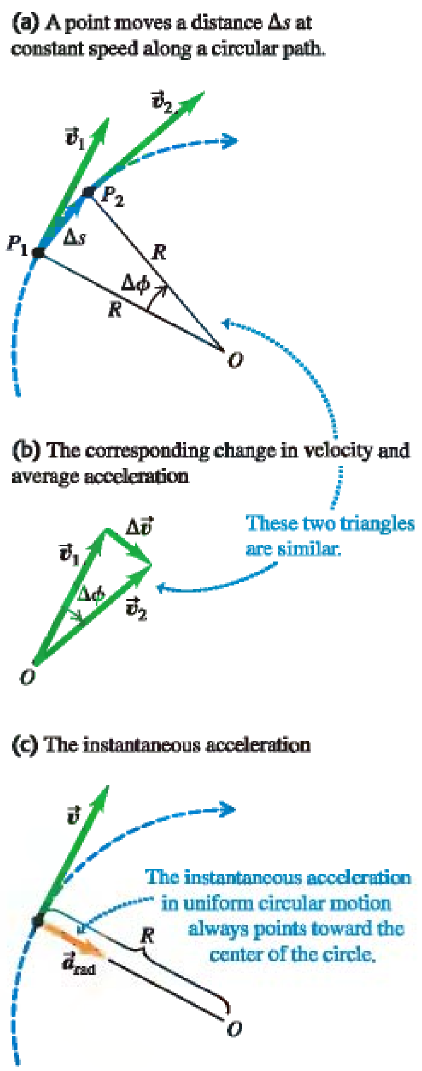
When a particle moves along a curved path, the direction of its velocity changes. As we saw in Section 3.2, this means that the particle *must* have a component of acceleration perpendicular to the path, even if its speed is constant (see Fig. 3.11b). In this section we'll calculate the acceleration for the important special case of motion in a circle.

ActivPhysics Online
4.1 Magnitude of Centripetal Acceleration

3.27 A car in uniform circular motion. The speed is constant and the acceleration is directed toward the center of the circular path.



3.28 Finding the velocity change $\Delta\vec{v}$, average acceleration $\vec{a}_{av}$, and instantaneous acceleration $\vec{a}_{rad}$ for a particle moving in a circle with constant speed.



Uniform Circular Motion

When a particle moves in a circle with *constant speed,* the motion is called **uniform circular motion.** A car rounding a curve with constant radius at constant speed, a satellite moving in a circular orbit, and an ice skater skating in a circle with constant speed are all examples of uniform circular motion (Fig. 3.27; compare Fig. 3.12). There is no component of acceleration parallel (tangent) to the path; otherwise, the speed would change. The acceleration vector is perpendicular (normal) to the path and hence directed inward (never outward!) toward the center of the circular path. This causes the direction of the velocity to change without changing the speed. Our next project is to show that the magnitude of the acceleration in uniform circular motion is related in a simple way to the speed of the particle and the radius of the circle.

Figure 3.28a shows a particle moving with constant speed in a circular path of radius R with center at O. The particle moves from P_1 to P_2 in a time Δt. The vector change in velocity $\Delta\vec{v}$ during this time is shown in Fig. 3.28b.

The angles labeled $\Delta\phi$ in Figs. 3.28a and 3.28b are the same because $\vec{v}_1$ is perpendicular to the line OP_1 and $\vec{v}_2$ is perpendicular to the line OP_2. Hence the triangles in Figs. 3.28a and 3.28b are *similar.* The ratios of corresponding sides of similar triangles are equal, so

$$\frac{|\Delta\vec{v}|}{v_1} = \frac{\Delta s}{R} \qquad \text{or} \qquad |\Delta\vec{v}| = \frac{v_1}{R}\Delta s$$

The magnitude a_{av} of the average acceleration during Δt is therefore

$$a_{av} = \frac{|\Delta\vec{v}|}{\Delta t} = \frac{v_1}{R}\frac{\Delta s}{\Delta t}$$

The magnitude a of the *instantaneous* acceleration $\vec{a}$ at point P_1 is the limit of this expression as we take point P_2 closer and closer to point P_1:

$$a = \lim_{\Delta t \to 0} \frac{v_1}{R}\frac{\Delta s}{\Delta t} = \frac{v_1}{R}\lim_{\Delta t \to 0}\frac{\Delta s}{\Delta t}$$

But the limit of $\Delta s/\Delta t$ is the speed v_1 at point P_1. Also, P_1 can be any point on the path, so we can drop the subscript and let v represent the speed at any point. Then

$$a_{rad} = \frac{v^2}{R} \qquad \text{(uniform circular motion)} \qquad (3.28)$$

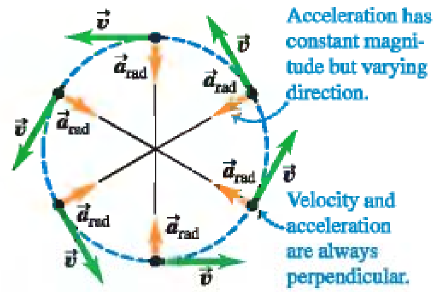
We have added the subscript "rad" as a reminder that the direction of the instantaneous acceleration at each point is always along a radius of the circle, toward its center. Because the speed is constant, the acceleration is always perpendicular to the instantaneous velocity. This is shown in Fig. 3.28c; compare with the right-hand illustration in Fig. 3.27.

We have found that *in uniform circular motion, the magnitude a of the instantaneous acceleration is equal to the square of the speed v divided by the radius R of the circle. Its direction is perpendicular to $\vec{v}$ and inward along the radius.*

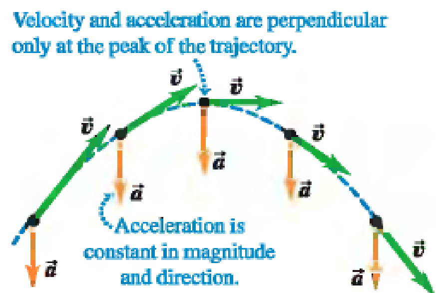
Because the acceleration is always directed toward the center of the circle, it is sometimes called **centripetal acceleration.** The word "centripetal" is derived from two Greek words meaning "seeking the center." Figure 3.29a shows the directions of the velocity and acceleration vectors at several points for a particle moving with uniform circular motion.

3.29 Acceleration and velocity (a) for a particle in uniform circular motion and (b) for a projectile with no air resistance.

(a) Uniform circular motion



(b) Projectile motion



CAUTION **Uniform circular motion vs. projectile motion** The acceleration in uniform circular motion has some similarities to the acceleration in projectile motion without air resistance, but there are also some important differences. In both uniform circular motion (Fig. 3.29a) and projectile motion (Fig. 3.29b) the *magnitude* of acceleration is the same at all times. However, in uniform circular motion the *direction* of $\vec{a}$ changes continuously so that it always points toward the center of the circle. (At the top of the circle the acceleration points down; at the bottom of the circle the acceleration points up.) In projectile motion, by contrast, the direction of $\vec{a}$ remains the same at all times.

We can also express the magnitude of the acceleration in uniform circular motion in terms of the **period** T of the motion, the time for one revolution (one complete trip around the circle). In a time T the particle travels a distance equal to the circumference $2\pi R$ of the circle, so its speed is

$$v = \frac{2\pi R}{T} \tag{3.29}$$

When we substitute this into Eq. (3.28), we obtain the alternative expression

$$a_{rad} = \frac{4\pi^2 R}{T^2} \quad \text{(uniform circular motion)} \tag{3.30}$$

Example 3.11 Centripetal acceleration on a curved road

An Aston Martin V8 Vantage sports car has a "lateral acceleration" of $0.96g$, which is $(0.96)(9.8\ \text{m/s}^2) = 9.4\ \text{m/s}^2$. This represents the maximum centripetal acceleration that the car can attain without skidding out of the circular path. If the car is traveling at a constant 40 m/s (about 89 mi/h, or 144 km/h), what is the minimum radius of curve it can negotiate? (Assume that the curve is unbanked.)

SOLUTION

IDENTIFY: Because the car is moving at a constant speed along a curve that is a segment of a circle, we can apply the ideas of uniform circular motion.

SET UP: We use Eq. (3.28) to find the target variable R (the radius of the curve) in terms of the given centripetal acceleration a_{rad} and speed v.

EXECUTE: We are given a_{rad} and v, so we solve Eq. (3.28) for R:

$$R = \frac{v^2}{a_{rad}} = \frac{(40\ \text{m/s})^2}{9.4\ \text{m/s}^2} = 170\ \text{m (about 560 ft)}$$

EVALUATE: Our result shows that the required turning radius R is proportional to the *square* of the speed. Hence even a small reduction in speed can make R substantially smaller. For example, reducing v by 20% (from 40 m/s to 32 m/s) would decrease R by 36% (from 170 m to 109 m).

Another way to make the required turning radius smaller is to *bank* the curve. We will investigate this option in Chapter 5.

Example 3.12 Centripetal acceleration on a carnival ride

In a carnival ride, the passengers travel at constant speed in a circle of radius 5.0 m. They make one complete circle in 4.0 s. What is their acceleration?

SOLUTION

IDENTIFY: The speed is constant, so this is a problem involving uniform circular motion.

SET UP: We are given the radius $R = 5.0\text{ m}$ and the period $T = 4.0\text{ s}$, so we can use Eq. (3.30) to calculate the acceleration. Alternatively, we can first calculate the speed v using Eq. (3.29) and then find the acceleration using Eq. (3.28).

EXECUTE: From Eq. (3.30),

$$a_{\text{rad}} = \frac{4\pi^2(5.0\text{ m})}{(4.0\text{ s})^2} = 12\text{ m/s}^2$$

We'll check this answer by using Eq. (3.28) after first determining the speed v. From Eq. (3.29), the speed is the circumference of the circle divided by the period T:

$$v = \frac{2\pi R}{T} = \frac{2\pi(5.0\text{ m})}{4.0\text{ s}} = 7.9\text{ m/s}$$

The centripetal acceleration is then

$$a_{\text{rad}} = \frac{v^2}{R} = \frac{(7.9\text{ m/s})^2}{5.0\text{ m}} = 12\text{ m/s}^2$$

Happily, we get the same answer for a_{rad} with both approaches.

EVALUATE: As in Example 3.11, the direction of $\vec{a}$ is always toward the center of the circle. The magnitude of $\vec{a}$ is greater than g, the acceleration due to gravity, so this is not a ride for the faint-hearted. (Some roller coasters subject their passengers to accelerations as great as $4g$.)

Nonuniform Circular Motion

We have assumed throughout this section that the particle's speed is constant. If the speed varies, we call the motion **nonuniform circular motion.** An example is a roller coaster car that slows down and speeds up as it moves around a vertical loop. In nonuniform circular motion, Eq. (3.28) still gives the *radial* component of acceleration $a_{\text{rad}} = v^2/R$, which is always *perpendicular* to the instantaneous velocity and directed toward the center of the circle. But since the speed v has different values at different points in the motion, the value of a_{rad} is not constant. The radial (centripetal) acceleration is greatest at the point in the circle where the speed is greatest.

In nonuniform circular motion there is also a component of acceleration that is *parallel* to the instantaneous velocity. This is the component $a_{\parallel}$ that we discussed in Section 3.2; here we call this component $a_{\tan}$ to emphasize that it is *tangent* to the circle. From the discussion at the end of Section 3.2 we see that the tangential component of acceleration $a_{\tan}$ is equal to the rate of change of *speed.* Thus

$$a_{\text{rad}} = \frac{v^2}{R} \quad \text{and} \quad a_{\tan} = \frac{d|\vec{v}|}{dt} \quad \text{(nonuniform circular motion)} \tag{3.31}$$

The vector acceleration of a particle moving in a circle with varying speed is the vector sum of the radial and tangential components of accelerations. The tangential component is in the same direction as the velocity if the particle is speeding up, and in the opposite direction if the particle is slowing down (Fig. 3.30).

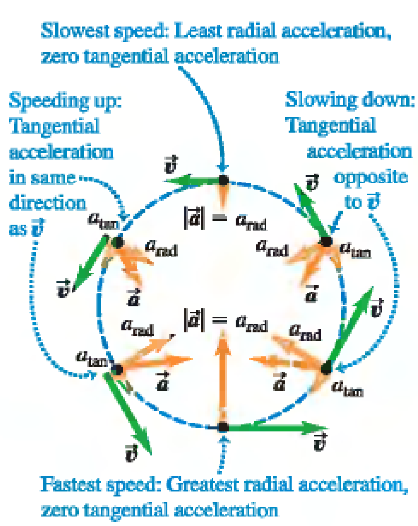
In *uniform* circular motion there is no tangential component of acceleration, but the radial component is the magnitude of $d\vec{v}/dt$.

CAUTION **Uniform vs. nonuniform circular motion** Note that the two quantities

$$\frac{d|\vec{v}|}{dt} \quad \text{and} \quad \left|\frac{d\vec{v}}{dt}\right|$$

are *not* the same. The first, equal to the tangential acceleration, is the rate of change of speed; it is zero whenever a particle moves with constant speed, even when its direction of motion changes (such as in *uniform* circular motion). The second is the magnitude of the vector acceleration; it is zero only when the particle's acceleration *vector* is zero—that is, when the particle moves in a straight line with constant speed. In *uniform* circular motion $|d\vec{v}/dt| = a_{\text{rad}} = v^2/r$; in *nonuniform* circular motion there is also a tangential component of acceleration, so $|d\vec{v}/dt| = \sqrt{a_{\text{rad}}^2 + a_{\tan}^2}$.

3.30 A particle moving in a vertical loop with a varying speed, like a roller coaster car.



Test Your Understanding of Section 3.4 Suppose that the particle in Fig. 3.30 experiences four times the acceleration at the bottom of the loop as it does at the top of the loop. Compared to its speed at the top of the loop, is its speed at the bottom of the loop (i) $\sqrt{2}$ times as great; (ii) 2 times as great; (iii) $2\sqrt{2}$ times as great; (iv) 4 times as great; or (v) 16 times as great.

3.5 Relative Velocity

You've no doubt observed how a car that is moving slowly forward appears to be moving backward when you pass it. In general, when two observers measure the velocity of a moving body, they get different results if one observer is moving relative to the other. The velocity seen by a particular observer is called the velocity *relative* to that observer, or simply **relative velocity.** Figure 3.31 shows a situation in which understanding relative velocity is extremely important.

We'll first consider relative velocity along a straight line, then generalize to relative velocity in a plane.

3.31 Airshow pilots face a complicated problem involving relative velocities. They must keep track of their motion relative to the air (to maintain enough airflow over the wings to sustain lift), relative to each other (to keep a tight formation without colliding), and relative to their audience (to remain in sight of the spectators).

Relative Velocity in One Dimension

A passenger walks with a velocity of 1.0 m/s along the aisle of a train that is moving with a velocity of 3.0 m/s (Fig. 3.32a). What is the passenger's velocity? It's a simple enough question, but it has no single answer. As seen by a second passenger sitting in the train, she is moving at 1.0 m/s. A person on a bicycle standing beside the train sees the walking passenger moving at 1.0 m/s + 3.0 m/s = 4.0 m/s. An observer in another train going in the opposite direction would give still another answer. We have to specify which observer we mean, and we speak of the velocity *relative* to a particular observer. The walking passenger's velocity relative to the train is 1.0 m/s, her velocity relative to the cyclist is 4.0 m/s, and so on. Each observer, equipped in principle with a meter stick and a stopwatch, forms what we call a **frame of reference.** Thus a frame of reference is a coordinate system plus a time scale.

Let's use the symbol A for the cyclist's frame of reference (at rest with respect to the ground) and the symbol B for the frame of reference of the moving train. In straight-line motion the position of a point P relative to frame A is given by $x_{P/A}$ (the position of P with respect to A), and the position of P relative to frame B is given by $x_{P/B}$ (see Fig. 3.32b). The position of the origin of A with respect to the origin of B is $x_{B/A}$. Figure 3.32b shows that

$$x_{P/A} = x_{P/B} + x_{B/A} \tag{3.32}$$

In words, the total distance from the origin of A to point P equals the distance from the origin of B to point P plus the distance from the origin of A to the origin of B.

The x-velocity of P relative to frame A, denoted by $v_{P/A\text{-}x}$, is the derivative of $x_{P/A}$ with respect to time. The other velocities are similarly obtained. So the time derivative of Eq. (3.32) gives us a relationship among the various velocities:

$$\frac{dx_{P/A}}{dt} = \frac{dx_{P/B}}{dt} + \frac{dx_{B/A}}{dt} \qquad \text{or}$$

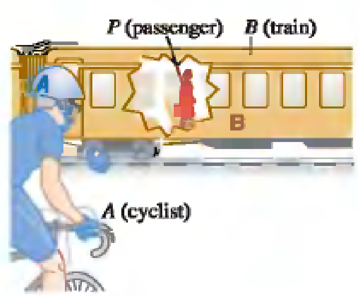
$$v_{P/A\text{-}x} = v_{P/B\text{-}x} + v_{B/A\text{-}x} \qquad \text{(relative velocity along a line)} \tag{3.33}$$

Getting back to the passenger on the train in Fig. 3.32, we see that A is the cyclist's frame of reference, B is the frame of reference of the train, and point P represents the passenger. Using the above notation, we have

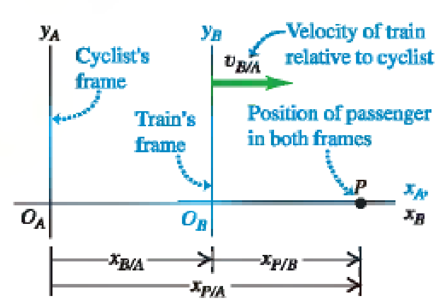
$$v_{P/B\text{-}x} = +1.0\ \text{m/s} \qquad v_{B/A\text{-}x} = +3.0\ \text{m/s}$$

3.32 (a) A passenger walking in a train. (b) The position of the passenger relative to the cyclist's frame of reference and the train's frame of reference.

(a)



(b)



From Eq. (3.33) the passenger's velocity $v_{P/A}$ relative to the cyclist is

$$v_{P/A\text{-}x} = +1.0\ \text{m/s} + 3.0\ \text{m/s} = +4.0\ \text{m/s}$$

as we already knew.

In this example, both velocities are toward the right, and we have taken this as the positive x-direction. If the passenger walks toward the *left* relative to the train, then $v_{P/B\text{-}x} = -1.0\ \text{m/s}$, and her x-velocity relative to the cyclist is $v_{P/A\text{-}x} = -1.0\ \text{m/s} + 3.0\ \text{m/s} = +2.0\ \text{m/s}$. The sum in Eq. (3.33) is always an algebraic sum, and any or all of the x-velocities may be negative.

When the passenger looks out the window, the stationary cyclist on the ground appears to her to be moving backward; we can call the cyclist's velocity relative to her $v_{A/P\text{-}x}$. Clearly, this is just the negative of $v_{P/A\text{-}x}$. In general, if A and B are any two points or frames of reference,

$$v_{A/B\text{-}x} = -v_{B/A\text{-}x} \tag{3.34}$$

Problem Solving Strategy 3.2 Relative Velocity

MP

IDENTIFY *the relevant concepts:* Whenever you see the phrase "velocity relative to" or "velocity with respect to," it's likely that the concepts of relative velocity will be helpful.

SET UP *the problem:* Label each frame of reference in the problem. Each moving body has its own frame of reference; in addition, you'll almost always have to include the frame of reference of the earth's surface. (Statements such as "The car is traveling north at 90 km/h" implicitly refer to the car's velocity relative to the surface of the earth.) Use the labels to help identify the target variable. For example, if you want to find the x-velocity of a car (C) with respect to a bus (B), your target variable is $v_{C/B\text{-}x}$.

EXECUTE *the solution:* Solve for the target variable using Eq. (3.33). (If the velocities are not along the same direction, you'll need to use the vector form of this equation, derived later in this section.) It's important to note the order of the double subscripts in Eq. (3.33): $v_{A/B\text{-}x}$ always means "x-velocity of A relative to B." These subscripts obey an interesting kind of algebra, as Eq. (3.33) shows. If we regard each one as a fraction, then the fraction on the left side is the *product* of the fractions on the right sides: $P/A = (P/B)(B/A)$. This is a handy rule you can use when applying Eq. (3.33) to any number of frames of reference. For example, if there are three different frames of reference A, B, and C, we can write immediately

$$v_{P/A\text{-}x} = v_{P/C\text{-}x} + v_{C/B\text{-}x} + v_{B/A\text{-}x}$$

EVALUATE *your answer:* Be on the lookout for stray minus signs in your answer. If the target variable is the x-velocity of a car relative to a bus $(v_{C/B\text{-}x})$, make sure that you haven't accidentally calculated the x-velocity of the *bus* relative to the *car* $(v_{B/C\text{-}x})$. If you have made this mistake, you can recover using Eq. (3.34).

Example 3.13 Relative velocity on a straight road

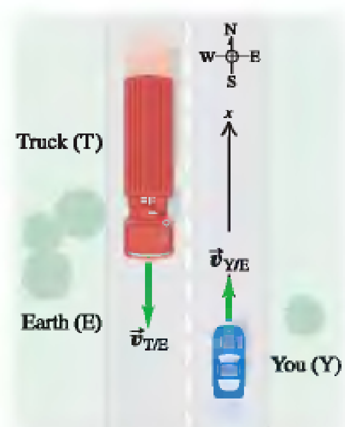
You are driving north on a straight two-lane road at a constant 88 km/h. A truck traveling at a constant 104 km/h approaches you (in the other lane, fortunately). (a) What is the truck's velocity relative to you? (b) What is your velocity with respect to the truck? (c) How do the relative velocities change after you and the truck have passed each other?

SOLUTION

IDENTIFY: This example is about relative velocities along a line.

SET UP: Let you be Y, the truck be T, and the earth's surface be E, and let the positive x-direction be north (Fig. 3.33). Then your x-velocity relative to the earth is $v_{Y/E\text{-}x} = +88\ \text{km/h}$. As the truck is initially approaching you, it must be moving south and its x-velocity with respect to the earth is $v_{T/E\text{-}x} = -104\ \text{km/h}$. The target variable in part (a) is $v_{T/Y\text{-}x}$; the target variable in part (b) is $v_{Y/T\text{-}x}$. We'll find both target variables by using Eq. (3.33) for relative velocity.

3.33 Reference frames for you and the truck.



EXECUTE: (a) To find $v_{T/Y\text{-}x}$, we first write Eq. (3.33) for the three frames Y, T, and E, and then rearrange:

$$v_{T/E\text{-}x} = v_{T/Y\text{-}x} + v_{Y/E\text{-}x}$$
$$v_{T/Y\text{-}x} = v_{T/E\text{-}x} - v_{Y/E\text{-}x}$$
$$= -104\ \text{km/h} - 88\ \text{km/h} = -192\ \text{km/h}$$

The truck is moving at 192 km/h in the negative x-direction (south) relative to you.

(b) From Eq. (3.34),

$$v_{Y/T\text{-}x} = -v_{T/Y\text{-}x} = -(-192\ \text{km/h}) = +192\ \text{km/h}$$

You are moving at 192 km/h in the positive x-direction (north) relative to the truck.

(c) The relative velocities do *not* change at all after you and the truck pass each other. The relative positions of the bodies don't matter. The truck is still moving at 192 km/h toward the south relative to you, but it is now moving away from you instead of toward you.

EVALUATE: To check your answer in part (b), try using Eq. (3.33) directly in the form $v_{Y/T\text{-}x} = v_{Y/E\text{-}x} + v_{E/T\text{-}x}$. (Remember that the x-velocity of the earth with respect to the truck is the opposite of the x-velocity of the truck with respect to the earth: $v_{E/T\text{-}x} = -v_{T/E\text{-}x}$.) Do you get the same result?

Relative Velocity in Two or Three Dimensions

We can extend the concept of relative velocity to include motion in a plane or in space by using vector addition to combine velocities. Suppose that the passenger in Fig. 3.32a is walking not down the aisle of the railroad car but from one side of the car to the other, with a speed of 1.0 m/s (Fig. 3.34a). We can again describe the passenger's position P in two different frames of reference: A for the stationary ground observer and B for the moving train. But instead of coordinates x, we use position vectors [&*lcacc*{~bfit~r~normal~}{lArarl}&] because the problem is now two-dimensional. Then, as Fig. 3.34b shows,

$$\vec{r}_{P/A} = \vec{r}_{P/B} + \vec{r}_{B/A} \tag{3.35}$$

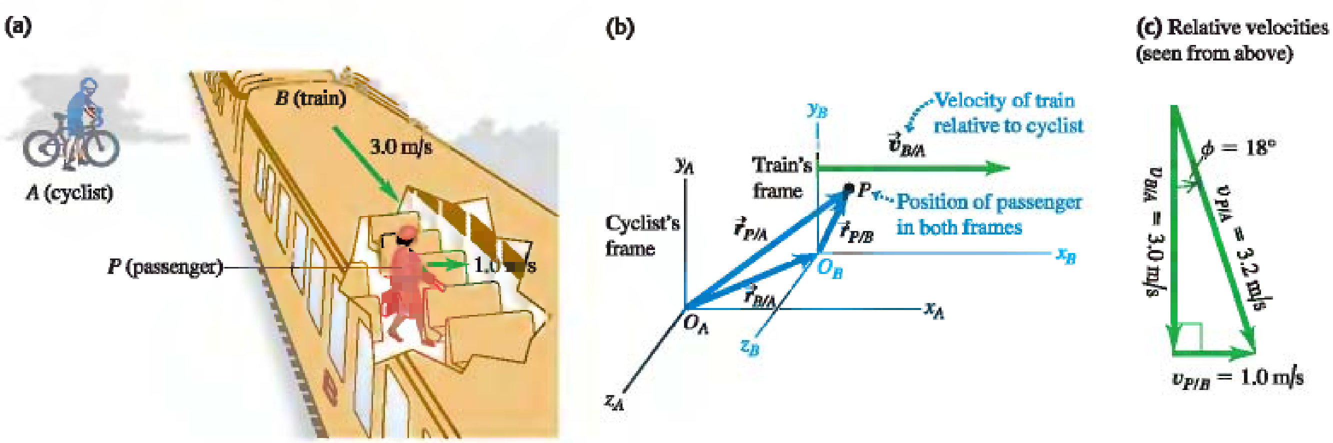
Just as we did before, we take the time derivative of this equation to get a relationship among the various velocities; the velocity of P relative to A is $\vec{v}_{P/A} = d\vec{r}_{P/A}/dt$ and so on for the other velocities. We get

$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A} \quad \text{(relative velocity in space)} \tag{3.36}$$

Equation (3.36) is known as the *Galilean velocity transformation*. It relates the velocity of a body P with respect to frame A and its velocity with respect to frame B ($\vec{v}_{P/A}$ and $\vec{v}_{P/B}$, respectively) to the velocity of frame B with respect to frame A ($\vec{v}_{B/A}$). If all three of these velocities lie along the same line, then Eq. (3.36) reduces to Eq. (3.33) for the components of the velocities along that line.

If the train is moving at $v_{B/A} = 3.0$ m/s relative to the ground and the passenger is moving at $v_{P/B} = 1.0$ m/s relative to the train, then the passenger's velocity

3.34 (a) A passenger walking across a railroad car. (b) Position of the passenger relative to the cyclist's frame and the train's frame. (c) Vector diagram for the velocity of the passenger relative to the ground (the cyclist's frame), $\vec{v}_{P/A}$.



vector $\vec{v}_{P/A}$ relative to the ground is as shown in Fig. 3.34c. The Pythagorean theorem then gives us

$$v_{P/A} = \sqrt{(3.0\ \text{m/s})^2 + (1.0\ \text{m/s})^2} = \sqrt{10\ \text{m}^2/\text{s}^2} = 3.2\ \text{m/s}$$

Figure 3.34c also shows that the *direction* of the passenger's velocity vector relative to the ground makes an angle ϕ with the train's velocity vector $\vec{v}_{B/A}$, where

$$\tan\phi = \frac{v_{P/B}}{v_{B/A}} = \frac{1.0\ \text{m/s}}{3.0\ \text{m/s}} \quad \text{and} \quad \phi = 18^\circ$$

As in the case of motion along a straight line, we have the general rule that if A and B are *any* two points or frames of reference,

$$\vec{v}_{A/B} = -\vec{v}_{B/A} \tag{3.37}$$

The velocity of the passenger relative to the train is the negative of the velocity of the train relative to the passenger, and so on.

In the early 20th century Albert Einstein showed in his special theory of relativity that the velocity-addition relationship given in Eq. (3.36) has to be modified when speeds approach the speed of light, denoted by c. It turns out that if the passenger in Fig. 3.32a could walk down the aisle at $0.30c$ and the train could move at $0.90c$, then her speed relative to the ground would be not $1.20c$ but $0.94c$; nothing can travel faster than light! We'll return to the special theory of relativity in Chapter 37.

Example 3.14 Flying in a crosswind

The compass of an airplane indicates that it is headed due north, and its airspeed indicator shows that it is moving through the air at 240 km/h. If there is a wind of 100 km/h from west to east, what is the velocity of the airplane relative to the earth?

SOLUTION

IDENTIFY: This problem involves velocities in two dimensions (northward and eastward), so it is a relative velocity problem using vectors.

SET UP: We are given the magnitude and direction of the velocity of the plane (P) relative to the air (A). We are also given the magnitude and direction of the wind velocity, which is the velocity of the air (A) with respect to the earth (E):

$$\vec{v}_{P/A} = 240\ \text{km/h} \qquad \text{due north}$$
$$\vec{v}_{A/E} = 100\ \text{km/h} \qquad \text{due east}$$

Our target variables are the magnitude and direction of the velocity of the plane (P) relative to the earth (E), $\vec{v}_{P/E}$. We'll find these using Eq. (3.36).

EXECUTE: Using Eq. (3.36), we have

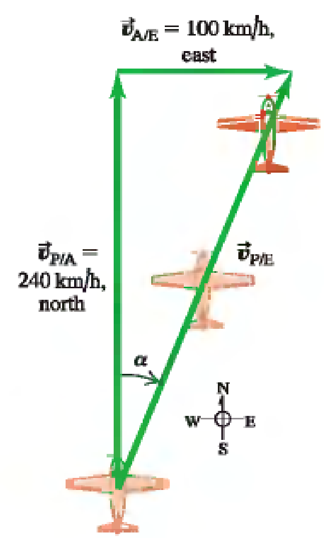
$$\vec{v}_{P/E} = \vec{v}_{P/A} + \vec{v}_{A/E}$$

Figure 3.35 shows the three relative velocities and their relationship; the unknowns are the speed $v_{P/E}$ and the angle α. From this diagram we find

$$v_{P/E} = \sqrt{(240\ \text{km/h})^2 + (100\ \text{km/h})^2} = 260\ \text{km/h}$$
$$\alpha = \arctan\left(\frac{100\ \text{km/h}}{240\ \text{km/h}}\right) = 23^\circ\ \text{E of N}$$

EVALUATE: The crosswind increases the speed of the airplane relative to the earth, but at the price of pushing the airplane off course.

3.35 The plane is pointed north, but the wind blows east, giving the resultant velocity $\vec{v}_{P/E}$ relative to the earth.



Example 3.15 Correcting for a crosswind

In Example 3.14, in what direction should the pilot head to travel due north? What will be her velocity relative to the earth? (Assume that her airspeed and the velocity of the wind are the same as in Example 3.14.)

SOLUTION

IDENTIFY: Like Example 3.14, this is a relative velocity problem with vectors.

SET UP: Figure 3.36 illustrates the situation. The vectors are arranged in accordance with the vector relative-velocity equation, Eq. (3.36):

$$\vec{v}_{P/E} = \vec{v}_{P/A} + \vec{v}_{A/E}$$

As Fig. 3.36 shows, the pilot points the nose of the airplane at an angle β into the wind to compensate for the crosswind. This angle, which tells us the direction of the vector $\vec{v}_{P/A}$ (the velocity of the airplane relative to the air), is one of our target variables. The other target variable is the speed of the airplane over the ground, which is the magnitude of the vector $\vec{v}_{P/E}$ (the velocity of the airplane relative to the earth). Here are the known and unknown quantities:

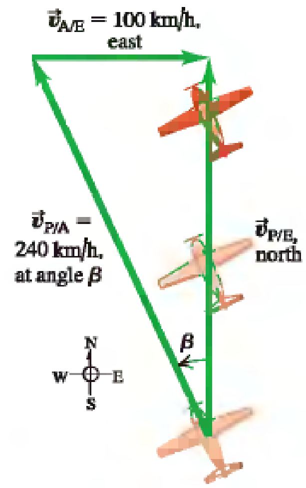
$\vec{v}_{P/E}$ = magnitude unknown	due north
$\vec{v}_{P/A} = 240\text{ km/h}$	direction unknown
$\vec{v}_{A/E} = 100\text{ km/h}$	due east

We can solve for the unknown target variables using Fig. 3.36 and trigonometry.

EXECUTE: From the diagram, the speed $v_{P/E}$ and the angle β are given by

$$v_{P/E} = \sqrt{(240\text{ km/h})^2 - (100\text{ km/h})^2} = 218\text{ km/h}^2$$

$$\beta = \arcsin\left(\frac{100\text{ km/h}}{240\text{ km/h}}\right) = 25°$$



3.36 The pilot must point the plane in the direction of the vector $\vec{v}_{P/A}$ to travel due north relative to the earth.

The pilot should point the airplane 25° west of north, and her ground speed is then 218 km/h.

EVALUATE: Note that there were two target variables—the magnitude of a vector and the direction of a vector—in both this example and Example 3.14. The difference is that in Example 3.14, the magnitude and direction referred to the *same* vector ($\vec{v}_{P/E}$), whereas in this example they referred to *different* vectors ($\vec{v}_{P/E}$ and $\vec{v}_{P/A}$).

It's no surprise that a headwind reduces an airplane's speed relative to the ground. This example shows that a *crosswind* also slows an airplane down—an unfortunate fact of aeronautical life.

Test Your Understanding of Section 3.5 Suppose the nose of an airplane is pointed due east and the airplane has an airspeed of 150 km/h. Due to the wind, the airplane is moving due *north* relative to the ground and its speed relative to the ground is 150 km/h. What is the velocity of the air relative to the earth? (i) 150 km/h from east to west; (ii) 150 km/h from south to north; (iii) 150 km/h from southeast to northwest; (iv) 212 km/h from east to west; (v) 212 km/h from south to north; (vi) 212 km/h from southeast to northwest; (vii) there is no possible wind velocity that could cause this.

CHAPTER 3 SUMMARY

Position, velocity, and acceleration vectors: The position vector $\vec{r}$ of a point P in space is the vector from the origin to P. Its components are the coordinates x, y, and z.

The average velocity vector $\vec{v}_{av}$ during the time interval Δt is the displacement $\Delta\vec{r}$ (the change in the position vector $\vec{r}$) divided by Δt. The instantaneous velocity vector $\vec{v}$ is the time derivative of $\vec{r}$, and its components are the time derivatives of x, y, and z. The instantaneous speed is the magnitude of $\vec{v}$. The velocity $\vec{v}$ of a particle is always tangent to the particle's path. (See Example 3.1.)

The average acceleration vector $\vec{a}_{av}$ during the time interval Δt equals $\Delta\vec{v}$ (the change in the velocity vector $\vec{v}$) divided by Δt. The instantaneous acceleration vector $\vec{a}$ is the time derivative of $\vec{v}$, and its components are the time derivatives of v_x, v_y, and v_z. (See Example 3.2.)

The component of acceleration parallel to the direction of the instantaneous velocity affects the speed, while the component of $\vec{a}$ perpendicular to $\vec{v}$ affects the direction of motion. (See Examples 3.3 and 3.4.)

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \tag{3.1}$$

$$\vec{v}_{av} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \frac{\Delta\vec{r}}{\Delta t} \tag{3.2}$$

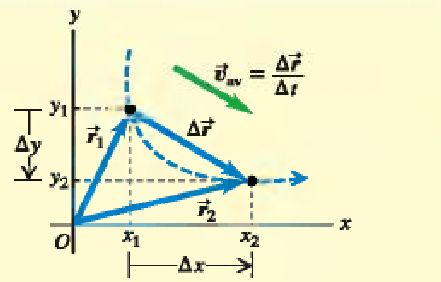
$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \tag{3.3}$$

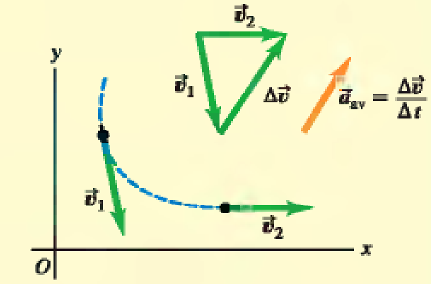
$$v_x = \frac{dx}{dt} \quad v_y = \frac{dy}{dt} \quad v_z = \frac{dz}{dt} \tag{3.4}$$

$$\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t} \tag{3.8}$$

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \tag{3.9}$$

$$a_x = \frac{dv_x}{dt} \quad a_y = \frac{dv_y}{dt} \quad a_z = \frac{dv_z}{dt} \tag{3.10}$$





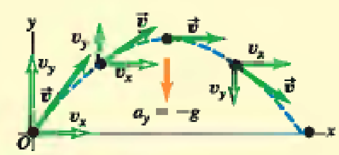
Projectile motion: In projectile motion with no air resistance, $a_x = 0$ and $a_y = -g$. The coordinates and velocity components are simple functions of time, and the shape of the path is always a parabola. We usually choose the origin to be at the initial position of the projectile. (See Examples 3.5–3.10.)

$$x = (v_0 \cos\alpha_0)t \tag{3.20}$$

$$y = (v_0 \sin\alpha_0)t - \frac{1}{2}gt^2 \tag{3.21}$$

$$v_x = v_0 \cos\alpha_0 \tag{3.22}$$

$$v_y = v_0 \sin\alpha_0 - gt \tag{3.23}$$

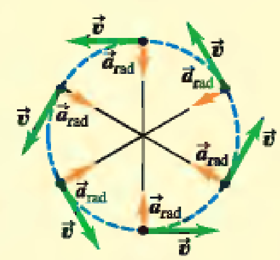


Uniform and nonuniform circular motion: When a particle moves in a circular path of radius R with constant speed v (uniform circular motion), its acceleration $\vec{a}$ is directed toward the center of the circle and perpendicular to $\vec{v}$. The magnitude a_{rad} of the acceleration can be expressed in terms of v and R or in terms of R and the period T (the time for one revolution), where $v = 2\pi R/T$. (See Examples 3.11 and 3.12.)

If the speed is not constant in circular motion (nonuniform circular motion), there is still a radial component of $\vec{a}$ given by Eq. (3.28) or (3.30), but there is also a component of $\vec{a}$ parallel (tangential) to the path. This tangential component is equal to the rate of change of speed, dv/dt.

$$a_{rad} = \frac{v^2}{R} \tag{3.28}$$

$$a_{rad} = \frac{4\pi^2 R}{T^2} \tag{3.30}$$



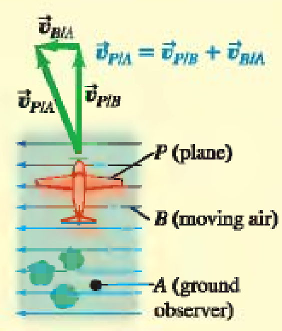
Relative velocity: When a body P moves relative to a body (or reference frame) B, and B moves relative to A, we denote the velocity of P relative to B by $\vec{v}_{P/B}$, the velocity of P relative to A by $\vec{v}_{P/A}$, and the velocity of B relative to A by $\vec{v}_{B/A}$. If these velocities are all along the same line, their components along that line are related by Eq. (3.33). More generally, these velocities are related by Eq. (3.36). (See Examples 3.13–3.15)

$$v_{P/A\text{-}x} = v_{P/B\text{-}x} + v_{B/A\text{-}x} \tag{3.33}$$

(relative velocity along a line)

$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A} \tag{3.36}$$

(relative velocity in space)



Key Terms

position vector, *72*
average velocity, *72*
instantaneous velocity, *72*
average acceleration, *75*
instantaneous acceleration, *75*
projectile, *79*
trajectory, *79*
uniform circular motion, *88*
centripetal acceleration, *89*
period, *89*
nonuniform circular motion, *90*
relative velocity, *91*
frame of reference, *91*

Answer to Chapter Opening Question ?

A car going around a curve at constant speed has an acceleration directed toward the inside of the curve (see Section 3.2, especially Fig. 3.12a).

Answers to Test Your Understanding Questions

3.1 Answer: (iii) If the instantaneous velocity $\vec{v}$ is constant over an interval, its value at any point (including the end of the interval) is the same as the average velocity $\vec{v}_{av}$ over the interval. In (i) and (ii) the direction of $\vec{v}$ at the end of the interval is tangent to the path at that point, while the direction of $\vec{v}_{av}$ points from the beginning of the path to its end (in the direction of the net displacement). In (iv) $\vec{v}$ and $\vec{v}_{av}$ are both directed along the straight line, but $\vec{v}$ has a greater magnitude because the speed has been increasing.

3.2 Answer: vector 7 At the high point of the sled's path, the speed is minimum. At that point the speed is neither increasing nor decreasing, and the parallel component of the acceleration (that is, the horizontal component) is zero. The acceleration has only a perpendicular component toward the inside of the sled's curved path. In other words, the acceleration is downward.

3.3 Answer: (i) If there were no gravity $(g = 0)$, the monkey would not fall and the dart would follow a straight-line path (shown as a dashed line). The effect of gravity is to make the monkey and the dart both fall the same distance $\frac{1}{2}gt^2$ below their $g = 0$ positions. Point A is the same distance below the monkey's initial position as point P is below the dashed straight line, so point A is where we would find the monkey at the time in question.

3.4 Answer: (ii) At both the top and bottom of the loop, the acceleration is purely radial and is given by Eq. (3.28). The radius R is the same at both points, so the difference in acceleration is due purely to differences in speed. Since a_{rad} is proportional to the square of v, the speed must be twice as great at the bottom of the loop as at the top.

3.5 Answer: (vi) The effect of the wind is to cancel the airplane's eastward motion and give it a northward motion. So the velocity of the air relative to the ground (the wind velocity) must have one 150-km/h component to the west and one 150-km/h component to the north. The combination of these is a vector of magnitude $\sqrt{(150\text{ km/h})^2 + (150\text{ km/h})^2} = 212\text{ km/h}$ that points to the northwest.

PROBLEMS

For instructor-assigned homework, go to www.masteringphysics.com

Discussion Questions

Q3.1. A simple pendulum (a mass swinging at the end of a string) swings back and forth in a circular arc. What is the direction of the acceleration off the mass at the ends of the swing? At the midpoint? In each case, explain how you obtain your answer.

Q3.2. Redraw Fig. 3.11a if $\vec{a}$ is antiparallel to $\vec{v}_1$. Does the particle move in a straight line? What happens to its speed?

Q3.3. A projectile moves in a parabolic path without air resistance. Is there any point at which $\vec{a}$ is parallel to $\vec{v}$? Perpendicular to $\vec{v}$? Explain.

Q3.4. When a rifle is fired at a distant target, the barrel is not lined up exactly on the target. Why not? Does the angle of correction depend on the distance of the target?

Q3.5. At the same instant that you fire a bullet horizontally from a gun, you drop a bullet from the height of the barrel. If there is no air resistance, which bullet hits the ground first? Explain.

Q3.6. A package falls out of an airplane that is flying in a straight line at a constant altitude and speed. If you could ignore air resistance, what would be the path of the package as observed by the pilot? As observed by a person on the ground?

Q3.7. Sketch the six graphs of the x- and y-components of position, velocity, and acceleration versus time for projectile motion with $x_0 = y_0 = 0$ and $0 < \alpha_0 < 90°$.

Q3.8. An object is thrown straight up into the air and feels no air resistance. How is it possible for it to have an acceleration when it has stopped moving at its highest point?

Q3.9. If a jumping frog can give itself the same initial speed regardless of the direction in which it jumps (forward or straight up), how is the maximum vertical height to which it can jump related to its maximum horizontal range $R_{max} = v_0^2/g$?

Q3.10. A projectile is fired upward at an angle θ above the horizontal with an initial speed v_0. At its maximum height, what are its velocity vector, its speed, and its acceleration vector?

Q3.11. In uniform circular motion, what are the *average* velocity and *average* acceleration for one revolution? Explain.

Q3.12. In uniform circular motion, how does the acceleration change when the speed is increased by a factor of 3? When the radius is decreased by a factor of 2?

Q3.13. In uniform circular motion, the acceleration is perpendicular to the velocity at every instant. Is this still true when the motion is not uniform—that is, when the speed is not constant?

Q3.14. Raindrops hitting the side windows of a car in motion often leave diagonal streaks even if there is no wind. Why? Is the explanation the same or different for diagonal streaks on the windshield?

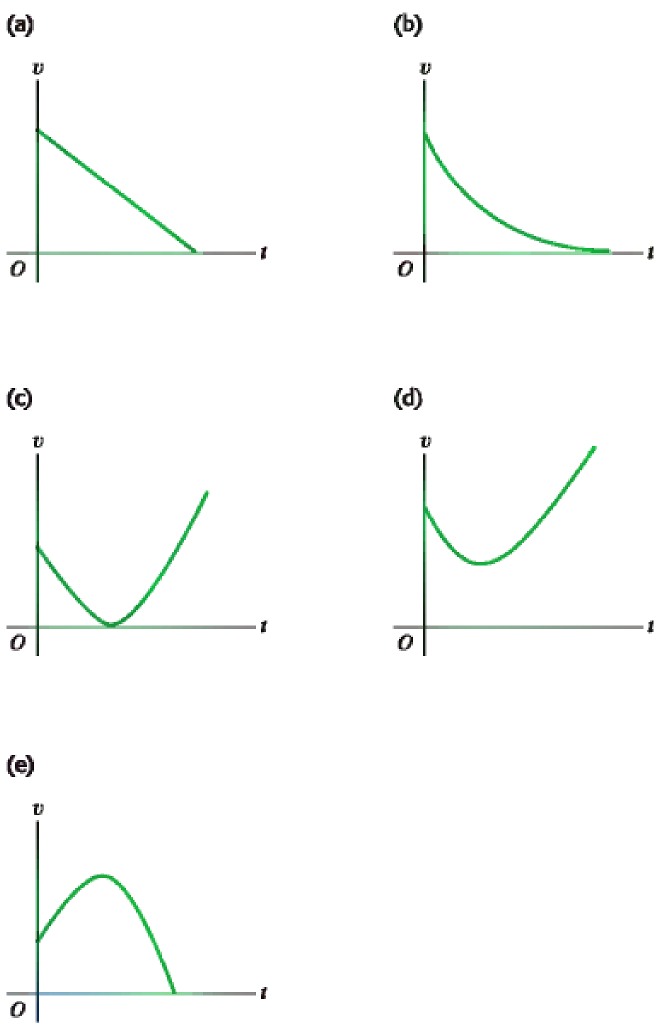
Q3.15. In a rainstorm with a strong wind, what determines the best position in which to hold an umbrella?

Q3.16. You are on the west bank of a river that is flowing north with a speed of 1.2 m/s. Your swimming speed relative to the water is 1.5 m/s, and the river is 60 m wide. What is your path relative to earth that allows you to cross the river in the shortest time? Explain your reasoning.

Q3.17. When you drop an object from a certain height, it takes time T to reach the ground with no air resistance. If you dropped it from three times that height, how long (in terms of T) would it take to reach the ground?

Q3.18. A stone is thrown into the air at an angle above the horizontal and feels negligible air resistance. Which graph in Fig. 3.37 best depicts the stone's *speed* v as a function of time t while it is in the air?

Figure 3.37 Question Q3.18.



Exercises

Section 3.1 Position and Velocity Vectors

3.1. A squirrel has x- and y-coordinates $(1.1\text{ m}, 3.4\text{ m})$ at time $t_1 = 0$ and coordinates $(5.3\text{ m}, -0.5\text{ m})$ at time $t_2 = 3.0\text{ s}$. For this time interval, find (a) the components of the average velocity, and (b) the magnitude and direction of the average velocity.

3.2. A rhinoceros is at the origin of coordinates at time $t_1 = 0$. For the time interval from $t_1 = 0$ to $t_2 = 12.0\text{ s}$, the rhino's average velocity has x-component -3.8 m/s and y-component 4.9 m/s. At time $t_2 = 12.0\text{ s}$, (a) what are the x- and y-coordinates of the rhino? (b) How far is the rhino from the origin?

3.3. A web page designer creates an animation in which a dot on a computer screen has a position of $\vec{r} = [4.0\text{ cm} + (2.5\text{ cm/s}^2)t^2]\hat{\imath} + (5.0\text{ cm/s})t\hat{\jmath}$. (a) Find the magnitude and direction of the dot's average velocity between $t = 0$ and $t = 2.0\text{ s}$. (b) Find the magnitude and direction of the instantaneous velocity at $t = 0$, $t = 1.0\text{ s}$, and $t = 2.0\text{ s}$. (c) Sketch the dot's trajectory from $t = 0$ to $t = 2.0\text{ s}$, and show the velocities calculated in part (b).

3.4. If $\vec{r} = bt^2\hat{\imath} + ct^3\hat{\jmath}$, where b and c are positive constants, when does the velocity vector make an angle of $45.0°$ with the x- and y-axes?

Section 3.2 The Acceleration Vector

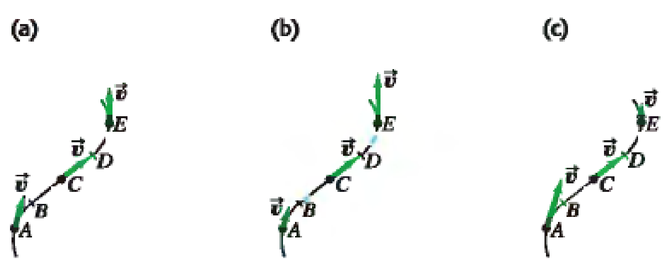
3.5. A jet plane is flying at a constant altitude. At time $t_1 = 0$ it has components of velocity $v_x = 90\text{ m/s}$, $v_y = 110\text{ m/s}$. At time $t_2 = 30.0\text{ s}$ the components are $v_x = -170\text{ m/s}$, $v_y = 40\text{ m/s}$. (a) Sketch the velocity vectors at t_1 and t_2. How do these two vectors differ? For this time interval calculate (b) the components of the average acceleration, and (c) the magnitude and direction of the average acceleration.

3.6. A dog running in an open field has components of velocity $v_x = 2.6\text{ m/s}$ and $v_y = -1.8\text{ m/s}$ at $t_1 = 10.0\text{ s}$. For the time interval from $t_1 = 10.0\text{ s}$ to $t_2 = 20.0\text{ s}$, the average acceleration of the dog has magnitude 0.45 m/s^2 and direction $31.0°$ measured from the $+x$-axis toward the $+y$-axis. At $t_2 = 20.0\text{ s}$, (a) what are the x- and y-components of the dog's velocity? (b) What are the magnitude and direction of the dog's velocity? (c) Sketch the velocity vectors at t_1 and t_2. How do these two vectors differ?

3.7. The coordinates of a bird flying in the xy-plane are given by $x(t) = \alpha t$ and $y(t) = 3.0\text{ m} - \beta t^2$, where $\alpha = 2.4\text{ m/s}$ and $\beta = 1.2\text{ m/s}^2$. (a) Sketch the path of the bird between $t = 0$ and $t = 2.0\text{ s}$. (b) Calculate the velocity and acceleration vectors of the bird as functions of time. (c) Calculate the magnitude and direction of the bird's velocity and acceleration at $t = 2.0\text{ s}$. (d) Sketch the velocity and acceleration vectors at $t = 2.0\text{ s}$. At this instant, is the bird speeding up, is it slowing down, or is its speed instantaneously not changing? Is the bird turning? If so, in what direction?

3.8. A particle moves along a path as shown in Fig. 3.38. Between points B and D, the path is a straight line. Sketch the acceleration vectors at A, C, and E in the cases in which (a) the particle moves with a constant speed; (b) the particle moves with a steadily increasing speed; (c) the particle moves with a steadily decreasing speed.

Figure 3.38 Exercise 3.8.



Section 3.3 Projectile Motion

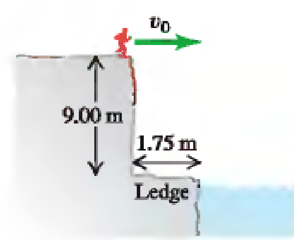
3.9. A physics book slides off a horizontal tabletop with a speed of 1.10 m/s. It strikes the floor in 0.350 s. Ignore air resistance. Find (a) the height of the tabletop above the floor; (b) the horizontal distance from the edge of the table to the point where the book strikes the floor; (c) the horizontal and vertical components of the book's velocity, and the magnitude and direction of its velocity, just before the book reaches the floor. (d) Draw x-t, y-t, v_x-t, and v_y-t graphs for the motion.

3.10. A military helicopter on a training mission is flying horizontally at a speed of 60.0 m/s and accidentally drops a bomb (fortunately not armed) at an elevation of 300 m. You can ignore air resistance. (a) How much time is required for the bomb to reach the earth? (b) How far does it travel horizontally while falling? (c) Find the horizontal and vertical components of its velocity just before it strikes the earth. (d) Draw x-t, y-t, v_x-t, and v_y-t graphs for the bomb's motion. (e) If the velocity of the helicopter remains constant, where is the helicopter when the bomb hits the ground?

3.11. Two crickets, Chirpy and Milada, jump from the top of a vertical cliff. Chirpy just drops and reaches the ground in 3.50 s, while Milada jumps horizontally with an initial speed of 95.0 cm/s. How far from the base of the cliff will Milada hit the ground?

3.12. A daring 510-N swimmer dives off a cliff with a running horizontal leap, as shown in Fig. 3.39. What must her minimum speed be just as she leaves the top of the cliff so that she will miss the ledge at the bottom, which is 1.75 m wide and 9.00 m below the top of the cliff?

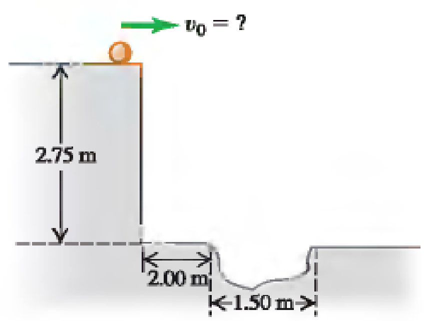
Figure 3.39 Exercise 3.12.



3.13. Leaping the River I. A car comes to a bridge during a storm and finds the bridge washed out. The driver must get to the other side, so he decides to try leaping it with his car. The side of the road the car is on is 21.3 m above the river, while the opposite side is a mere 1.8 m above the river. The river itself is a raging torrent 61.0 m wide. (a) How fast should the car be traveling at the time it leaves the road in order just to clear the river and land safely on the opposite side? (b) What is the speed of the car just before it lands on the other side?

3.14. A small marble rolls horizontally with speed v_0 off the top of a platform 2.75 m tall and feels no appreciable air resistance. On the level ground, 2.00 m from the base of the platform, there is a gaping hole in the ground (Fig. 3.40.) For what range of marble speeds v_0 will the marble land in the hole?

Figure 3.40 Exercise 3.14.



3.15. Inside a starship at rest on the earth, a ball rolls off the top of a horizontal table and lands a distance D from the foot of the table. This starship now lands on the unexplored Planet X. The commander, Captain Curious, rolls the same ball off the same table with the same initial speed as on earth and finds that it lands a distance $2.76D$ from the foot of the table. What is the acceleration due to gravity on Planet X?

3.16. A rookie quarterback throws a football with an initial upward velocity component of 16.0 m/s and a horizontal velocity component of 20.0 m/s. Ignore air resistance. (a) How much time is required for the football to reach the highest point of the trajectory? (b) How high is this point? (c) How much time (after it is thrown) is required for the football to return to its original level? How does this compare with the time calculated in part (a)? (d) How far has the football traveled horizontally during this time? (e) Draw x-t, y-t, v_x-t, and v_y-t graphs for the motion.

3.17. On level ground a shell is fired with an initial velocity of 80.0 m/s at 60.0° above the horizontal and feels no appreciable air resistance. (a) Find the horizontal and vertical components of the shell's initial velocity. (b) How long does it take the shell to reach its highest point? (c) Find its maximum height above the ground. (d) How far from its firing point does the shell land? (e) At its highest point, find the horizontal and vertical components of its acceleration and velocity.

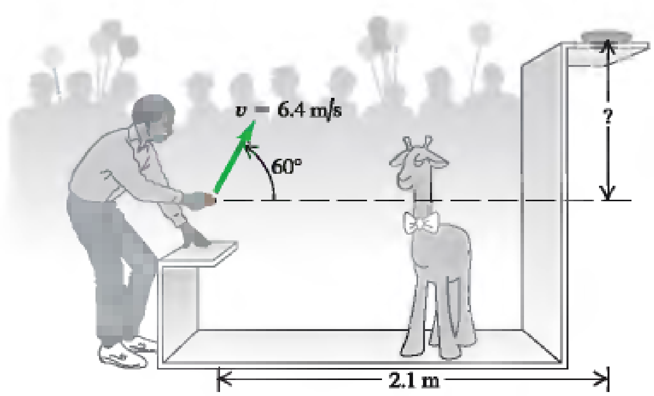
3.18. A pistol that fires a signal flare gives it an initial velocity (muzzle velocity) of 125 m/s at an angle of 55.0° above the horizontal. You can ignore air resistance. Find the flare's maximum height and the distance from its firing point to its landing point if it is fired (a) on the level salt flats of Utah, and (b) over the flat Sea of Tranquility on the Moon, where $g = 1.67\ \text{m/s}^2$.

3.19. A major leaguer hits a baseball so that it leaves the bat at a speed of 30.0 m/s and at an angle of 36.9° above the horizontal. You can ignore air resistance. (a) At what *two* times is the baseball at a height of 10.0 m above the point at which it left the bat? (b) Calculate the horizontal and vertical components of the baseball's velocity at each of the two times calculated in part (a). (c) What are the magnitude and direction of the baseball's velocity when it returns to the level at which it left the bat?

3.20. A shot putter releases the shot some distance above the level ground with a velocity of 12.0 m/s, 51.0° above the horizontal. The shot hits the ground 2.08 s later. You can ignore air resistance. (a) What are the components of the shot's acceleration while in flight? (b) What are the components of the shot's velocity at the beginning and at the end of its trajectory? (c) How far did she throw the shot horizontally? (d) Why does the expression for R in Example 3.8 *not* give the correct answer for part (c)? (e) How high was the shot above the ground when she released it? (f) Draw x-t, y-t, v_x-t, and v_y-t graphs for the motion.

3.21. Win the Prize. In a carnival booth, you win a stuffed giraffe if you toss a quarter into a small dish. The dish is on a shelf above the point where the quarter leaves your hand and is a horizontal distance of 2.1 m from this point (Fig. 3.41). If you toss the coin with a velocity of 6.4 m/s at an angle of 60° above the horizontal, the coin lands in the dish. You can ignore air resistance. (a) What is the height of the shelf above the point where the quarter leaves your hand? (b) What is the vertical component of the velocity of the quarter just before it lands in the dish?

Figure 3.41 Exercise 3.21.



3.22. Suppose the departure angle α_0 in Fig. 3.26 is 42.0° and the distance d is 3.00 m. Where will the dart and monkey meet if the initial speed of the dart is (a) 12.0 m/s? (b) 8.0 m/s? (c) What will happen if the initial speed of the dart is 4.0 m/s? Sketch the trajectory in each case.

3.23. A man stands on the roof of a 15.0-m-tall building and throws a rock with a velocity of magnitude 30.0 m/s at an angle of 33.0° above the horizontal. You can ignore air resistance. Calculate

(a) the maximum height above the roof reached by the rock; (b) the magnitude of the velocity of the rock just before it strikes the ground; and (c) the horizontal range from the base of the building to the point where the rock strikes the ground. (d) Draw x-t, y-t, v_x-t, and v_y-t graphs for the motion.

3.24. Firemen are shooting a stream of water at a burning building using a high-pressure hose that shoots out the water with a speed of 25.0 m/s as it leaves the end of the hose. Once it leaves the hose, the water moves in projectile motion. The firemen adjust the angle of elevation α of the hose until the water takes 3.00 s to reach a building 45.0 m away. You can ignore air resistance; assume that the end of the hose is at ground level. (a) Find the angle of elevation α. (b) Find the speed and acceleration of the water at the highest point in its trajectory. (c) How high above the ground does the water strike the building, and how fast is it moving just before it hits the building?

3.25. A 124-kg balloon carrying a 22-kg basket is descending with a constant downward velocity of 20.0 m/s. A 1.0-kg stone is thrown from the basket with an initial velocity of 15.0 m/s perpendicular to the path of the descending balloon, as measured relative to a person at rest in the basket. The person in the basket sees the stone hit the ground 6.00 s after being thrown. Assume that the balloon continues its downward descent with the same constant speed of 20.0 m/s. (a) How high was the balloon when the rock was thrown out? (b) How high is the balloon when the rock hits the ground? (c) At the instant the rock hits the ground, how far is it from the basket? (d) Just before the rock hits the ground, find its horizontal and vertical velocity components as measured by an observer (i) at rest in the basket and (ii) at rest on the ground.

3.26. A cannon, located 60.0 m from the base of a vertical 25.0-m-tall cliff, shoots a 15-kg shell at 43.0° above the horizontal toward the cliff. (a) What must the minimum muzzle velocity be for the shell to clear the top of the cliff? (b) The ground at the top of the cliff is level, with a constant elevation of 25.0 m above the cannon. Under the conditions of part (a), how far does the shell land past the edge of the cliff?

3.27. An airplane is flying with a velocity of 90.0 m/s at an angle of 23.0° above the horizontal. When the plane is 114 m directly above a dog that is standing on level ground, a suitcase drops out of the luggage compartment. How far from the dog will the suitcase land? You can ignore air resistance.

Section 3.4 Motion in a Circle

3.28. On your first day at work for an appliance manufacturer, you are told to figure out what to do to the period of rotation during a washer spin cycle to triple the centripetal acceleration. You impress your boss by answering immediately. What do you tell her?

3.29. The earth has a radius of 6380 km and turns around once on its axis in 24 h. (a) What is the radial acceleration of an object at the earth's equator? Give your answer in m/s^2 and as a fraction of g. (b) If a_{rad} at the equator is greater than g, objects would fly off the earth's surface and into space. (We will see the reason for this in Chapter 5.) What would the period of the earth's rotation have to be for this to occur?

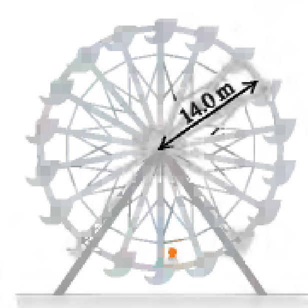
3.30. A model of a helicopter rotor has four blades, each 3.40 m long from the central shaft to the blade tip. The model is rotated in a wind tunnel at 550 rev/min. (a) What is the linear speed of the blade tip, in m/s? (b) What is the radial acceleration of the blade tip expressed as a multiple of the acceleration of gravity, g?

3.31. In a test of a "g-suit," a volunteer is rotated in a horizontal circle of radius 7.0 m. What must the period of rotation be so that the centripetal acceleration has a magnitude of (a) $3.0g$? (b) $10g$?

3.32. The radius of the earth's orbit around the sun (assumed to be circular) is 1.50×10^8 km, and the earth travels around this orbit in 365 days. (a) What is the magnitude of the orbital velocity of the earth, in m/s? (b) What is the radial acceleration of the earth toward the sun, in m/s^2? (c) Repeat parts (a) and (b) for the motion of the planet Mercury (orbit radius $= 5.79 \times 10^7$ km, orbital period $= 88.0$ days).

3.33. A Ferris wheel with radius 14.0 m is turning about a horizontal axis through its center (Fig. 3.42). The linear speed of a passenger on the rim is constant and equal to 7.00 m/s. What are the magnitude and direction of the passenger's acceleration as she passes through (a) the lowest point in her circular motion? (b) The highest point in her circular motion? (c) How much time does it take the Ferris wheel to make one revolution?

Figure 3.42 Exercises 3.33 and 3.34.



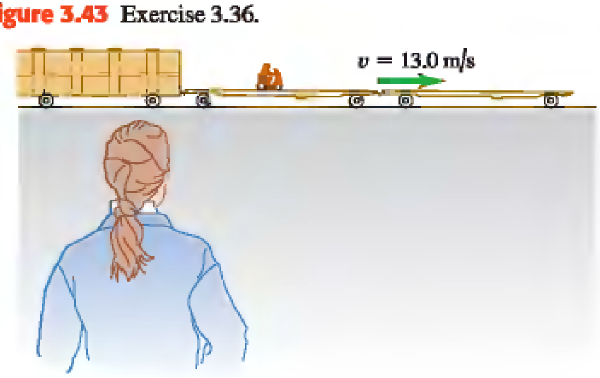
3.34. The Ferris wheel in Fig. 3.42, which rotates counterclockwise, is just starting up. At a given instant, a passenger on the rim of the wheel and passing through the lowest point of his circular motion is moving at 3.00 m/s and is gaining speed at a rate of 0.500 m/s^2. (a) Find the magnitude and the direction of the passenger's acceleration at this instant. (b) Sketch the Ferris wheel and the passenger, showing his velocity and acceleration vectors.

3.35. Hypergravity. At its Ames Research Center, NASA uses its large "20-G" centrifuge to test the effects of very large accelerations ("hypergravity") on test pilots and astronauts. In this device, an arm 8.84 m long rotates about one end in a horizontal plane, and the astronaut is strapped in at the other end. Suppose that he is aligned along the arm with his head at the outermost end. The maximum sustained acceleration to which humans are subjected in this machine is typically $12.5g$. (a) How fast must the astronaut's head be moving to experience this maximum acceleration? (b) What is the *difference* between the acceleration of his head and feet if the astronaut is 2.00 m tall? (c) How fast in rpm (rev/min) is the arm turning to produce the maximum sustained acceleration?

Section 3.5 Relative Velocity

3.36. A railroad flatcar is traveling to the right at a speed of 13.0 m/s relative to an observer standing on the ground. Someone is riding a motor scooter on the flatcar (Fig. 3.43). What is the velocity (magnitude and direction) of the motor scooter relative to the flatcar if its velocity relative to the observer on the ground is (a) 18.0 m/s to the right? (b) 3.0 m/s to the left? (c) zero?

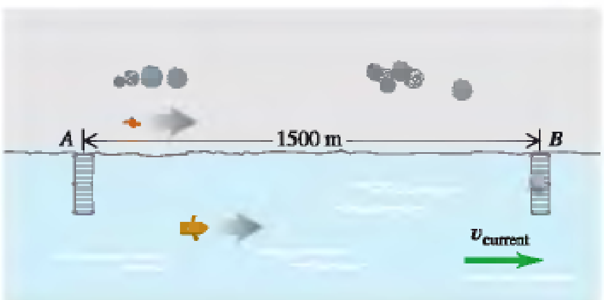
Figure 3.43 Exercise 3.36.



3.37. A "moving sidewalk" in an airport terminal building moves at 1.0 m/s and is 35.0 m long. If a woman steps on at one end and walks at 1.5 m/s relative to the moving sidewalk, how much time does she require to reach the opposite end if she walks (a) in the same direction the sidewalk is moving? (b) In the opposite direction?

3.38. Two piers, A and B, are located on a river: B is 1500 m downstream from A (Fig. 3.44). Two friends must make round trips from pier A to pier B and return. One rows a boat at a constant speed of 4.00 km/h relative to the water; the other walks on the shore at a constant speed of 4.00 km/h. The velocity of the river is 2.80 km/h in the direction from A to B. How much time does it take each person to make the round trip?

Figure 3.44 Exercise 3.38.



3.39. A canoe has a velocity of 0.40 m/s southeast relative to the earth. The canoe is on a river that is flowing 0.50 m/s east relative to the earth. Find the velocity (magnitude and direction) of the canoe relative to the river.

3.40. An airplane pilot wishes to fly due west. A wind of 80.0 km/h (about 50 mi/h) is blowing toward the south. (a) If the airspeed of the plane (its speed in still air) is 320.0 km/h (about 200 mi/h), in which direction should the pilot head? (b) What is the speed of the plane over the ground? Illustrate with a vector diagram.

3.41. Crossing the River I. A river flows due south with a speed of 2.0 m/s. A man steers a motorboat across the river; his velocity relative to the water is 4.2 m/s due east. The river is 800 m wide. (a) What is his velocity (magnitude and direction) relative to the earth? (b) How much time is required to cross the river? (c) How far south of his starting point will he reach the opposite bank?

3.42. Crossing the River II. (a) In which direction should the motorboat in Exercise 3.41 head in order to reach a point on the opposite bank directly east from the starting point? (The boat's speed relative to the water remains 4.2 m/s.) (b) What is the velocity of the boat relative to the earth? (c) How much time is required to cross the river?

3.43. The nose of an ultralight plane is pointed south, and its airspeed indicator shows 35 m/s. The plane is in a 10-m/s wind blowing toward the southwest relative to the earth. (a) In a vector-addition diagram, show the relationship of $\vec{v}_{P/E}$ (the velocity of the plane relative to the earth) to the two given vectors. (b) Letting x be east and y be north, find the components of $\vec{v}_{P/E}$. (c) Find the magnitude and direction of $\vec{v}_{P/E}$.

Problems

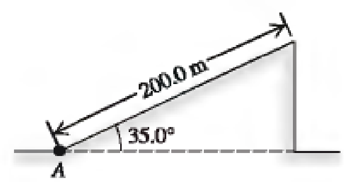
3.44. A faulty model rocket moves in the xy-plane (the positive y-direction is vertically upward). The rocket's acceleration has components $a_x(t) = \alpha t^2$ and $a_y(t) = \beta - \gamma t$, where $\alpha = 2.50\ \text{m/s}^4$, $\beta = 9.00\ \text{m/s}^2$, and $\gamma = 1.40\ \text{m/s}^3$. At $t = 0$ the rocket is at the origin and has velocity $\vec{v}_0 = v_{0x}\hat{\imath} + v_{0y}\hat{\jmath}$ with $v_{0x} = 1.00$ m/s and $v_{0y} = 7.00$ m/s. (a) Calculate the velocity and position vectors as functions of time. (b) What is the maximum height reached by the rocket? (c) Sketch the path of the rocket. (d) What is the horizontal displacement of the rocket when it returns to $y = 0$?

3.45. A rocket is fired at an angle from the top of a tower of height $h_0 = 50.0$ m. Because of the design of the engines, its position coordinates are of the form $x(t) = A + Bt^2$ and $y(t) = C + Dt^3$, where A, B, C, and D are constants. Furthermore, the acceleration of the rocket 1.00 s after firing is $\vec{a} = (4.00\hat{\imath} + 3.00\hat{\jmath})\ \text{m/s}^2$. Take the origin of coordinates to be at the base of the tower. (a) Find the constants A, B, C, and D, including their SI units. (b) At the instant after the rocket is fired, what are its acceleration vector and its velocity? (c) What are the x- and y-components of the rocket's velocity 10.0 s after it is fired, and how fast is it moving? (d) What is the position vector of the rocket 10.0 s after it is fired?

3.46. A bird flies in the xy-plane with a velocity vector given by $\vec{v} = (\alpha - \beta t^2)\hat{\imath} + \gamma t\hat{\jmath}$, with $\alpha = 2.4$ m/s, $\beta = 1.6\ \text{m/s}^3$, and $\gamma = 4.0\ \text{m/s}^2$. The positive y-direction is vertically upward. At $t = 0$ the bird is at the origin. (a) Calculate the position and acceleration vectors of the bird as functions of time. (b) What is the bird's altitude (y-coordinate) as it flies over $x = 0$ for the first time after $t = 0$?

3.47. A test rocket is launched by accelerating it along a 200.0-m incline at 1.25 m/s^2 starting from rest at point A (Figure 3.45.) The incline rises at 35.0° above the horizontal, and at the instant the rocket leaves it, its engines turn off and it is subject only to gravity (air resistance can be ignored). Find (a) the maximum height above the ground that the rocket reaches, and (b) the greatest horizontal range of the rocket beyond point A.

Figure 3.45 Problem 3.47.



3.40. Martian Athletics. In the long jump, an athlete launches herself at an angle above the ground and lands at the same height, trying to travel the greatest horizontal distance. Suppose that on earth she is in the air for time T, reaches a maximum height h, and achieves a horizontal distance D. If she jumped in *exactly* the same way during a competition on Mars, where g_{Mars} is 0.379 of its earth value, find her time in the air, maximum height, and horizontal distance. Express each of these three quantities in terms of its earth value. Air resistance can be neglected on both planets.

3.49. Dynamite! A demolition crew uses dynamite to blow an old building apart. Debris from the explosion flies off in all directions and is later found at distances as far as 50 m from the explosion. Estimate the maximum speed at which debris was blown outward by the explosion. Describe any assumptions that you make.

3.50. Spiraling Up. It is common to see birds of prey rising upward on thermals. The paths they take may be spiral-like. You can model the spiral motion as uniform circular motion combined with a constant upward velocity. Assume a bird completes a circle of radius 8.00 m every 5.00 s and rises vertically at a rate of 3.00 m/s. Determine: (a) the speed of the bird relative to the ground; (b) the bird's acceleration (magnitude and direction); and (c) the angle between the bird's velocity vector and the horizontal.

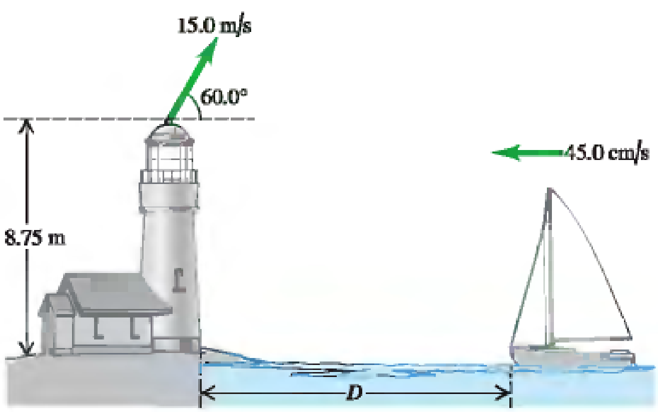
3.51. A jungle veterinarian with a blow-gun loaded with a tranquilizer dart and a sly 1.5-kg monkey are each 25 m above the ground in trees 90 m apart. Just as the hunter shoots horizontally at the monkey, the monkey drops from the tree in a vain attempt to escape being hit. What must the minimum muzzle velocity of the dart have been for the hunter to hit the monkey before it reached the ground?

3.52. A movie stuntwoman drops from a helicopter that is 30.0 m above the ground and moving with a constant velocity whose components are 10.0 m/s upward and 15.0 m/s horizontal and toward the south. You can ignore air resistance. (a) Where on the ground (relative to the position of the helicopter when she drops) should the stuntwoman have placed the foam mats that break her fall? (b) Draw *x-t*, *y-t*, v_x-*t*, and v_y-*t* graphs of her motion.

3.53. In fighting forest fires, airplanes work in support of ground crews by dropping water on the fires. A pilot is practicing by dropping a canister of red dye, hoping to hit a target on the ground below. If the plane is flying in a horizontal path 90.0 m above the ground and with a speed of 64.0 m/s (143 mi/h), at what horizontal distance from the target should the pilot release the canister? Ignore air resistance.

3.54. As a ship is approaching the dock at 45.0 cm/s, an important piece of landing equipment needs to be thrown to it before it can dock. This equipment is thrown at 15.0 m/s at 60.0° above the horizontal from the top of a tower at the edge of the water, 8.75 m above the ship's deck (Fig. 3.46.) For this equipment to land at the front of the ship, at what distance *D* from the dock should the ship be when the equipment is thrown? Air resistance can be neglected.

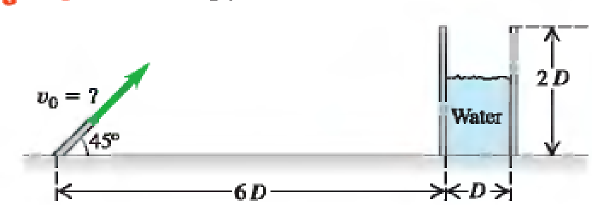
Figure 3.46 Problem 3.54.



3.55. The Longest Home Run. According to the *Guinness Book of World Records*, the longest home run ever measured was hit by Roy "Dizzy" Carlyle in a minor league game. The ball traveled 188 m (618 ft) before landing on the ground outside the ballpark. (a) Assuming the ball's initial velocity was 45° above the horizontal and ignoring air resistance, what did the initial speed of the ball need to be to produce such a home run if the ball was hit at a point 0.9 m (3.0 ft) above ground level? Assume that the ground was perfectly flat. (b) How far would the ball be above a fence 3.0 m (10 ft) high if the fence was 116 m (380 ft) from home plate?

3.56. A water hose is used to fill a large cylindrical storage tank of diameter *D* and height 2*D*. The hose shoots the water at 45° above the horizontal from the same level as the base of the tank and is a distance 6*D* away (Fig. 3.47). For what *range* of launch speeds (v_0) will the water enter the tank? Ignore air resistance, and express your answer in terms of *D* and *g*.

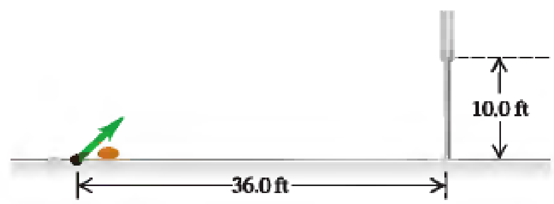
Figure 3.47 Problem 3.56.



3.57. A projectile is being launched from ground level with no air resistance. You want to avoid having it enter a temperature inversion layer in the atmosphere a height *h* above the ground. (a) What is the maximum launch speed you could give this projectile if you shot it straight up? Express your answer in terms of *h* and *g*. (b) Suppose the launcher available shoots projectiles at twice the maximum launch speed you found in part (a). At what maximum angle above the horizontal should you launch the projectile? (c) How far (in terms of *h*) from the launcher does the projectile in part (b) land?

3.58. Kicking a Field Goal. In U.S. football, after a touchdown the team has the opportunity to earn one more point by kicking the ball over the bar between the goal posts. The bar is 10.0 ft above the ground, and the ball is kicked from ground level, 36.0 ft horizontally from the bar (Fig. 3.48). Football regulations are stated in English units, but convert to SI units for this problem. (a) There is a minimum angle above the ground such that if the ball is launched below this angle, it can never clear the bar, no matter how fast it is kicked. What is this angle? (b) If the ball is kicked at 45.0° above the horizontal, what must its initial speed be if it to just clear the bar? Express your answer in m/s and km/h.

Figure 3.48 Problem 3.58.



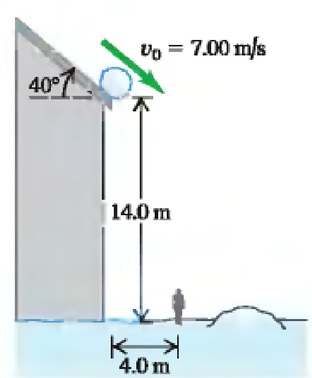
3.59. A projectile is launched with speed v_0 at an angle α_0 above the horizontal. The launch point is a height *h* above the ground. (a) Show that if air resistance is ignored, the horizontal distance that the projectile travels before striking the ground is

$$x = \frac{v_0 \cos\alpha_0}{g}\left(v_0 \sin\alpha_0 + \sqrt{v_0^2 \sin^2\alpha_0 + 2gh}\right)$$

Verify that if the launch point is at ground level so that $h = 0$, this is equal to the horizontal range *R* found in Example 3.8. (b) For the case where $v_0 = 10$ m/s and $h = 5.0$ m, graph *x* as a function of launch angle α_0 for values of α_0 from 0° to 90°. Your graph should show that *x* is zero if $\alpha_0 = 90°$, but *x* is nonzero if $\alpha_0 = 0$; explain why this is so. (c) We saw in Example 3.8 that for a projectile that lands at the same height from which it is launched, the horizontal range is maximum for $\alpha_0 = 45°$. For the case graphed in part (b), is the angle for maximum horizontal distance equal to, less than, or greater than 45°? (This is a general result for the situation where a projectile is launched from a point higher than where it lands.)

3.60. Look Out! A snowball rolls off a barn roof that slopes downward at an angle of 40° (Fig. 3.49). The edge of the roof is 14.0 m above the ground, and the snowball has a speed of 7.00 m/s as it rolls off the roof. Ignore air resistance. (a) How far from the edge of the barn does the snowball strike the ground if it doesn't strike anything else while falling? (b) Draw *x-t*, *y-t*, v_x-*t*, and v_y-*t* graphs for the motion in part (a). (c) A man 1.9 m tall is standing 4.0 m from the edge of the barn. Will he be hit by the snowball?

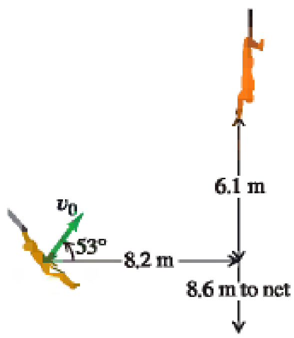
Figure 3.49 Problem 3.60.



3.61. (a) Prove that a projectile launched at angle α_0 has the same horizontal range as one launched with the same speed at angle $(90° - \alpha_0)$. (b) A frog jumps at a speed of 2.2 m/s and lands 25 cm from its starting point. At which angles above the horizontal could it have jumped?

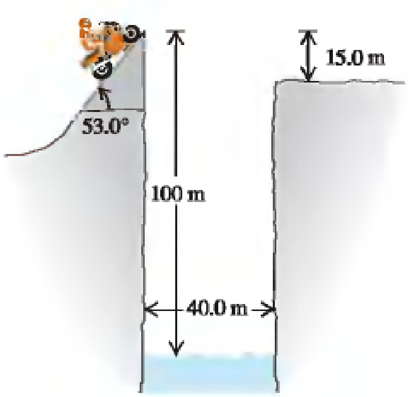
3.62. On the Flying Trapeze. A new circus act is called the Texas Tumblers. Lovely Mary Belle swings from a trapeze, projects herself at an angle of 53°, and is supposed to be caught by Joe Bob, whose hands are 6.1 m above and 8.2 m horizontally from her launch point (Fig. 3.50). You can ignore air resistance. (a) What initial speed v_0 must Mary Belle have just to reach Joe Bob? (b) For the initial speed calculated in part (a), what are the magnitude and direction of her velocity when Mary Belle reaches Joe Bob? (c) Assuming that Mary Belle has the initial speed calculated in part (a), draw x-t, y-t, v_x-t, and v_y-t graphs showing the motion of both tumblers. Your graphs should show the motion up until the point where Mary Belle reaches Joe Bob. (d) The night of their debut performance, Joe Bob misses her completely as she flies past. How far horizontally does Mary Belle travel, from her initial launch point, before landing in the safety net 8.6 m below her starting point?

Figure 3.50 Problem 3.62.



3.63. Leaping the River II. A physics professor did daredevil stunts in his spare time. His last stunt was an attempt to jump across a river on a motorcycle (Fig. 3.51). The takeoff ramp was inclined at 53.0°, the river was 40.0 m wide, and the far bank was 15.0 m lower than the top of the ramp. The river itself was 100 m below the ramp. You can ignore air resistance. (a) What should his speed have been at the top of the ramp to have just made it to the edge of the far bank? (b) If his speed was only half the value found in (a), where did he land?

Figure 3.51 Problem 3.63.



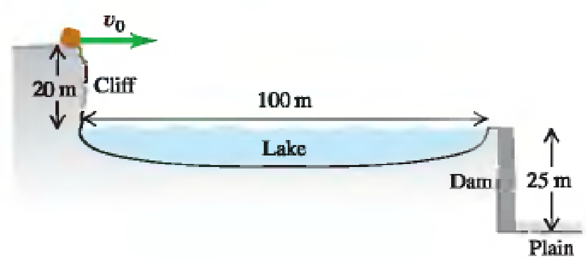
3.64. A rock is thrown from the roof of a building with a velocity v_0 at an angle of α_0 from the horizontal. The building has height h. You can ignore air resistance. Calculate the magnitude of the velocity of the rock just before it strikes the ground, and show that this speed is independent of α_0.

3.65. A 5500-kg cart carrying a vertical rocket launcher moves to the right at a constant speed of 30.0 m/s along a horizontal track. It launches a 45.0-kg rocket vertically upward with an initial speed of 40.0 m/s relative to the cart. (a) How high will the rocket go? (b) Where, relative to the cart, will the rocket land? (c) How far does the cart move while the rocket is in the air? (d) At what angle, relative to the horizontal, is the rocket traveling just as it leaves the cart, as measured by an observer at rest on the ground? (e) Sketch the rocket's trajectory as seen by an observer (i) stationary on the cart and (ii) stationary on the ground.

3.66. A 2.7-kg ball is thrown upward with an initial speed of 20.0 m/s from the edge of a 45.0-m-high cliff. At the instant the ball is thrown, a woman starts running away from the base of the cliff with a constant speed of 6.00 m/s. The woman runs in a straight line on level ground, and air resistance acting on the ball can be ignored. (a) At what angle above the horizontal should the ball be thrown so that the runner will catch it just before it hits the ground, and how far does the woman run before she catches the ball? (b) Carefully sketch the ball's trajectory as viewed by (i) a person at rest on the ground and (ii) the runner.

3.67. A 76.0-kg boulder is rolling horizontally at the top of a vertical cliff that is 20 m above the surface of a lake, as shown in Fig. 3.52. The top of the vertical face of a dam is located 100 m from the foot of the cliff, with the top of the dam level with the surface of the water in the lake. A level plain is 25 m below the top of the dam. (a) What must be the minimum speed of the rock just as it leaves the cliff so it will travel to the plain without striking the dam? (b) How far from the foot of the dam does the rock hit the plain?

Figure 3.52 Problem 3.67.



3.68. Tossing Your Lunch. Henrietta is going off to her physics class, jogging down the sidewalk at 3.05 m/s. Her husband Bruce suddenly realizes that she left in such a hurry that she forgot her lunch of bagels, so he runs to the window of their apartment, which is 43.9 m above the street level and directly above the sidewalk, to throw them to her. Bruce throws them horizontally 9.00 s after Henrietta has passed below the window, and she catches them on the run. You can ignore air resistance. (a) With what initial speed must Bruce throw the bagels so Henrietta can catch them just before they hit the ground? (b) Where is Henrietta when she catches the bagels?

3.69. Two tanks are engaged in a training exercise on level ground. The first tank fires a paint-filled training round with a muzzle speed of 250 m/s at 10.0° above the horizontal while advancing toward the second tank with a speed of 15.0 m/s relative to the ground. The second tank is retreating at 35.0 m/s relative to the ground, but is hit by the shell. You can ignore air resistance and assume the shell hits at the same height above ground from which it was fired. Find the distance between the tanks (a) when the round was first fired and (b) at the time of impact.

3.70. Bang! A student sits atop a platform a distance h above the ground. He throws a large firecracker horizontally with a speed v. However, a wind blowing parallel to the ground gives the firecracker a constant horizontal acceleration with magnitude a. This results in the firecracker reaching the ground directly under the student. Determine the height h in terms of v, a, and g. You can ignore the effect of air resistance on the vertical motion.

3.71. A rocket is launched vertically from rest with a constant upward acceleration of 1.75 m/s^2. Suddenly 22.0 s after launch, an unneeded fuel tank is jettisoned by shooting it away from the rocket. A crew member riding in the rocket measures that the initial speed of the tank is 25.0 m/s and that it moves perpendicular to the rocket's path. The fuel tank feels no appreciable air resistance and feels only the force of gravity once it leaves the rocket. (a) How fast is the rocket moving at the instant the fuel tank is jettisoned? (b) What are the horizontal and vertical components of the fuel tank's velocity just as it is jettisoned as measured by (i) a crew member in the rocket and (ii) a technician standing on the ground? (c) At what angle with respect to the horizontal does the jettisoned fuel tank initially move, as measured by (i) a crew member in the rocket and (ii) a technician standing on the ground? (d) What maximum height above the launch pad does the jettisoned tank reach?

3.72. When it is 145 m above the ground, a rocket traveling vertically upward at a constant 8.50 m/s relative to the ground launches a secondary rocket at a speed of 12.0 m/s at an angle of 53.0° above the horizontal, both quantities being measured by an astronaut sitting in the rocket. Air resistance is too small to worry about. (a) Just as the secondary rocket is launched, what are the horizontal and vertical components of its velocity relative to (i) the astronaut sitting in the rocket and (ii) Mission Control on the ground? (b) Find the initial speed and launch angle of the secondary rocket as measured by Mission Control. (c) What maximum height above the ground does the secondary rocket reach?

3.73. In a Fourth of July celebration, a firecracker is launched from ground level with an initial velocity of 25.0 m/s at 30.0° from the *vertical*. At its maximum height it explodes in a starburst into many fragments, two of which travel forward initially at 20.0 m/s at ±53.0° with respect to the horizontal, both quantities measured *relative to the original firecracker just before it exploded.* With what angles with respect to the horizontal do the two fragments initially move right after the explosion, as measured by a spectator standing on the ground?

3.74. In an action-adventure film, the hero is supposed to throw a grenade from his car, which is going 90.0 km/h, to his enemy's car, which is going 110 km/h. The enemy's car is 15.8 m in front of the hero's when he lets go of the grenade. If the hero throws the grenade so its initial velocity relative to him is at an angle of 45° above the horizontal, what should the magnitude of the initial velocity be? The cars are both traveling in the same direction on a level road. You can ignore air resistance. Find the magnitude of the velocity both relative to the hero and relative to the earth.

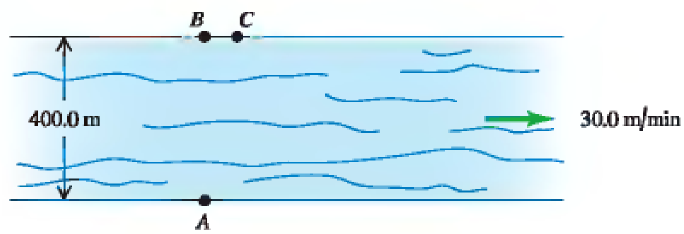
3.75. A rock tied to a rope moves in the *xy*-plane. Its coordinates are given as functions of time by

$$x(t) = R\cos\omega t \qquad y(t) = R\sin\omega t$$

where R and ω are constants. (a) Show that the rock's distance from the origin is constant and equal to R—that is, that its path is a circle of radius R. (b) Show that at every point the rock's velocity is perpendicular to its position vector. (c) Show that the rock's acceleration is always opposite in direction to its position vector and has magnitude $\omega^2 R$. (d) Show that the magnitude of the rock's velocity is constant and equal to ωR. (e) Combine the results of parts (c) and (d) to show that the rock's acceleration has constant magnitude v^2/R.

3.76. A 400.0-m-wide river flows from west to east at 30.0 m/min. Your boat moves at 100.0 m/min relative to the water no matter which direction you point it. To cross this river, you start from a dock at point A on the south bank. There is a boat landing directly opposite at point B on the north bank, and also one at point C, 75.0 m downstream from B (Fig. 3.53). (a) Where on the north shore will you land if you point your boat perpendicular to the water current, and what distance will you have traveled? (b) If you initially aim your boat directly toward point C and do not change that bearing relative to the shore, where on the north shore will you land? (c) To reach point C: (i) at what bearing must you aim your boat, (ii) how long will it take to cross the river, (iii) what distance do you travel, and (iv) and what is the speed of your boat as measured by an observer standing on the river bank?

Figure 3.53 Problem 3.76.



3.77. Cycloid. A particle moves in the *xy*-plane. Its coordinates are given as functions of time by

$$x(t) = R(\omega t - \sin\omega t) \qquad y(t) = R(1 - \cos\omega t)$$

where R and ω are constants. (a) Sketch the trajectory of the particle. (This is the trajectory of a point on the rim of a wheel that is rolling at a constant speed on a horizontal surface. The curve traced out by such a point as it moves through space is called a *cycloid*.) (b) Determine the velocity components and the acceleration components of the particle at any time t. (c) At which times is the particle momentarily at rest? What are the coordinates of the particle at these times? What are the magnitude and direction of the acceleration at these times? (d) Does the magnitude of the acceleration depend on time? Compare to uniform circular motion.

3.78. A projectile is fired from point A at an angle above the horizontal. At its highest point, after having traveled a horizontal distance D from its launch point, it suddenly explodes into two identical fragments that travel horizontally with equal but opposite velocities as measured *relative to the projectile just before it exploded.* If one fragment lands back at point A, how far from A (in terms of D) does the other fragment land?

3.79. Centrifuge on Mercury. A laboratory centrifuge on earth makes n rpm (rev/min) and produces an acceleration of 5.00g at its outer end. (a) What is the acceleration (in g's) at a point halfway out to the end? (b) This centrifuge is now used in a space capsule on the planet Mercury, where g_{Mercury} is 0.378 what it is on earth. How many rpm (in terms of n) should it make to produce $5g_{\text{Mercury}}$ at its outer end?

3.80. Raindrops. When a train's velocity is 12.0 m/s eastward, raindrops that are falling vertically with respect to the earth make traces that are inclined 30.0° to the vertical on the windows of the train. (a) What is the horizontal component of a drop's velocity with respect to the earth? With respect to the train? (b) What is the magnitude of the velocity of the raindrop with respect to the earth? With respect to the train?

3.81. An airplane pilot sets a compass course due west and maintains an airspeed of 220 km/h. After flying for 0.500 h, she finds herself over a town 120 km west and 20 km south of her starting point. (a) Find the wind velocity (magnitude and direction). (b) If the wind velocity is 40 km/h due south, in what direction should the pilot set her course to travel due west? Use the same airspeed of 220 km/h.

3.62. An elevator is moving upward at a constant speed of 2.50 m/s. A bolt in the elevator ceiling 3.00 m above the elevator floor works loose and falls. (a) How long does it take for the bolt to fall to the elevator floor? What is the speed of the bolt just as it hits the elevator floor (b) according to an observer in the elevator? (c) According to an observer standing on one of the floor landings of the building? (d) According to the observer in part (c), what distance did the bolt travel between the ceiling and the floor of the elevator?

3.83. Suppose the elevator in Problem 3.82 starts from rest and maintains a constant upward acceleration of 4.00 m/s^2, and the bolt falls out the instant the elevator begins to move. (a) How long does it take for the bolt to reach the floor of the elevator? (b) Just as it reaches the floor, how fast is the bolt moving according to an observer (i) in the elevator? (ii) Standing on the floor landings of the building? (c) According to each observer in part (b), how far has the bolt traveled between the ceiling and floor of the elevator?

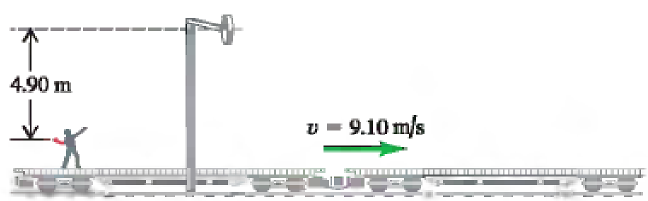
3.84. City A lies directly west of city B. When there is no wind, an airliner makes the 5550-km round-trip flight between them in 6.60 h of flying time while traveling at the same speed in both directions. When a strong, steady 225-km/h wind is blowing from west to east and the airliner has the same airspeed as before, how long will the trip take?

3.85. In a World Cup soccer match, Juan is running due north toward the goal with a speed of 8.00 m/s relative to the ground. A teammate passes the ball to him. The ball has a speed of 12.0 m/s and is moving in a direction of 37.0° east of north, relative to the ground. What are the magnitude and direction of the ball's velocity relative to Juan?

Challenge Problems

3.86. A man is riding on a flatcar traveling at a constant speed of 9.10 m/s (Fig. 3.54). He wishes to throw a ball through a stationary hoop 4.90 m above the height of his hands in such a manner that the ball will move horizontally as it passes through the hoop. He throws the ball with a speed of 10.8 m/s with respect to himself. (a) What must the vertical component of the initial velocity of the ball be? (b) How many seconds after he releases the ball will it pass through the hoop? (c) At what horizontal distance in front of the hoop must he release the ball? (d) When the ball leaves the man's hands, what is the direction of its velocity relative to the frame of reference of the flatcar? Relative to the frame of reference of an observer standing on the ground?

Figure 3.54 Challenge Problem 3.86.

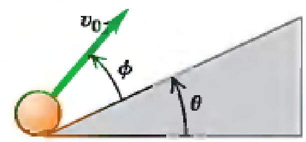


3.87. A shotgun fires a large number of pellets upward, with some pellets traveling very nearly vertically and others as much as 1.0° from the vertical. Assume that the initial speed of the pellets is uniformly 150 m/s, and ignore air resistance. (a) Within what radius from the point of firing will the pellets land? (b) If there are 1000 pellets, and they fall in a uniform distribution over a circle with the radius calculated in part (a), what is the probability that at least one pellet will fall on the head of the person who fires the shotgun? Assume that his head has a radius of 10 cm. (c) Air resistance, in fact, has several effects. It slows down the rising pellets, decreases their horizontal component of velocity, and limits the speed with which they fall. Which of these effects will tend to make the radius larger than calculated in part (a), and which will tend to make it smaller? What do you think the overall effect of air resistance will be? (The effect of air resistance on a velocity component increases as the magnitude of the component increases.)

3.80. A projectile is thrown from a point P. It moves in such a way that its distance from P is always increasing. Find the maximum angle above the horizontal with which the projectile could have been thrown. You can ignore air resistance.

3.89. Projectile Motion on an Incline I. A baseball is given an initial velocity with magnitude v_0 at an angle ϕ above the surface of an incline, which is in turn inclined at an angle θ above the horizontal (Fig. 3.55) (a) Calculate the distance, measured along the incline, from the launch point to where the baseball strikes the incline. Your answer will be in terms of v_0, g, θ, and ϕ. (b) What angle ϕ gives the maximum range, measured along the incline? (*Note:* You might be interested in the three different methods of solution presented by I. R. Lapidus in *Amer. Jour. of Phys.*, Vol. 51 (1983), pp. 806 and 847. See also H. A. Buckmaster in *Amer. Jour. of Phys.*, Vol. 53 (1985), pp. 638–641, for a thorough study of this and some similar problems.)

Figure 3.55 Challenge Problem 3.89.



3.90. Projectile Motion on an Incline II. Refer to Challenge Problem 3.89. (a) An archer on ground that has a constant upward slope of 30.0° aims at a target 60.0 m farther up the incline. The arrow in the bow and the bull's-eye at the center of the target are each 1.50 m above the ground. The initial velocity of the arrow just after it leaves the bow has magnitude 32.0 m/s. At what angle above the *horizontal* should the archer aim to hit the bull's-eye? If there are two such angles, calculate the smaller of the two. You might have to solve the equation for the angle by iteration—that is, by trial and error. How does the angle compare to that required when the ground is level, with 0 slope? (b) Repeat the above for ground that has a constant *downward* slope of 30.0°.

3.91. For no apparent reason, a poodle is running at a constant speed of $v = 5.00$ m/s in a circle with radius $R = 2.50$ m. Let $\vec{v}_1$ be the velocity vector at time t_1, and let $\vec{v}_2$ be the velocity vector at time t_2. Consider $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$ and $\Delta t = t_2 - t_1$. Recall that $\vec{a}_{av} = \Delta\vec{v}/\Delta t$. For $\Delta t = 0.5$ s, 0.1 s, and 0.05 s, calculate the magnitude (to four significant figures) and direction (relative to $\vec{v}_1$) of the average acceleration $\vec{a}_{av}$. Compare your results to the general expression for the instantaneous acceleration $\vec{a}$ for uniform circular motion that is derived in the text.

3.92. A rocket designed to place small payloads into orbit is carried to an altitude of 12.0 km above sea level by a converted airliner. When the airliner is flying in a straight line at a constant speed of 850 km/h, the rocket is dropped. After the drop, the airliner maintains the same altitude and speed and continues to fly in a straight line. The rocket falls for a brief time, after which its rocket motor turns on. Once its rocket motor is on, the combined effects of thrust and gravity give the rocket a constant acceleration of magnitude 3.00g directed at an angle of 30.0° above the horizontal. For reasons of safety, the rocket should be at least 1.00 km in front of the airliner when it climbs through the airliner's altitude. Your job is to determine the minimum time that the rocket must fall before its engine starts. You can ignore air resistance.

Your answer should include (i) a diagram showing the flight paths of both the rocket and the airliner, labeled at several points with vectors for their velocities and accelerations; (ii) an x-t graph showing the motions of both the rocket and the airliner; and (iii) a y-t graph showing the motions of both the rocket and the airliner. In the diagram and the graphs, indicate when the rocket is dropped, when the rocket motor turns on, and when the rocket climbs through the altitude of the airliner.

3.93. Two students are canoeing on a river. While heading upstream, they accidentally drop an empty bottle overboard. They then continue paddling for 60 minutes, reaching a point 2.0 km farther upstream. At this point they realize that the bottle is missing and, driven by ecological awareness, they turn around and head downstream. They catch up with and retrieve the bottle (which has been moving along with the current) 5.0 km downstream from the turn-around point. (a) Assuming a constant paddling effort throughout, how fast is the river flowing? (b) What would the canoe speed in a still lake be for the same paddling effort?

NEWTON'S LAWS OF MOTION

4

?The standing child is pushing the child seated on the swing. Is the seated child pushing back? If so, is he pushing with the same amount of force or a different amount?

LEARNING GOALS

By studying this chapter, you will learn:

- What the concept of force means in physics, and why forces are vectors.
- The significance of the net force on an object, and what happens when the net force is zero.
- The relationship among the net force on an object, the object's mass, and its acceleration.
- How the forces that two bodies exert on each other are related.

We've seen in the last two chapters how to describe motion in one, two, or three dimensions. But what are the underlying *causes* of motion? For example, how can a tugboat push a cruise ship that's much heavier than the tug? Why is it harder to control a car on wet ice than on dry concrete? The answers to these and similar questions take us into the subject of **dynamics,** the relationship of motion to the forces that cause it. In the two preceding chapters we studied *kinematics,* the language for *describing* motion. Now we are ready to think about what makes bodies move the way they do.

In this chapter we will use two new concepts, *force* and *mass,* to analyze the principles of dynamics. These principles can be wrapped up in just three statements that were clearly stated for the first time by Sir Isaac Newton (1642–1727), who published them in 1687 in his *Philosophiae Naturalis Principia Mathematica* ("Mathematical Principles of Natural Philosophy"). These three statements are called **Newton's laws of motion.** The first law states that when the net force on a body is zero, its motion doesn't change. The second law relates force to acceleration when the net force is *not* zero. The third law is a relationship between the forces that two interacting bodies exert on each other.

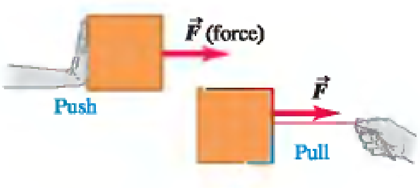
Newton's laws are not the product of mathematical derivations, but rather a synthesis of what physicists have learned from a multitude of *experiments* about how objects move. (Newton used the ideas and observations of many scientists before him, including Copernicus, Brahe, Kepler, and especially Galileo Galilei, who died the same year Newton was born.) These laws are truly fundamental, for they cannot be deduced or proved from other principles. Newton's laws are the foundation of **classical mechanics** (also called **Newtonian mechanics**); using them we can understand most familiar kinds of motion. Newton's laws need modification only for situations involving extremely high speeds (near the speed of light) or very small sizes (such as within the atom).

Newton's laws are very simple to state, yet many students find these laws difficult to grasp and to work with. The reason is that before studying physics, you've spent years walking, throwing balls, pushing boxes, and doing dozens of things that involve motion. Along the way, you've developed a set of "common sense" ideas about motion and its causes. But many of these "common sense" ideas don't stand up to logical analysis. A big part of the job of this chapter—and of the rest of our study of physics—is helping you to recognize how "common sense" ideas can sometimes lead you astray, and how to adjust your understanding of the physical world to make it consistent with what experiments tell us.

4.1 Force and Interactions

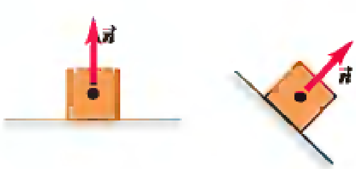
4.1 Some properties of forces.

- A force is a push or a pull.
- A force is an interaction between two objects or between an object and its environment.
- A force is a vector quantity, with magnitude and direction.

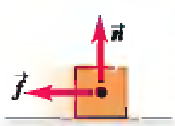


4.2 Four common types of forces.

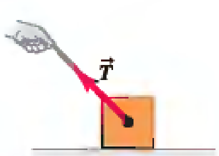
(a) Normal force $\vec{n}$: When an object rests or pushes on a surface, the surface exerts a push on it that is directed perpendicular to the surface.



(b) Friction force $\vec{f}$: In addition to the normal force, a surface may exert a frictional force on an object, directed parallel to the surface.



(c) Tension force $\vec{T}$: A pulling force exerted on an object by a rope, cord, etc.



(d) Weight $\vec{w}$: The pull of gravity on an object is a long-range force (a force that acts over a distance).

In everyday language, a **force** is a push or a pull. A better definition is that a force is an *interaction* between two bodies or between a body and its environment (Fig. 4.1). That's why we always refer to the force that one body *exerts* on a second body. When you push on a car that is stuck in the snow, you exert a force on the car; a steel cable exerts a force on the beam it is hoisting at a construction site; and so on. As Fig. 4.1 shows, force is a *vector* quantity; you can push or pull a body in different directions.

When a force involves direct contact between two bodies, such as a push or pull that you exert on an object with your hand, we call it a **contact force.** Figures 4.2a, 4.2b, and 4.2c show three common types of contact forces. The **normal force** (Fig. 4.2a) is exerted on an object by any surface with which it is contact. The adjective *normal* means that the force always acts perpendicular to the surface of contact, no matter what the angle of that surface. By contrast, the **friction force** (Fig. 4.2b) exerted on an object by a surface acts *parallel* to the surface, in the direction that opposes sliding. The pulling force exerted by a stretched rope or cord on an object to which it's attached is called a **tension force** (Fig. 4.2c). When you tug on your dog's leash, the force that pulls on her collar is a tension force.

In addition to contact forces, there are **long-range forces** that act even when the bodies are separated by empty space. The force between two magnets is an example of a long-range force, as is the force of gravity (Fig. 4.2d); the earth pulls a dropped object toward it even though there is no direct contact between the object and the earth. The gravitational force that the earth exerts on your body is called your **weight.**

To describe a force vector $\vec{F}$, we need to describe the *direction* in which it acts as well as its *magnitude,* the quantity that describes "how much" or "how hard" the force pushes or pulls. The SI unit of the magnitude of force is the *newton,* abbreviated N. (We'll give a precise definition of the newton in Section 4.3.) Table 4.1 lists some typical force magnitudes.

Table 4.1 Typical Force Magnitudes

Sun's gravitational force on the earth	3.5×10^{22} N
Thrust of a space shuttle during launch	3.1×10^{7} N
Weight of a large blue whale	1.9×10^{6} N
Maximum pulling force of a locomotive	8.9×10^{5} N
Weight of a 250-lb linebacker	1.1×10^{3} N
Weight of a medium apple	1 N
Weight of smallest insect eggs	2×10^{-6} N
Electric attraction between the proton and the electron in a hydrogen atom	8.2×10^{-8} N
Weight of a very small bacterium	1×10^{-18} N
Weight of a hydrogen atom	1.6×10^{-26} N
Weight of an electron	8.9×10^{-30} N
Gravitational attraction between the proton and the electron in a hydrogen atom	3.6×10^{-47} N

A common instrument for measuring force magnitudes is the *spring balance*. It consists of a coil spring enclosed in a case with a pointer attached to one end. When forces are applied to the ends of the spring, it stretches by an amount that depends on the force. We can make a scale for the pointer by using a number of identical bodies with weights of exactly 1 N each. When one, two, or more of these are suspended simultaneously from the balance, the total force stretching the spring is 1 N, 2 N, and so on, and we can label the corresponding positions of the pointer 1 N, 2 N, and so on. Then we cau use this instrument to measure the magnitude of an unknown force. We can also make a similar instrument that measures pushes instead of pulls.

Figure 4.3 shows a spring balance being used to measure a pull or push that we apply to a box. In each case we draw a vector to represent the applied force. The labels indicate the magnitude and direction of the force. The length of the vector also shows the magnitude; the longer the vector, the greater the force magnitude.

Superposition of Forces

When you throw a ball, there are at least two forces acting on it: the push of your hand and the downward pull of gravity. Experiment shows that when two forces $\vec{F}_1$ and $\vec{F}_2$ act at the same time at a point A of a body (Fig. 4.4), the effect on the body's motion is the same as if a single force $\vec{R}$ were acting equal to the *vector sum* of the original forces: $\vec{R} = \vec{F}_1 + \vec{F}_2$. More generally, *any number of forces applied at a point on a body have the same effect as a single force equal to the vector sum of the forces.* This important principle is called **superposition of forces.**

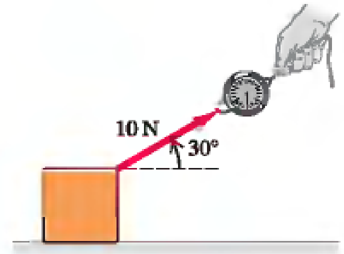
The experimental discovery that forces combine according to vector addition is of the utmost importance, and we will use this fact throughout our study of physics. It allows us to replace a force by its component vectors, as we did with displacements in Section 1.8. For example, in Fig. 4.5a, force $\vec{F}$ acts on a body at point O. The component vectors of $\vec{F}$ in the directions Ox and Oy are $\vec{F}_x$ and $\vec{F}_y$. When $\vec{F}_x$ and $\vec{F}_y$ are applied simultaneously, as in Fig. 4.5b, the effect is exactly the same as the effect of the original force $\vec{F}$. Hence *any force can be replaced by its component vectors, acting at the same point.*

It's frequently more convenient to describe a force $\vec{F}$ in terms of its x- and y-components F_x and F_y rather than by its component vectors (recall from Section 1.8 that *component vectors* are vectors, but *components* are just numbers). For the case shown in Fig. 4.5, both F_x and F_y are positive; for other orientations of the force $\vec{F}$, either F_x or F_y can be negative or zero.

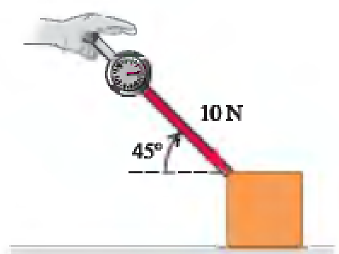
There is no law that says our coordinate axes have to be vertical and horizontal. Figure 4.6 shows a crate being pulled up a ramp by a force $\vec{F}$, represented by its components F_x and F_y parallel and perpendicular to the sloping surface of the ramp.

4.3 Using a vector arrow to denote the force that we exert when (a) pulling a block with a string or (b) pushing a block with a stick.

(a) A 10-N pull directed 30° above the horizontal

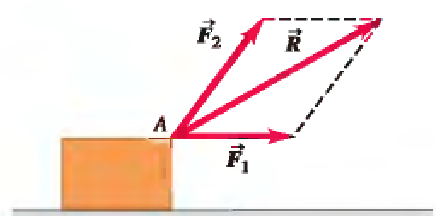


(b) A 10-N push directed 45° below the horizontal



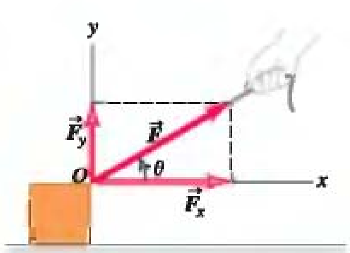
4.4 Superposition of forces.

Two forces $\vec{F}_1$ and $\vec{F}_2$ acting on a point A have the same effect as a single force $\vec{R}$ equal to their vector sum.

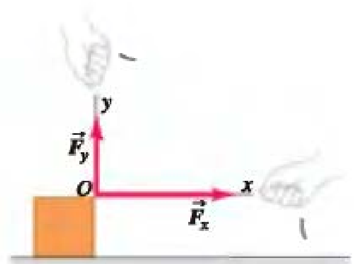


4.5 The force $\vec{F}$, which acts at an angle θ from the x-axis, may be replaced by its rectangular component vectors $\vec{F}_x$ and $\vec{F}_y$.

(a) Component vectors: $\vec{F}_x$ and $\vec{F}_y$
Components: $F_x = F\cos\theta$ and $F_y = F\sin\theta$

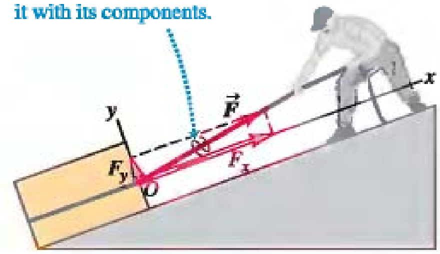


(b) Component vectors $\vec{F}_x$ and $\vec{F}_y$ together have the same effect as original force $\vec{F}$.

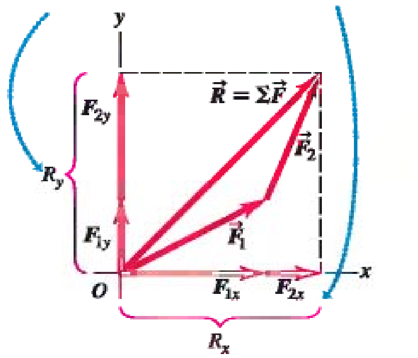


4.6 F_x and F_y are the components of $\vec{F}$ parallel and perpendicular to the sloping surface of the inclined plane.

We cross out a vector when we replace it with its components.



4.7 Finding the components of the vector sum (resultant) $\vec{R}$ of two forces $\vec{F}_1$ and $\vec{F}_2$.



CAUTION **Using a wiggly line in force diagrams** In Fig. 4.6 we draw a wiggly line through the force vector $\vec{F}$ to show that we have replaced it by its x- and y-components. Otherwise, the diagram would include the same force twice. We will draw such a wiggly line in any force diagram where a force is replaced by its components. Look for this wiggly line in other figures in this and subsequent chapters.

We will often need to find the vector sum (resultant) of *all* the forces acting on a body. We call this the **net force** acting on the body. We will use the Greek letter Σ (capital sigma, equivalent to the Roman S) as a shorthand notation for a sum. If the forces are labeled $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$, and so on, we abbreviate the sum as

$$\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots = \sum \vec{F} \tag{4.1}$$

We read $\sum\vec{F}$ as "the vector sum of the forces" or "the net force." The component version of Eq. (4.1) is the pair of component equations

$$R_x = \sum F_x \qquad R_y = \sum F_y \tag{4.2}$$

Here $\sum F_x$ is the sum of the x-components and $\sum F_y$ is the sum of the y-components (Fig. 4.7). Each component may be positive or negative, so be careful with signs when you evaluate the sums in Eq. (4.2).

Once we have R_x and R_y, we can find the magnitude and direction of the net force $\vec{R} = \sum\vec{F}$ acting on the body. The magnitude is

$$R = \sqrt{R_x^2 + R_y^2}$$

and the angle θ between $\vec{R}$ and the $+x$-axis can be found from the relation $\tan\theta = R_y/R_x$. The components R_x and R_y may be positive, negative, or zero, and the angle θ may be in any of the four quadrants.

In three-dimensional problems, forces may also have z-components; then we add the equation $R_z = \sum F_z$ to Eq. (4.2). The magnitude of the net force is then

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

Example 4.1 Superposition of forces

Three professional wrestlers are fighting over the same champion's belt. As viewed from above, they apply the three horizontal forces to the belt that are shown in Fig. 4.8a. The magnitudes of the three forces are $F_1 = 250\text{ N}$, $F_2 = 50\text{ N}$, and $F_3 = 120\text{ N}$. Find the x- and y-components of the net force on the belt, and find the magnitude and direction of the net force.

SOLUTION

IDENTIFY: This example is just a problem in vector addition. The only new feature is that the vectors represent forces.

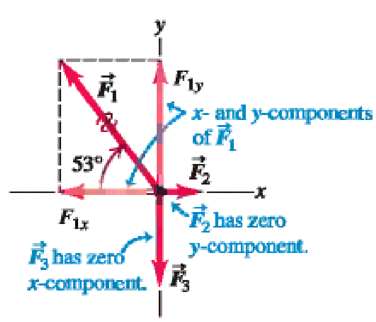
SET UP: We need to find the x- and y-components of the net force $\vec{R}$, so we'll use the component method of vector addition expressed by Eq. (4.2). Once we have the components of $\vec{R}$, we can find its magnitude and direction.

EXECUTE: From Fig. 4.8a, the angles between the three forces $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$ and the $+x$-axis are $\theta_1 = 180° - 53° = 127°$, $\theta_2 = 0°$, and $\theta_3 = 270°$. The x- and y-components of the three forces are

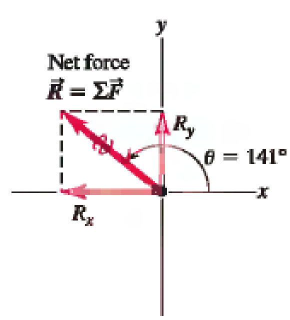
$$F_{1x} = (250\text{ N})\cos 127° = -150\text{ N}$$
$$F_{1y} = (250\text{ N})\sin 127° = 200\text{ N}$$
$$F_{2x} = (50\text{ N})\cos 0° = 50\text{ N}$$
$$F_{2y} = (50\text{ N})\sin 0° = 0\text{ N}$$

4.8 (a) Three forces acting on a belt. (b) The net force $\vec{R} = \Sigma\vec{F}$ and its components.

(a)



(b)



$$F_{3x} = (120\text{ N})\cos 270° = 0\text{ N}$$
$$F_{3y} = (120\text{ N})\sin 270° = -120\text{ N}$$

From Eq. (4.2) the net force $\vec{R} = \Sigma\vec{F}$ has components

$$R_x = F_{1x} + F_{2x} + F_{3x} = (-150\text{ N}) + 50\text{ N} + 0\text{ N} = -100\text{ N}$$
$$R_y = F_{1y} + F_{2y} + F_{3y} = 200\text{ N} + 0\text{ N} + (-120\text{ N}) = 80\text{ N}$$

The net force has a negative x-component and a positive y-component, so it points to the left and toward the top of the page in Fig. 4.8b (that is, in the second quadrant).

The magnitude of the net force $\vec{R} = \sum\vec{F}$ is

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(-100\ \text{N})^2 + (80\ \text{N})^2} = 128\ \text{N}$$

To find the angle between the net force and the $+x$-axis, we use the relation $\tan\theta = R_y/R_x$, or

$$\theta = \arctan\frac{R_y}{R_x} = \arctan\left(\frac{80\ \text{N}}{-100\ \text{N}}\right) = \arctan(-0.80)$$

The two possible solutions are $\theta = -39^\circ$ and $\theta = -39^\circ + 180^\circ = 141^\circ$. Since the net force lies in the second quadrant, as mentioned earlier, the correct answer is 141° (see Fig. 4.8b).

EVALUATE: In this situation the net force is *not* zero, and you can see intuitively that wrestler 1 (who exerts the largest force, $\vec{F}_1$, on the belt) is likely to walk away with the belt at the end of the struggle. In Section 4.2 we will explore in detail what happens in situations in which the net force *is* zero.

Test Your Understanding of Section 4.1 Figure 4.6 shows a force $\vec{F}$ acting on a crate. With the x- and y-axes shown in the figure, which statement about the components of the *gravitational* force that the earth exerts on the crate (the crate's weight) is *correct?* (i) The x- and y-components are both positive. (ii) The x-component is zero and the y-component is positive. (iii) The x-component is negative and the y-component is positive. (iv) The x- and y-components are both negative. (v) The x-component is zero and the y-component is negative. (vi) The x-component is positive and the y-component is negative.

4.2 Newton's First Law

We have discussed some of the properties of forces, but so far have said nothing about how forces affect motion. To begin, let's consider what happens when the net force on a body is *zero.* You would almost certainly agree that if a body is at rest, and if no net force acts on it (that is, no net push or pull), that body will remain at rest. But what if there is zero net force acting on a body in *motion?*

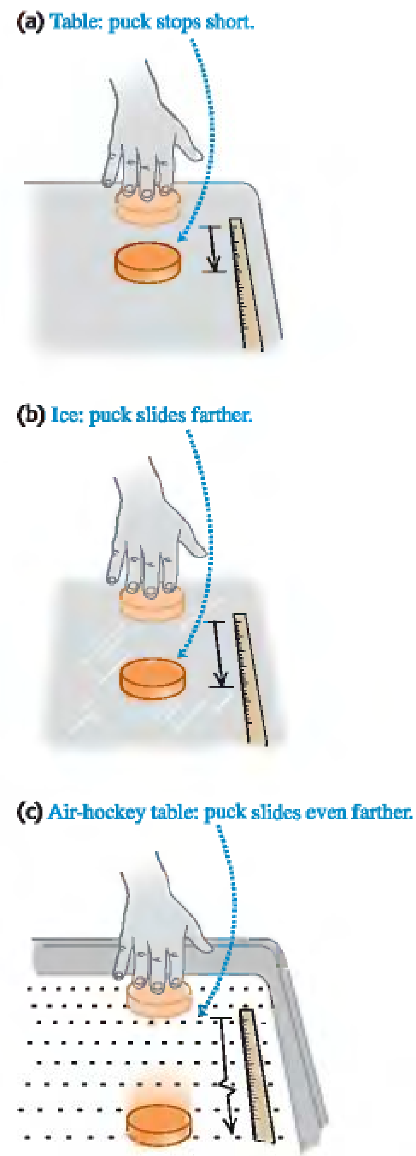
To see what happens in this case, suppose you slide a hockey puck along a horizontal tabletop, applying a horizontal force to it with your hand (Fig. 4.9a). After you stop pushing, the puck *does not* continue to move indefinitely; it slows down and stops. To keep it moving, you have to keep pushing (that is, applying a force). You might come to the "common sense" conclusions that bodies in motion naturally come to rest and that a force is required to sustain motion.

But now imagine pushing the puck across a smooth surface of ice (Fig. 4.9b). After you quit pushing, the puck will slide a lot farther before it stops. Put it on an air-hockey table, where it floats on a thin cushion of air, and it moves still farther (Fig. 4.9c). In each case, what slows the puck down is *friction,* an interaction between the lower surface of the puck and the surface on which it slides. Each surface exerts a frictional force on the puck that resists the puck's motion; the difference in the three cases is the magnitude of the frictional force. The ice exerts less friction than the tabletop, so the puck travels farther. The gas molecules of the air-hockey table exert the least friction of all. If we could eliminate friction completely, the puck would never slow down, and we would need no force at all to keep the puck moving once it had been started. Thus the "common sense" idea that a force is required to sustain motion is *incorrect.*

Experiments like the ones we've just described show that when no net force acts on a body, the body either remains at rest or moves with constant velocity in a straight line. Once a body has been set in motion, no net force is needed to keep it moving. We now call this observation *Newton's first law of motion:*

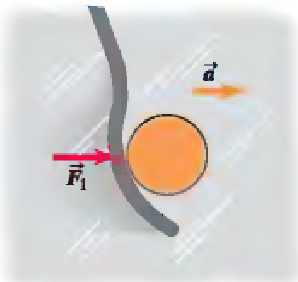
Newton's first law of motion: A body acted on by no net force moves with constant velocity (which may be zero) and zero acceleration.

4.9 The slicker the surface, the farther a puck slides after being given an initial velocity. On an air-hockey table (c) the friction force is practically zero, so the puck continues with almost constant velocity.

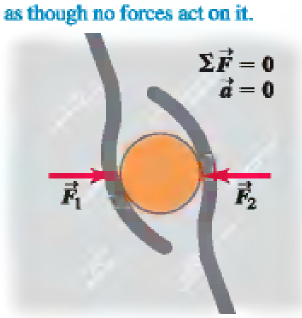


4.10 (a) A hockey puck accelerates in the direction of a net applied force $\vec{F}_1$. (b) When the net force is zero, the acceleration is zero, and the puck is in equilibrium.

(a) A puck on a frictionless surface accelerates when acted on by a single horizontal force.



(b) An object acted on by forces whose vector sum is zero behaves as though no forces act on it.



The tendency of a body to keep moving once it is set in motion results from a property called **inertia.** You use inertia when you try to get ketchup out of a bottle by shaking it. First you start the bottle (and the ketchup inside) moving forward; when you jerk the bottle back, the ketchup tends to keep moving forward and, you hope, ends up on your burger. The tendency of a body at rest to remain at rest is also due to inertia. You may have seen a tablecloth yanked out from under the china without breaking anything. The force on the china isn't great enough to make it move appreciably during the short time it takes to pull the tablecloth away.

It's important to note that the *net* force is what matters in Newton's first law. For example, a physics book at rest on a horizontal tabletop has two forces acting on it: an upward supporting force, or normal force, exerted by the tabletop (see Fig. 4.2a) and the downward force of the earth's gravitational attraction (a long-range force that acts even if the tabletop is elevated above the ground; see Fig. 4.2d). The upward push of the surface is just as great as the downward pull of gravity, so the *net* force acting on the book (that is, the vector sum of the two forces) is zero. In agreement with Newton's first law, if the book is at rest on the tabletop, it remains at rest. The same principle applies to a hockey puck sliding on a horizontal, frictionless surface: The vector sum of the upward push of the surface and the downward pull of gravity is zero. Once the puck is in motion, it continues to move with constant velocity because the *net* force acting on it is zero.

Here's another example. Suppose a hockey puck rests on a horizontal surface with negligible friction, such as an air-hockey table or a slab of wet ice. If the puck is initially at rest and a single horizontal force $\vec{F}_1$ acts on it (Fig. 4.10a), the puck starts to move. If the puck is in motion to begin with, the force changes its speed, its direction, or both, depending on the direction of the force. In this case the net force is equal to $\vec{F}_1$, which is *not* zero. (There are also two vertical forces: the earth's gravitational attraction and the upward normal force exerted by the surface. But as we mentioned earlier, these two forces cancel.)

Now suppose we apply a second force $\vec{F}_2$ (Fig. 4.10b), equal in magnitude to $\vec{F}_1$ but opposite in direction. The two forces are negatives of each other, $\vec{F}_2 = -\vec{F}_1$, and their vector sum is zero:

$$\sum\vec{F} = \vec{F}_1 + \vec{F}_2 = \vec{F}_1 + (-\vec{F}_1) = \mathbf{0}$$

Again, we find that if the body is at rest at the start, it remains at rest; if it is initially moving, it continues to move in the same direction with constant speed. These results show that in Newton's first law, *zero net force is equivalent to no force at all.* This is just the principle of superposition of forces that we saw in Section 4.1.

When a body is either at rest or moving with constant velocity (in a straight line with constant speed), we say that the body is in **equilibrium.** For a body to be in equilibrium, it must be acted on by no forces, or by several forces such that their vector sum—that is, the net force—is zero:

$$\sum\vec{F} = \mathbf{0} \qquad \text{(body in equilibrium)} \qquad (4.3)$$

For this to be true, each component of the net force must be zero, so

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \text{(body in equilibrium)} \qquad (4.4)$$

We are assuming that the body can be represented adequately as a point particle. When the body has finite size, we also have to consider *where* on the body the forces are applied. We will return to this point in Chapter 11.

Conceptual Example 4.2 Zero net force means constant velocity

In the classic 1950 science fiction film *Rocketship X-M,* a spaceship is moving in the vacuum of outer space, far from any planet, when its engine dies. As a result, the spaceship slows down and stops. What does Newton's first law say about this event?

SOLUTION

In this situation there are no forces acting on the spaceship, so according to Newton's first law, it will *not* stop. It continues to move in a straight line with constant speed. Some science fiction movies have made use of very accurate science; this was not one of them.

Conceptual Example 4.3 Constant velocity means zero net force

You are driving a Porsche Carrera GT on a straight testing track at a constant speed of 150 km/h. You pass a 1971 Volkswagen Beetle doing a constant 75 km/h. For which car is the net force greater?

SOLUTION

The key word in this question is "net." Both cars are in equilibrium because their velocities are both constant; therefore the *net* force on each car is *zero.*

This conclusion seems to contradict the "common sense" idea that the faster car must have a greater force pushing it. It's true that the forward force on your Porsche is much greater than that on the Volkswagen (thanks to your Porsche's high-power engine). But there is also a *backward* force acting on each car due to road friction and air resistance. The only reason these cars need engines is to counteract this backward force so that the vector sum of the forward and backward forces will be zero and the car will travel with constant velocity. The backward force on your Porsche is greater because of its greater speed, so its engine has to be more powerful than the Volkswagen's.

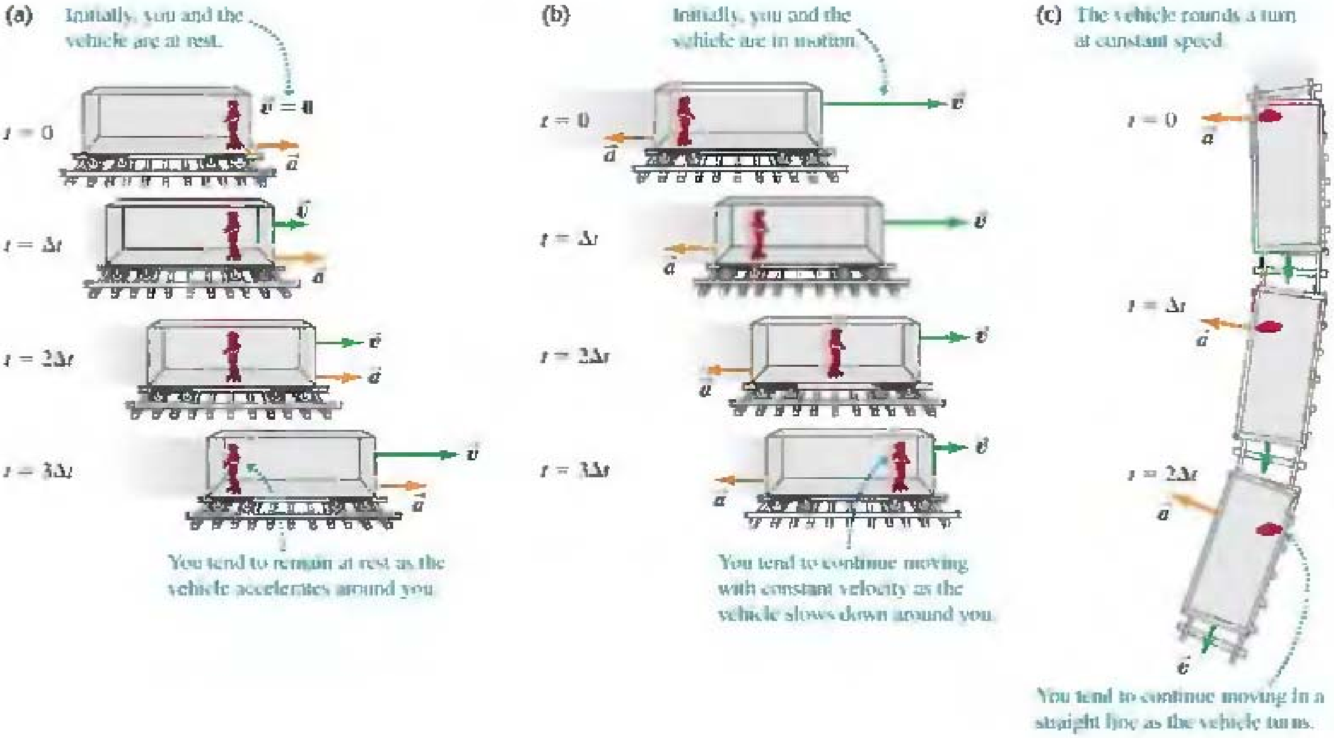
Inertial Frames of Reference

In discussing relative velocity in Section 3.5, we introduced the concept of *frame of reference.* This concept is central to Newton's laws of motion. Suppose you are in a bus that is traveling on a straight road and speeding up. If you could stand in the aisle on roller skates, you would start moving *backward* relative to the bus as the bus gains speed. If instead the bus was slowing to a stop, you would start moving forward down the aisle. In either case, it looks as though Newton's first law is not obeyed; there is no net force acting on you, yet your velocity changes. What's wrong?

The point is that the bus is accelerating with respect to the earth and is *not* a suitable frame of reference for Newton's first law. This law is valid in some frames of reference and not valid in others. A frame of reference in which Newton's first law *is* valid is called an **inertial frame of reference.** The earth is at least approximately an inertial frame of reference, but the bus is not. (The earth is not a completely inertial frame, owing to the acceleration associated with its rotation and its motion around the sun. These effects are quite small, however; see Exercises 3.29 and 3.32.) Because Newton's first law is used to define what we mean by an inertial frame of reference, it is sometimes called the *law of inertia.*

Figure 4.11 helps us understand what you experience when riding in a vehicle that's accelerating. In Fig. 4.11a, a vehicle is initially at rest and then begins to accelerate to the right. A passenger on roller skates (which nearly eliminate the effects of friction) has virtually no net force acting on her, so she tends to remain at rest relative to the inertial frame of the earth. As the vehicle accelerates around her, she moves backward relative to the vehicle. In the same way, a passenger in a vehicle that is slowing down tends to continue moving with constant velocity relative to the earth, and so moves forward relative to the vehicle (Fig. 4.11b). A vehicle is also accelerating if it moves at a constant speed but is turning (Fig. 4.11c). In this case a passenger tends to continue moving relative to the earth at constant speed in a straight line; relative to the vehicle, the passenger moves to the side of the vehicle on the outside of the turn.

4.11 Riding in an accelerating vehicle.



In each case shown in Fig. 4.11, an observer in the vehicle's frame of reference might be tempted to conclude that there *is* a net force acting on the passenger, since the passenger's velocity *relative to the vehicle* changes in each case. This conclusion is simply wrong; the net force on the passenger is indeed zero. The vehicle observer's mistake is in trying to apply Newton's first law in the vehicle's frame of reference, which is *not* an inertial frame and in which Newton's first law isn't valid (Fig. 4.12). In this book we will use *only* inertial frames of reference.

We've mentioned only one (approximately) inertial frame of reference: the earth's surface. But there are many inertial frames. If we have an inertial frame of reference A, in which Newton's first law is obeyed, then *any* second frame of reference B will also be inertial if it moves relative to A with constant velocity $\vec{v}_{B/A}$. We can prove this using the relative velocity relation Eq. (3.36) from Section 3.5:

$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$$

Suppose that P is a body that moves with constant velocity $\vec{v}_{P/A}$ with respect to an inertial frame A. By Newton's first law the net force on this body is zero. The velocity of P relative to another frame B has a different value, $\vec{v}_{P/B} = \vec{v}_{P/A} - \vec{v}_{B/A}$. But if the relative velocity $\vec{v}_{B/A}$ of the two frames is constant, then $\vec{v}_{P/B}$ is constant as well. Thus B is also an inertial frame; the velocity of P in this frame is constant, and the net force on P is zero, so Newton's first law is obeyed in B. Observers in frames A and B will disagree about the velocity of P, but they will agree that P has a constant velocity (zero acceleration) and has zero net force acting on it.

There is no single inertial frame of reference that is preferred over all others for formulating Newton's laws. If one frame is inertial, then every other frame moving relative to it with constant velocity is also inertial. Viewed in this light, the state of rest and the state of motion with constant velocity are not very different; both occur when the vector sum of forces acting on the body is zero.

4.12 From the frame of reference of the car, it seems as though a force is pushing the crash test dummies forward as the car comes to a sudden stop. But there is really no such force: As the car stops, the dummies keep moving forward as a consequence of Newton's first law.

Test Your Understanding of Section 4.2 In which of the following situations is there zero net force on the body? (i) an airplane flying due north at a steady 120 m/s and at a constant altitude; (ii) a car driving straight up a hill with a 3° slope at a constant 90 km/h; (iii) a hawk circling at a constant 20 km/h at a constant height of 15 m above an open field; (iv) a box with slick, frictionless surfaces in the back of a truck as the truck accelerates forward on a level road at 5 m/s^2.

4.3 Newton's Second Law

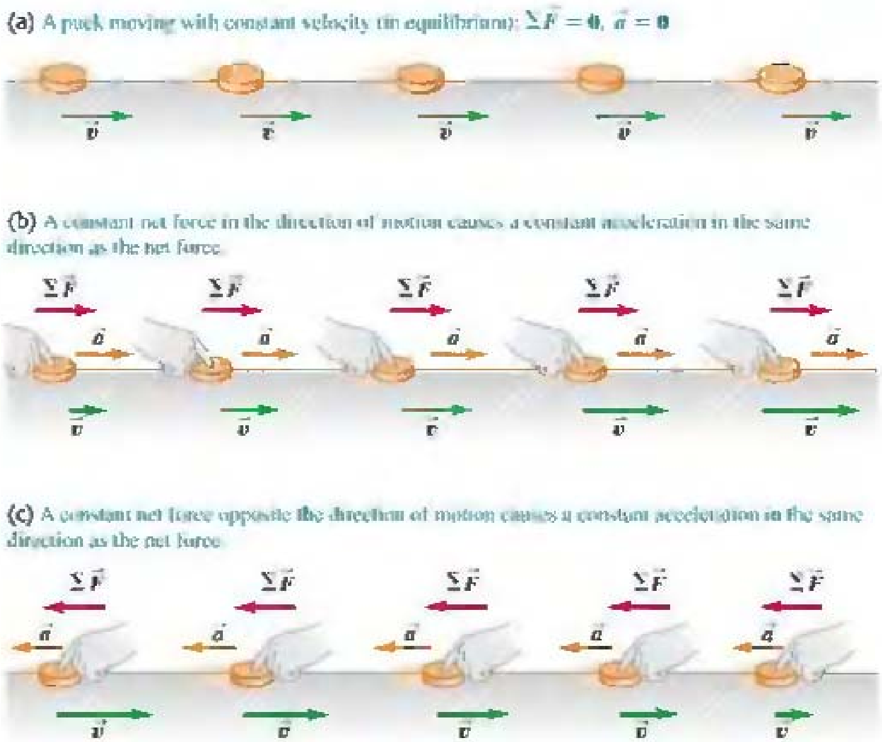
Newton's first law tells us that when a body is acted on by zero net force, it moves with constant velocity and zero acceleration. In Fig. 4.13a, a hockey puck is sliding to the right on wet ice. There is negligible friction, so there are no horizontal forces acting on the puck; the downward force of gravity and the upward normal force exerted by the ice surface sum to zero. So the net force $\sum \vec{F}$ acting on the puck is zero, the puck has zero acceleration, and its velocity is constant.

But what happens when the net force is *not* zero? In Fig. 4.13b we apply a constant horizontal force to a sliding puck in the same direction that the puck is moving. Then $\sum \vec{F}$ is constant and in the same horizontal direction as $\vec{v}$. We find that during the time the force is acting, the velocity of the puck changes at a constant rate; that is, the puck moves with constant acceleration. The speed of the puck increases, so the acceleration $\vec{a}$ is in the same direction as $\vec{v}$ and $\sum \vec{F}$.

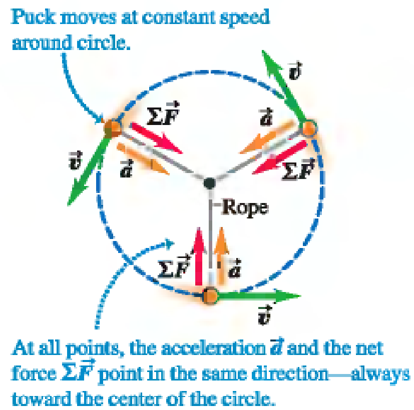
In Fig. 4.13c we reverse the direction of the force on the puck so that $\sum \vec{F}$ acts opposite to $\vec{v}$. In this case as well, the puck has an acceleration; the puck moves more and more slowly to the right. The acceleration $\vec{a}$ in this case is to the left, in the same direction as $\sum \vec{F}$. As in the previous case, experiment shows that the acceleration is constant if $\sum \vec{F}$ is constant.

We conclude that *a net force acting on a body causes the body to accelerate in the same direction as the net force.* If the magnitude of the net force is constant, as in Figs. 4.13b and 4.13c, then so is the magnitude of the acceleration.

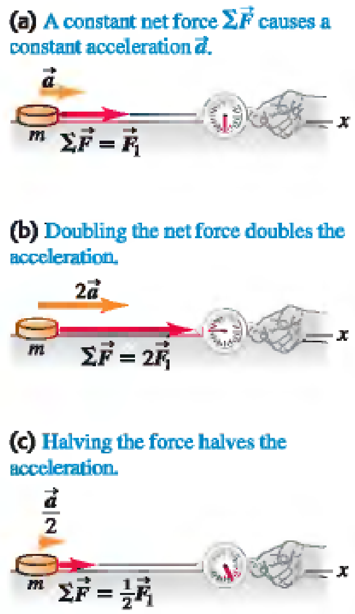
4.13 Exploring the relationship between the acceleration of a body and the net force acting on the body (in this case, a hockey puck on a frictionless surface).



4.14 A top view of a hockey puck in uniform circular motion on a frictionless horizontal surface.



4.15 For a body of a given mass m, the magnitude of the body's acceleration is directly proportional to the magnitude of the net force acting on the body.



These conclusions about net force and acceleration also apply to a body moving along a curved path. For example, Fig. 4.14 shows a hockey puck moving in a horizontal circle on an ice surface of negligible friction. A rope attaching the puck to the ice exerts a tension force of constant magnitude toward the center of the circle. The result is a net force and an acceleration that are constant in magnitude and directed toward the center of the circle. The speed of the puck is constant, so this is uniform circular motion, as discussed in Section 3.4.

Figure 4.15a shows another experiment to explore the relationship between acceleration and net force. We apply a constant horizontal force to a puck on a frictionless horizontal surface, using the spring balance described in Section 4.1 with the spring stretched a constant amount. As in Figs. 4.13b and 4.13c, this horizontal force equals the net force on the puck. If we change the magnitude of the net force, the acceleration changes in the same proportion. Doubling the net force doubles the acceleration (Fig. 4.15b), halving the net force halves the acceleration (Fig. 4.15c), and so on. Many such experiments show that *for any given body, the magnitude of the acceleration is directly proportional to the magnitude of the net force acting on the body.*

Mass and Force

Our results mean that for a given body, the *ratio* of the magnitude $|\sum\vec{F}|$ of the net force to the magnitude $a = |\vec{a}|$ of the acceleration is constant, regardless of the magnitude of the net force. We call this ratio the *inertial mass,* or simply the **mass,** of the body and denote it by m. That is,

$$m = \frac{|\sum\vec{F}|}{a} \quad \text{or} \quad |\sum\vec{F}| = ma \quad \text{or} \quad a = \frac{|\sum\vec{F}|}{m} \tag{4.5}$$

Mass is a quantitative measure of inertia, which we discussed in Section 4.2. The last of the equations in Eq. (4.5) says that the greater its mass, the more a body "resists" being accelerated. When you hold a piece of fruit in your hand at the supermarket and move it slightly up and down to estimate its heft, you're applying a force and seeing how much the fruit accelerates up and down in response. If a force causes a large acceleration, the fruit has a small mass; if the same force causes only a small acceleration, the fruit has a large mass. In the same way, if you hit a table-tennis ball and then a basketball with the same force, the basketball has much smaller acceleration because it has much greater mass.

The SI unit of mass is the **kilogram.** We mentioned in Section 1.3 that the kilogram is officially defined to be the mass of a cylinder of platinum–iridium alloy kept in a vault near Paris. We can use this standard kilogram, along with Eq. (4.5), to define the **newton:**

One newton is the amount of net force that gives an acceleration of 1 meter per second squared to a body with a mass of 1 kilogram.

This definition allows us to calibrate the spring balances and other instruments used to measure forces. Because of the way we have defined the newton, it is related to the units of mass, length, and time. For Eq. (4.5) to be dimensionally consistent, it must be true that

$$1\text{ newton} = (1\text{ kilogram})(1\text{ meter per second squared})$$

or

$$1\text{ N} = 1\text{ kg}\cdot\text{m/s}^2$$

We will use this relationship many times in the next few chapters, so keep it in mind.

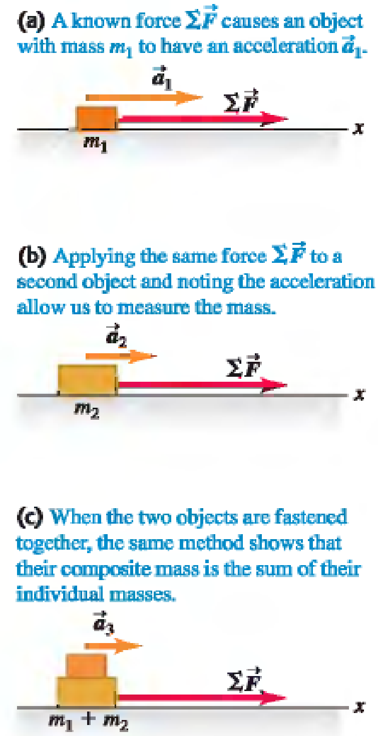
We can also use Eq. (4.5) to compare a mass with the standard mass and thus to *measure* masses. Suppose we apply a constant net force $\sum\vec{F}$ to a body having a known mass m_1 and we find an acceleration of magnitude a_1 (Fig. 4.16a). We then apply the same force to another body having an unknown mass m_2, and we find an acceleration of magnitude a_2 (Fig. 4.16b). Then, according to Eq. (4.5),

$$m_1 a_1 = m_2 a_2$$
$$\frac{m_2}{m_1} = \frac{a_1}{a_2} \qquad \text{(same net force)} \tag{4.6}$$

For the same net force, the ratio of the masses of two bodies is the inverse of the ratio of their accelerations. In principle we could use Eq. (4.6) to measure an unknown mass m_2, but it is usually easier to determine mass indirectly by measuring the body's *weight.* We'll return to this point in Section 4.4.

When two bodies with masses m_1 and m_2 are fastened together, we find that the mass of the composite body is always $m_1 + m_2$ (Fig. 4.16c). This additive property of mass may seem obvious, but it has to be verified experimentally. Ultimately, the mass of a body is related to the number of protons, electrons, and neutrons it contains. This wouldn't be a good way to *define* mass because there is no practical way to count these particles. But the concept of mass is the most fundamental way to characterize the quantity of matter in a body.

4.16 For a given net force $\sum\vec{F}$ acting on a body, the acceleration is inversely proportional to the mass of the body. Masses add like ordinary scalars.



Stating Newton's Second Law

We've been careful to state that the *net* force on a body is what causes that body to accelerate. Experiment shows that if a combination of forces $\vec{F}_1, \vec{F}_2, \vec{F}_3$, and so on is applied to a body, the body will have the same acceleration (magnitude and direction) as when only a single force is applied, if that single force is equal to the vector sum $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots$. In other words, the principle of superposition of forces (see Fig. 4.4) also holds true when the net force is not zero and the body is accelerating.

Equation (4.5) relates the magnitude of the net force on a body to the magnitude of the acceleration that it produces. We have also seen that the direction of the net force is the same as the direction of the acceleration, whether the body's path is straight or curved. Newton wrapped up all these relationships and experimental results in a single concise statement that we now call *Newton's second law of motion:*

Newton's second law of motion: If a net external force acts on a body, the body accelerates. The direction of acceleration is the same as the direction of the net force. The mass of the body times the acceleration of the body equals the net force vector.

In symbols,

$$\sum\vec{F} = m\vec{a} \qquad \text{(Newton's second law of motion)} \tag{4.7}$$

An alternative statement is that the acceleration of a body is in the same direction as the net force acting on the body, and is equal to the net force divided by the body's mass:

$$\vec{a} = \frac{\sum\vec{F}}{m}$$

Newton's second law is a fundamental law of nature, the basic relationship between force and motion. Most of the remainder of this chapter and all of the next are devoted to learning how to apply this principle in various situations.

Equation (4.7) has many practical applications (Fig. 4.17). You've actually been using it all your life to measure your body's acceleration. In your inner ear, microscopic hair cells sense the magnitude and direction of the force that they must exert to cause small membranes to accelerate along with the rest of your body. By Newton's second law, the acceleration of the membranes—and hence

4.17 The design of high-performance motorcycles depends fundamentally on Newton's second law. To maximize the forward acceleration, the designer makes the motorcycle as light as possible (that is, minimizes the mass) and uses the most powerful engine possible (thus maximizing the forward force).

that of your body as a whole—is proportional to this force and has the same direction. In this way, you can sense the magnitude and direction of your acceleration even with your eyes closed!

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2.1.3 Tension Change
2.1.4 Sliding on an Incline

Using Newton's Second Law

There are at least four aspects of Newton's second law that deserve special attention. First, Eq. (4.7) is a *vector* equation. Usually we will use it in component form, with a separate equation for each component of force and the corresponding acceleration:

$$\sum F_x = ma_x \qquad \sum F_y = ma_y \qquad \sum F_z = ma_z \qquad \text{(Newton's second law of motion)} \tag{4.8}$$

This set of component equations is equivalent to the single vector equation (4.7). Each component of the net force equals the mass times the corresponding component of acceleration.

Second, the statement of Newton's second law refers to *external* forces. By this we mean forces exerted on the body by other bodies in its environment. It's impossible for a body to affect its own motion by exerting a force on itself; if it were possible, you could lift yourself to the ceiling by pulling up on your belt! That's why only external forces are included in the sum $\sum \vec{F}$ in Eqs. (4.7) and (4.8).

Third, Eqs. (4.7) and (4.8) are valid only when the mass m is *constant*. It's easy to think of systems whose masses change, such as a leaking tank truck, a rocket ship, or a moving railroad car being loaded with coal. But such systems are better handled by using the concept of momentum; we'll get to that in Chapter 8.

Finally, Newton's second law is valid only in inertial frames of reference, just like the first law. Thus it is not valid in the reference frame of any of the accelerating vehicles in Fig. 4.11; relative to any of these frames, the passenger accelerates even though the net force on the passenger is zero. We will usually assume that the earth is an adequate approximation to an inertial frame, although because of its rotation and orbital motion it is not precisely inertial.

CAUTION **$m\vec{a}$ is not a force** You must keep in mind that even though the vector $m\vec{a}$ is equal to the vector sum $\sum \vec{F}$ of all the forces acting on the body, the vector $m\vec{a}$ is *not* a force. Acceleration is a *result* of a nonzero net force; it is not a force itself. It's "common sense" to think that there is a "force of acceleration" that pushes you back into your seat when your car accelerates forward from rest. But *there is no such force;* instead, your inertia causes you to tend to stay at rest relative to the earth, and the car accelerates around you (see Fig. 4.11a). The "common sense" confusion arises from trying to apply Newton's second law where it isn't valid, in the noninertial reference frame of an accelerating car. We will always examine motion relative to *inertial* frames of reference only.

In learning how to use Newton's second law, we will begin in this chapter with examples of straight-line motion. Then in Chapter 5 we will consider more general cases and develop more detailed problem-solving strategies.

Example 4.4 Determining acceleration from force

A worker applies a constant horizontal force with magnitude 20 N to a box with mass 40 kg resting on a level floor with negligible friction. What is the acceleration of the box?

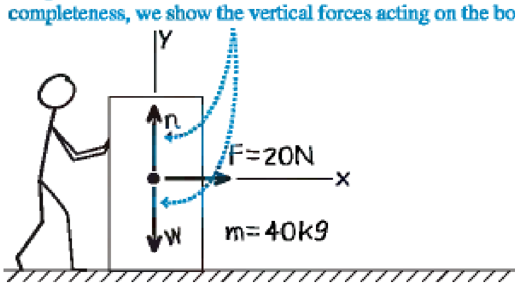
SOLUTION

IDENTIFY: This problem involves force and acceleration. Whenever you encounter a problem of this kind, you should approach it using Newton's second law.

SET UP: In *any* problem involving forces, the first steps are to choose a coordinate system and then identify all of the forces acting on the body in question.

It's usually convenient to take one axis either along or opposite the direction of the body's acceleration, which in this case is horizontal. Hence we take the $+x$-axis to be in the direction of the applied horizontal force (that is, the direction in which the box accelerates) and the $+y$-axis to be upward (Fig. 4.18). In most

4.18 Our sketch for this problem. The tiles under the box are freshly waxed, so we assume that friction is negligible.



force problems that you'll encounter (including this one), the force vectors all lie in a plane, so the z-axis isn't used.

The forces acting on the box are (i) the horizontal force $\vec{F}$ exerted by the worker, of magnitude 20 N; (ii) the weight $\vec{w}$ of the box—that is, the downward gravitational force exerted by the earth; and (iii) the upward supporting force $\vec{n}$ exerted by the floor. As in Section 4.2, we call $\vec{n}$ a *normal* force because it is normal (perpendicular) to the surface of contact. (We use an italic letter n to avoid confusion with the abbreviation N for newton.) We are told that friction is negligible, so no friction force is present.

Since the box doesn't move vertically at all, the y-acceleration is zero: $a_y = 0$. Our target variable is the x-component of acceleration, a_x. We'll find it using Newton's second law in component form as given by Eq. (4.8).

EXECUTE: From Fig. 4.18, only the 20-N force has a nonzero x-component. Hence the first relation in Eqs. (4.8) tells us that

$$\sum F_x = F = 20\ \text{N} = ma_x$$

Hence the x-component of acceleration is

$$a_x = \frac{\sum F_x}{m} = \frac{20\ \text{N}}{40\ \text{kg}} = \frac{20\ \text{kg}\cdot\text{m/s}^2}{40\ \text{kg}} = 0.50\ \text{m/s}^2$$

EVALUATE: The acceleration is in the $+x$-direction, the same direction as the net force. The net force is constant, so the acceleration is also constant. If we are given the initial position and velocity of the box, we can find the position and velocity at any later time from the equations of motion with constant acceleration we derived in Chapter 2.

Notice that to determine a_x, we didn't have to use the y-component of Newton's second law from Eq. (4.8), $\sum F_y = ma_y$. By using this equation, can you show that the magnitude n of the normal force in this situation is equal to the weight of the box?

Example 4.5 Determining force from acceleration

A waitress shoves a ketchup bottle with mass 0.45 kg to the right along a smooth, level lunch counter. The bottle leaves her hand moving at 2.8 m/s, then slows down as it slides because of the constant horizontal friction force exerted on it by the counter top. It slides a distance of 1.0 m before coming to rest. What are the magnitude and direction of the friction force acting on the bottle?

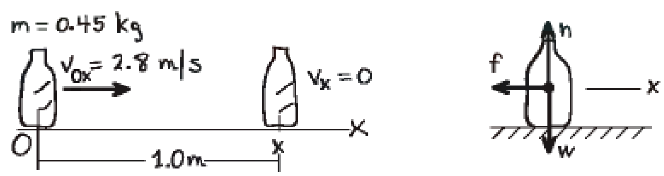
SOLUTION

IDENTIFY: Like Example 4.4, this problem involves forces and acceleration (the slowing of the ketchup bottle), so we'll use Newton's second law to solve it.

SET UP: As in Example 4.4, we first choose a coordinate system and then identify the forces acting on the body (in this case, the ketchup bottle). As Fig. 4.19 shows, we choose the $+x$-axis to be in the direction that the bottle slides, and we take the origin to be where the bottle leaves the waitress's hand moving at 2.8 m/s. Figure 4.19 also shows the forces acting on the bottle. The friction force $\vec{f}$ acts to slow the bottle down, so its direction must be opposite the direction of velocity (see Fig. 4.13c).

Our target variable is the magnitude f of the friction force. We'll find it using the x-component of Newton's second law from Eq. (4.8). To do so, we'll first need to know the x-component of the bottle's acceleration, a_x. We aren't told the value of a_x in the problem, but we are told that the friction force is constant. Hence the acceleration is constant as well, and we can calculate a_x by using one of the constant-acceleration formulas from Section 2.4. Since we know the bottle's initial x-coordinate and x-velocity $(x_0 = 0,$ $v_{0x} = 2.8\ \text{m/s})$ as well as its final x-coordinate and x-velocity $(x = 1.0\ \text{m}, v_x = 0)$, the easiest equation to use to determine a_x is Eq. (2.13), $v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$.

4.19 Our sketch for this problem.



EXECUTE: From Eq. (2.13),

$$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$$

$$a_x = \frac{v_x^2 - v_{0x}^2}{2(x - x_0)} = \frac{(0\ \text{m/s})^2 - (2.8\ \text{m/s})^2}{2(1.0\ \text{m} - 0\ \text{m})} = -3.9\ \text{m/s}^2$$

The negative sign means that the acceleration is toward the *left;* the velocity is in the opposite direction to acceleration, as it must be, since the bottle is slowing down. The net force in the x-direction is the x-component $-f$ of the friction force, so

$$\sum F_x = -f = ma_x = (0.45\ \text{kg})(-3.9\ \text{m/s}^2)$$
$$= -1.8\ \text{kg}\cdot\text{m/s}^2 = -1.8\ \text{N}$$

Continued

Again the negative sign shows that the force on the bottle is directed toward the left. The magnitude of the friction force is $f = 1.8$ N. Remember that magnitudes are *always* positive!

EVALUATE: We chose the $+x$-axis to be in the direction of the bottle's motion, so that a_x was negative. As a check on the result, try repeating the calculation with the $+x$-axis directed *opposite* to the motion (to the left in Fig. 4.19) so that a_x is positive. In this case you should find that $\sum F_x$ is equal to $+f$ (because the friction force is now in the $+x$-direction), which in turn is equal to $+1.8$ N. Your answers for the *magnitudes* of forces (which are always positive numbers) should never depend on your choice of coordinate axes!

4.20 Despite its name, the English unit of mass has nothing to do with the type of slug shown here. A common garden slug has a mass of about 15 grams, or about 10^{-3} slug.

Some Notes on Units

A few words about units are in order. In the cgs metric system (not used in this book), the unit of mass is the gram, equal to 10^{-3} kg, and the unit of distance is the centimeter, equal to 10^{-2} m. The cgs unit of force is called the *dyne:*

$$1 \text{ dyne} = 1 \text{ g} \cdot \text{cm/s}^2 = 10^{-5} \text{ N}$$

In the British system, the unit of force is the *pound* (or pound-force) and the unit of mass is the *slug* (Fig. 4.20). The unit of acceleration is 1 foot per second squared, so

$$1 \text{ pound} = 1 \text{ slug} \cdot \text{ft/s}^2$$

The official definition of the pound is

$$1 \text{ pound} = 4.448221615260 \text{ newtons}$$

It is handy to remember that a pound is about 4.4 N and a newton is about 0.22 pound. Next time you want to order a "quarter-pounder," try asking for a "one-newtoner" and see what happens. Another useful fact: A body with a mass of 1 kg has a weight of about 2.2 lb at the earth's surface.

Table 4.2 summarizes the units of force, mass, and acceleration in the three systems.

Table 4.2 Units of Force, Mass, and Acceleration

System of Units	Force	Mass	Acceleration
SI	newton (N)	kilogram (kg)	m/s^2
cgs	dyne (dyn)	gram (g)	cm/s^2
British	pound (lb)	slug	ft/s^2

Test Your Understanding of Section 4.3 Rank the following situations in order of the magnitude of the object's acceleration, from lowest to highest. Are there any cases that have the same magnitude of acceleration? (i) a 2.0-kg object acted on by a 2.0-N net force; (ii) a 2.0-kg object acted on by an 8.0-N net force; (iii) an 8.0-kg object acted on by a 2.0-N net force; (iv) an 8.0-kg object acted on by a 8.0-N net force.

4.4 Mass and Weight

One of the most familiar forces is the *weight* of a body, which is the gravitational force that the earth exerts on the body. (If you are on another planet, your weight is the gravitational force that planet exerts on you.) Unfortunately, the terms *mass* and *weight* are often misused and interchanged in everyday conversation. It is absolutely essential for you to understand clearly the distinctions between these two physical quantities.

Mass characterizes the *inertial* properties of a body. Mass is what keeps the china on the table when you yank the tablecloth out from under it. The greater the mass, the greater the force needed to cause a given acceleration; this is reflected in Newton's second law, $\sum \vec{F} = m\vec{a}$.

Weight, on the other hand, is a *force* exerted on a body by the pull of the earth. Mass and weight are related: Bodies having large mass also have large weight. A large stone is hard to throw because of its large *mass,* and hard to lift off the ground because of its large *weight.*

To understand the relationship between mass and weight, note that a freely falling body has an acceleration of magnitude g. Newton's second law tells us that a force must act to produce this acceleration. If a 1-kg body falls with an acceleration of 9.8 m/s^2, the required force has magnitude

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2.9 Pole-Vaulter Vaults

$$F = ma = (1 \text{ kg})(9.8 \text{ m/s}^2) = 9.8 \text{ kg} \cdot \text{m/s}^2 = 9.8 \text{ N}$$

The force that makes the body accelerate downward is its weight. Any body near the surface of the earth that has a mass of 1 kg *must* have a weight of 9.8 N to give it the acceleration we observe when it is in free fall. More generally, a body with mass m must have weight with magnitude w given by

$$w = mg \qquad \text{(magnitude of the weight of a body of mass } m\text{)} \tag{4.9}$$

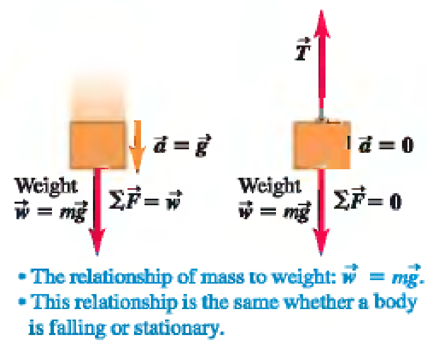
Hence the magnitude w of a body's weight is directly proportional to its mass m. The weight of a body is a force, a vector quantity, and we can write Eq. (4.9) as a vector equation (Fig. 4.21):

$$\vec{w} = m\vec{g} \tag{4.10}$$

Remember that g is the *magnitude* of $\vec{g}$, the acceleration due to gravity, so g is always a positive number, by definition. Thus w, given by Eq. (4.9), is the *magnitude* of the weight and is also always positive.

CAUTION **A body's weight acts at all times** It is important to understand that the weight of a body acts on the body *all the time,* whether it is in free fall or not. If we suspend an object from a chain, it is in equilibrium, and its acceleration is zero. But its weight, given by Eq. (4.10), is still pulling down on it (Fig. 4.21). In this case the chain pulls up on the object, applying an upward force. The *vector sum* of the forces is zero, but the weight still acts.

4.21 The relationship of mass and weight.



Conceptual Example 4.6 Net force and acceleration in free fall

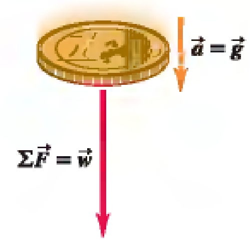
In Example 2.6 (Section 2.5) a one-euro coin was dropped from rest from the Leaning Tower of Pisa. If the coin falls freely, so that the effects of the air are negligible, how does the net force on the coin vary as it falls?

SOLUTION

In free fall, the acceleration $\vec{a}$ of the coin is constant and equal to $\vec{g}$. Hence by Newton's second law the net force $\Sigma\vec{F} = m\vec{a}$ is also constant and equal to $m\vec{g}$, which is the coin's weight $\vec{w}$ (Fig. 4.22). The coin's velocity changes as it falls, but the net force acting on it remains constant. If this surprises you, perhaps you still believe in the erroneous "common sense" idea that greater speed implies greater force. If so, you should reread Conceptual Example 4.3.

The net force on a freely falling coin is constant even if you initially toss it upward. The force that your hand exerts on the coin to toss it is a contact force, and it disappears the instant that the coin loses contact with your hand. From then on, the only force acting on the coin is its weight $\vec{w}$.

4.22 The acceleration of a freely falling object is constant, and so is the net force acting on the object.

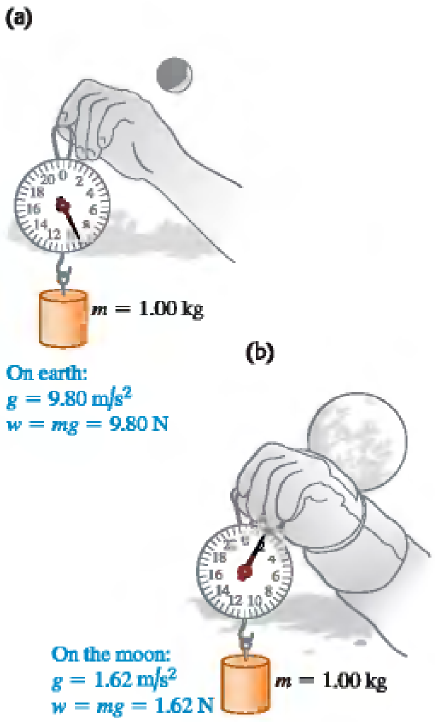


Variation of g with Location

We will use $g = 9.80\ \text{m/s}^2$ for problems on the earth (or, if the other data in the problem are given to only two significant figures, $g = 9.8\ \text{m/s}^2$). In fact, the value of g varies somewhat from point to point on the earth's surface, from about 9.78 to $9.82\ \text{m/s}^2$, because the earth is not perfectly spherical and because of effects due to its rotation and orbital motion. At a point where $g = 9.80\ \text{m/s}^2$, the weight of a standard kilogram is $w = 9.80\ \text{N}$. At a different point, where $g = 9.78\ \text{m/s}^2$, the weight is $w = 9.78\ \text{N}$ but the mass is still 1 kg. The weight of a body varies from one location to another; the mass does not.

If we take a standard kilogram to the surface of the moon, where the acceleration of free fall (equal to the value of g at the moon's surface) is $1.62\ \text{m/s}^2$, its

4.23 The weight of a 1-kilogram mass (a) on earth and (b) on the moon.



weight is 1.62 N, but its mass is still 1 kg (Fig. 4.23). An 80.0-kg astronaut has a weight on earth of $(80.0\ \mathrm{kg})(9.80\ \mathrm{m/s^2}) = 784$ N, but on the moon the astronaut's weight would be only $(80.0\ \mathrm{kg})(1.62\ \mathrm{m/s^2}) = 130$ N. In Chapter 12 we'll see how to calculate the value of g at the surface of the moon or on other worlds.

Measuring Mass and Weight

In Section 4.3 we described a way to compare masses by comparing their accelerations when subjected to the same net force. Usually, however, the easiest way to measure the mass of a body is to measure its weight, often by comparing with a standard. Equation (4.9) says that two bodies that have the same weight at a particular location also have the same mass. We can compare weights very precisely; the familiar equal-arm balance (Fig. 4.24) can determine with great precision (up to 1 part in 10^6) when the weights of two bodies are equal and hence when their masses are equal. (This method doesn't work in the apparent "zero-gravity" environment of outer space. Instead, we apply a known force to the body, measure its acceleration, and compute the mass as the ratio of force to acceleration. This method, or a variation of it, is used to measure the masses of astronauts in orbiting space stations as well as the masses of atomic and subatomic particles.)

The concept of mass plays two rather different roles in mechanics. The weight of a body (the gravitational force acting on it) is proportional to its mass; we call the property related to gravitational interactions *gravitational mass.* On the other hand, we call the inertial property that appears in Newton's second law the *inertial mass.* If these two quantities were different, the acceleration due to gravity might well be different for different bodies. However, extraordinarily precise experiments have established that in fact the two *are* the same to a precision of better than one part in 10^{12}.

CAUTION **Don't confuse mass and weight** The SI units for mass and weight are often misused in everyday life. Incorrect expressions such as "This box weighs 6 kg" are nearly universal. What is meant is that the *mass* of the box, probably determined indirectly by *weighing,* is 6 kg. Be careful to avoid this sloppy usage in your own work! In SI units, weight (a force) is measured in newtons, while mass is measured in kilograms.

Example 4.7 Mass and weight

A 2.49×10^4 N Rolls-Royce Phantom traveling in the $+x$-direction makes a fast stop; the x-component of the net force acting on it is -1.83×10^4 N. What is its acceleration?

SOLUTION

IDENTIFY: Again we will use Newton's second law to relate force and acceleration. To use this relationship, we need to know the car's mass. However, because the newton is a unit for force, we know that 2.49×10^4 N is the car's weight, not its mass. So we'll also have to use the relationship between a body's mass and its weight.

SET UP: Our target variable is the x-component of acceleration of the car, a_x. (The motion is purely in the x-direction.) We use Eq. (4.9) to determine the car's mass from its weight and then use the x-component of Newton's second law from Eq. (4.8) to determine a_x.

EXECUTE: The mass m of the car is

$$m = \frac{w}{g} = \frac{2.49 \times 10^4\ \mathrm{N}}{9.80\ \mathrm{m/s^2}} = \frac{2.49 \times 10^4\ \mathrm{kg \cdot m/s^2}}{9.80\ \mathrm{m/s^2}} = 2540\ \mathrm{kg}$$

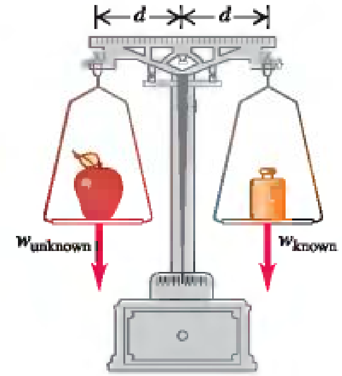
Then $\sum F_x = ma_x$ gives

$$a_x = \frac{\sum F_x}{m} = \frac{-1.83 \times 10^4\ \mathrm{N}}{2540\ \mathrm{kg}} = \frac{-1.83 \times 10^4\ \mathrm{kg \cdot m/s^2}}{2540\ \mathrm{kg}} = -7.20\ \mathrm{m/s^2}$$

EVALUATE: The negative sign means that the acceleration vector points in the negative x-direction. This makes sense: The car is moving in the positive x-direction and is slowing down.

Note that the acceleration can alternatively be written as $-0.735g$. It's of interest that -0.735 is also the ratio of -1.83×10^4 N (the x-component of the net force) to 2.49×10^4 N (the weight). Indeed, the acceleration of a body expressed as a multiple of g is always equal to the ratio of the net force on the body to its weight. Can you see why?

Test Your Understanding of Section 4.4 Suppose an astronaut landed on a planet where $g = 19.6\ \text{m/s}^2$. Compared to earth, would it be easier, harder, or just as easy for her to walk around? Would it be easier, harder, or just as easy for her to catch a ball that is moving horizontally at $12\ \text{m/s}$? (Assume that the astronaut's spacesuit is a lightweight model that doesn't impede her movements in any way.)



4.24 An equal-arm balance determines the mass of a body (such as an apple) by comparing its weight to a known weight.

4.5 Newton's Third Law

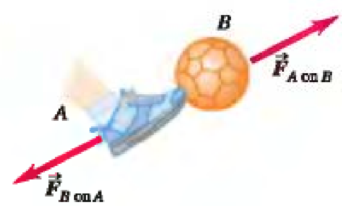
A force acting on a body is always the result of its interaction with another body, so forces always come in pairs. You can't pull on a doorknob without the doorknob pulling back on you. When you kick a football, the forward force that your foot exerts on the ball launches it into its trajectory, but you also feel the force the ball exerts back on your foot. If you kick a boulder, the pain you feel is due to the force that the boulder exerts on your foot.

In each of these cases, the force that you exert on the other body is in the opposite direction to the force that body exerts on you. Experiments show that whenever two bodies interact, the two forces that they exert on each other are always *equal in magnitude* and *opposite in direction*. This fact is called *Newton's third law of motion:*

Newton's third law of motion: If body *A* exerts a force on body *B* (an "action"), then body *B* exerts a force on body *A* (a "reaction"). These two forces have the same magnitude but are opposite in direction. These two forces act on *different* bodies.

For example, in Fig. 4.25 $\vec{F}_{A\text{ on }B}$ is the force applied *by* body *A* (first subscript) *on* body *B* (second subscript), and $\vec{F}_{B\text{ on }A}$ is the force applied *by* body *B* (first subscript) *on* body *A* (second subscript). The mathematical statement of Newton's third law is

$$\vec{F}_{A\text{ on }B} = -\vec{F}_{B\text{ on }A} \qquad \text{(Newton's third law of motion)} \qquad (4.11)$$



4.25 If body *A* exerts a force $\vec{F}_{A\text{ on }B}$ on body *B*, then body *B* exerts a force $\vec{F}_{B\text{ on }A}$ on body *A* that is equal in magnitude and opposite in direction: $\vec{F}_{A\text{ on }B} = -\vec{F}_{B\text{ on }A}$.

It doesn't matter whether one body is inanimate (like the soccer ball in Fig. 4.25) and the other is not (like the kicker): They necessarily exert forces on each other that obey Eq. (4.11). ?

In the statement of Newton's third law, "action" and "reaction" are the two opposite forces (in Fig. 4.25, $\vec{F}_{A\text{ on }B}$ and $\vec{F}_{B\text{ on }A}$); we sometimes refer to them as an **action–reaction pair.** This is *not* meant to imply any cause-and-effect relationship; we can consider either force as the "action" and the other as the "reaction." We often say simply that the forces are "equal and opposite," meaning that they have equal magnitudes and opposite directions.

CAUTION **The two forces in an action–reaction pair act on different bodies** We stress that the two forces described in Newton's third law act on *different* bodies. This is important in problems involving Newton's first or second law, which involve the forces that act on a body. For instance, the net force acting on the soccer ball in Fig. 4.25 is the vector sum of the weight of the ball and the force $\vec{F}_{A\text{ on }B}$ exerted by the kicker. You would not include the force $\vec{F}_{B\text{ on }A}$ because this force acts on the kicker, not on the ball.

In Fig. 4.25 the action and reaction forces are *contact* forces that are present only when the two bodies are touching. But Newton's third law also applies to *long-range* forces that do not require physical contact, such as the force of gravitational attraction. A table-tennis ball exerts an upward gravitational force on the earth that's equal in magnitude to the downward gravitational force the earth exerts on the ball. When you drop the ball, both the ball and the earth accelerate toward each other. The net force on each body has the same magnitude, but the earth's acceleration is microscopically small because its mass is so great. Nevertheless, it does move!

Conceptual Example 4.8 Which force is greater?

After your sports car breaks down, you start to push it to the nearest repair shop. While the car is starting to move, how does the force you exert on the car compare to the force the car exerts on you? How do these forces compare when you are pushing the car along at a constant speed?

SOLUTION

In *both* cases, the force you exert on the car is equal in magnitude and opposite in direction to the force the car exerts on you. It's true that you have to push harder to get the car going than to keep it going. But no matter how hard you push on the car, the car pushes just as hard back on you. Newton's third law gives the same result whether the two bodies are at rest, moving with constant velocity, or accelerating.

You may wonder how the car "knows" to push back on you with the same magnitude of force that you exert on it. It may help to remember that the forces you and the car exert on each other are really interactions between the atoms at the surface of your hand and the atoms at the surface of the car. These interactions are analogous to miniature springs between adjacent atoms, and a compressed spring exerts equally strong forces on both of its ends.

Fundamentally, though, the reason we know that objects of different masses exert equally strong forces on each other is that *experiment tells us so.* Never forget that physics isn't merely a collection of rules and equations; rather, it's a systematic description of the natural world based on experiment and observation.

Conceptual Example 4.9 Applying Newton's third law: Objects at rest

An apple sits on a table in equilibrium. What forces act on it? What is the reaction force to each of the forces acting on the apple? What are the action–reaction pairs?

SOLUTION

Figure 4.26a shows the forces acting on the apple. In the diagram, $\vec{F}_{\text{earth on apple}}$ is the weight of the apple—that is, the downward gravitational force exerted *by* the earth (first subscript) *on* the apple (second subscript). Similarly, $\vec{F}_{\text{table on apple}}$ is the upward force exerted *by* the table (first subscript) *on* the apple (second subscript).

As the earth pulls down on the apple, the apple exerts an equally strong upward pull $\vec{F}_{\text{apple on earth}}$ on the earth, as shown in Fig. 4.26b. $\vec{F}_{\text{apple on earth}}$ and $\vec{F}_{\text{earth on apple}}$ are an action–reaction pair, representing the mutual interaction of the apple and the earth, so

$$\vec{F}_{\text{apple on earth}} = -\vec{F}_{\text{earth on apple}}$$

Also, as the table pushes up on the apple with force $\vec{F}_{\text{table on apple}}$, the corresponding reaction is the downward force $\vec{F}_{\text{apple on table}}$ exerted by the apple on the table (Fig. 4.26c). So we have

$$\vec{F}_{\text{apple on table}} = -\vec{F}_{\text{table on apple}}$$

The two forces acting on the apple are $\vec{F}_{\text{table on apple}}$ and $\vec{F}_{\text{earth on apple}}$. Are they an action–reaction pair? No, they aren't, despite being equal and opposite. They do not represent the mutual interaction of two bodies; they are two different forces acting on the *same* body. *The two forces in an action–reaction pair **never** act on the same body.* Here's another way to look at it. Suppose we suddenly yank the table out from under the apple (Fig. 4.26d). The two forces $\vec{F}_{\text{apple on table}}$ and $\vec{F}_{\text{table on apple}}$ then become zero, but $\vec{F}_{\text{apple on earth}}$ and $\vec{F}_{\text{earth on apple}}$ are still there (the gravitational interaction is still present). Since $\vec{F}_{\text{table on apple}}$ is now zero, it can't be the negative of $\vec{F}_{\text{earth on apple}}$, and these two forces can't be an action–reaction pair.

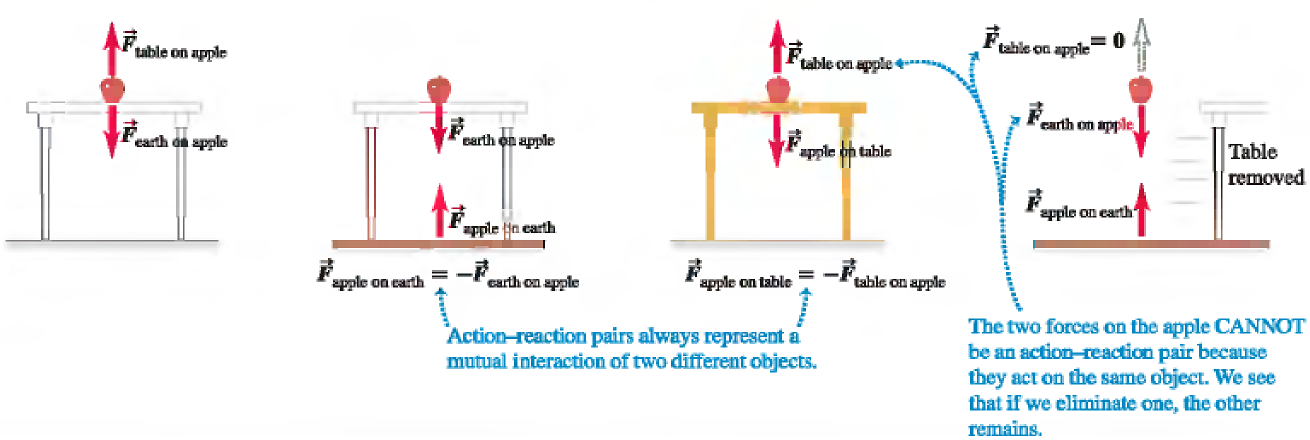
4.26 The two forces in an action–reaction pair always act on different bodies.

(a) The forces acting on the apple

(b) The action–reaction pair for the interaction between the apple and the earth

(c) The action–reaction pair for the interaction between the apple and the table

(d) We eliminate one of the forces acting on the apple



Conceptual Example 4.10 Applying Newton's third law: Objects in motion

A stonemason drags a marble block across a floor by pulling on a rope attached to the block (Fig. 4.27a). The block may or may not be in equilibrium. How are the various forces related? What are the action–reaction pairs?

SOLUTION

We'll use subscripts on all the forces to help explain things: B for the block, R for the rope, and M for the mason. Vector $\vec{F}_{\mathrm{M\,on\,R}}$ represents the force exerted by the *mason* on the *rope*. Its reaction is the equal and opposite force $\vec{F}_{\mathrm{R\,on\,M}}$ exerted by the *rope* on the *mason*. Vector $\vec{F}_{\mathrm{R\,on\,B}}$ represents the force exerted by the *rope* on the *block*. The reaction to it is the equal and opposite force $\vec{F}_{\mathrm{B\,on\,R}}$ exerted by the *block* on the *rope*. For these two action-reaction pairs (Fig. 4.27b), we have

$$\vec{F}_{\mathrm{R\,on\,M}} = -\vec{F}_{\mathrm{M\,on\,R}} \qquad \text{and} \qquad \vec{F}_{\mathrm{B\,on\,R}} = -\vec{F}_{\mathrm{R\,on\,B}}$$

Be sure you understand that the forces $\vec{F}_{\mathrm{M\,on\,R}}$ and $\vec{F}_{\mathrm{B\,on\,R}}$ are *not* an action–reaction pair (Fig. 4.27c) because both of these forces act on the *same* body (the rope); an action and its reaction *must* always act on *different* bodies. Furthermore, the forces $\vec{F}_{\mathrm{M\,on\,R}}$ and $\vec{F}_{\mathrm{B\,on\,R}}$ are not necessarily equal in magnitude. Applying Newton's second law to the rope, we get

$$\sum\vec{F} = \vec{F}_{\mathrm{M\,on\,R}} + \vec{F}_{\mathrm{B\,on\,R}} = m_{\mathrm{rope}}\vec{a}_{\mathrm{rope}}$$

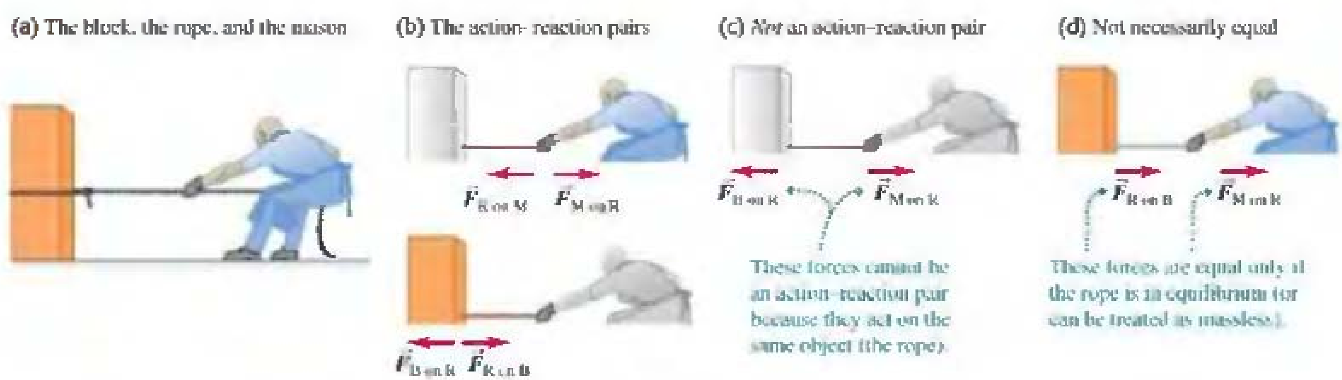
If the block and rope are accelerating (that is, speeding up or slowing down), the rope is not in equilibrium, and $\vec{F}_{\mathrm{M\,on\,R}}$ must have a different magnitude than $\vec{F}_{\mathrm{B\,on\,R}}$. By contrast, the action–reaction forces $\vec{F}_{\mathrm{M\,on\,R}}$ and $\vec{F}_{\mathrm{R\,on\,M}}$ are always equal in magnitude, as are $\vec{F}_{\mathrm{R\,on\,B}}$ and $\vec{F}_{\mathrm{B\,on\,R}}$. Newton's third law holds whether or not the bodies are accelerating.

In the special case in which the rope is in equilibrium, the forces $\vec{F}_{\mathrm{M\,on\,R}}$ and $\vec{F}_{\mathrm{B\,on\,R}}$ are equal in magnitude. But this is an example of Newton's *first* law, not his *third*. Another way to look at this is that in equilibrium, $\vec{a}_{\mathrm{rope}} = \mathbf{0}$ in the preceding equation. Then $\vec{F}_{\mathrm{B\,on\,R}} = -\vec{F}_{\mathrm{M\,on\,R}}$ because of Newton's first or second law.

This is also true if the rope is accelerating but has negligibly small mass compared to the block or the mason. In this case, $m_{\mathrm{rope}} = 0$ in the above equation, so again $\vec{F}_{\mathrm{B\,on\,R}} = -\vec{F}_{\mathrm{M\,on\,R}}$. Since $\vec{F}_{\mathrm{B\,on\,R}}$ *always* equals $-\vec{F}_{\mathrm{R\,on\,B}}$ by Newton's third law (they are an action–reaction pair), in these same special cases, $\vec{F}_{\mathrm{R\,on\,B}}$ also equals $\vec{F}_{\mathrm{M\,on\,R}}$ (Fig. 4.27d). In other words, in these cases the force of the rope on the block equals the force of the mason on the rope, and we can then think of the rope as "transmitting" to the block, without change, the force the person exerts on the rope. This is a useful point of view, but you have to remember that it is valid *only* when the rope has negligibly small mass or is in equilibrium.

If you feel as though you're drowning in subscripts at this point, take heart. Go over this discussion again, comparing the symbols with the vector diagrams, until you're sure you see what's going on.

4.27 Identifying the forces that act when a mason pulls on a rope attached to a block.



Conceptual Example 4.11 A Newton's third law paradox?

We saw in Conceptual Example 4.10 that the stonemason pulls as hard on the rope–block combination as that combination pulls back on him. Why, then, does the block move while the stonemason remains stationary?

SOLUTION

The way out of this seeming conundrum is to keep in mind the difference between Newton's *second* law and his *third* law. The only forces involved in Newton's second law are those that act *on* that body. The vector sum of these forces determines how the body accelerates (and whether it accelerates at all). By contrast, Newton's third law relates the forces that two *different* bodies exert on each other. The third law alone tells you nothing about the motion of either body.

If the rope–block combination is initially at rest, it begins to slide if the stonemason exerts a force $\vec{F}_{\mathrm{M\,on\,R}}$ that is *greater* in magnitude than the friction force that the floor exerts on the block (Fig. 4.28). (The marble block has a smooth underside, which helps to minimize friction.) Hence there is a net force on the rope–block combination to the right, and so it accelerates to the right. By contrast, the stonemason *doesn't* move because the net force acting on him is *zero*. His shoes have nonskid soles that don't

Continued

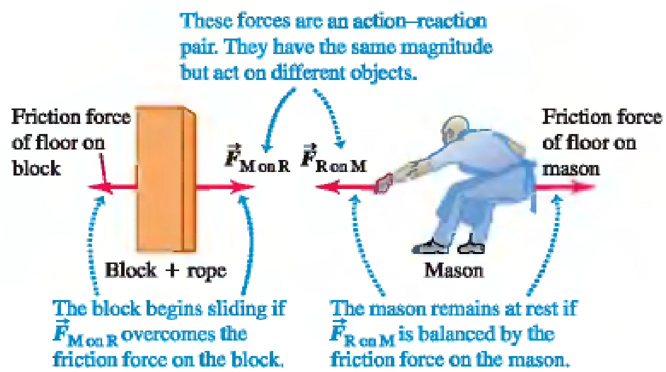
slip on the floor, so the friction force that the floor exerts on him is strong enough to exactly balance the pull of the rope, $\vec{F}_{\text{R on M}}$. (Both the block and the stonemason also experience a downward force of gravity and an upward normal force exerted by the floor. These balance each other and cancel out, so we haven't included them in Fig. 4.28.)

Once the block is moving, the stonemason doesn't need to pull quite so hard; he need exert only enough force to exactly balance the friction force on the block. Then the net force on the moving block is zero, and the block continues to move toward the mason at a constant velocity in accordance with Newton's first law.

We conclude that the block moves while the stonemason doesn't because different amounts of friction act on them. If the floor were freshly waxed, so that there was little friction between the floor and the stonemason's shoes, pulling on the rope would start the block sliding to the right *and* start him sliding to the left.

The moral of this example is that when analyzing the motion of a body, remember that only forces acting *on* the body determine its motion. From this perspective, Newton's third law is merely a tool that can help you determine what those forces are.

4.28 The horizontal forces acting on the block–rope combination (left) and the mason (right). (The vertical forces are not shown.)



A body, such as the rope in Fig. 4.27, that has pulling forces applied at its ends is said to be in *tension*. The **tension** at any point is the magnitude of force acting at that point (see Fig. 4.2c). In Fig. 4.27b the tension at the right end of the rope is the magnitude of $\vec{F}_{\text{M on R}}$ (or of $\vec{F}_{\text{R on M}}$), and the tension at the left end equals the magnitude of $\vec{F}_{\text{B on R}}$ (or of $\vec{F}_{\text{R on B}}$). If the rope is in equilibrium and if no forces act except at its ends, the tension is the *same* at both ends and throughout the rope. Thus, if the magnitudes of $\vec{F}_{\text{B on R}}$ and $\vec{F}_{\text{M on R}}$ are 50 N each, the tension in the rope is 50 N (*not* 100 N). The *total* force vector $\vec{F}_{\text{B on R}} + \vec{F}_{\text{M on R}}$ acting on the rope in this case is zero!

We emphasize once more a fundamental truth: The two forces in an action–reaction pair *never* act on the same body. Remembering this simple fact can often help you avoid confusion about action–reaction pairs and Newton's third law.

Test Your Understanding of Section 4.5 You are driving your car on a country road when a mosquito splatters itself on the windshield. Which has the greater magnitude, the force that the car exerted on the mosquito or the force that the mosquito exerted on the car? Or are the magnitudes the same? If they are different, how can you reconcile this fact with Newton's third law? If they are equal, why is the mosquito splattered while the car is undamaged?

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2.1.1 Force Magnitudes

4.6 Free-Body Diagrams

Newton's three laws of motion contain all the basic principles we need to solve a wide variety of problems in mechanics. These laws are very simple in form, but the process of applying them to specific situations can pose real challenges. In this brief section we'll point out three key ideas and techniques to use in any problems involving Newton's laws. You'll learn others in Chapter 5, which also extends the use of Newton's laws to cover more complex situations.

1. *Newton's first and second laws apply to a specific body.* Whenever you use Newton's first law, $\sum\vec{F} = \mathbf{0}$, for an equilibrium situation or Newton's second law, $\sum\vec{F} = m\vec{a}$, for a nonequilibrium situation, you must decide at the

beginning to which body you are referring. This decision may sound trivial, but it isn't.

2. *Only forces acting on the body matter.* The sum $\sum\vec{F}$ includes all the forces that act *on* the body in question. Hence, once you've chosen the body to analyze, you have to identify all the forces acting on it. Don't get confused between the forces acting on a body and the forces exerted by that body on some other body. For example, to analyze a person walking, you would include in $\sum\vec{F}$ the force that the ground exerts on the person as he walks, but *not* the force that the person exerts on the ground (Fig. 4.29). These forces form an action–reaction pair and are related by Newton's third law, but only the member of the pair that acts on the body you're working with goes into $\sum\vec{F}$.
3. *Free-body diagrams are essential to help identify the relevant forces.* A **free-body diagram** is a diagram showing the chosen body by itself, "free" of its surroundings, with vectors drawn to show the magnitudes and directions of all the forces applied to the body by the various other bodies that interact with it. We have already shown some free-body diagrams in Figs. 4.18, 4.19, 4.21, and 4.26a. Be careful to include all the forces acting *on* the body, but be equally careful *not* to include any forces that the body exerts on any other body. In particular, the two forces in an action–reaction pair must *never* appear in the same free-body diagram because they never act on the same body. Furthermore, forces that a body exerts on itself are never included, since these can't affect the body's motion.

CAUTION **Forces in free-body diagrams** When you have a complete free-body diagram, you *must* be able to answer for each force the question: What other body is applying this force? If you can't answer that question, you may be dealing with a nonexistent force. Be especially on your guard to avoid nonexistent forces such as "the force of acceleration" or "the $m\vec{a}$ force," discussed in Section 4.3.

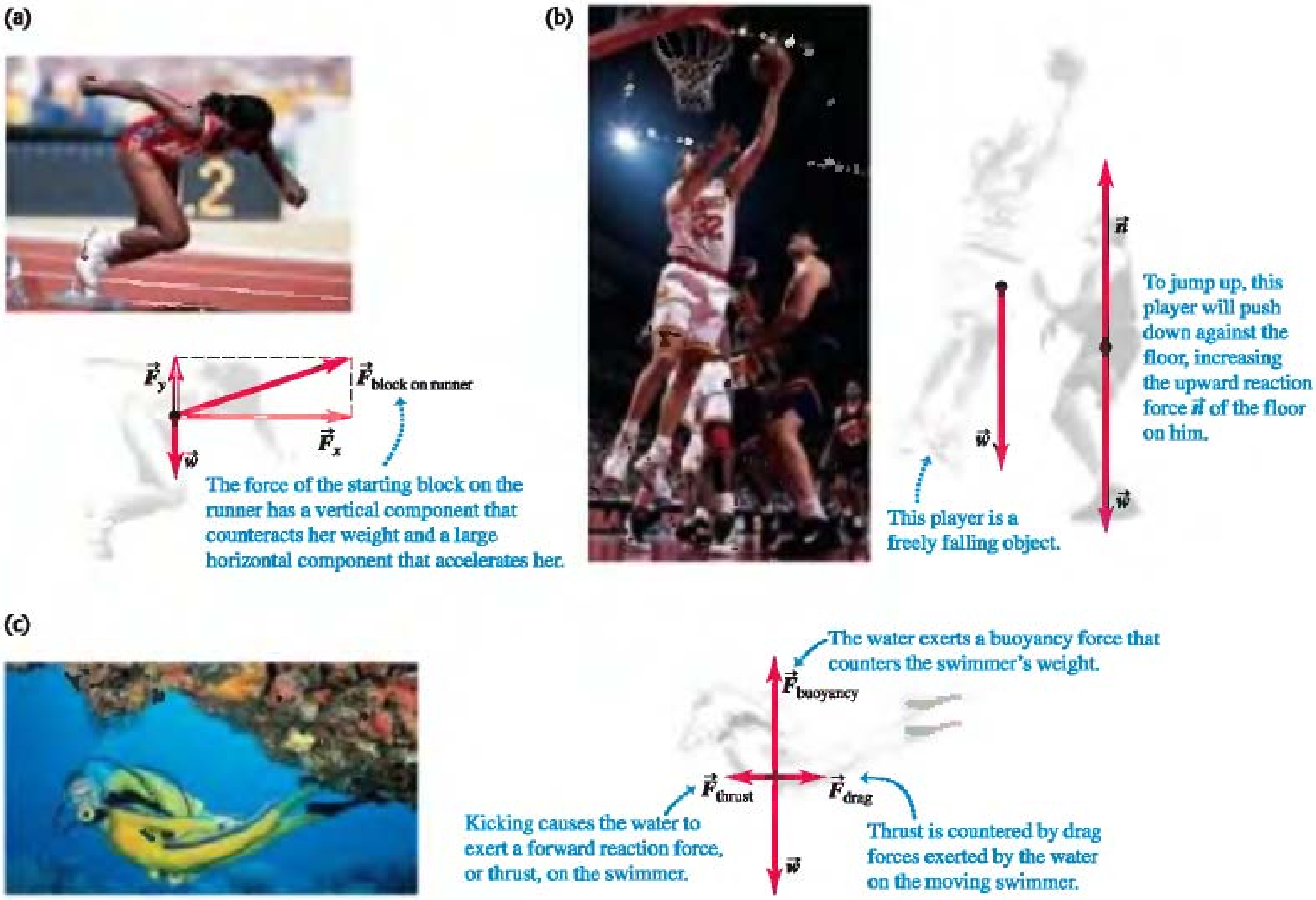
When a problem involves more than one body, you have to take the problem apart and draw a separate free-body diagram for each body. For example, Fig. 4.27c shows a separate free-body diagram for the rope in the case in which the rope is considered massless (so that no gravitational force acts on it). Figure 4.28 also shows diagrams for the block and the mason, but these are *not* complete free-body diagrams because they don't show all the forces acting on each body. (We left out the vertical forces—the weight force exerted by the earth and the upward normal force exerted by the floor.)

Figure 4.30 on page 128 presents some real-life situations and the corresponding complete free-body diagrams. Note that in each situation a person exerts a force on something in his or her surroundings, but the force that shows up in the person's free-body diagram is the surroundings pushing back *on* the person.

Test Your Understanding of Section 4.6 The buoyancy force shown in Fig. 4.30c is one half of an action–reaction pair. What force is the other half of this pair? (i) the weight of the swimmer; (ii) the forward thrust force; (iii) the backward drag force; (iv) the downward force that the swimmer exerts on the water; (v) the backward force that the swimmer exerts on the water by kicking.

4.29 The simple act of walking depends crucially on Newton's third law. To start moving forward, you push backward on the ground with your foot. As a reaction, the ground pushes forward on your foot (and hence on your body as a whole) with a force of the same magnitude. This *external* force provided by the ground is what accelerates your body forward.

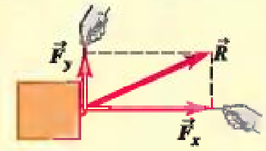
4.30 Examples of free-body diagrams. In each case, the free-body diagram shows all the external forces that act on the object in question.



CHAPTER 4 SUMMARY

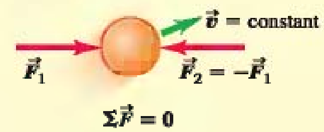
Force as a vector: Force is a quantitative measure of the interaction between two bodies. It is a vector quantity. When several forces act on a body, the effect on its motion is the same as when a single force, equal to the vector sum (resultant) of the forces, acts on the body. (See Example 4.1.)

$$\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots = \sum \vec{F} \quad (4.1)$$



The net force on a body end Newton's first law: Newton's first law states that when the vector sum of all forces acting on a body (the *net force*) is zero, the body is in equilibrium and has zero acceleration. If the body is initially at rest, it remains at rest; if it is initially in motion, it continues to move with constant velocity. This law is valid only in inertial frames of reference. (See Examples 4.2 and 4.3.)

$$\sum \vec{F} = 0 \quad (4.3)$$



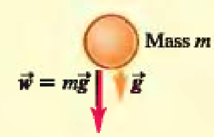
Mess, acceleretion, end Newton's second law: The inertial properties of a body are characterized by its *mass*. The acceleration of a body under the action of a given set of forces is directly proportional to the vector sum of the forces (the *net force*) and inversely proportional to the mass of the body. This relationship is Newton's second law. Like Newton's first law, this law is valid only in inertial frames of reference. The unit of force is defined in terms of the units of mass and acceleration. In SI units, the unit of force is the newton (N), equal to 1 kg · m/s². (See Examples 4.4 and 4.5.)

$$\sum \vec{F} = m\vec{a} \quad (4.7)$$

$$\sum F_x = ma_x \qquad \sum F_y = ma_y \qquad \sum F_z = ma_z \quad (4.8)$$

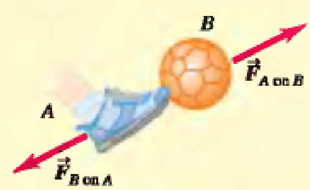
Weight: The weight $\vec{w}$ of a body is the gravitational force exerted on it by the earth. Weight is a vector quantity. The magnitude of the weight of a body at any specific location is equal to the product of its mass m and the magnitude of the acceleration due to gravity g at that location. While the weight of a body depends on its location, the mass is independent of location. (See Examples 4.6 and 4.7.)

$$w = mg \quad (4.9)$$



Newton's third law and action–reaction pairs: Newton's third law states that when two bodies interact, they exert forces on each other that at each instant are equal in magnitude and opposite in direction. These forces are called action and reaction forces. Each of these two forces acts on only one of the two bodies; they never act on the same body. (See Examples 4.8–4.11.)

$$\vec{F}_{A \text{ on } B} = -\vec{F}_{B \text{ on } A} \quad (4.11)$$



Key Terms

dynamics, *107*
Newton's laws of motion, *107*
classical (Newtonian) mechanics, *107*
force, *108*
contact force, *108*
normal force, *108*
friction force, *108*
tension force, *108*
long-range forces, *108*
weight, *108*
superposition of forces, *109*
net force, *110*
Newton's first law of motion, *111*
inertia, *112*
equilibrium, *112*
inertial frame of reference, *113*
mass, *116*
kilogram, *116*
newton, *116*
Newton's second law of motion, *117*
Newton's third law of motion, *123*
action–reaction pair, *123*
tension, *126*
free-body diagram, *127*

Answer to Chapter Opening Question ?

Newton's third law tells us that the seated child (who we'll call Ryder) pushes on the standing child (who we'll call Stan) just as hard as Stan pushes on Ryder, but in the opposite direction. This is true whether Ryder pushes on Stan "actively" (for instance, if Ryder pushed his hand against Stan's) or "passively" (if Ryder's back does the pushing, as in the photograph that opens the chapter). The force magnitudes would be greater in the "active" case than in the "passive" case, but either way Ryder's push on Stan is just as strong as Stan's push on Ryder.

Answers to Test Your Understanding Questions

4.1 Answer: (iv) The gravitational force on the crate points straight downward. In Fig. 4.6 the x-axis points up and to the right, and the y-axis points up and to the left. Hence the gravitational force has both an x-component and a y-component, and both are negative.

4.2 Answer: (i), (ii), and (iv) In (i), (ii), and (iv) the body is not accelerating, so the net force on the body is zero. [In (iv), the box remains stationary as seen in the inertial reference frame of the ground as the truck accelerates forward, like the skater in Fig. 4.11a.] In (iii), the hawk is moving in a circle; hence it is accelerating and is *not* in equilibrium.

4.3 Answer: (iii), (i) and (iv) (tie), (ii) The acceleration is equal to the net force divided by the mass. Hence the magnitude of the acceleration in each situation is
(i) $a = (2.0\,\text{N})/(2.0\,\text{kg}) = 1.0\,\text{m/s}^2$;
(ii) $a = (8.0\,\text{N})/(2.0\,\text{N}) = 4.0\,\text{m/s}^2$;
(iii) $a = (2.0\,\text{N})/(8.0\,\text{kg}) = 0.25\,\text{m/s}^2$;
(iv) $a = (8.0\,\text{N})/(8.0\,\text{kg}) = 1.0\,\text{m/s}^2$.

4.4 It would take twice the effort for the astronaut to walk around because her weight on the planet would be twice as much as on the earth. But it would be just as easy to catch a ball moving horizontally. The ball's *mass* is the same as on earth, so the horizontal force the astronaut would have to exert to bring it to a stop (i.e., to give it the same acceleration) would also be the same as on earth.

4.5 By Newton's third law, the two forces have equal magnitudes. Because the car has much greater mass than the mosquito, it undergoes only a tiny, imperceptible acceleration in response to the force of the impact. By contrast, the mosquito, with its minuscule mass, undergoes a catastrophically large acceleration.

4.6 Answer: (iv) The buoyancy force is an *upward* force that the *water* exerts on the *swimmer.* By Newton's third law, the other half of the action–reaction pair is a *downward* force that the *swimmer* exerts on the *water* and has the same magnitude as the buoyancy force. It's true that the weight of the swimmer is also downward and has the same magnitude as the buoyancy force; however, the weight acts on the same body (the swimmer) as the buoyancy force, and so these forces aren't an action–reaction pair.

PROBLEMS

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Discussion Questions

Q4.1. Can a body be in equilibrium when only one force acts on it? Explain.

Q4.2. A ball thrown straight up has zero velocity at its highest point. Is the ball in equilibrium at this point? Why or why not?

Q4.3. A helium balloon hovers in midair, neither ascending nor descending. Is it in equilibrium? What forces act on it?

Q4.4. When you fly in an airplane at night in smooth air, there is no sensation of motion, even though the plane may be moving at 800 km/h (500 mi/h). Why is this?

Q4.5. If the two ends of a rope in equilibrium are pulled with forces of equal magnitude and opposite direction, why is the total tension in the rope not zero?

Q4.6. You tie a brick to the end of a rope and whirl the brick around you in a horizontal circle. Describe the path of the brick after you suddenly let go of the rope.

Q4.7. When a car stops suddenly, the passengers tend to move forward relative to their seats. Why? When a car makes a sharp turn, the passengers tend to slide to one side of the car. Why?

Q4.8. Some people say that the "force of inertia" (or "force of momentum") throws the passengers forward when a car brakes sharply. What is wrong with this explanation?

Q4.9. A passenger in a moving bus with no windows notices that a ball that has been at rest in the aisle suddenly starts to move toward the rear of the bus. Think of two different possible explanations, and devise a way to decide which is correct.

Q4.10. Suppose you chose the fundamental SI units to be force, length, and time instead of mass, length, and time. What would be the units of mass in terms of those fundamental units?

Q4.11. Some of the ancient Greeks thought that the "natural state" of an object was to be at rest, so objects would seek their natural state by coming to rest if left alone. Explain why this view can actually seem quite plausible in the everyday world.

Q4.12. Why is the earth only approximately an inertial reference frame?

Q4.13. Does Newton's second law hold true for an observer in a van as it speeds up, slows down, or rounds a corner? Explain.

Q4.14. Some students refer to the quantity $m\vec{a}$ as "the force of acceleration." Is it correct to refer to this quantity as a force? If so, what exerts this force? If not, what is a better description of this quantity?

Q4.15. The acceleration of a falling body is measured in an elevator traveling upward at a constant speed of 9.8 m/s. What result is obtained?

Q4.16. You can play catch with a softball in a bus moving with constant speed on a straight road, just as though the bus were at rest. Is this still possible when the bus is making a turn at constant speed on a level road? Why or why not?

Q4.17. Students sometimes say that the force of gravity on an object is 9.8 m/s^2. What is wrong with this view?

Q4.18. The head of a hammer begins to come loose from its wooden handle. How should you strike the handle on a concrete sidewalk to reset the head? Why does this work?

Q4.19. Why can it hurt your foot more to kick a big rock than a small pebble? *Must* the big rock hurt more? Explain.

Q4.20. "It's not the fall that hurts you; it's the sudden stop at the bottom." Translate this saying into the language of Newton's laws of motion.

Q4.21. A person can dive into water from a height of 10 m without injury, but a person who jumps off the roof of a 10-m-tall building and lands on a concrete street is likely to be seriously injured. Why is there a difference?

Q4.22. Why are cars designed to crumple up in front and back for safety? Why not for side collisions and rollovers?

Q4.23. When a bullet is fired from a gun, what is the origin of the force that accelerates the bullet?

Q4.24. When a string barely strong enough lifts a heavy weight, it can lift the weight by a steady pull; but if you jerk the string, it will break. Explain in terms of Newton's laws of motion.

Q4.25. A large crate is suspended from the end of a vertical rope. Is the tension in the rope greater when the crate is at rest or when it is moving upward at constant speed? If the crate is traveling upward, is the tension in the rope greater when the crate is speeding up or when it is slowing down? In each case explain in terms of Newton's laws of motion.

Q4.26. Which feels a greater pull due to the earth's gravity, a 10-kg stone or a 20-kg stone? If you drop them, why does the 20-kg stone not fall with twice the acceleration of the 10-kg stone? Explain your reasoning.

Q4.27. Why is it incorrect to say that 1.0 kg *equals* 2.2 lb?

Q4.28. A horse is hitched to a wagon. Since the wagon pulls back on the horse just as hard as the horse pulls on the wagon, why does the wagon not remain in equilibrium, no matter how hard the horse pulls?

Q4.29. True or false? You exert a push P on an object and it pushes back on you with a force F. If the object is moving at constant velocity, then F is equal to P, but if the object is being accelerated, then P must be greater than F.

Q4.30. A large truck and a small compact car have a head-on collision. During the collision, the truck exerts a force $\vec{F}_{\mathrm{T\,on\,C}}$ on the car, and the car exerts a force $\vec{F}_{\mathrm{C\,on\,T}}$ on the truck. Which force has the larger magnitude, or are they the same? Does your answer depend on how fast each vehicle was moving before the collision? Why or why not?

Q4.31. When a car comes to a stop on a level highway, what force causes it to slow down? When the car increases its speed on the same highway, what force causes it to speed up? Explain.

Q4.32. A small compact car is pushing a large van that has broken down, and they travel along the road with equal velocities and accelerations. While the car is speeding up, is the force it exerts on the van larger than, smaller than, or the same magnitude as the force the van exerts on it? Which object, the car or the van, has the larger net force on it, or are the net forces the same? Explain.

Q4.33. Consider a tug-of-war between two people who pull in opposite directions on the ends of a rope. By Newton's third law, the force that A exerts on B is just as great as the force that B exerts on A. So what determines who wins? (*Hint:* Draw a free-body diagram showing all the forces that act on each person.)

Q4.34. On the moon, $g = 1.62$ m/s^2. If a 2-kg brick drops on your foot from a height of 2 m, will this hurt more, or less, or the same if it happens on the moon instead of on the earth? Explain. If a 2-kg brick is thrown and hits you when it is moving horizontally at 6 m/s, will this hurt more, less, or the same if it happens on the moon instead of on the earth? Explain. (On the moon, assume that you are inside a pressurized structure, so you are not wearing a spacesuit.)

Q4.35. A manual for student pilots contains the following passage: "When an airplane flies at a steady altitude, neither climbing nor descending, the upward lift force from the wings equals the airplane's weight. When the airplane is climbing at a steady rate, the upward lift is greater than the weight; when the airplane is descending at a steady rate, the upward lift is less than the weight." Are these statements correct? Explain.

Q4.36. If your hands are wet and no towel is handy, you can remove some of the excess water by shaking them. Why does this get rid of the water?

Q4.37. If you are squatting down (such as when you are examining the books on the bottom shelf in a library or bookstore) and suddenly get up, you can temporarily feel light-headed. What do Newton's laws of motion have to say about why this happens?

Q4.38. When a car is hit from behind, the passengers can receive a whiplash. Use Newton's laws of motion to explain what causes this to occur.

Q4.39. In a head-on auto collision, passengers not wearing seat belts can be thrown through the windshield. Use Newton's laws of motion to explain why this happens.

Q4.40. In a head-on collision between a compact 1000-kg car and a large 2500-kg car, which one experiences the greater force? Explain. Which one experiences the greater acceleration? Explain why. Now explain why passengers in the smaller car are more likely to be injured than those in the large car, even if the bodies of both cars are equally strong.

Q4.41. Suppose you are in a rocket with no windows, traveling in deep space far from any other objects. Without looking outside the rocket or making any contact with the outside world, explain how you could determine if the rocket is (a) moving forward at a constant 80% of the speed of light and (b) accelerating in the forward direction.

Exercises

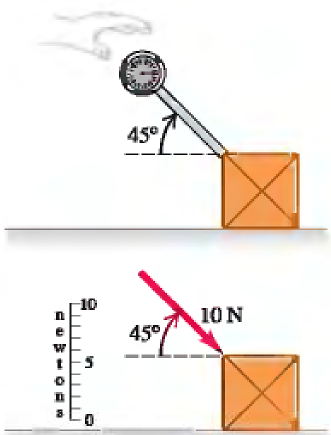
Section 4.1 Force and Interactions

4.1. Two forces have the same magnitude F. What is the angle between the two vectors if their sum has a magnitude of (a) $2F$? (b) $\sqrt{2}F$? (c) zero? Sketch the three vectors in each case.

4.2. Instead of using the x- and y-axes of Fig. 4.8 to analyze the situation of Example 4.1, use axes rotated 37.0° counterclockwise, so the y-axis is parallel to the 250-N force. (a) For these axes find the x- and y-components of the net force on the belt. (b) From the components computed in part (a) find the magnitude and direction of the net force. Compare your results to Example 4.1.

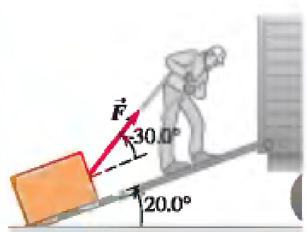
4.3. A warehouse worker pushes a crate along the floor, as shown in Fig. 4.31, with a force of 10 N that points downward at an angle of 45° below the horizontal. Find the horizontal and vertical components of the force.

Figure 4.31 Exercise 4.3.



4.4. A man is dragging a trunk up the loading ramp of a mover's truck. The ramp has a slope angle of 20.0°, and the man pulls upward with a force $\vec{F}$ whose direction makes an angle of 30.0° with the ramp (Fig. 4.32). (a) How large a force $\vec{F}$ is necessary for the component F_x parallel to the ramp to be 60.0 N? (b) How large will the component F_y perpendicular to the ramp then be?

Figure 4.32 Exercise 4.4.



4.5. Two dogs pull horizontally on ropes attached to a post; the angle between the ropes is 60.0°. If dog *A* exerts a force of 270 N and dog *B* exerts a force of 300 N, find the magnitude of the resultant force and the angle it makes with dog *A*'s rope.

4.6. Two forces, $\vec{F}_1$ and $\vec{F}_2$, act at a point. The magnitude of $\vec{F}_1$ is 9.00 N, and its direction is 60.0° above the *x*-axis in the second quadrant. The magnitude of $\vec{F}_2$ is 6.00 N, and its direction is 53.1° below the *x*-axis in the third quadrant. (a) What are the *x*- and *y*-components of the resultant force? (b) What is the magnitude of the resultant force?

Section 4.3 Newton's Second Law

4.7. If a net horizontal force of 132 N is applied to a person with mass 60 kg who is resting on the edge of a swimming pool, what horizontal acceleration is produced?

4.8. What magnitude of net force is required to give a 135-kg refrigerator an acceleration of magnitude 1.40 m/s^2?

4.9. A box rests on a frozen pond, which serves as a frictionless horizontal surface. If a fisherman applies a horizontal force with magnitude 48.0 N to the box and produces an acceleration of magnitude 3.00 m/s^2, what is the mass of the box?

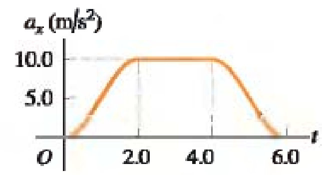
4.10. A dockworker applies a constant horizontal force of 80.0 N to a block of ice on a smooth horizontal floor. The frictional force is negligible. The block starts from rest and moves 11.0 m in 5.00 s. (a) What is the mass of the block of ice? (b) If the worker stops pushing at the end of 5.00 s, how far does the block move in the next 5.00 s?

4.11. A hockey puck with mass 0.160 kg is at rest at the origin $(x = 0)$ on the horizontal, frictionless surface of the rink. At time $t = 0$ a player applies a force of 0.250 N to the puck, parallel to the *x*-axis; he continues to apply this force until $t = 2.00$ s. (a) What are the position and speed of the puck at $t = 2.00$ s? (b) If the same force is again applied at $t = 5.00$ s, what are the position and speed of the puck at $t = 7.00$ s?

4.12. A crate with mass 32.5 kg initially at rest on a warehouse floor is acted on by a net horizontal force of 140 N. (a) What acceleration is produced? (b) How far does the crate travel in 10.0 s? (c) What is its speed at the end of 10.0 s?

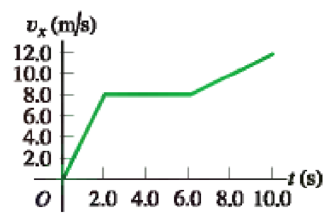
4.13. A 4.50-kg toy cart undergoes an acceleration in a straight line (the *x*-axis). The graph in Fig. 4.33 shows this acceleration as a function of time. (a) Find the maximum net force on this cart. When does this maximum force occur? (b) During what times is the net force on the cart a constant? (c) When is the net force equal to zero?

Figure 4.33 Exercise 4.13.



4.14. A 2.75-kg cat moves in a straight line (the *x*-axis). Figure 4.34 shows a graph of the *x*-component of this cat's velocity as a function of time. (a) Find the maximum net force on this cat. When does this force occur? (b) When is the net force on the cat equal to zero? (c) What is the net force at time 8.5 s?

Figure 4.34 Exercise 4.14.



4.15. A small 8.00-kg rocket burns fuel that exerts a time-varying upward force on the rocket. This force obeys the equation $F = A + Bt^2$. Measurements show that at $t = 0$, the force is 100.0 N, and at the end of the first 2.00 s, it is 150.0 N. (a) Find the constants A and B, including their SI units. (b) Find the *net* force on this rocket and its acceleration (i) the instant after the fuel ignites and (ii) 3.00 s after fuel ignition. (c) Suppose you were using this rocket in outer space, far from all gravity. What would its acceleration be 3.00 s after fuel ignition?

4.16. An electron $(\text{mass} = 9.11 \times 10^{-31}\ \text{kg})$ leaves one end of a TV picture tube with zero initial speed and travels in a straight line to the accelerating grid, which is 1.80 cm away. It reaches the grid with a speed of 3.00×10^6 m/s. If the accelerating force is constant, compute (a) the acceleration; (b) the time to reach the grid; (c) the net force, in newtons. (You can ignore the gravitational force on the electron.)

Section 4.4 Mass and Weight

4.17. Superman throws a 2400-N boulder at an adversary. What horizontal force must Superman apply to the boulder to give it a horizontal acceleration of 12.0 m/s^2?

4.18. A bowling ball weighs 71.2 N (16.0 lb). The bowler applies a horizontal force of 160 N (36.0 lb) to the ball. What is the magnitude of the horizontal acceleration of the ball?

4.19. At the surface of Jupiter's moon Io, the acceleration due to gravity is $g = 1.81\ \text{m/s}^2$. A watermelon weighs 44.0 N at the surface of the earth. (a) What is the watermelon's mass on the earth's surface? (b) What are its mass and weight on the surface of Io?

4.20. An astronaut's pack weighs 17.5 N when she is on earth but only 3.24 N when she is at the surface of an asteroid. (a) What is the acceleration due to gravity on this asteroid? (b) What is the mass of the pack on the asteroid?

Section 4.5 Newton's Third Law

4.21. World-class sprinters can accelerate out of the starting blocks with an acceleration that is nearly horizontal and has magnitude $15\ \text{m/s}^2$. How much horizontal force must a 55-kg sprinter exert on the starting blocks during a start to produce this acceleration? Which body exerts the force that propels the sprinter: the blocks or the sprinter herself?

4.22. Imagine that you are holding a book weighing 4 N at rest on the palm of your hand. Complete the following sentences: (a) A downward force of magnitude 4 N is exerted on the book by ___________. (b) An upward force of magnitude ___________ is exerted on ___________ by your hand. (c) Is the upward force in part (b) the reaction to the downward force in part (a)? (d) The reaction to the force in part (a) is a force of magnitude ___________, exerted on ___________ by ___________. Its direction is ___________. (e) The reaction to the force in part (b) is a force of magnitude ___________, exerted on ___________ by ___________. Its direction is ___________. (f) The forces in parts (a) and (b) are equal and opposite because of Newton's ___________ law. (g) The forces in parts (b) and (e) are equal and opposite because of Newton's ___________ law. Now suppose that you exert an upward force of magnitude 5 N on the book. (h) Does the book remain in equilibrium? (i) Is the force exerted on the book by your hand equal and opposite to the force exerted on the book by the earth? (j) Is the force exerted on the book by the earth equal and opposite to the force exerted on the earth by the book? (k) Is the force exerted on the book by your hand equal and opposite to the force exerted on your hand by the book? Finally, suppose you snatch your hand away while the book is moving upward. (l) How many forces then act on the book? (m) Is the book in equilibrium?

4.23. A bottle is given a push along a tabletop and slides off the edge of the table. Do *not* ignore air resistance. (a) What forces are exerted on the bottle while it is falling from the table to the floor? (b) What is the reaction to each force; that is, on which body and by which body is the reaction exerted?

4.24. The upward normal force exerted by the floor is 620 N on an elevator passenger who weighs 650 N. What are the reaction forces to these two forces? Is the passenger accelerating? If so, what are the magnitude and direction of the acceleration?

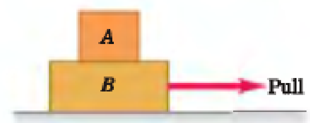
4.25. A student with mass 45 kg jumps off a high diving board. Using 6.0×10^{24} kg for the mass of the earth, what is the acceleration of the earth toward her as she accelerates toward the earth with an acceleration of $9.8\ \text{m/s}^2$? Assume that the net force on the earth is the force of gravity she exerts on it.

Section 4.6 Free-Body Diagrams

4.26. An athlete throws a ball of mass m directly upward, and it feels no appreciable air resistance. Draw a free-body diagram of this ball while it is free of the athlete's hand and (a) moving upward; (b) at its highest point; (c) moving downward. (d) Repeat parts (a), (b), and (c) if the athlete throws the ball at a 60° angle above the horizontal instead of directly upward.

4.27. Two crates, A and B, sit at rest side by side on a frictionless horizontal surface. The crates have masses m_A and m_B. A horizontal force $\vec{F}$ is applied to crate A and the two crates move off to the right. (a) Draw clearly labeled free-body diagrams for crate A and for crate B. Indicate which pairs of forces, if any, are third-law action–reaction pairs. (b) If the magnitude of force $\vec{F}$ is less than the total weight of the two crates, will it cause the crates to move? Explain.

Figure 4.35 Exercise 4.28.



4.20. A person pulls horizontally on block B in Fig. 4.35, causing both blocks to move together as a unit. While this system is moving, make a carefully labeled free-body diagram of block A if (a) the table is frictionless and (b) there is friction between block B and the table and the pull is equal to the friction force on block B due to the table.

4.29. A ball is hanging from a long string that is tied to the ceiling of a train car traveling eastward on horizontal tracks. An observer inside the train car sees the ball hang motionless. Draw a clearly labeled free-body diagram for the ball if (a) the train has a uniform velocity, and (b) the train is speeding up uniformly. Is the net force on the ball zero in either case? Explain.

4.30. A large box containing your new computer sits on the bed of your pickup truck. You are stopped at a red light. The light turns green and you stomp on the gas and the truck accelerates. To your horror, the box starts to slide toward the back of the truck. Draw clearly labeled free-body diagrams for the truck and for the box. Indicate pairs of forces, if any, that are third-law action–reaction pairs. (The bed of the truck is *not* frictionless.)

4.31. A chair of mass 12.0 kg is sitting on the horizontal floor; the floor is not frictionless. You push on the chair with a force $F = 40.0$ N that is directed at an angle of 37.0° below the horizontal and the chair slides along the floor. (a) Draw a clearly labeled free-body diagram for the chair. (b) Use your diagram and Newton's laws to calculate the normal force that the floor exerts on the chair.

4.32. A skier of mass 65.0 kg is pulled up a snow-covered slope at constant speed by a tow rope that is parallel to the ground. The ground slopes upward at a constant angle of 26.0° above the horizontal, and you can ignore friction. (a) Draw a clearly labeled free-body diagram for the skier. (b) Calculate the tension in the tow rope.

4.33. A truck is pulling a car on a horizontal highway using a horizontal rope. The car is in neutral gear, so we can assume that there is no appreciable friction between its tires and the highway. As the truck is accelerating to highway speeds, draw a free-body diagram of (a) the car and (b) the truck. (c) What force accelerates this system forward? Explain how this force originates.

Problems

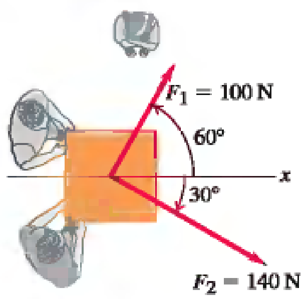
4.34. A .22 rifle bullet, traveling at 350 m/s, strikes a large tree, which it penetrates to a depth of 0.130 m. The mass of the bullet is 1.80 g. Assume a constant retarding force. (a) How much time is required for the bullet to stop? (b) What force, in newtons, does the tree exert on the bullet?

4.35. Two horses pull horizontally on ropes attached to a stump. The two forces $\vec{F}_1$ and $\vec{F}_2$ that they apply to the stump are such that the net (resultant) force $\vec{R}$ has a magnitude equal to that of $\vec{F}_1$ and makes an angle of 90° with $\vec{F}_1$. Let $F_1 = 1300$ N and $R = 1300$ N also. Find the magnitude of $\vec{F}_2$ and its direction (relative to $\vec{F}_1$).

4.36. You have just landed on Planet X. You take out a 100-g ball, release it from rest from a height of 10.0 m, and measure that it takes 2.2 s to reach the ground. You can ignore any force on the ball from the atmosphere of the planet. How much does the 100-g ball weigh on the surface of Planet X?

4.37. Two adults and a child want to push a wheeled cart in the direction marked x in Fig. 4.36. The two adults push with horizontal forces $\vec{F}_1$ and $\vec{F}_2$ as shown in the figure. (a) Find the magnitude and direction of the *smallest* force that the child should exert. You can ignore the effects of friction. (b) If the child exerts the minimum force found in part (a), the cart accelerates at $2.0\ \mathrm{m/s^2}$ in the $+x$-direction. What is the weight of the cart?

Figure 4.36 Problem 4.37.



4.38. An oil tanker's engines have broken down, and the wind is blowing the tanker straight toward a reef at a constant speed of 1.5 m/s (Fig. 4.37). When the tanker is 500 m from the reef, the wind dies down just as the engineer gets the engines going again. The rudder is stuck, so the only choice is to try to accelerate straight backward away from the reef. The mass of the tanker and cargo is 3.6×10^7 kg, and the engines produce a net horizontal force of 8.0×10^4 N on the tanker. Will the ship hit the reef? If it does, will the oil be safe? The hull can withstand an impact at a speed of 0.2 m/s or less. You can ignore the retarding force of the water on the tanker's hull.

Figure 4.37 Problem 4.38.

4.39. A Standing Vertical Jump. Basketball player Darrell Griffith is on record as attaining a standing vertical jump of 1.2 m (4 ft). (This means that he moved upward by 1.2 m after his feet left the floor.) Griffith weighed 890 N (200 lb). (a) What is his speed as he leaves the floor? (b) If the time of the part of the jump before his feet left the floor was 0.300 s, what was his average acceleration (magnitude and direction) while he was pushing against the floor? (c) Draw his free-body diagram (see Section 4.6). In terms of the forces on the diagram, what is the net force on him? Use Newton's laws and the results of part (b) to calculate the average force he applied to the ground.

4.40. An advertisement claims that a particular automobile can "stop on a dime." What net force would actually be necessary to stop a 850-kg automobile traveling initially at 45.0 km/h in a distance equal to the diameter of a dime, which is 1.8 cm?

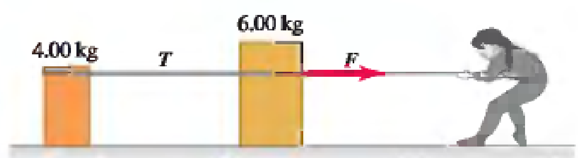
4.41. A 4.80-kg bucket of water is accelerated upward by a cord of negligible mass whose breaking strength is 75.0 N. (a) Draw the free-body force diagram for the bucket. In terms of the forces on your diagram, what is the net force on the bucket? (b) Apply Newton's second law to the bucket and find the maximum upward acceleration that can be given to the bucket without breaking the cord.

4.42. A parachutist relies on air resistance (mainly on her parachute) to decrease her downward velocity. She and her parachute have a mass of 55.0 kg, and air resistance exerts a total upward force of 620 N on her and her parachute. (a) What is the weight of the parachutist? (b) Draw a free-body diagram for the parachutist (see Section 4.6). Use that diagram to calculate the net force on the parachutist. Is the net force upward or downward? (c) What is the acceleration (magnitude and direction) of the parachutist?

4.43. Two crates, one with mass 4.00 kg and the other with mass 6.00 kg, sit on the frictionless surface of a frozen pond, connected by a light rope (Fig. 4.38). A woman wearing golf shoes (so she can get traction on the ice) pulls horizontally on the 6.00-kg crate with a force F that gives the crate an acceleration of $2.50\ \mathrm{m/s^2}$. (a) What is the acceleration of the 4.00-kg crate? (b) Draw a free-body diagram for the 4.00-kg crate. Use that diagram and Newton's second law to find the tension T in the rope that connects the two crates. (c) Draw a free-body diagram for the 6.00-kg crate. What is the direction of the net force on the 6.00-kg crate? Which is larger in magnitude, force T or force F? (d) Use part (c) and Newton's second law to calculate the magnitude of the force F.

4.44. An astronaut is tethered by a strong cable to a spacecraft. The astronaut and her spacesuit have a total mass of 105 kg, while the mass of the cable is negligible. The mass of the spacecraft is 9.05×10^4 kg. The spacecraft is far from any large astronomical bodies, so we can ignore the gravitational forces on it and the astronaut. We also assume that both the spacecraft and the astronaut are initially at rest in an inertial reference frame. The astronaut then pulls on the cable with a force of 80.0 N. (a) What force does the cable exert on the astronaut? (b) Since $\sum \vec{F} = m\vec{a}$, how can a "massless" $(m = 0)$ cable exert a force? (c) What is the astronaut's acceleration? (d) What force does the cable exert on the spacecraft? (e) What is the acceleration of the spacecraft?

Figure 4.38 Problem 4.43.



4.45. To study damage to aircraft that collide with large birds, you design a test gun that will accelerate chicken-sized objects so that their displacement along the gun barrel is given by $x = (9.0 \times 10^3\ \mathrm{m/s^2})t^2 - (8.0 \times 10^4\ \mathrm{m/s^3})t^3$. The object leaves the end of the barrel at $t = 0.025$ s. (a) How long must the gun barrel be? (b) What will be the speed of the objects as they leave the end of the barrel? (c) What net force must be exerted on a 1.50-kg object at (i) $t = 0$ and (ii) $t = 0.025$ s?

4.46. A spacecraft descends vertically near the surface of Planet X. An upward thrust of 25.0 kN from its engines slows it down at a rate of $1.20\ \mathrm{m/s^2}$, but it speeds up at a rate of $0.80\ \mathrm{m/s^2}$ with an upward thrust of 10.0 kN. (a) In each case, what is the direction of the acceleration of the spacecraft? (b) Draw a free-body diagram for the spacecraft. In each case, speeding up or slowing down, what is the direction of the net force on the spacecraft? (c) Apply Newton's second law to each case, slowing down or speeding up, and use this to find the spacecraft's weight near the surface of Planet X.

4.47. A 6.50-kg instrument is hanging by a vertical wire inside a space ship that is blasting off at the surface of the earth. This ship starts from rest and reaches an altitude of 276 m in 15.0 s with constant acceleration. (a) Draw a free-body diagram for the instrument during this time. Indicate which force is greater. (b) Find the force that the wire exerts on the instrument.

4.40. Suppose the rocket in Problem 4.47 is coming in for a vertical landing instead of blasting off. The captain adjusts the engine thrust so that the magnitude of the rocket's acceleration is the same as it was during blast-off. Repeat parts (a) and (b).

4.49. A gymnast of mass m climbs a vertical rope attached to the ceiling. You can ignore the weight of the rope. Draw a free-body diagram for the gymnast. Calculate the tension in the rope if the gymnast (a) climbs at a constant rate; (b) hangs motionless on the rope; (c) accelerates up the rope with an acceleration of magnitude $|\vec{a}|$; (d) slides down the rope with a downward acceleration of magnitude $|\vec{a}|$.

4.50. A loaded elevator with very worn cables has a total mass of 2200 kg, and the cables can withstand a maximum tension of 28,000 N. (a) Draw the free-body force diagram for the elevator. In terms of the forces on your diagram, what is the net force on the elevator? Apply Newton's second law to the elevator and find the maximum upward acceleration for the elevator if the cables are not to break. (b) What would be the answer to part (a) if the elevator were on the moon, where $g = 1.62\ \mathrm{m/s^2}$?

4.51. Jumping to the Ground. A 75.0-kg man steps off a platform 3.10 m above the ground. He keeps his legs straight as he falls, but at the moment his feet touch the ground his knees begin to bend, and, treated as a particle, he moves an additional 0.60 m before coming to rest. (a) What is his speed at the instant his feet touch the ground? (b) Treating him as a particle, what is his acceleration (magnitude and direction) as he slows down, if the acceleration is assumed to be constant? (c) Draw his free-body diagram (see Section 4.6). In terms of the forces on the diagram, what is the net force on him? Use Newton's laws and the results of part (b) to calculate the average force his feet exert on the ground while he slows down. Express this force in newtons and also as a multiple of his weight.

4.52. A 4.9-N hammer head is stopped from an initial downward velocity of 3.2 m/s in a distance of 0.45 cm by a nail in a pine board. In addition to its weight, there is a 15-N downward force on the hammer head applied by the person using the hammer. Assume that the acceleration of the hammer head is constant while it is in contact with the nail and moving downward. (a) Draw a free-body diagram for the hammer head. Identify the reaction force to each action force in the diagram. (b) Calculate the downward force $\vec{F}$ exerted by the hammer head on the nail while the hammer head is in contact with the nail and moving downward. (c) Suppose the nail is in hardwood and the distance the hammer head travels in coming to rest is only 0.12 cm. The downward forces on the hammer head are the same as on part (b). What then is the force $\vec{F}$ exerted by the hammer head on the nail while the hammer head is in contact with the nail and moving downward?

4.53. A uniform cable of weight w hangs vertically downward, supported by an upward force of magnitude w at its top end. What is the tension in the cable (a) at its top end; (b) at its bottom end; (c) at its middle? Your answer to each part must include a free-body diagram. (*Hint:* For each question choose the body to analyze to be a section of the cable or a point along the cable.) (d) Graph the tension in the rope versus the distance from its top end.

4.54. The two blocks in Fig. 4.39 are connected by a heavy uniform rope with a mass of 4.00 kg. An upward force of 200 N is applied as shown. (a) Draw three free-body diagrams, one for the 6.00-kg block, one for the 4.00-kg rope, and another one for the 5.00-kg block. For each force, indicate what body exerts that force. (b) What is the acceleration of the system? (c) What is the tension at the top of the heavy rope? (d) What is the tension at the midpoint of the rope?

Figure 4.39 Problem 4.54.

4.55. An athlete whose mass is 90.0 kg is performing weight-lifting exercises. Starting from the rest position, he lifts, with constant acceleration, a barbell that weighs 490 N. He lifts the barbell a distance of 0.60 m in 1.6 s. (a) Draw a clearly labeled free-body force diagram for the barbell and for the athlete. (b) Use the diagrams in part (a) and Newton's laws to find the total force that his feet exert on the ground as he lifts the barbell.

4.56. A hot-air balloon consists of a basket, one passenger, and some cargo. Let the total mass be M. Even though there is an upward lift force on the balloon, the balloon is initially accelerating downward at a rate of $g/3$. (a) Draw a free-body diagram for the descending balloon. (b) Find the upward lift force in terms of the initial total weight Mg. (c) The passenger notices that he is heading straight for a waterfall and decides he needs to go up. What fraction of the total weight must he drop overboard so that the balloon accelerates *upward* at a rate of $g/2$? Assume that the upward lift force remains the same.

4.57. A student tries to raise a chain consisting of three identical links. Each link has a mass of 300 g. The three-piece chain is connected to a string and then suspended vertically, with the student holding the upper end of the string and pulling upward. Because of the student's pull, an upward force of 12 N is applied to the chain by the string. (a) Draw a free-body diagram for *each* of the links in the chain and also for the entire chain considered as a single body. (b) Use the results of part (a) and Newton's laws to find (i) the acceleration of the chain and (ii) the force exerted by the top link on the middle link.

4.56. The position of a 2.75×10^5 N training helicopter under test is given by $\vec{r} = (0.020\ \mathrm{m/s^3})t^3\hat{\imath} + (2.2\ \mathrm{m/s})t\hat{\jmath} - (0.060\ \mathrm{m/s^2})t^2\hat{k}$. Find the net force on the helicopter at $t = 5.0$ s.

4.59. An object with mass m moves along the x-axis. Its position as a function of time is given by $x(t) = At - Bt^3$, where A and B are constants. Calculate the net force on the object as a function of time.

4.60. An object with mass m initially at rest is acted on by a force $\vec{F} = k_1\hat{\imath} + k_2t^3\hat{\jmath}$, where k_1 and k_2 are constants. Calculate the velocity $\vec{v}(t)$ of the object as a function of time.

Challenge Problems

4.61. If we know $F(t)$, the force as a function of time, for straight-line motion, Newton's second law gives us $a(t)$, the acceleration as a function of time. We can then integrate $a(t)$ to find $v(t)$ and $x(t)$. However, suppose we know $F(v)$ instead. (a) The net force on a body moving along the x-axis equals $-Cv^2$. Use Newton's second law written as $\sum F = m\,dv/dt$ and two integrations to show that $x - x_0 = (m/C)\ln(v_0/v)$. (b) Show that Newton's second law can be written as $\sum F = mv\,dv/dx$. Derive the same expression as in part (a) using this form of the second law and one integration.

4.62. An object of mass m is at rest in equilibrium at the origin. At $t = 0$ a new force $\vec{F}(t)$ is applied that has components

$$F_x(t) = k_1 + k_2y \qquad F_y(t) = k_3t$$

where k_1, k_2, and k_3 are constants. Calculate the position $\vec{r}(t)$ and velocity $\vec{v}(t)$ vectors as functions of time.

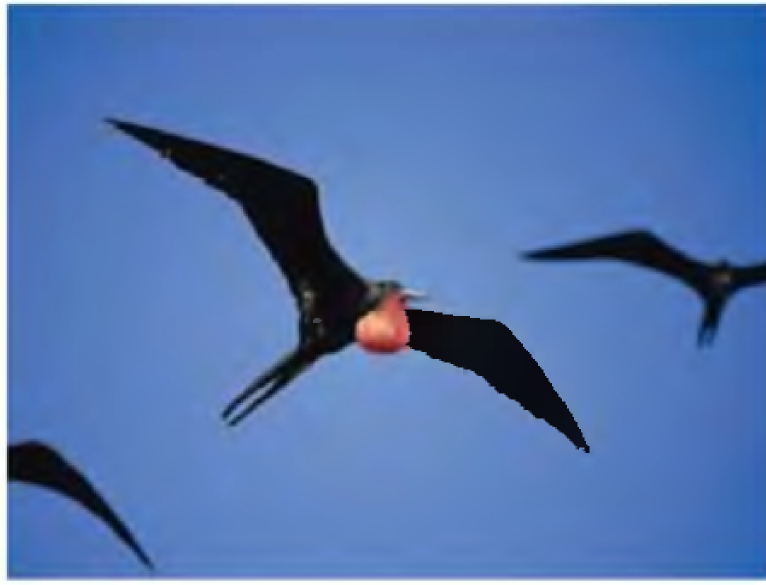
5 APPLYING NEWTON'S LAWS

LEARNING GOALS

By studying this chapter, you will learn:

- How to use Newton's first law to solve problems involving the forces that act on a body in equilibrium.
- How to use Newton's second law to solve problems involving the forces that act on an accelerating body.
- The nature of the different types of friction forces—static friction, kinetic friction, rolling friction, and fluid resistance—and how to solve problems that involve these forces.
- How to solve problems involving the forces that act on a body moving along a circular path.
- The key properties of the four fundamental forces of nature.

? Suppose a gliding bird is caught in an updraft so that it ascends at a steady rate. In this situation, which has a greater magnitude: the force of gravity or the upward force of the air on the bird?



We saw in Chapter 4 that Newton's three laws of motion, the foundation of classical mechanics, can be stated very simply. But *applying* these laws to situations such as an iceboat skating across a frozen lake, a toboggan sliding down a hill, or an airplane making a steep turn requires analytical skills and problem-solving technique. In this chapter we'll help you extend the problem-solving skills you began to develop in Chapter 4.

We begin with equilibrium problems, in which a body is at rest or moving with constant velocity. Then we generalize our problem-solving techniques to include bodies that are not in equilibrium, for which we need to deal precisely with the relationships between forces and motion. We will learn how to describe and analyze the contact force acting on a body when it rests or slides on a surface. Finally, we study the important case of uniform circular motion, in which a body moves in a circle with constant speed.

All these situations involve the concept of force, a concept we'll use throughout our study of physics. We close the chapter with a brief look at the fundamental nature of force and the classes of forces found in our physical universe.

5.1 Using Newton's First Law: Particles in Equilibrium

We learned in Chapter 4 that a body is in *equilibrium* when it is at rest or moving with constant velocity in an inertial frame of reference. A hanging lamp, a suspension bridge, an airplane flying straight and level at a constant speed—all are examples of equilibrium situations. In this section we consider only equilibrium of a body that can be modeled as a particle. (In Chapter 11, we'll consider the additional principles needed when the body can't be represented adequately as a particle.) The essential physical principle is Newton's first law: When a particle is at rest or is moving with constant velocity in an inertial frame of reference, the net force acting on it—that is, the vector sum of all the forces acting on it—must be zero:

$$\sum \vec{F} = \mathbf{0} \qquad \text{(particle in equilibrium, vector form)} \qquad (5.1)$$

We most often use this equation in component form:

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \text{(particle in equilibrium, component form)} \qquad (5.2)$$

This section is about using Newton's first law to solve problems dealing with bodies in equilibrium. Some of these problems may seem complicated, but the important thing to remember is that *all* problems involving particles in equilibrium are done in the same way. Problem-Solving Strategy 5.1 details the steps you need to follow for any and all such problems. Study this strategy carefully, look at how it's applied in the worked-out examples, and try to apply it yourself when you solve assigned problems.

Problem-Solving Strategy 5.1 Newton's First Law: Equilibrium of a Particle

MP

IDENTIFY *the relevant concepts:* You must use Newton's first law for any problem that involves forces acting on a body in equilibrium—that is, either at rest or moving with constant velocity. For example, a car is in equilibrium when it's parked, but also when it's traveling down a straight road at a steady speed.

If the problem involves more than one body and the bodies interact with each other, you'll also need to use Newton's *third* law. This law allows you to relate the force that one body exerts on a second body to the force that the second body exerts on the first one.

Be certain that you identify the target variable(s). Common target variables in equilibrium problems include the magnitude of one of the forces, the components of a force, or the direction (angle) of a force.

SET UP *the problem* using the following steps:

1. Draw a very simple sketch of the physical situation, showing dimensions and angles. You don't have to be an artist!
2. Draw a free-body diagram for each body that is in equilibrium. For the present, we consider the body as a particle, so you can represent it as a large dot. In your free-body diagram, *do not* include the other bodies that interact with it, such as a surface it may be resting on, or a rope pulling on it.
3. Ask yourself what is interacting with the body by touching it or in any other way. On your free-body diagram, draw a force vector for each interaction and label each force with a symbol representing the *magnitude* of the force. If you know the angle at which a force is directed, draw the angle accurately and label it. Include the body's weight, except in cases where the body has negligible mass (and hence negligible weight). If the mass is given, use $w = mg$ to find the weight. A surface in contact with the body exerts a normal force perpendicular to the surface and possibly a friction force parallel to the surface. A rope or chain exerts a pull (never a push) in a direction along its length.
4. *Do not* show in the free-body diagram any forces exerted *by* the body on any other body. The sums in Eqs. (5.1) and (5.2) include only forces that act *on* the body. For each force on the body, ask yourself "What other body causes that force?" If you can't answer that question, you may be imagining a force that isn't there.
5. Choose a set of coordinate axes and include them in your free-body diagram. (If there is more than one body in the problem, choose axes for each body separately.) Label the positive direction for each axis. If a body rests or slides on a plane surface, it usually simplifies the solution to take the axes in the directions parallel and perpendicular to this surface, even when the plane is tilted.

EXECUTE *the solution* as follows:

1. Find the components of each force along each of the body's coordinate axes. Draw a wiggly line through each force vector that has been replaced by its components, so you don't count it twice. Remember that while the *magnitude* of a force is always positive, the *component* of a force along a particular direction may be positive or negative.
2. Set the algebraic sum of all x-components of force equal to zero. In a separate equation, set the algebraic sum of all y-components equal to zero. (*Never* add x- and y-components in a single equation.)
3. If there are two or more bodies, repeat all of the above steps for each body. If the bodies interact with each other, use Newton's third law to relate the forces they exert on each other.
4. Make sure that you have as many independent equations as the number of unknown quantities. Then solve these equations to obtain the target variables.

EVALUATE *your answer:* Look at your results and ask whether they make sense. When the result is a symbolic expression or formula, try to think of special cases (particular values or extreme cases for the various quantities) for which you can guess what the results ought to be. Check to see that your formula works in these particular cases.

Example 5.1 One-dimensional equilibrium: Tension in a massless rope

A gymnast with mass $m_G = 50.0\text{ kg}$ suspends herself from the lower end of a hanging rope. The upper end of the rope is attached to the gymnasium ceiling. What is the gymnast's weight? What force (magnitude and direction) does the rope exert on her? What is the tension at the top of the rope? Assume that the mass of the rope itself is negligible.

SOLUTION

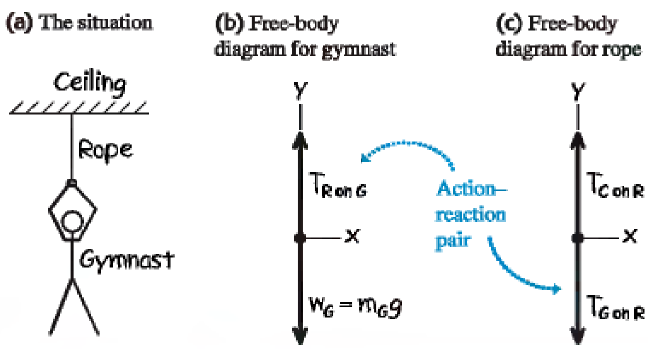
IDENTIFY: The gymnast and the rope are in equilibrium, so we can apply Newton's first law to both bodies. We'll also use Newton's third law to relate the forces that the gymnast and the rope exert on each other. the target variables are the weight of the gymnast, w_G; the force that the rope exerts on the gymnast (call it $T_{\text{R on G}}$); and the tension that the ceiling exerts on the top of the rope (call it $T_{\text{C on R}}$).

SET UP: We sketch the situation (Fig. 5.1a) and draw separate free-body diagrams for the gymnast (Fig. 5.1b) and the rope (Fig. 5.1c). We take the positive y-axis to be upward, as shown. Each force acts in the vertical direction and so has only a y-component.

The two forces $T_{\text{R on G}}$ and $T_{\text{G on R}}$ are the upward force of the rope on the gymnast (in Fig. 5.1b) and the downward force of the gymnast on the rope (in Fig. 5.1c). These forces form an action–reaction pair, so they must have the same magnitude.

Note also that the gymnast's weight w_G is the attractive (downward) force exerted on the *gymnast* by the *earth*. Its reaction force is the equal to and opposite the attractive (upward) force exerted on the *earth* by the *gymnast*. This force acts on the earth, not on the gymnast, so it doesn't appear in her free-body diagram (Fig. 5.1b). Compare the discussion of the apple in Conceptual Example 4.9 (Section 4.5). Similarly, the force that the rope exerts on the ceiling doesn't appear in Fig. 5.1c.

5.1 Our sketches for this problem.



EXECUTE: The magnitude of the gymnast's weight is the product of her mass and the acceleration due to gravity, g:

$$w_G = m_G g = (50.0\text{ kg})(9.80\text{ m/s}^2) = 490\text{ N}$$

This force points in the negative y-direction, so its y-component is $-w_G$. The upward force exerted by the rope has unknown magnitude $T_{\text{R on G}}$ and positive y-component $+T_{\text{R on G}}$. Because the gymnast is in equilibrium, the sum of the y-components of force acting on her must be zero:

$$\text{Gymnast:}\quad \sum F_y = T_{\text{R on G}} + (-w_G) = 0 \quad \text{so}$$
$$T_{\text{R on G}} = w_G = 490\text{ N}$$

The rope pulls *up* on the gymnast with a force $T_{\text{R on G}}$ of magnitude 490 N. By Newton's third law, the gymnast pulls *down* on the rope with a force of the same magnitude, $T_{\text{G on R}} = 490\text{ N}$.

The rope is also in equilibrium. We have assumed that it is weightless, so the upward force of magnitude $T_{\text{C on R}}$ that the ceiling exerts on its top end must make the *net* vertical force on the rope equal to zero. Expressed as an equation, this says

$$\text{Rope:}\quad \sum F_y = T_{\text{C on R}} + (-T_{\text{G on R}}) = 0 \quad \text{so}$$
$$T_{\text{C on R}} = T_{\text{G on R}} = 490\text{ N}$$

EVALUATE: The *tension* at any point in the rope is the force that acts at that point. For this weightless rope, the tension $T_{\text{G on R}}$ at the lower end has the same value as the tension $T_{\text{C on R}}$ at the upper end. Indeed, for an ideal weightless rope, the tension has the same value at any point along the rope's length. (Compare the discussion of Conceptual Example 4.10 in Section 4.5.)

Note that we have defined tension to be the *magnitude* of a force, so it is always positive. But the y-component of force acting on the rope at its lower end is $-T_{\text{G on R}} = -490\text{ N}$.

Example 5.2 One-dimensional equilibrium: Tension in a rope with mass

Suppose that in Example 5.1, the weight of the rope is not negligible but is 120 N. Find the tension at each end of the rope.

SOLUTION

IDENTIFY: As in Example 5.1, the target variables are the magnitudes $T_{\text{G on R}}$ and $T_{\text{C on R}}$ of the forces that act at the bottom and top of the rope, respectively. Once again, we'll apply Newton's first law to the gymnast and to the rope, and use Newton's third law to relate the forces that the gymnast and rope exert on each other.

SET UP: Again we draw separate free-body diagrams for the gymnast (Fig. 5.2a) and the rope (Fig. 5.2b). The only difference from Example 5.1 is that there are now *three* forces acting on the rope: the downward force exerted by the gymnast $(T_{\text{G on R}})$, the upward force exerted by the ceiling $(T_{\text{C on R}})$, and the weight of the rope, of magnitude $w_R = 120\text{ N}$.

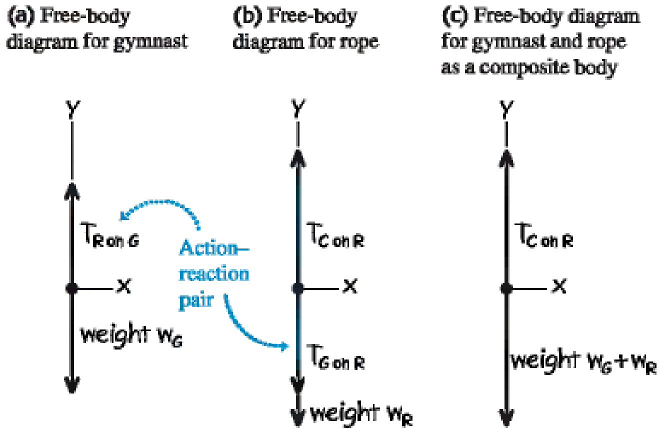
EXECUTE: The gymnast's free-body diagram is the same as in Example 5.1, so her equilibrium condition is also the same. From Newton's third law, $T_{\text{R on G}} = T_{\text{G on R}}$, and we have

$$\text{Gymnast:}\quad \sum F_y = T_{\text{R on G}} + (-w_G) = 0 \quad \text{so}$$
$$T_{\text{R on G}} = T_{\text{G on R}} = w_G = 490\text{ N}$$

The equilibrium condition $\sum F_y = 0$ for the rope is

$$\text{Rope:}\quad \sum F_y = T_{\text{C on R}} + (-T_{\text{G on R}}) + (-w_R) = 0$$

Note that the y-component of $T_{\text{C on R}}$ is positive because it points in the $+y$-direction, but the y-components of both $T_{\text{G on R}}$ and w_R are



5.2 Our sketches for this problem, including the weight of the rope.

negative. When we solve for $T_{C\,on\,R}$ and substitute the values $T_{G\,on\,R} = T_{R\,on\,G} = 490\text{ N}$ and $w_R = 120\text{ N}$, we find

$$T_{C\,on\,R} = T_{G\,on\,R} + w_R = 490\text{ N} + 120\text{ N} = 610\text{ N}$$

EVALUATE: When we include the weight of the rope, the tension is *different* at the rope's two ends. The force $T_{C\,on\,R}$ exerted by the ceiling has to hold up both the 490-N weight of the gymnast and the 120-N weight of the rope, so $T_{C\,on\,R} = 610\text{ N}$.

To see this more explicitly, draw a free-body diagram for a composite body consisting of the gymnast and rope considered as a unit (Fig. 5.2c). Only two external forces act on this composite body: the force $T_{C\,on\,R}$ exerted by the ceiling and the total weight $w_G + w_R = 490\text{ N} + 120\text{ N} = 610\text{ N}$. (The forces $T_{G\,on\,R}$ and $T_{R\,on\,G}$ are *internal* to the composite body. Since Newton's first law involves only *external* forces, the internal forces play no role.) Hence Newton's first law applied to this composite body is

$$\text{Composite body:} \qquad \sum F_y = T_{C\,on\,R} + [-(w_G + w_R)] = 0$$

and so $T_{C\,on\,R} = w_G + w_R = 610\text{ N}$.

This method of treating the gymnast and rope as a composite body seems a lot simpler, and you may be wondering why we didn't use it first. The answer is that we can't find the tension $T_{G\,on\,R}$ at the bottom of the rope by this method. *Moral: Whenever you have more than one body in a problem involving Newton's laws, the safest approach is to treat each body separately.*

Example 5.3 Two-dimensional equilibrium

In Fig. 5.3a, a car engine with weight w hangs from a chain that is linked at ring O to two other chains, one fastened to the ceiling and the other to the wall. Find the tension in each of the three chains in terms of w. The weights of the ring and chains are negligible.

SOLUTION

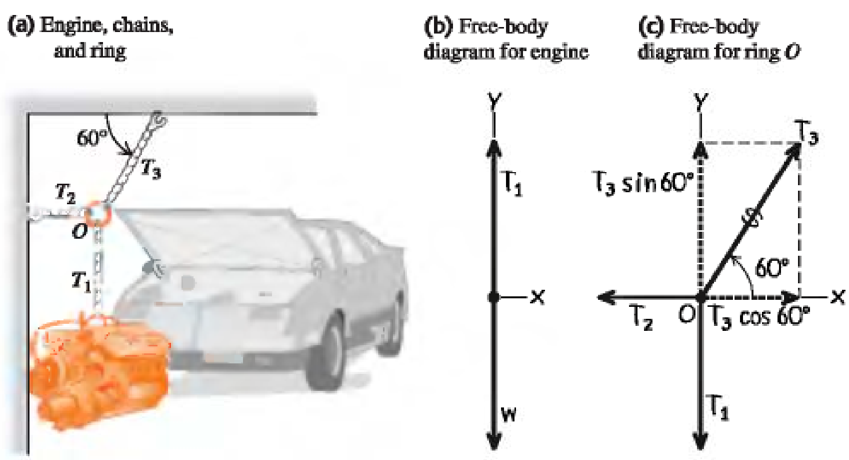
IDENTIFY: The target variables are the tensions T_1, T_2, and T_3 in the three chains (Fig. 5.3a). It may seem strange that we neglect the weight of the chains and ring in this example, while in Example 5.2 we did *not* neglect the weight of a mere rope. The reason is that the weight of the chains or ring is very small compared to the weight of the massive engine. By contrast, in Example 5.2 the rope weighed a reasonable fraction of the gymnast's weight (120 N compared to 490 N).

All the bodies in the example are in equilibrium, so we'll use Newton's first law to determine T_1, T_2, and T_3. We need three simultaneous equations, one for each target variable. However, applying Newton's first law to just one body gives us just *two* equations, as in Eq. (5.2). So to solve the problem, we'll have to consider more than one body in equilibrium. We'll look at the engine (which is acted on by T_1) and the ring (which is connected to all three chains and so is acted on by all three tensions).

SET UP: Figures 5.3b and 5.3c show our free-body diagrams, including an x-y coordinate system, for the engine and the ring, respectively.

The two forces acting on the engine are its weight w and the upward force T_1 exerted by the vertical chain; the three forces acting on the ring are the tensions from the vertical chain (T_1), the horizontal chain (T_2), and the slanted chain (T_3). Because the vertical chain has negligible weight, it exerts forces of the same magnitude T_1 at both of its ends: upward on the engine in Fig. 5.3b and

5.3 (a) The situation. (b), (c) Our free-body diagrams.



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downward on the ring in Fig. 5.3c (see Example 5.1). If the weight were not negligible, these two forces would have different magnitudes, as was the case for the rope in Example 5.2. We're also neglecting the weight of the ring, which is why it isn't included in the forces in Fig. 5.3c.

EXECUTE: The forces acting on the engine are along the y-axis only, so Newton's first law says

$$\text{Engine:} \quad \sum F_y = T_1 + (-w) = 0 \qquad \text{and} \qquad T_1 = w$$

The horizontal and slanted chains do not exert forces on the engine itself because they are not attached to it. These forces appear when we apply Newton's first law to the ring, however.

In the free-body diagram for the ring (Fig. 5.3c), remember that T_1, T_2, and T_3 are the *magnitudes* of the forces. We first resolve the force with magnitude T_3 into its x- and y-components. The ring is in equilibrium, so we then write separate equations stating that the x- and y-components of the net force on the ring are zero. (Remember from Problem-Solving Strategy 5.1 that we *never* add x- and y-components together in a single equation.) We find

$$\text{Ring:} \quad \sum F_x = T_3\cos 60° + (-T_2) = 0$$
$$\text{Ring:} \quad \sum F_y = T_3\sin 60° + (-T_1) = 0$$

Because $T_1 = w$ (from the engine equation), we can rewrite the second ring equation as

$$T_3 = \frac{T_1}{\sin 60°} = \frac{w}{\sin 60°} = 1.155w$$

We can now use this result in the first ring equation:

$$T_2 = T_3\cos 60° = w\frac{\cos 60°}{\sin 60°} = 0.577w$$

So we can express all three tensions as multiples of the weight w of the engine, which we assume is known. To summarize,

$$T_1 = w$$
$$T_2 = 0.577w$$
$$T_3 = 1.155w$$

EVALUATE: Our results show that the chain attached to the ceiling exerts a force on the ring of magnitude T_3, which is *greater* than the weight of the engine. If this seems strange, note that the vertical component of this force is equal to T_1, which in turn is equal to w. But since this force also has a horizontal component, its magnitude T_3 must be somewhat larger than w. Hence the chain attached to the ceiling is under the greatest tension and is the one most susceptible to breaking.

You may have thought at first that the most important body in this problem was the engine. But to get enough equations to solve the problem, we also had to consider the forces acting on a second body (the ring connecting the chains). Situations like this are fairly common in equilibrium problems, so keep this technique in mind.

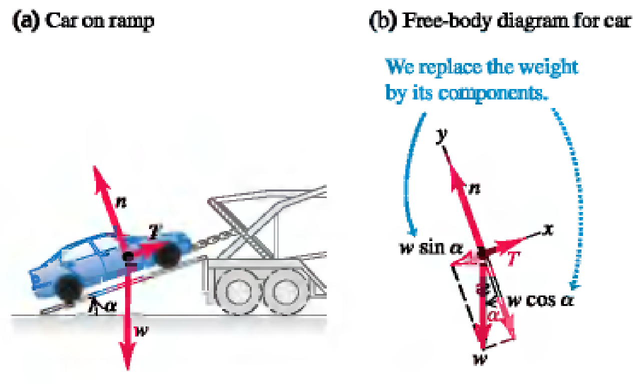
Example 5.4 An inclined plane

A car of weight w rests on a slanted ramp leading to a car-transporter trailer (Fig. 5.4a). Only a cable running from the trailer to the car prevents the car from rolling backward off the ramp. (The car's brakes are off and its transmission is in neutral.) Find the tension in the cable and the force that the tracks exert on the car's tires.

SOLUTION

IDENTIFY: The car is in equilibrium, so once again we use Newton's first law. The ramp exerts a separate force on each of the car's tires, but for simplicity we lump all of these together into a single force. For a further simplification, we'll assume that there's very little friction on the car, and so we ignore the component of this force on the car that acts *parallel* to the ramp (see Fig. 4.2b).

5.4 A cable holds a car at rest on a ramp.



(We'll return to the friction force in Section 5.3.) Hence we can say that the ramp only exerts a force on the car that is *perpendicular* to the tracks. This force appears because the atoms on the surface of the track resist having the atoms of the tires squeezed into them. As in Section 4.1, we call this force the *normal* force (see Fig. 4.2a). The two target variables are the magnitude n of the normal force and the magnitude T of the tension in the cable.

SET UP: Figure 5.4b shows a free-body diagram for the car. The three forces acting on the car are its weight (magnitude w), the tension in the cable (magnitude T), and the normal force (magnitude n). Note that the normal force acts up and to the left because it's preventing the car from penetrating into the solid tracks.

We choose the x- and y-axes to be parallel and perpendicular to the ramp as shown. This choice makes the problem easier to analyze because only the weight force has both an x- and y-component. If we chose axes that were horizontal and vertical, our job would be harder because we'd need to find x- and y-components for *two* forces (the normal force and the tension).

Note that the angle α between the ramp and the horizontal is equal to the angle α between the weight vector $\vec{w}$ and the normal to the plane of the ramp.

EXECUTE: To write down the x- and y-components of Newton's first law, we need to find the components of the weight. One complication is that the angle α in Fig. 5.4b is *not* measured from the $+x$-axis toward the $+y$-axis. Hence we *cannot* use Eqs. (1.6) directly to find the components. (You may want to review Section 1.8 to make sure that you understand this important point.)

One approach to finding the components of $\vec{w}$ is to consider the right triangles in Fig. 5.4b. The sine of α is the magnitude of the x-component of $\vec{w}$ (that is, the side of the triangle opposite α) divided by the magnitude w (the hypotenuse of the triangle). Similarly, the cosine of α is the magnitude of the y-component (the side of the triangle adjacent to α) divided by w. Both components are negative, so $w_x = -w\sin\alpha$ and $w_y = -w\cos\alpha$.

Another approach is to recognize that one component of $\vec{w}$ must involve $\sin\alpha$ while the other component involves $\cos\alpha$. To decide which is which, draw the free-body diagram so that the angle α is noticeably smaller or larger than 45°. (You'll have to fight the natural tendency to draw such angles as being close to 45°.) We've drawn Fig. 5.4b so that α is smaller than 45°, so $\sin\alpha$ is less than $\cos\alpha$. The figure shows that the x-component of $\vec{w}$ is smaller than the y-component, so the x-component must involve $\sin\alpha$ and the y-component must involve $\cos\alpha$. We again find $w_x = -w\sin\alpha$ and $w_y = -w\cos\alpha$.

In Fig. 5.4b we draw a wiggly line through the original vector representing the weight to remind us not to count it twice. The equilibrium conditions then give us

$$\sum F_x = T + (-w\sin\alpha) = 0$$
$$\sum F_y = n + (-w\cos\alpha) = 0$$

Be sure you understand how these signs are related to our choice of coordinates. Remember that, by definition, T, w, and n are all *magnitudes* of vectors and are therefore all positive.

Solving these equations for T and n, we find

$$T = w\sin\alpha$$
$$n = w\cos\alpha$$

EVALUATE: Our answers for T and n depend on the value of α; we can check this dependence by looking at some special cases. If the angle α is zero, then $\sin\alpha = 0$ and $\cos\alpha = 1$. In this case, the ramp is horizontal; our answers tell us that no cable tension T is needed to hold the car, and the normal force n is equal in magnitude to the weight. If the angle is 90°, then $\sin\alpha = 1$ and $\cos\alpha = 0$. Then the cable tension T equals the weight w, and the normal force n is zero. Are these the results you would expect for these particular cases?

CAUTION **Normal force and weight may not be equal** It's a common error to automatically assume that the magnitude n of the normal force is equal to the weight w. But our result shows that this is *not* true in general. It's always best to treat n as a variable and solve for its value, as we have done here.

? How would the answers for T and n be affected if the car were not stationary but were being pulled up the ramp at a constant speed? This, too, is an equilibrium situation, since the car's velocity is constant. So the calculation is exactly the same, and T and n have the same values as when the car is at rest. (It's true that T must be greater than $w\sin\alpha$ to *start* the car moving up the ramp, but that's not what we asked.)

Example 5.5 Tension over a frictionless pulley

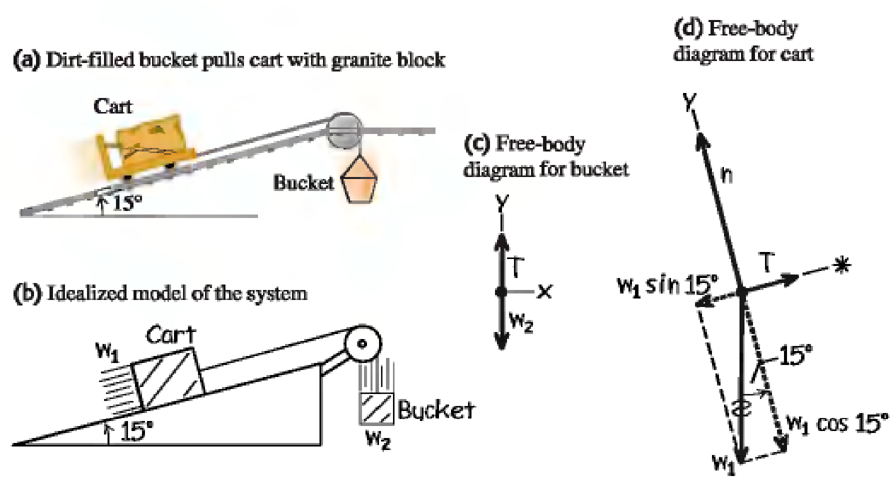
Blocks of granite are to be hauled up a 15° slope out of a quarry, and dirt is to be dumped into the quarry to fill up old holes. To simplify the process, you design a system in which a granite block on a cart with steel wheels (weight w_1, including both block and cart) is pulled uphill on steel rails by a dirt-filled bucket (weight w_2, including both dirt and bucket) dropping vertically into the quarry (Fig. 5.5a). How must the weights w_1 and w_2 be related in order for the system to move with constant speed? Ignore friction in the pulley and wheels and the weight of the cable.

SOLUTION

IDENTIFY: The cart and bucket each move with a constant velocity (that is, in a straight line at constant speed). Hence each body is in equilibrium, and we can apply Newton's first law to each.

Our two target variables are the weights w_1 and w_2. The forces that act on the bucket are its weight w_2 and an upward tension exerted by the cable. The cart has *three* forces acting on it: its weight w_1, a normal force of magnitude n exerted by the rails, and

5.5 (a) The situation. (b) Our idealized model. (c), (d) Our free-body diagrams.



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a tension force from the cable. (We're ignoring friction, so we're assuming that the rails exert no force parallel to the incline.) This is exactly like the situation for the car on the ramp in Example 5.4. As in that example, the forces on the cart are not all along the same direction, so we'll need to use both components of Newton's first law in Eq. (5.2).

We're assuming that the cable has negligible weight, so the tension forces that the rope exerts on the cart and on the bucket have the same magnitude T.

SET UP: Figure 5.5b shows our idealized model for the system, and Figs. 5.5c and 5.5d show the free-body diagrams we draw. Note that we're free to orient the axes differently for each body; the choices shown are the most convenient ones. As we did for the car in Example 5.4, we represent the weight of the granite block in terms of its x- and y-components.

EXECUTE: Applying $\sum F_y = 0$ to the dirt-filled bucket in Fig. 5.5c, we find

$$\sum F_y = T + (-w_2) = 0 \qquad \text{so} \qquad T = w_2$$

Applying $\sum F_x = 0$ to the block and cart in Fig. 5.5d, we get

$$\sum F_x = T + (-w_1 \sin 15°) = 0 \qquad \text{so} \qquad T = w_1 \sin 15°$$

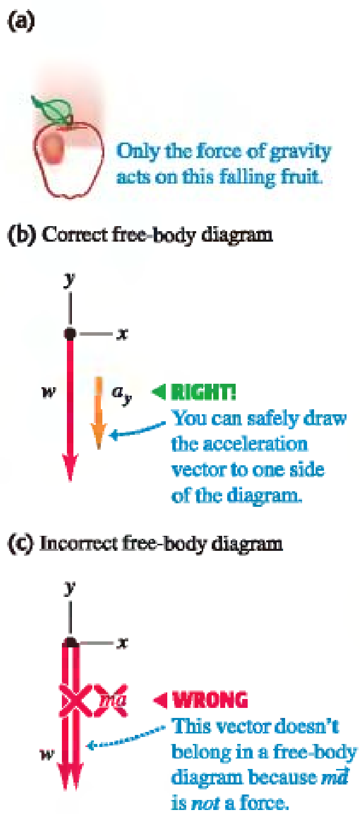
Equating the two expressions for T, we find

$$w_2 = w_1 \sin 15° = 0.26 w_1$$

EVALUATE: Our analysis doesn't depend on the direction of motion, only on the velocity being constant. Hence the system can move with constant speed in *either* direction if the weight of dirt and bucket totals 26% of the weight of the granite block and cart. What would happen if w_2 were greater than $0.26w_1$? If it were less than $0.26w_1$?

Notice that we didn't need to apply the equation $\sum F_y = 0$ to the cart and block; this would be useful only if we wanted to find the value of n. Can you show that $n = w_1 \cos 15°$?

5.6 Correct and incorrect free-body diagrams for a falling body.



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2.1.5 Car Race
2.2 Lifting a Crate
2.3 Lowering a Crate
2.4 Rocket Blasts Off
2.5 Modified Atwood Machine

Test Your Understanding of Section 5.1 A traffic light of weight w hangs from two lightweight cables, one on each side of the light. Each cable hangs at a 45° angle from the horizontal. What is the tension in each cable? (i) $w/2$; (ii) $w/\sqrt{2}$; (iii) w; (iv) $w\sqrt{2}$; (v) $2w$.

5.2 Using Newton's Second Law: Dynamics of Particles

We are now ready to discuss *dynamics* problems. In these problems, we apply Newton's second law to bodies on which the net force is *not* zero, so the bodies are *not* in equilibrium and hence are accelerating. The net force is equal to the mass of the body times its acceleration:

$$\sum \vec{F} = m\vec{a} \qquad \text{(Newton's second law, vector form)} \qquad (5.3)$$

We most often use this relationship in component form:

$$\sum F_x = ma_x \qquad \sum F_y = ma_y \qquad \text{(Newton's second law, component form)} \qquad (5.4)$$

The following problem-solving strategy is very similar to Problem-Solving Strategy 5.1 for equilibrium problems in Section 5.1. We urge you to study it carefully, watch how we apply it in our examples, and use it when you tackle the end-of-chapter problems. Remember that you can solve *any* dynamics problem using this strategy.

CAUTION **$m\vec{a}$ doesn't belong in free-body diagrams** Remember that the quantity $m\vec{a}$ is the *result* of forces acting on a body, *not* a force itself; it's not a push or a pull exerted by anything in the body's environment. When you draw the free-body diagram for an accelerating body (like the fruit in Fig. 5.6a), make sure you *never* include the "$m\vec{a}$ force" because *there is no such force* (Fig. 5.6c). You should review Section 4.3 if you're not clear on this point. Sometimes we draw the acceleration vector $\vec{a}$ *alongside* a free-body diagram, as in Fig. 5.6b. But we *never* draw the acceleration vector with its tail touching the body (a position reserved exclusively for the forces that act on the body).

Problem-Solving Strategy 5.2 Newton's Second Law: Dynamics of Particles

MP

IDENTIFY *the relevant concepts:* You have to use Newton's second law for *any* problem that involves forces acting on an accelerating body.

Identify the target variable—usually an acceleration or a force. If the target variable is something else, you'll need to select another concept to use. For example, suppose you want to find how fast a sled is moving when it reaches the bottom of a hill. This means your target variable is the sled's final velocity. Newton's second law will let you find the sled's acceleration; you'll then use the constant-acceleration relationships from Section 2.4 to find velocity from acceleration.

SET UP *the problem* using the following steps:

1. Draw a simple sketch of the situation. Identify one or more moving bodies to which you'll apply Newton's second law.
2. For each body you identified, draw a free-body diagram that shows all the forces acting *on* the body. Remember, the acceleration of a body is determined by the forces that act on it, *not* by the forces that it exerts on anything else. Make sure you can answer the question "What other body is applying this force?" for each force in your diagram. Never include the quantity $m\vec{a}$ in your free-body diagram; it's not a force!
3. Label each force with an algebraic symbol for the force's *magnitude.* (Remember that magnitudes are always positive. Minus signs show up later when you take components of the forces.) Usually, one of the forces will be the body's weight; it's usually best to label this as $w = mg$. If a numerical value of mass is given, you can compute the corresponding weight.
4. Choose your x- and y-coordinate axes for each body, and show them in its free-body diagram. Be sure to indicate the positive direction for each axis. If you know the direction of the acceleration, it usually simplifies things to take one positive axis along that direction. If your problem involves more than one object and the objects accelerate in different directions, you can use a different set of axes for each object.
5. In addition to Newton's second law, $\sum\vec{F} = m\vec{a}$, identify any other equations you might need. (You need as many equations as there are target variables.) For example, you might need one or more of the equations for motion with constant acceleration. If more than one body is involved, there may be relationships among their motions; for example, they may be connected by a rope. Express any such relationships as equations relating the accelerations of the various bodies.

EXECUTE *the solution* as follows:

1. For each object, determine the components of the forces along each of the object's coordinate axes. When you represent a force in terms of its components, draw a wiggly line through the original force vector to remind you not to include it twice.
2. For each object, write a separate equation for each component of Newton's second law, as in Eq. (5.4).
3. Make a list of all the known and unknown quantities. In your list, identify the target variable or variables.
4. Check that you have as many equations as there are unknowns. If you have too few equations, go back to step 5 of "Set up the problem." If you have too many equations, perhaps there is an unknown quantity that you haven't identified as such.
5. Do the easy part—the math! Solve the equations to find the target variable(s).

EVALUATE *your answer:* Does your answer have the correct units? (When appropriate, use the conversion $1\text{ N} = 1\text{ kg}\cdot\text{m/s}^2$.) Does it have the correct algebraic sign? (If the problem is about a sled sliding downhill, you probably took the positive x-axis to point down the hill. If you then find that the sled has a negative acceleration—that is, the acceleration is uphill—something went wrong in your calculations.) When possible, consider particular values or extreme cases of quantities and compare the results with your intuitive expectations. Ask, "Does this result make sense?"

Example 5.6 Straight-line motion with a constant force

An iceboat is at rest on a perfectly frictionless horizontal surface (Fig. 5.7a). A wind is blowing (along the direction of the runners) so that 4.0 s after the iceboat is released, it attains a velocity of 6.0 m/s (about 22 km/h, or 13 mi/h). What constant horizontal force F_W does the wind exert on the iceboat? The mass of iceboat and rider is 200 kg.

SOLUTION

IDENTIFY: Our target variable is one of the forces (F_W) acting on the iceboat, so we'll need to use Newton's second law. This law involves forces and acceleration, but the acceleration isn't given; we'll need to find it. Since the wind is assumed to exert a constant force, the resulting acceleration is constant and we'll be able to use one of the constant-acceleration formulas from Section 2.4.

SET UP: Figure 5.7b shows our free-body diagram for the iceboat and rider considered as a unit. The forces acting on this body are the weight w, the normal force n exerted by the surface, and the

5.7 (a) The situation. (b) Our free-body diagram.

(a) Iceboat and rider on frictionless ice

(b) Free-body diagram for iceboat and rider

Continued

horizontal force F_W (our target variable). The net force and hence the acceleration are to the right, so we chose the positive x-axis in this direction.

To find the x-acceleration, note what we are told about the iceboat's motion: It starts at rest so that its initial x-velocity is $v_{0x} = 0$, and it attains an x-velocity $v_x = 6.0\ \text{m/s}$ after an elapsed time $t = 4.0\ \text{s}$. An equation we can use to relate the x-acceleration a_x to these quantities is Eq. (2.8), $v_x = v_{0x} + a_x t$.

EXECUTE: The *known* quantities are the mass $m = 200\ \text{kg}$, the initial and final x-velocities $v_{0x} = 0$ and $v_x = 6.0\ \text{m/s}$, and the elapsed time $t = 4.0\ \text{s}$. The three *unknown* quantities are the acceleration a_x, the normal force n, and the horizontal force F_W (the target variable). Hence we need three equations.

The first two equations are the x- and y-equations for Newton's second law. The force F_W is in the positive x-direction, while the forces n and mg are in the positive and negative y-directions, respectively. Hence we have

$$\sum F_x = F_W = ma_x$$
$$\sum F_y = n + (-mg) = 0$$

The third equation we need is the constant-acceleration relationship

$$v_x = v_{0x} + a_x t$$

To find F_W, we first solve the constant-acceleration equation for a_x and then substitute this into the $\sum F_x$ equation:

$$a_x = \frac{v_x - v_{0x}}{t} = \frac{6.0\ \text{m/s} - 0\ \text{m/s}}{4.0\ \text{s}} = 1.5\ \text{m/s}^2$$
$$F_W = ma_x = (200\ \text{kg})(1.5\ \text{m/s}^2) = 300\ \text{kg}\cdot\text{m/s}^2$$

One $\text{kg}\cdot\text{m/s}^2$ is the same as 1 newton (N), so the final answer is

$$F_W = 300\ \text{N} \qquad \text{(about 67 lb)}$$

Note that we did not need the $\sum F_y$ equation at all to find F_W. We would need this equation if we wanted to find the normal force n:

$$n - mg = 0$$
$$n = mg = (200\ \text{kg})(9.8\ \text{m/s}^2)$$
$$= 2.0 \times 10^3\ \text{N} \qquad \text{(about 450 lb)}$$

EVALUATE: Our answers for F_W and n have the correct units for a force, as they should. The magnitude n of the normal force is equal to mg, the combined weight of the iceboat and rider, because the surface is horizontal and these are the only vertical forces that act. Does it seem reasonable that the force F_W is substantially *less* than mg?

Example 5.7 Straight-line motion with friction

Suppose a constant horizontal friction force with magnitude 100 N opposes the motion of the iceboat in Example 5.6. In this case, what constant force F_W must the wind exert on the iceboat to cause the same constant x-acceleration $a_x = 1.5\ \text{m/s}^2$?

SOLUTION

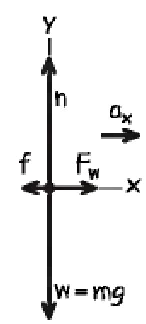
IDENTIFY: Once again the target variable is F_W. We are given the x-acceleration, so to find F_W all we need is Newton's second law.

SET UP: Figure 5.8 shows our new free-body diagram. The only difference from Fig. 5.7b is the addition of the friction force $\vec{f}$, which points opposite to the motion. (Note that its *magnitude*, $f = 100\ \text{N}$, is a positive quantity but that its *component* in the x-direction is negative, equal to $-f$ or $-100\ \text{N}$.)

EXECUTE: Two forces now have x-components: the force of the wind and the friction force. The x-component of Newton's second law gives

$$\sum F_x = F_W + (-f) = ma_x$$
$$F_W = ma_x + f = (200\ \text{kg})(1.5\ \text{m/s}^2) + (100\ \text{N}) = 400\ \text{N}$$

5.8 Our free-body diagram for the iceboat and rider with a friction force $\vec{f}$ opposing the motion.



EVALUATE: Because there is friction, a greater force F_W is needed than in Example 5.6. We need 100 N to overcome friction and 300 N more to give the iceboat the necessary acceleration.

Example 5.8 Tension in an elevator cable

An elevator and its load have a total mass of 800 kg (Fig. 5.9a). The elevator is originally moving downward at 10.0 m/s; it slows to a stop with constant acceleration in a distance of 25.0 m. Find the tension T in the supporting cable while the elevator is being brought to rest.

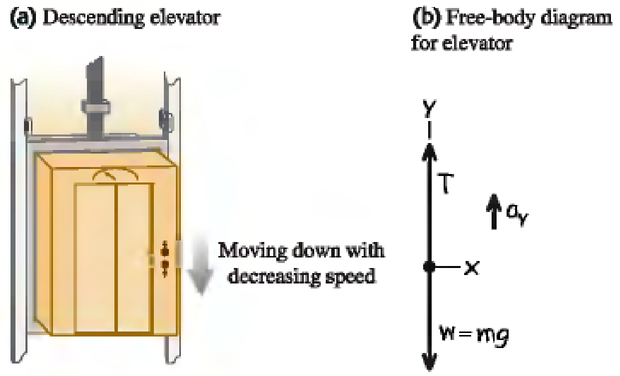
SOLUTION

IDENTIFY: The target variable is the tension T, which we will find using Newton's second law. As in Example 5.6, we'll have to determine the acceleration using the constant-acceleration formulas.

SET UP: Our free-body diagram in Fig. 5.9b shows the two forces acting on the elevator: its weight w and the tension force T of the cable. The elevator is moving downward with decreasing speed, so its acceleration is upward; we chose the positive y-axis to be in this direction.

The elevator is moving in the negative y-direction, so its initial y-velocity v_{0y} and its y-displacement $y - y_0$ are both negative: $v_{0y} = -10.0\ \text{m/s}$ and $y - y_0 = -25.0\ \text{m}$. The final y-velocity is $v_y = 0$. To find the y-acceleration a_y from this information, we'll use Eq. (2.13) in the form $v_y^2 = v_{0y}^2 + 2a_y(y - y_0)$. Once we

5.9 (a) The situation. (b) Our free-body diagram.



have a_y, we'll substitute it into the y-component of Newton's second law from Eq. (5.4).

EXECUTE: First let's write out Newton's second law. The tension force acts upward and the weight acts downward, so

$$\sum F_y = T + (-w) = ma_y$$

We solve for the target variable T:

$$T = w + ma_y = mg + ma_y = m(g + a_y)$$

To determine a_y, we rewrite the constant-acceleration equation $v_y^2 = v_{0y}^2 + 2a_y(y - y_0)$:

$$a_y = \frac{v_y^2 - v_{0y}^2}{2(y - y_0)} = \frac{(0)^2 - (-10.0\text{ m/s})^2}{2(-25.0\text{ m})} = +2.00\text{ m/s}^2$$

The acceleration is upward (positive), just as it should be for downward motion with decreasing speed.

Now we can substitute the acceleration into the equation for the tension:

$$T = m(g + a_y) = (800\text{ kg})(9.80\text{ m/s}^2 + 2.00\text{ m/s}^2)$$
$$= 9440\text{ N}$$

EVALUATE: The tension is 1600 N *greater* than the weight. This makes sense: The net force must be upward to provide the upward acceleration that brings the elevator to a halt. Can you see that we would get the same answers for a_y and T if the elevator were moving *upward* and *gaining* speed at a rate of 2.00 m/s²?

Example 5.9 Apparent weight in an accelerating elevator

A 50.0-kg woman stands on a bathroom scale while riding in the elevator in Example 5.8 (Fig. 5.10a). What is the reading on the scale?

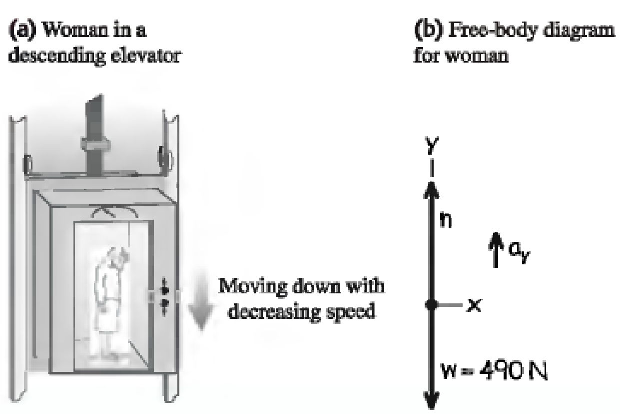
SOLUTION

IDENTIFY: The scale reads the magnitude of the downward force exerted *by* the woman *on* the scale. By Newton's third law, this equals the magnitude of the upward normal force exerted *by* the scale *on* the woman. Hence our target variable is the magnitude n of the normal force.

We'll find n by applying Newton's second law to the woman. We already know her acceleration; it's the same as the acceleration of the elevator, which we calculated in Example 5.8.

SET UP: Figure 5.10b shows our free-body diagram for the woman. The forces acting on her are the normal force n exerted by the scale and her weight $w = mg = (50.0\text{ kg})(9.80\text{ m/s}^2) =$

5.10 (a) The situation. (b) Our free-body diagram.



490 N. (The tension force, which played a major role in Example 5.8, doesn't appear here because it doesn't act directly on the woman. What pushes upward on her feet is the scale, *not* the elevator cable.) From Example 5.8, the y-acceleration of the elevator and of the woman is $a_y = +2.00\text{ m/s}^2$.

EXECUTE: Newton's second law gives

$$\sum F_y = n + (-mg) = ma_y$$
$$n = mg + ma_y = m(g + a_y)$$
$$= (50.0\text{ kg})(9.80\text{ m/s}^2 + 2.00\text{ m/s}^2) = 590\text{ N}$$

EVALUATE: Our answer for n means that while the elevator is stopping, the scale pushes up on the woman with a force of 590 N. By Newton's third law, she pushes down on the scale with the same force; so the scale reads 590 N, which is 100 N more than her actual weight. The scale reading is called the passenger's **apparent weight.** The woman *feels* the floor pushing up harder on her feet than when the elevator is stationary or moving with constant velocity.

What would the woman feel if the elevator were accelerating *downward* so that $a_y = -2.00\text{ m/s}^2$? This would be the case if the elevator were moving upward with decreasing speed or moving downward with increasing speed. To find the answer for this situation, we just insert the new value of a_y in our equation for n:

$$n = m(g + a_y) = (50.0\text{ kg})[9.80\text{ m/s}^2 + (-2.00\text{ m/s}^2)]$$
$$= 390\text{ N}$$

Now the woman feels as though she weighs only 390 N, or 100 N *less* than her actual weight.

You can feel these effects yourself; try taking a few steps in an elevator that is coming to a stop after descending (when your apparent weight is greater than your true weight w) or coming to a stop after ascending (when your apparent weight is less than w).

5.11 Astronauts in orbit feel "weightless" because they have the same acceleration as their spacecraft—*not* because they are "outside the pull of the earth's gravity." (If no gravity acted on them, the astronauts and their spacecraft wouldn't remain in orbit, but would fly off into deep space.)

Apparent Weight and Apparent Weightlessness

Let's generalize the result of Example 5.9. When a passenger with mass m rides in an elevator with y-acceleration a_y, a scale shows the passenger's apparent weight to be

$$n = m(g + a_y)$$

When the elevator is accelerating upward, a_y is positive and n is greater than the passenger's weight $w = mg$. When the elevator is accelerating downward, a_y is negative and n is less than the weight. If the passenger doesn't know the elevator is accelerating, she may feel as though her weight is changing; indeed, this is just what the scale shows.

The extreme case occurs when the elevator has a downward acceleration $a_y = -g$, that is, when it is in free fall. In that case, $n = 0$ and the passenger *seems* to be weightless. Similarly, an astronaut orbiting the earth in a spacecraft experiences *apparent weightlessness* (Fig. 5.11). In each case, the person is not truly weightless because there is still a gravitational force acting. But the person's sensations in this free-fall condition are exactly the same as though the person were in outer space with no gravitational force at all. In both cases the person and the vehicle (elevator or spacecraft) are falling together with the same acceleration g, so nothing pushes the person against the floor or walls of the vehicle.

Example 5.10 Acceleration down a hill

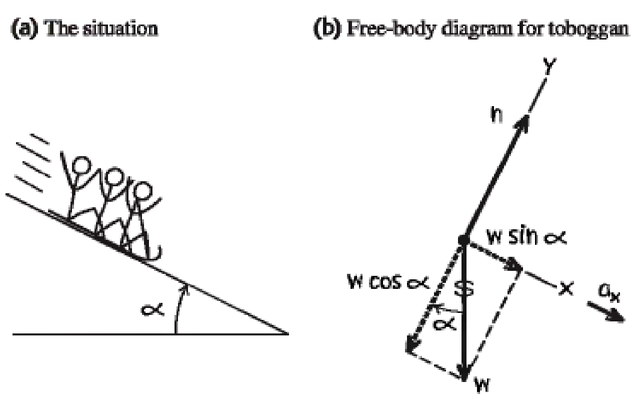
A toboggan loaded with vacationing students (total weight w) slides down a long, snow-covered slope. The hill slopes at a constant angle α, and the toboggan is so well waxed that there is virtually no friction. What is its acceleration?

SOLUTION

IDENTIFY: Our target variable is the acceleration, which we'll find using Newton's second law. There is no friction, so only two forces act on the toboggan: its weight w and the normal force n exerted by the hill. As in Example 5.4 (Section 5.1), the surface is inclined so the normal force is not vertical and is not opposite to the weight. Hence we must use both components of $\sum\vec{F} = m\vec{a}$ in Eq. (5.4).

SET UP: Figure 5.12 shows our sketch and free-body diagram. We take axes parallel and perpendicular to the surface of the hill, so that the acceleration (which is parallel to the hill) is along the positive x-direction.

5.12 Our sketches for this problem.



EXECUTE: The normal force has only a y-component, but the weight has both x- and y-components: $w_x = w\sin\alpha$ and $w_y = -w\cos\alpha$. (Compare to Example 5.4, in which the x-component of weight was $-w\sin\alpha$. The difference is that the positive x-axis was uphill in Example 5.4, while in Fig. 5.12b it is downhill.) The wiggly line in Fig. 5.12b reminds us that we have resolved the weight into its components.

The acceleration is purely in the $+x$-direction, so $a_y = 0$. Newton's second law in component form then tells us that

$$\sum F_x = w\sin\alpha = ma_x$$
$$\sum F_y = n - w\cos\alpha = ma_y = 0$$

Since $w = mg$, the x-component equation tells us that $mg\sin\alpha = ma_x$, or

$$a_x = g\sin\alpha$$

Note that we didn't need the y-component equation to find the acceleration. That's the beauty of choosing the x-axis to lie along the acceleration direction! What the y-components tell us is the magnitude of the normal force that the hill exerts on the toboggan:

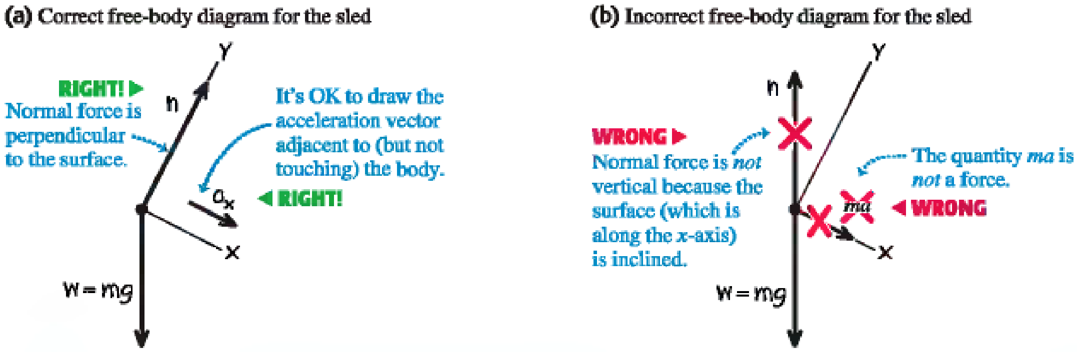
$$n = w\cos\alpha = mg\cos\alpha$$

EVALUATE: Notice that the mass does not appear in our answer for the acceleration. This means that *any* toboggan, regardless of its mass or number of passengers, slides down a frictionless hill with an acceleration of $g\sin\alpha$. In particular, if the plane is horizontal, $\alpha = 0$ and $a_x = 0$ (the toboggan does not accelerate); if the plane is vertical, $\alpha = 90°$ and $a_x = g$ (the toboggan is in free fall).

Notice also that the normal force n is not equal to the toboggan's weight (compare Example 5.4 in Section 5.1). We don't need this result here, but it will be useful in a later example.

CAUTION **Common free-body diagram errors** Figure 5.13 shows both the correct way (Fig. 5.13a) and a common *incorrect* way (Fig. 5.13b) to draw the free-body diagram for the toboggan. The diagram in Fig. 5.13b is wrong for two reasons: the normal force must be drawn perpendicular to the surface, and there's no such thing as the "$m\vec{a}$ force." If you remember that "normal" means "perpendicular" and that $m\vec{a}$ is not itself a force, you'll be well on your way to always drawing correct free-body diagrams.

5.13 Correct and incorrect diagrams for a toboggan on a frictionless hill.



Example 5.11 Two bodies with the same acceleration

You push a 1.00-kg food tray through the cafeteria line with a constant 9.0-N force. As the tray moves, it pushes on a 0.50-kg carton of milk (Fig. 5.14a). The tray and carton slide on a horizontal surface that is so greasy that friction can be neglected. Find the acceleration of the tray and carton and the horizontal force that the tray exerts on the carton.

SOLUTION

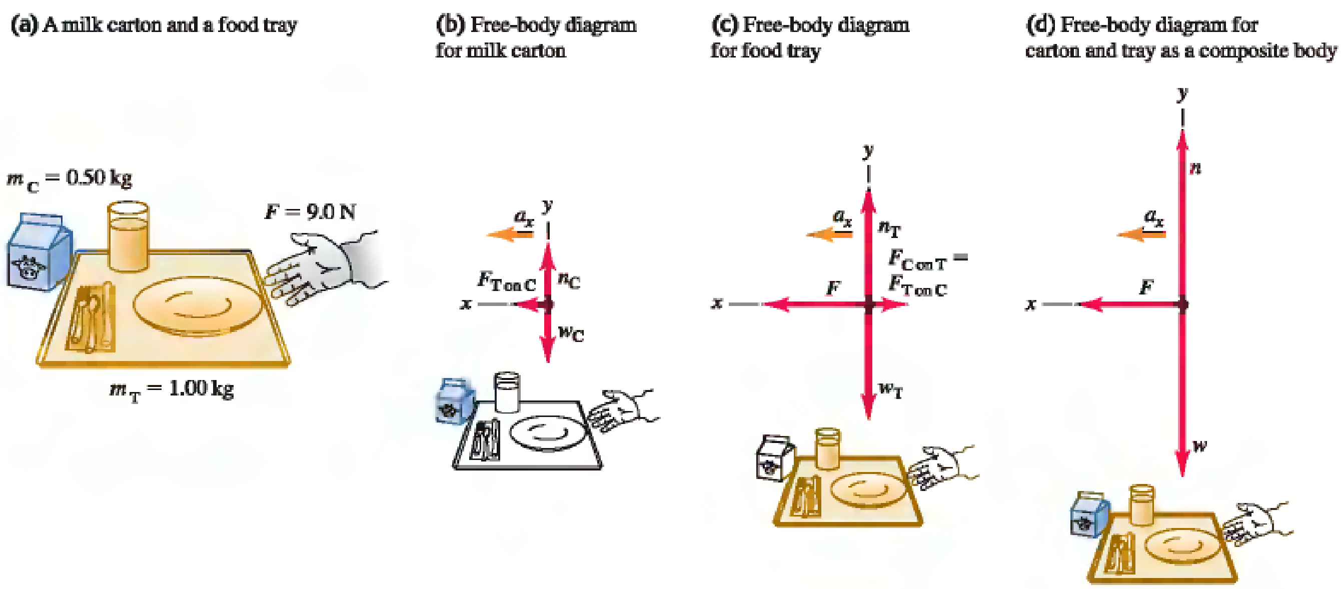
IDENTIFY: Our *two* target variables are the acceleration of the system of tray and carton and the force of the tray on the carton. Again we will use Newton's second law, but we'll have to apply it to two different bodies to get two equations (one for each target variable).

SET UP: We can set up the problem in two ways.

Method 1: We can treat the milk carton (mass m_C) and tray (mass m_T) as separate bodies, each with its own free-body diagram (Figs. 5.14b and 5.14c). Note that the force F that you exert on the tray doesn't appear in the free-body diagram for the milk carton. Instead, what makes the carton accelerate is the force of magnitude $F_{T\,on\,C}$ exerted on it by the tray. By Newton's third law, the carton exerts a force of equal magnitude on the tray: $F_{C\,on\,T} = F_{T\,on\,C}$. We take the acceleration to be in the positive x-direction; both the tray and milk carton move with the same x-acceleration a_x.

Method 2: We can treat the tray and milk carton as a composite body of mass $m = m_T + m_C = 1.50$ kg (Fig. 5.14d). The only

5.14 Pushing a food tray and milk carton in the cafeteria line.



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horizontal force acting on this composite body is the force F that you exert. The forces $F_{\mathrm{T\,on\,C}}$ and $F_{\mathrm{C\,on\,T}}$ don't come into play because they're *internal* to this composite body, and Newton's second law tells us that only *external* forces affect a body's acceleration (see Section 4.3). Hence we'll need an additional equation to find the magnitude $F_{\mathrm{T\,on\,C}}$ using this method; we'll get this equation by also applying Newton's second law to the milk carton, as in Method 1.

EXECUTE: *Method 1:* The x-component equations of Newton's second law for the tray and for the carton are

$$\text{Tray:} \quad \sum F_x = F - F_{\mathrm{C\,on\,T}} = F - F_{\mathrm{T\,on\,C}} = m_{\mathrm{T}}a_x$$
$$\text{Carton:} \quad \sum F_x = F_{\mathrm{T\,on\,C}} = m_{\mathrm{C}}a_x$$

These are two simultaneous equations for the two target variables a_x and $F_{\mathrm{T\,on\,C}}$. (Two equations are all we need, which means that the y-components don't play a role in this example.) An easy way to solve the two equations for a_x is to add them; this eliminates $F_{\mathrm{T\,on\,C}}$, giving

$$F = m_{\mathrm{T}}a_x + m_{\mathrm{C}}a_x = (m_{\mathrm{T}} + m_{\mathrm{C}})a_x$$

and

$$a_x = \frac{F}{m_{\mathrm{T}} + m_{\mathrm{C}}} = \frac{9.0\ \mathrm{N}}{1.00\ \mathrm{kg} + 0.50\ \mathrm{kg}} = 6.0\ \mathrm{m/s^2}$$

Substituting this back into the equation for the carton gives

$$F_{\mathrm{T\,on\,C}} = m_{\mathrm{C}}a_x = (0.50\ \mathrm{kg})(6.0\ \mathrm{m/s^2}) = 3.0\ \mathrm{N}$$

Method 2: The x-component of Newton's second law for the composite body of mass m is

$$\sum F_x = F = ma_x$$

and the acceleration of this composite body is

$$a_x = \frac{F}{m} = \frac{9.0\ \mathrm{N}}{1.50\ \mathrm{kg}} = 6.0\ \mathrm{m/s^2}$$

Then, looking at the milk carton by itself, we see that to give it an acceleration of 6.0 $\mathrm{m/s^2}$ requires that the tray exert a force:

$$F_{\mathrm{T\,on\,C}} = m_{\mathrm{C}}a_x = (0.50\ \mathrm{kg})(6.0\ \mathrm{m/s^2}) = 3.0\ \mathrm{N}$$

EVALUATE: The answers are the same with either method, as they should be. To check the answers, note that there are different forces on the two sides of the tray: $F = 9.0\ \mathrm{N}$ on the right and $F_{\mathrm{C\,on\,T}} = 3.0\ \mathrm{N}$ on the left. Hence the net horizontal force on the tray is $F - F_{\mathrm{C\,on\,T}} = 6.0\ \mathrm{N}$, exactly enough to accelerate a 1.00-kg tray at 6.0 $\mathrm{m/s^2}$.

The method of treating the two bodies as a single composite body works *only* if the two bodies have the same magnitude *and* direction of acceleration. If the accelerations are different, we must treat the two bodies separately, as in the next example.

Example 5.12 Two bodies with the same magnitude of acceleration

Figure 5.15a shows an air-track glider with mass m_1 moving on a level, frictionless air track in the physics lab. The glider is connected to a lab weight with mass m_2 by a light, flexible, nonstretching string that passes over a small frictionless pulley. Find the acceleration of each body and the tension in the string.

SOLUTION

IDENTIFY: The glider and weight are accelerating, so again we must use Newton's second law. Our three target variables are the tension T in the string and the accelerations of the two bodies.

SET UP: The two bodies move in different directions—one horizontal, one vertical—so we can't consider them together as we did the bodies in Example 5.11. Figures 5.15b and 5.15c show our free-body diagrams and coordinate systems. It's convenient to have both bodies accelerate in the positive axis directions, so we chose the positive y-direction for the lab weight to be downward. (It's perfectly all right to use different coordinate axes for the two bodies.)

There is no friction in the pulley and we consider the string to be massless, so the tension T in the string is the same throughout; it applies a force of the same magnitude T to each body. (You may want to review Conceptual Example 4.10 in Section 4.5, where we discussed the tension force exerted by a massless string.) The weights are m_1g and m_2g.

While the *directions* of the two accelerations are different, their *magnitudes* are the same. That's because the string doesn't stretch. Hence the two bodies must move equal distances in equal times, and so their speeds at any instant must be equal. When the speeds change, they change by equal amounts in a given time, so the accelerations of the two bodies must have the same magnitude a. We can express this relationship as

$$a_{1x} = a_{2y} = a$$

Thanks to this relationship, we actually have only *two* target variables: a and the tension T.

EXECUTE: For the glider on the track, Newton's second law gives

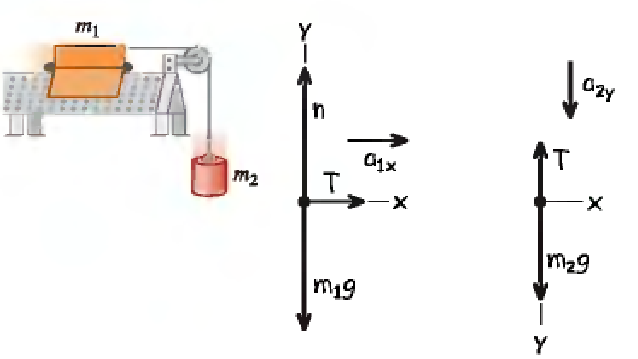
$$\text{Glider:} \quad \sum F_x = T = m_1a_{1x} = m_1a$$
$$\text{Glider:} \quad \sum F_y = n + (-m_1g) = m_1a_{1y} = 0$$

For the lab weight, the only forces are in the y-direction, and

$$\text{Lab weight:} \quad \sum F_y = m_2g + (-T) = m_2a_{2y} = m_2a$$

5.15 (a) The situation. (b), (c) Our free-body diagrams.

(a) Apparatus (b) Free-body diagram for glider (c) Free-body diagram for weight



In these equations we've used the relationships $a_{1y} = 0$ (the glider doesn't accelerate vertically) and $a_{1x} = a_{2y} = a$ (the two objects have the same magnitude of acceleration).

The x-equation for the glider and the equation for the lab weight give us two simultaneous equations for the target variables T and a:

$$\text{Glider:} \qquad T = m_1 a$$

$$\text{Lab weight:} \qquad m_2 g - T = m_2 a$$

We add the two equations to eliminate T, giving

$$m_2 g = m_1 a + m_2 a = (m_1 + m_2)a$$

and so the magnitude of each body's acceleration is

$$a = \frac{m_2}{m_1 + m_2} g$$

Substituting this back into the first equation (for the glider), we get

$$T = \frac{m_1 m_2}{m_1 + m_2} g$$

EVALUATE: The acceleration is less than g, as you might expect; the lab weight accelerates more slowly because the string tension pulls it back.

The tension T is *not* equal to the weight $m_2 g$ of the lab weight, but is *less* by a factor of $m_1/(m_1 + m_2)$. If T *were* equal to $m_2 g$, then the lab weight would be in equilibrium, and it isn't.

CAUTION **Tension and weight may not be equal** It's a common mistake to assume that if an object is attached to a vertical string, the string tension must be equal to the object's weight. That was the case in Example 5.5, where the acceleration was zero, but it would certainly be wrong in this example! The only safe approach is to *always* treat the tension as a variable, as we have done here.

Finally, let's check some special cases. If $m_1 = 0$, then the lab weight would fall freely and there would be no tension in the string. The equations do give $T = 0$ and $a = g$ when $m_1 = 0$. Also, if $m_2 = 0$, we expect no tension and no acceleration; for this case the equations do indeed tell us that $T = 0$ and $a = 0$.

Test Your Understanding of Section 5.2 Suppose you hold the glider in Example 5.12 so that it and the weight are initially at rest. You give the glider a push to the left in Fig. 5.15a and then release it. The string remains taut as the glider moves to the left, comes instantaneously to rest, then moves to the right. At the instant the glider has zero velocity, what is the tension in the string? (i) greater than in Example 5.12; (ii) the same as in Example 5.12; (iii) less than in Example 5.12, but greater than zero; (iv) zero.

5.3 Frictional Forces

We have seen several problems where a body rests or slides on a surface that exerts forces on the body. Whenever two bodies interact by direct contact (touching) of their surfaces, we describe the interaction in terms of *contact forces.* The normal force is one example of a contact force; in this section we'll look in detail at another contact force, the force of friction.

Friction is important in many aspects of everyday life. The oil in a car engine minimizes friction between moving parts, but without friction between the tires and the road we couldn't drive or turn the car. Air drag—the frictional force exerted by the air on a body moving through it—decreases automotive fuel economy but makes parachutes work. Without friction, nails would pull out, light bulbs would unscrew effortlessly, and ice hockey would be hopeless (Fig. 5.16).

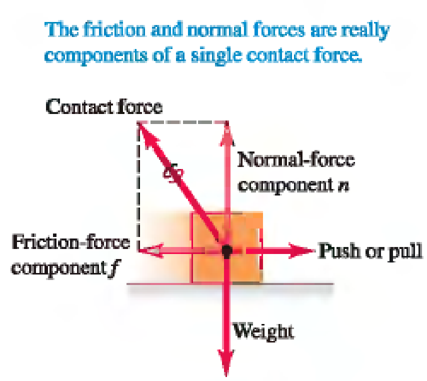
5.16 The sport of ice hockey depends on having the right amount of friction between a player's skates and the ice. If there were too much friction, the players would move too slowly; if there were too little friction, they would fall over.

Kinetic and Static Friction

When you try to slide a heavy box of books across the floor, the box doesn't move at all unless you push with a certain minimum force. Then the box starts moving, and you can usually keep it moving with less force than you needed to get it started. If you take some of the books out, you need less force than before to get it started or keep it moving. What general statements can we make about this behavior?

First, when a body rests or slides on a surface, we can think of the surface as exerting a single contact force on the body, with force components perpendicular and parallel to the surface (Fig. 5.17). The perpendicular component vector is the normal force, denoted by $\vec{n}$. The component vector parallel to the surface (and perpendicular to $\vec{n}$) is the **friction force,** denoted by $\vec{f}$. If the surface is frictionless, then $\vec{f}$ is zero but there is still a normal force. (Frictionless surfaces are an unattainable idealization, like a massless rope. But we can approximate a surface

5.17 When a block is pushed or pulled over a surface, the surface exerts a contact force on it.



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2.5 Truck Pulls Crate
2.6 Pushing a Crate Up a Wall
2.7 Skier Goes Down a Slope
2.8 Skier and Rope Tow
2.10 Truck Pulls Two Crates

as frictionless if the effects of friction are negligibly small.) The direction of the friction force is always such as to oppose relative motion of the two surfaces.

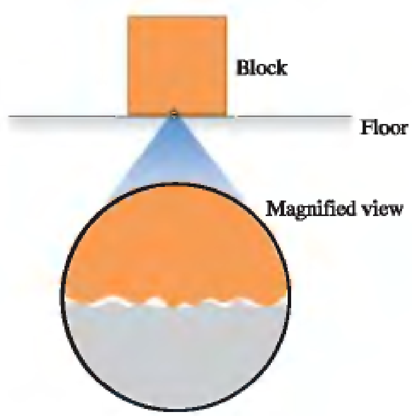
The kind of friction that acts when a body slides over a surface is called a **kinetic friction force** $\vec{f}_k$. The adjective "kinetic" and the subscript "k" remind us that the two surfaces are moving relative to each other. The *magnitude* of the kinetic friction force usually increases when the normal force increases. This is why it takes more force to slide a box full of books across the floor than to slide the same box when it is empty. This principle is also used in automotive braking systems: The harder the brake pads are squeezed against the rotating brake disks, the greater the braking effect. In many cases the magnitude of the kinetic friction force f_k is found experimentally to be approximately *proportional* to the magnitude n of the normal force. In such cases we represent the relationship by the equation

$$f_k = \mu_k n \qquad \text{(magnitude of kinetic friction force)} \qquad (5.5)$$

where μ_k (pronounced "mu-sub-k") is a constant called the **coefficient of kinetic friction.** The more slippery the surface, the smaller the coefficient of friction. Because it is a quotient of two force magnitudes, μ_k is a pure number, without units.

CAUTION **Friction and normal forces are always perpendicular** Remember that Eq. (5.5) is *not* a vector equation because $\vec{f}_k$ and $\vec{n}$ are always perpendicular. Rather, it is a scalar relationship between the magnitudes of the two forces.

5.18 The normal and friction forces arise from interactions between molecules at high points on the surfaces of the block and the floor.



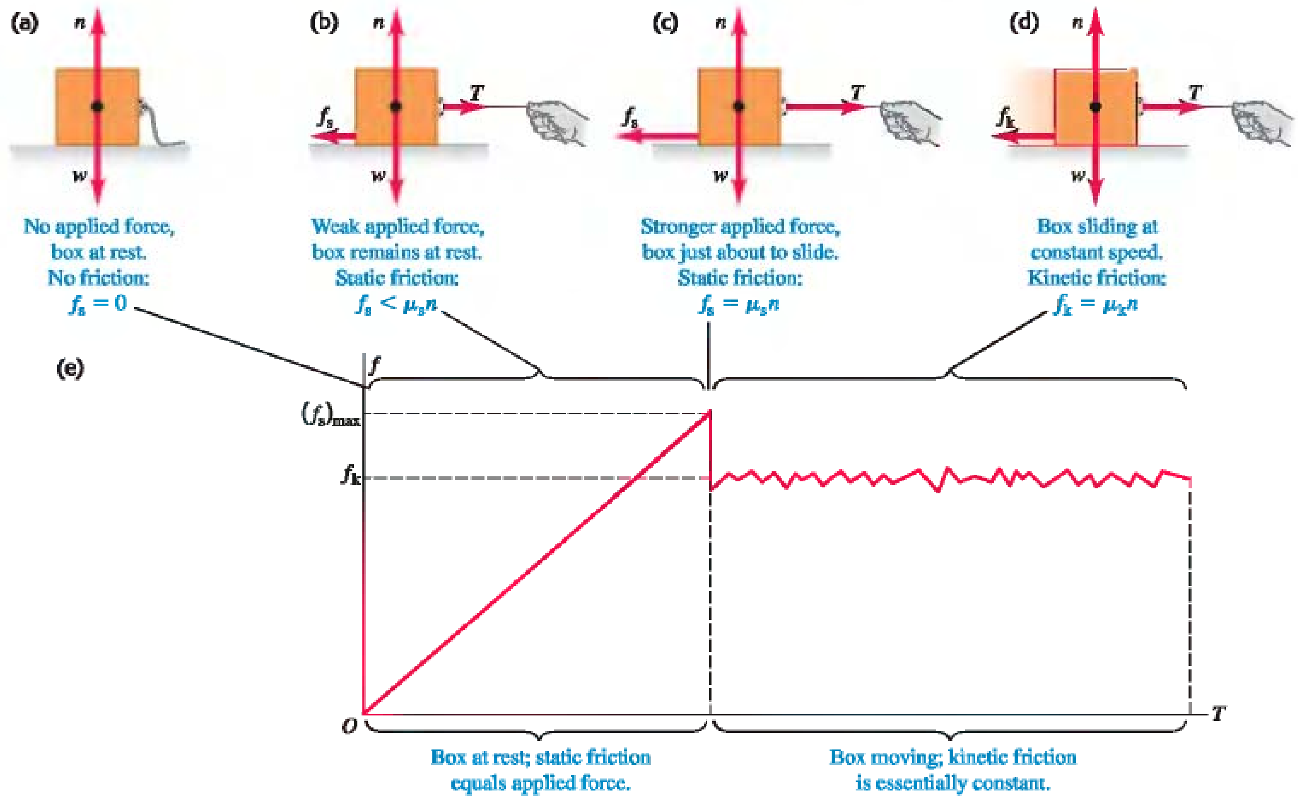
Equation (5.5) is only an approximate representation of a complex phenomenon. On a microscopic level, friction and normal forces result from the intermolecular forces (fundamentally electrical in nature) between two rough surfaces at points where they come into contact (Fig. 5.18). As a box slides over the floor, bonds between the two surfaces form and break, and the total number of such bonds varies; hence the kinetic friction force is not perfectly constant. Smoothing the surfaces can actually increase friction, since more molecules are able to interact and bond; bringing two smooth surfaces of the same metal together can cause a "cold weld." Lubricating oils work because an oil film between two surfaces (such as the pistons and cylinder walls in a car engine) prevents them from coming into actual contact.

Table 5.1 lists some representative values of μ_k. Although these values are given with two significant figures, they are only approximate, since friction forces

Table 5.1 Approximate Coefficients of Friction

Materials	Coefficient of Static Friction, μ_s	Coefficient of Kinetic Friction, μ_k
Steel on steel	0.74	0.57
Aluminum on steel	0.61	0.47
Copper on steel	0.53	0.36
Brass on steel	0.51	0.44
Zinc on cast iron	0.85	0.21
Copper on cast iron	1.05	0.29
Glass on glass	0.94	0.40
Copper on glass	0.68	0.53
Teflon on Teflon	0.04	0.04
Teflon on steel	0.04	0.04
Rubber on concrete (dry)	1.0	0.8
Rubber on concrete (wet)	0.30	0.25

5.19 (a), (b), (c) When there is no relative motion, the magnitude of the static friction force f_s is less than or equal to $\mu_s n$. (d) When there is relative motion, the magnitude of the kinetic friction force f_k equals $\mu_k n$. (e) A graph of the friction force magnitude f as a function of the magnitude T of the applied force T. The kinetic friction force varies somewhat as intermolecular bonds form and break.



can also depend on the speed of the body relative to the surface. For now we'll ignore this effect and assume that μ_k and f_k are independent of speed, in order to concentrate on the simplest cases. Table 5.1 also lists coefficients of static friction; we'll define these shortly.

Friction forces may also act when there is *no* relative motion. If you try to slide a box across the floor, the box may not move at all because the floor exerts an equal and opposite friction force on the box. This is called a **static friction force** $\vec{f}_s$. In Fig. 5.19a, the box is at rest, in equilibrium, under the action of its weight $\vec{w}$ and the upward normal force $\vec{n}$. The normal force is equal in magnitude to the weight $(n = w)$ and is exerted on the box by the floor. Now we tie a rope to the box (Fig. 5.19b) and gradually increase the tension T in the rope. At first the box remains at rest because, as T increases, the force of static friction f_s also increases (staying equal in magnitude to T).

At some point T becomes greater than the maximum static friction force f_s the surface can exert. Then the box "breaks loose" (the tension T is able to break the bonds between molecules in the surfaces of the box and floor) and starts to slide. Figure 5.19c shows the forces when T is at this critical value. If T exceeds this value, the box is no longer in equilibrium. For a given pair of surfaces the maximum value of f_s depends on the normal force. Experiment shows that in many cases this maximum value, called $(f_s)_{max}$, is approximately *proportional* to n; we call the proportionality factor μ_s the **coefficient of static friction.** Table 5.1 lists some representative values of μ_s. In a particular situation, the actual force of static friction can have any magnitude between zero (when there is no other force parallel to the surface) and a maximum value given by $\mu_s n$. In symbols,

$$f_s \leq \mu_s n \qquad \text{(magnitude of static friction force)} \qquad (5.6)$$

Like Eq. (5.5), this is a relationship between magnitudes, *not* a vector relationship. The equality sign holds only when the applied force T has reached the critical value at which motion is about to start (Fig. 5.19c). When T is less than this value (Fig. 5.19b), the inequality sign holds. In that case we have to use the equilibrium conditions $(\sum \vec{F} = \mathbf{0})$ to find f_s. If there is no applied force $(T = 0)$ as in Fig. 5.19a, then there is no static friction force either $(f_s = 0)$.

As soon as the box starts to slide (Fig. 5.19d), the friction force usually *decreases;* it's easier to keep the box moving than to start it moving. Hence the coefficient of kinetic friction is usually *less* than the coefficient of static friction for any given pair of surfaces, as Table 5.1 shows. If we start with no applied force $(T = 0)$ and gradually increase the force, the friction force varies somewhat, as shown in Fig. 5.19e.

In some situations the surfaces will alternately stick (static friction) and slip (kinetic friction). This is what causes the horrible sound made by chalk held at the wrong angle while writing on the blackboard. Other stick-slip phenomena are the squeak of windshield wipers on dry glass and the shriek of tires sliding on asphalt pavement. A more positive example is the motion of a violin bow against the string.

When a body slides on a layer of gas, friction can be made very small. In the linear air track used in physics laboratories, the gliders are supported on a layer of air. The frictional force is velocity dependent, but at typical speeds the effective coefficient of friction is of the order of 0.001.

Example 5.13 Friction in horizontal motion

You are trying to move a 500-N crate across a level floor. To start the crate moving, you have to pull with a 230-N horizontal force. Once the crate "breaks loose" and starts to move, you can keep it moving at constant velocity with only 200 N. What are the coefficients of static and kinetic friction?

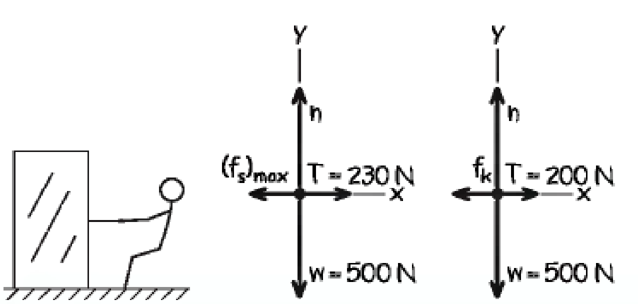
SOLUTION

IDENTIFY: The crate is in equilibrium whether it is at rest or moving with constant velocity, so we use Newton's first law as expressed by Eq. (5.2). We'll also need the relationships in Eqs. (5.5) and (5.6) to find the target variables μ_s and μ_k.

SET UP: In either situation there are four forces acting on the crate: the downward weight force (magnitude $w = 500\text{ N}$), the upward normal force (magnitude n) exerted by the ground, a tension force (magnitude T) to the right exerted by the rope, and a friction force to the left exerted by the ground. Figures 5.20a and 5.20b show our sketch and free-body diagram for the instant just before the crate starts to move, when the static friction force has its maximum possible value $(f_s)_{max} = \mu_s n$. Once the crate is moving to the right at constant velocity, the friction force changes to its kinetic form (Fig. 5.20c). Because the rope in Fig. 5.20a is in equilibrium, the tension is the same at both ends. Hence the tension force that the rope exerts on the crate has the same magnitude as the force you exert on the rope.

5.20 Our sketches for this problem.

(a) Pulling a crate (b) Free-body diagram for crate just before it starts to move (c) Free-body diagram for crate moving at constant speed



EXECUTE: Just before the crate starts to move (Fig. 5.20b), we have

$$\sum F_x = T + (-(f_s)_{max}) = 0 \quad \text{so} \quad (f_s)_{max} = T = 230\text{ N}$$
$$\sum F_y = n + (-w) = 0 \quad \text{so} \quad n = w = 500\text{ N}$$

Then we use Eq. (5.6), $(f_s)_{max} = \mu_s n$, to find the value of μ_s:

$$\mu_s = \frac{(f_s)_{max}}{n} = \frac{230\text{ N}}{500\text{ N}} = 0.46$$

After the crate starts to move, the forces are as shown in Fig. 5.20c, and we have

$$\sum F_x = T + (-f_k) = 0 \quad \text{so} \quad f_k = T = 200\text{ N}$$
$$\sum F_y = n + (-w) = 0 \quad \text{so} \quad n = w = 500\text{ N}$$

Using $f_k = \mu_k n$ from Eq. (5.5), we find

$$\mu_k = \frac{f_k}{n} = \frac{200\text{ N}}{500\text{ N}} = 0.40$$

EVALUATE: It's easier to keep the crate moving than to start it moving, and so the coefficient of kinetic friction is less than the coefficient of static friction.

Example 5.14 Static friction can be less than the maximum

In Example 5.13, what is the friction force if the crate is at rest on the surface and a horizontal force of 50 N is applied to it?

SOLUTION

IDENTIFY: The applied force is less than the maximum force of static friction, $(f_s)_{max} = 230\text{ N}$. Hence the crate remains at rest and the net force acting on it is zero. The target variable is the magnitude f_s of the friction force.

SET UP: The free-body diagram is the same as in Fig. 5.20b, but with $(f_s)_{max}$ replaced by f_s and $T = 230\text{ N}$ replaced by $T = 50\text{ N}$.

EXECUTE: From the equilibrium conditions, Eq. (5.2), we have

$$\sum F_x = T + (-f_s) = 0 \quad \text{so} \quad f_s = T = 50\text{ N}$$

EVALUATE: In this case, f_s is less than the maximum value $(f_s)_{max} = \mu_s n$. The frictional force can prevent motion for any horizontal applied force up to 230 N.

Example 5.15 Minimizing kinetic friction

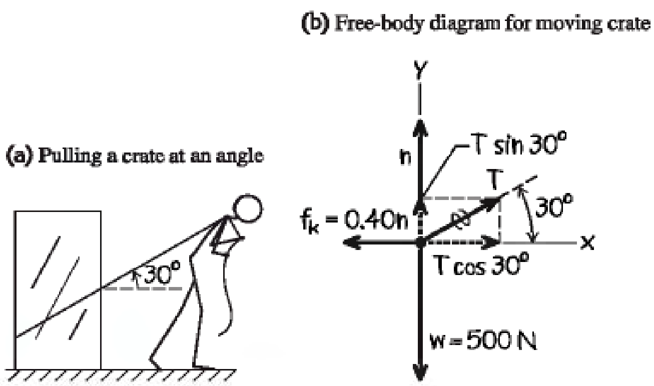
In Example 5.13, suppose you try to move the crate by tying a rope around it and pulling upward on the rope at an angle of 30° above the horizontal. How hard do you have to pull to keep the crate moving with constant velocity? Is this easier or harder than pulling horizontally? Assume $w = 500\text{ N}$ and $\mu_k = 0.40$.

SOLUTION

IDENTIFY: The crate is in equilibrium because its velocity is constant, so we again apply Newton's first law. Since the crate is in motion, the ground exerts a *kinetic* friction force. The target variable is the magnitude T of the tension force.

SET UP: Figure 5.21 shows our sketch and free-body diagram. The kinetic friction force f_k is still equal to $\mu_k n$, but now the normal force n is *not* equal in magnitude to the weight of the crate. The force exerted by the rope has an additional vertical component that tends to lift the crate off the floor.

5.21 Our sketches for this problem.



EXECUTE: From the equilibrium conditions and the equation $f_k = \mu_k n$, we have

$$\sum F_x = T\cos 30° + (-f_k) = 0 \quad \text{so} \quad T\cos 30° = \mu_k n$$
$$\sum F_y = T\sin 30° + n + (-w) = 0 \quad \text{so} \quad n = w - T\sin 30°$$

These are two equations for the two unknown quantities T and n. To solve them, we can eliminate one unknown and solve for the other. There are many ways to do this; one way is to substitute the expression for n in the second equation back into the first equation:

$$T\cos 30° = \mu_k(w - T\sin 30°)$$

Then we solve this equation for T, with the result

$$T = \frac{\mu_k w}{\cos 30° + \mu_k \sin 30°} = 188\text{ N}$$

We can substitute this result back into either of the original equations to obtain n. If we use the second equation to do this, we get

$$n = w - T\sin 30° = (500\text{ N}) - (188\text{ N})\sin 30° = 406\text{ N}$$

EVALUATE: The normal force is *less* than the weight of the box $(w = 500\text{ N})$ because the vertical component of tension pulls upward on the crate. Despite this, the tension required is a little less than the 200-N force needed when you pulled horizontally in Example 5.13. Try pulling at 22°; you'll find you need even less force (see Challenge Problem 5.123).

Example 5.16 Toboggan ride with friction I

Let's go back to the toboggan we studied in Example 5.10 (Section 5.2). The wax has worn off and there is now a nonzero coefficient of kinetic friction μ_k. The slope has just the right angle to make the toboggan slide with constant speed. Derive an expression for the slope angle in terms of w and μ_k.

SOLUTION

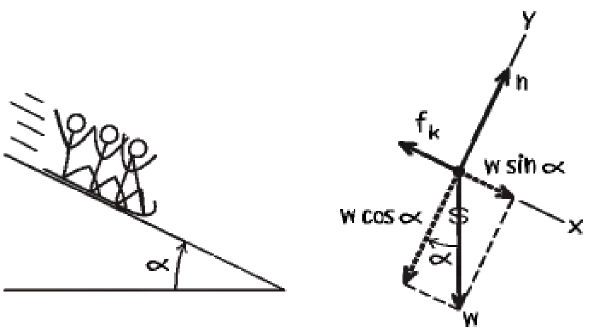
IDENTIFY: Our target variable is the slope angle α. The toboggan is in equilibrium because its velocity is constant, so we use Newton's first law. There are three forces acting on the toboggan: its weight, the normal force, and the kinetic friction force. Since the motion is downhill, the kinetic friction force (which opposes the motion of the toboggan over the hill) is directed uphill.

SET UP: Figure 5.22 shows our sketch and free-body diagram. We take axes perpendicular and parallel to the surface and represent the weight in terms of its components in these two directions, as shown. (Compare Fig. 5.12b in Example 5.10.) The magnitude of the friction force is given by Eq. (5.5), $f_k = \mu_k n$.

Continued

5.22 Our sketches for this problem.

(a) The situation (b) Free-body diagram for toboggan



EXECUTE: The equilibrium conditions are

$$\sum F_x = w\sin\alpha + (-f_k) = w\sin\alpha - \mu_k n = 0$$
$$\sum F_y = n + (-w\cos\alpha) = 0$$

(We used the relationship $f_k = \mu_k n$ in the equation for the x-components.) Rearranging, we get

$$\mu_k n = w\sin\alpha \qquad \text{and} \qquad n = w\cos\alpha$$

Just as in Example 5.10, the normal force n is *not* equal to the weight w. When we divide the first of these equations by the second, we find

$$\mu_k = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha \qquad \text{so} \qquad \alpha = \arctan\mu_k$$

EVALUATE: The weight w doesn't appear in this expression. *Any* toboggan, regardless of its weight, slides down an incline with constant speed if the coefficient of kinetic friction equals the tangent of the slope angle of the incline. The greater the coefficient of friction, the steeper the slope has to be for the toboggan to slide with constant velocity.

Example 5.17 Toboggan ride with friction II

The same toboggan with the same coefficient of friction as in Example 5.16 *accelerates* down a steeper hill. Derive an expression for the acceleration in terms of g, α, μ_k, and w.

SOLUTION

IDENTIFY: The toboggan is accelerating and hence not in equilibrium, so we must use Newton's second law, $\sum\vec{F} = m\vec{a}$, in its component form as given in Eq. (5.4). Our target variable is the downhill acceleration.

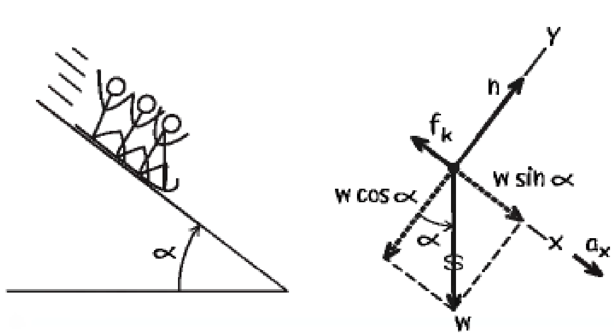
SET UP: Figure 5.23 shows our sketches. The free-body diagram (Fig. 5.23b) is almost the same as for Example 5.16. The toboggan's y-component of acceleration a_y is still zero, but the x-component a_x is not.

EXECUTE: It's convenient to express the weight as $w = mg$. Then Newton's second law in component form says

$$\sum F_x = mg\sin\alpha + (-f_k) = ma_x$$
$$\sum F_y = n + (-mg\cos\alpha) = 0$$

5.23 Our sketches for this problem.

(a) The situation (b) Free-body diagram for toboggan



From the second equation and Eq. (5.5) we get an expression for f_k:

$$n = mg\cos\alpha$$
$$f_k = \mu_k n = \mu_k mg\cos\alpha$$

We substitute this back into the x-component equation:

$$mg\sin\alpha + (-\mu_k mg\cos\alpha) = ma_x$$
$$a_x = g(\sin\alpha - \mu_k\cos\alpha)$$

EVALUATE: Does this result make sense? Let's check some special cases. First, if the hill is vertical, $\alpha = 90°$; then $\sin\alpha = 1$, $\cos\alpha = 0$, and $a_x = g$. This is free fall, just what we would expect. Second, on a hill at angle α with no friction, $\mu_k = 0$. Then $a_x = g\sin\alpha$. The situation is the same as in Example 5.10; happily, we get the same result. Next, suppose that there is just enough friction to make the toboggan move with constant velocity. In that case $a_x = 0$, so it must be that

$$\sin\alpha = \mu_k\cos\alpha \qquad \text{and} \qquad \mu_k = \tan\alpha$$

This agrees with our result from Example 5.16. Finally, note that there may be so much friction that $\mu_k\cos\alpha$ is actually greater than $\sin\alpha$. In that case, a_x is negative; if we give the toboggan an initial downhill push to start it moving, it will slow down and eventually stop.

We have pretty much beaten the toboggan problem to death, but look what we've done: Starting with a simple problem, we extended it to more and more general situations. The general result we found in this example includes *all* the previous ones as special cases. Don't memorize this general result; it is useful only for this one set of problems. But make sure you understand how we obtained it and what it means.

One final variation that you may want to try out is the case in which we give the toboggan an initial push up the hill. The direction of the kinetic friction force is now reversed, so the acceleration is different from the downhill value. It turns out that the expression for a_x is the same as for downhill motion except that the minus sign becomes plus. Can you prove this?

Rolling Friction

It's a lot easier to move a loaded filing cabinet across a horizontal floor using a cart with wheels than to slide it. How much easier? We can define a **coefficient of rolling friction** μ_r, which is the horizontal force needed for constant speed on a flat surface divided by the upward normal force exerted by the surface. Transportation engineers call μ_r the *tractive resistance*. Typical values of μ_r are 0.002 to 0.003 for steel wheels on steel rails and 0.01 to 0.02 for rubber tires on concrete. These values show one reason railroad trains are generally much more fuel efficient than highway trucks.

Example 5.18 Motion with rolling friction

A typical car weighs about 12,000 N (about 2700 lb). If the coefficient of rolling friction is $\mu_r = 0.015$, what horizontal force is needed to make the car move with constant speed on a level road? Neglect air resistance.

SOLUTION

IDENTIFY: The car is moving with constant velocity, so this is an equilibrium problem that uses Newton's first law. The four forces on the car are the weight, the upward normal force, the backward force of rolling friction, and the unknown forward force F (our target variable).

SET UP: The free-body diagram is much like the one in Fig. 5.20c of Example 5.13, but with the kinetic friction force replaced by the rolling friction force f_r and with the tension force replaced by the unknown force F.

EXECUTE: As in Example 5.13, Newton's first law for the *vertical* components tells us that the normal force is equal in magnitude to the car's weight. Hence, from the definition of μ_r, the rolling friction force f_r is

$$f_r = \mu_r n = (0.015)(12{,}000\ \text{N}) = 180\ \text{N} \qquad (\text{about } 40\ \text{lb})$$

Newton's first law for the *horizontal* components tells us that a forward force with this magnitude is needed to keep the car moving with constant speed.

EVALUATE: The required force is rather small, which is why it's possible to push a car by hand. (As in the case of sliding, it's easier to keep a car rolling than it is to start it rolling.) We've ignored the effects of air resistance, which is a pretty good approximation if the car is moving slowly. But at typical highway speeds, air resistance is a larger effect than rolling friction.

Try applying this analysis to the crate in Example 5.13. If the crate is on a rubber-wheeled dolly with $\mu_r = 0.02$, only a 10-N force is needed to keep it moving at constant velocity. Can you verify this?

Fluid Resistance and Terminal Speed

ActivPhysics Online 2.1.2 Skydiver

Sticking your hand out the window of a fast-moving car will convince you of the existence of **fluid resistance,** the force that a fluid (a gas or liquid) exerts on a body moving through it. The moving body exerts a force on the fluid to push it out of the way. By Newton's third law, the fluid pushes back on the body with an equal and opposite force.

The *direction* of the fluid resistance force acting on a body is always opposite the direction of the body's velocity relative to the fluid. The *magnitude* of the fluid resistance force usually increases with the speed of the body through the fluid. This is very different from the kinetic friction force between two surfaces in contact, which we can usually regard as independent of speed. For very low speeds, the magnitude f of the fluid resistance force is approximately proportional to the body's speed v:

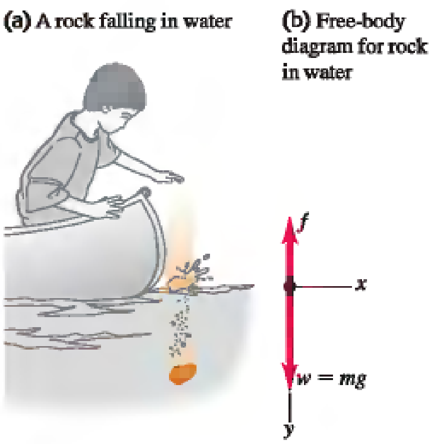
$$f = kv \qquad \text{(fluid resistance at low speed)} \tag{5.7}$$

where k is a proportionality constant that depends on the shape and size of the body and the properties of the fluid. In motion through air at the speed of a tossed tennis ball or faster, the resisting force is approximately proportional to v^2 rather than to v. It is then called **air drag** or simply *drag*. Airplanes, falling raindrops, and bicyclists all experience air drag. In this case we replace Eq. (5.7) by

$$f = Dv^2 \qquad \text{(fluid resistance at high speed)} \tag{5.8}$$

Because of the v^2 dependence, air drag increases rapidly with increasing speed. The air drag on a typical car is negligible at low speeds but comparable to or greater than rolling resistance at highway speeds. The value of D depends on the shape and size of the body and on the density of the air. You should verify that the units of the constant k in Eq. (5.7) are $N \cdot s/m$ or kg/s, and that the units of the constant D in Eq. (5.8) are $N \cdot s^2/m^2$ or kg/m.

5.24 A rock falling through a fluid (water).



Because of the effects of fluid resistance, an object falling in a fluid does *not* have a constant acceleration. To describe its motion, we can't use the constant-acceleration relationships from Chapter 2; instead, we have to start over using Newton's second law. As an example, suppose you drop a rock at the surface of a pond and let it fall to the bottom (Fig. 5.24a). The fluid resistance force in this situation is given by Eq. (5.7). What are the acceleration, velocity, and position of the rock as functions of time?

Figure 5.24b shows the free-body diagram. We take the positive y-direction to be downward and neglect any force associated with buoyancy in the water. Since the rock is moving downward, its speed v is equal to its y-velocity v_y and the fluid resistance force is in the $-y$-direction. There are no x-components, so Newton's second law gives

$$\sum F_y = mg + (-kv_y) = ma_y$$

When the rock first starts to move, $v_y = 0$, the resisting force is zero, and the initial acceleration is $a_y = g$. As the speed increases, the resisting force also increases, until finally it is equal in magnitude to the weight. At this time $mg - kv_y = 0$, the acceleration becomes zero, and there is no further increase in speed. The final speed v_t, called the **terminal speed,** is given by $mg - kv_t = 0$, or

$$v_t = \frac{mg}{k} \qquad \text{(terminal speed, fluid resistance } f = kv\text{)} \qquad (5.9)$$

Figure 5.25 shows how the acceleration, velocity, and position vary with time. As time goes by, the acceleration approaches zero and the velocity approaches v_t (remember that we chose the positive y-direction to be down). The slope of the graph of y versus t becomes constant as the velocity becomes constant.

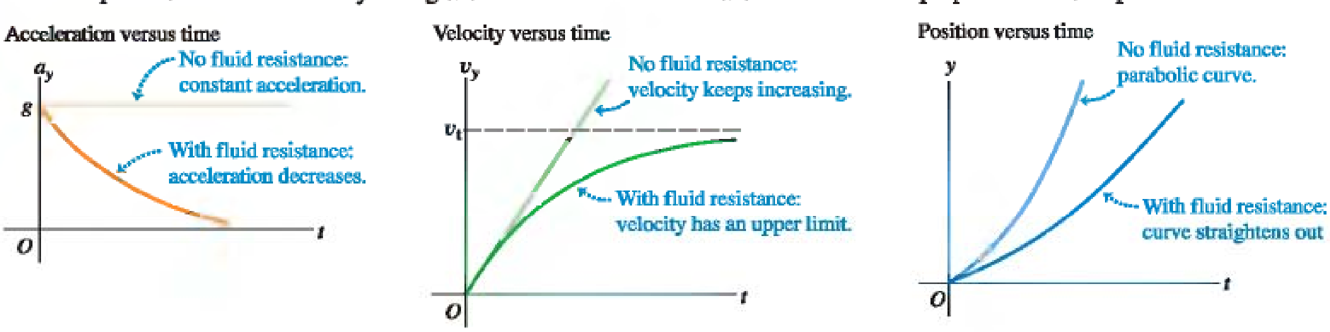
To see how the graphs in Fig. 5.25 are derived, we must find the relationship between speed and time during the interval before the terminal speed is reached. We go back to Newton's second law, which we rewrite using $a_y = dv_y/dt$:

$$m\frac{dv_y}{dt} = mg - kv_y$$

After rearranging terms and replacing mg/k by v_t, we integrate both sides, noting that $v_y = 0$ when $t = 0$:

$$\int_0^v \frac{dv_y}{v_y - v_t} = -\frac{k}{m}\int_0^t dt$$

5.25 Graphs of the motion of a body falling without fluid resistance and with fluid resistance proportional to the speed.



which integrates to

$$\ln\frac{v_t - v_y}{v_t} = -\frac{k}{m}t \qquad \text{or} \qquad 1 - \frac{v_y}{v_t} = e^{-(k/m)t}$$

and finally

$$v_y = v_t[1 - e^{-(k/m)t}] \tag{5.10}$$

Note that v_y becomes equal to the terminal speed v_t only in the limit that $t \rightarrow \infty$; the rock cannot attain terminal speed in any finite length of time.

The derivative of v_y gives a_y as a function of time, and the integral of v_y gives y as a function of time. We leave the derivations for you to complete (see Exercise 5.46); the results are

$$a_y = ge^{-(k/m)t} \tag{5.11}$$

$$y = v_t\left[t - \frac{m}{k}(1 - e^{-(k/m)t})\right] \tag{5.12}$$

Now look again at Fig. 5.25, which shows graphs of these three relationships.

In deriving the terminal speed in Eq. (5.9), we assumed that the fluid resistance force is proportional to the speed. For an object falling through the air at high speeds, so that the fluid resistance is equal to Dv^2 as in Eq. (5.8), the terminal speed is reached when Dv^2 equals the weight mg (Fig. 5.26a). You can show that the terminal speed v_t is given by

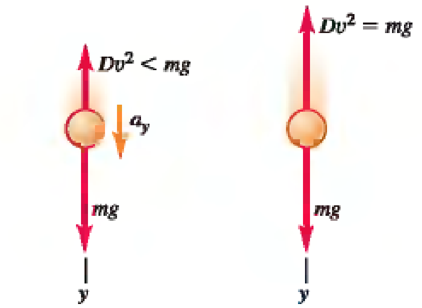
$$v_t = \sqrt{\frac{mg}{D}} \qquad \text{(terminal speed, fluid resistance } f = Dv^2\text{)} \tag{5.13}$$

This expression for terminal speed explains why heavy objects in air tend to fall faster than light objects. Two objects with the same physical size but different mass (say, a table-tennis ball and a lead ball with the same radius) have the same value of D but different values of m. The more massive object has a higher terminal speed and falls faster. The same idea explains why a sheet of paper falls faster if you first crumple it into a ball; the mass m is the same, but the smaller size makes D smaller (less air drag for a given speed) and v_t larger. Skydivers use the same principle to control their descent (Fig. 5.26b).

Figure 5.27 shows the trajectories of a baseball with and without air drag, assuming a coefficient $D = 1.3 \times 10^{-3}$ kg/m (appropriate for a batted ball at sea level). You can see that both the range of the baseball and the maximum height reached are substantially less than the zero-drag calculation would lead you to believe. Hence the baseball trajectory we calculated in Example 3.8 (Section 3.3) by ignoring air drag is quite unrealistic. Air drag is an important part of the game of baseball!

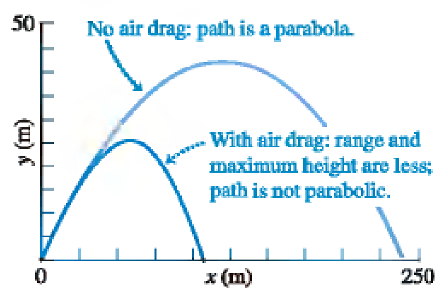
5.26 (a) Air drag and terminal speed. (b) By changing the positions of their arms and legs while falling, skydivers can change the value of the constant D in Eq. (5.8) and hence adjust the terminal speed of their fall [Eq. (5.13)].

(a) Free-body diagrams for falling with air drag



(b) A skydiver falling at terminal speed

5.27 Computer-generated trajectories of a baseball launched at 50 m/s at 35° above the horizontal. Note that the scales are different on the horizontal and vertical axes.



Example 5.19 Terminal speed of a skydiver

For a human body falling through air in a spread-eagle position (Fig. 5.26b), the numerical value of the constant D in Eq. (5.8) is about 0.25 kg/m. Find the terminal speed for a lightweight 50-kg skydiver.

SOLUTION

IDENTIFY: This example uses the relationship among terminal speed, mass, and drag coefficient.

SET UP: We use Eq. (5.13) to find the target variable v_t.

EXECUTE: We find for $m = 50$ kg:

$$v_t = \sqrt{\frac{mg}{D}} = \sqrt{\frac{(50\text{ kg})(9.8\text{ m/s}^2)}{0.25\text{ kg/m}}}$$
$$= 44\text{ m/s} \qquad \text{(about 160 km/h, or 99 mi/h)}$$

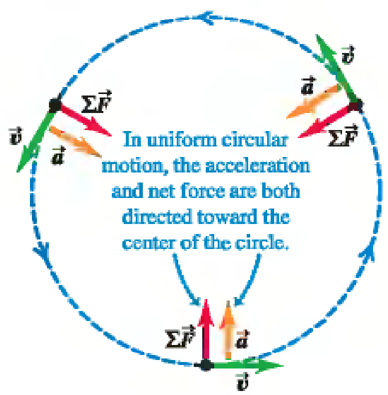
Continued

EVALUATE: The terminal speed is proportional to the square root of the skydiver's mass, so a more robust skydiver with the same drag coefficient D but twice the mass would have a terminal speed $\sqrt{2} = 1.41$ times greater, or 63 m/s. (A skydiver with more mass would also have more frontal area and hence a larger drag coefficient, so his terminal speed would be a bit less than 63 m/s.) Even the lightweight skydiver's terminal speed is quite high, so skydives don't last very long. A drop from 2800 m (9200 ft) to the surface at the terminal speed takes only $(2800\ \text{m})/(44\ \text{m/s}) = 64\ \text{s}$.

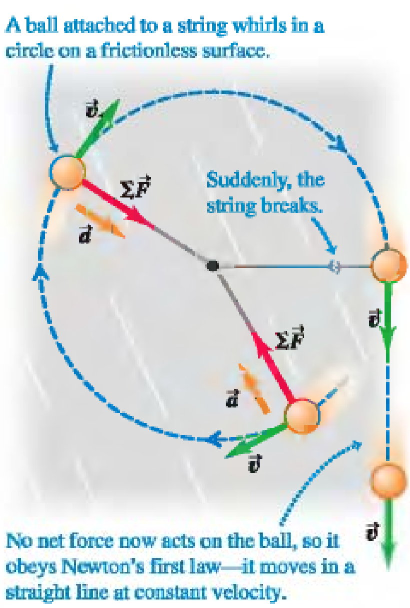
When the sky diver deploys the parachute, the value of D increases greatly. Hence the terminal speed of the skydiver and parachute decreases dramatically to a much slower value.

Test Your Understanding of Section 5.3 Consider a box that is placed on different surfaces. (a) In which situation(s) is there *no* friction force acting on the box? (b) In which situation(s) is there a *static* friction force acting on the box? (c) In which situation(s) is there a *kinetic* friction force on the box? (i) The box is at rest on a rough horizontal surface. (ii) The box is at rest on a rough tilted surface. (iii) The box is on the rough-surfaced flat bed of a truck—the truck is moving at a constant velocity on a straight, level road, and the box remains in the same place in the middle of the truck bed. (iv) The box is on the rough-surfaced flat bed of a truck—the truck is speeding up on a straight, level road, and the box remains in the same place in the middle of the truck bed. (v) The box is on the rough-surfaced flat bed of a truck—the truck is climbing a hill, and the box is sliding toward the back of the truck.

5.28 In uniform circular motion, both the acceleration and the net force are directed toward the center of the circle.



5.29 What happens if the inward radial force suddenly ceases to act on a body in circular motion?



5.4 Dynamics of Circular Motion

We talked about uniform circular motion in Section 3.4. We showed that when a particle moves in a circular path with constant speed, the particle's acceleration is always directed toward the center of the circle (perpendicular to the instantaneous velocity). The magnitude a_{rad} of the acceleration is constant and is given in terms of the speed v and the radius R of the circle by

$$a_{\text{rad}} = \frac{v^2}{R} \quad \text{(uniform circular motion)} \tag{5.14}$$

The subscript "rad" is a reminder that at each point the acceleration is radially inward toward the center of the circle, perpendicular to the instantaneous velocity. We explained in Section 3.4 why this acceleration is often called *centripetal acceleration.*

We can also express the centripetal acceleration a_{rad} in terms of the *period T*, the time for one revolution:

$$T = \frac{2\pi R}{v} \tag{5.15}$$

In terms of the period, a_{rad} is

$$a_{\text{rad}} = \frac{4\pi^2 R}{T^2} \quad \text{(uniform circular motion)} \tag{5.16}$$

Uniform circular motion, like all other motion of a particle, is governed by Newton's second law. To make the particle accelerate toward the center of the circle, the net force $\sum \vec{F}$ on the particle must always be directed toward the center (Fig. 5.28). The magnitude of the acceleration is constant, so the magnitude F_{net} of the net force must also be constant. If the inward net force stops acting, the particle flies off in a straight line tangent to the circle (Fig. 5.29).

The magnitude of the radial acceleration is given by $a_{\text{rad}} = v^2/R$, so the magnitude F_{net} of the net force on a particle with mass m in uniform circular motion must be

$$F_{\text{net}} = ma_{\text{rad}} = m\frac{v^2}{R} \quad \text{(uniform circular motion)} \tag{5.17}$$

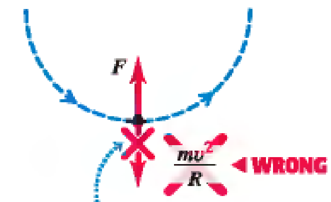
Uniform circular motion can result from *any* combination of forces, just so the net force $\sum\vec{F}$ is always directed toward the center of the circle and has a constant magnitude. Note that the body need not move around a complete circle: Equation (5.17) is valid for *any* path that can be regarded as part of a circular arc.

CAUTION **Avoid using "centrifugal force"** Figure 5.30 shows both a correct free-body diagram for uniform circular motion (Fig. 5.30a) and a common *incorrect* diagram (Fig. 5.30b). Figure 5.30b is incorrect because it includes an extra outward force of magnitude $m(v^2/R)$ to "keep the body out there" or to "keep it in equilibrium." There are three reasons not to include such an outward force, usually called *centrifugal force* ("centrifugal" means "fleeing from the center") First, the body does *not* "stay out there": It is in constant motion around its circular path. Because its velocity is constantly changing in direction, the body accelerates and is *not* in equilibrium. Second, if there *were* an additional outward force that balanced the inward force, the net force would be zero and the body would move in a straight line, not a circle (Fig. 5.29). And third, the quantity $m(v^2/R)$ is *not* a force; it corresponds to the $m\vec{a}$ side of $\sum\vec{F} = m\vec{a}$ and does not appear in $\sum\vec{F}$ (Fig. 5.30a). It's true that when you ride in a car that goes around a circular path, you tend to slide to the outside of the turn as though there was a "centrifugal force." But we saw in Section 4.2 that what really happens is that you tend to keep moving in a straight line, and the outer side of the car "runs into" you as the car turns (Fig. 4.11c). *In an inertial frame of reference there is no such thing as "centrifugal force."* We won't mention this term again, and we strongly advise you to avoid using it as well.

5.30 (a) Correct and (b) incorrect free-body diagrams for a body in uniform circular motion.

(a) Correct free-body diagram

(b) Incorrect free-body diagram



Example 5.20 Force in uniform circular motion

A sled with a mass of 25.0 kg rests on a horizontal sheet of essentially frictionless ice. It is attached by a 5.00-m rope to a post set in the ice. Once given a push, the sled revolves uniformly in a circle around the post (Fig. 5.31a). If the sled makes five complete revolutions every minute, find the force F exerted on it by the rope.

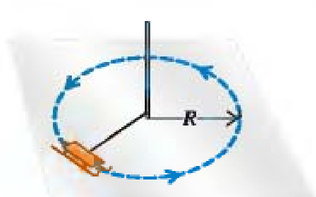
SOLUTION

IDENTIFY: The sled is moving in uniform circular motion, so it has a radial acceleration. We will apply Newton's second law to the sled to find the magnitude F of the force exerted by the rope (our target variable).

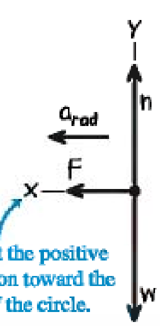
SET UP: Figure 5.31b shows our free-body diagram for the sled. The acceleration has only an x-component; this is toward the center of the circle, so we denote it as a_{rad}. The acceleration isn't given, so we'll need to determine its value using either Eq. (5.14) or Eq. (5.16).

5.31 (a) The situation. (b) Our free-body diagram.

(a) A sled in uniform circular motion



(b) Free-body diagram for the sled



EXECUTE: The acceleration in the y-direction is zero, so the net force in that direction is zero and the normal force and weight have the same magnitude. For the x-direction, Newton's second law gives

$$\sum F_x = F = ma_{rad}$$

We can find the centripetal acceleration a_{rad} using Eq. (5.16). The sled moves in a circle of radius $R = 5.00\text{ m}$ with a period $T = (60.0\text{ s})/(5\text{ rev}) = 12.0\text{ s}$, so

$$a_{rad} = \frac{4\pi^2 R}{T^2} = \frac{4\pi^2(5.00\text{ m})}{(12.0\text{ s})^2} = 1.37\text{ m/s}^2$$

Alternatively, we can first use Eq. (5.15) to find the speed v:

$$v = \frac{2\pi R}{T} = \frac{2\pi(5.00\text{ m})}{12.0\text{ s}} = 2.62\text{ m/s}$$

Then, using Eq. (5.14),

$$a_{rad} = \frac{v^2}{R} = \frac{(2.62\text{ m/s})^2}{5.00\text{ m}} = 1.37\text{ m/s}^2$$

Hence the magnitude F of the force exerted by the rope is

$$F = ma_{rad} = (25.0\text{ kg})(1.37\text{ m/s}^2) = 34.3\text{ kg}\cdot\text{m/s}^2 = 34.3\text{ N}$$

EVALUATE: A greater force would be needed if the sled moved around the circle at a higher speed v. In fact, if v were doubled while R remained the same, F would be four times greater. Can you show this? How would F change if v remained the same but the radius R were doubled?

Example 5.21 The conical pendulum

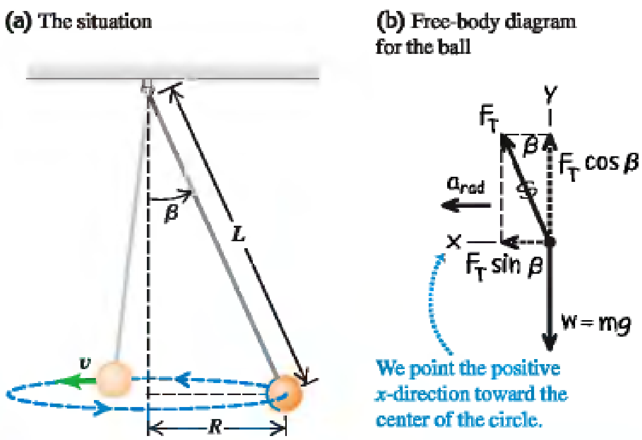
An inventor proposes to make a pendulum clock using a pendulum bob with mass m at the end of a thin wire of length L. Instead of swinging back and forth, the bob moves in a horizontal circle with constant speed v, with the wire making a constant angle β with the vertical direction (Fig. 5.32a). This system is called a *conical pendulum* because the suspending wire traces out a cone. Find the tension F in the wire and the period T (the time for one revolution of the bob) in terms of β.

SOLUTION

IDENTIFY: To find our two target variables, the tension F and period T, we need two equations. These will be the horizontal and vertical components of Newton's second law applied to the bob. We'll find the acceleration of the bob toward the center of the circle using one of the circular motion equations.

SET UP: Figure 5.32b shows our free-body diagram for the bob as well as a coordinate system. The forces on the bob in the position shown are the weight mg and the tension F in the wire. Note that the center of the circular path is in the same horizontal plane as the bob, *not* at the top end of the wire. The horizontal component of tension is the force that produces the horizontal acceleration a_{rad} toward the center of the circle.

5.32 (a) The situation. (b) Our free-body diagram.



EXECUTE: The bob has zero vertical acceleration; the horizontal acceleration is toward the center of the circle, which is why we use the symbol a_{rad}. The $\sum \vec{F} = m\vec{a}$ equations are

$$\sum F_x = F\sin\beta = ma_{rad}$$
$$\sum F_y = F\cos\beta + (-mg) = 0$$

These are two equations for the two unknowns F and β. The equation for $\sum F_y$ gives $F = mg/\cos\beta$; substituting this result into the equation for $\sum F_x$ and using $\sin\beta/\cos\beta = \tan\beta$, we find

$$\tan\beta = \frac{a_{rad}}{g}$$

To relate β to the period T, we use Eq. (5.16) for a_{rad}. The radius of the circle is $R = L\sin\beta$, so

$$a_{rad} = \frac{4\pi^2 R}{T^2} = \frac{4\pi^2 L\sin\beta}{T^2}$$

Substituting this into $\tan\beta = a_{rad}/g$, we obtain

$$\tan\beta = \frac{4\pi^2 L\sin\beta}{gT^2}$$

which we can rewrite as

$$T = 2\pi\sqrt{\frac{L\cos\beta}{g}}$$

EVALUATE: For a given length L, as the angle β increases, $\cos\beta$ decreases, the period T becomes smaller, and the tension $F = mg/\cos\beta$ increases. The angle can never be 90°, however; this would require that $T = 0$, $F = \infty$, and $v = \infty$. A conical pendulum would not make a very good clock because the period depends on the angle β in such a direct way.

Example 5.22 Rounding a flat curve

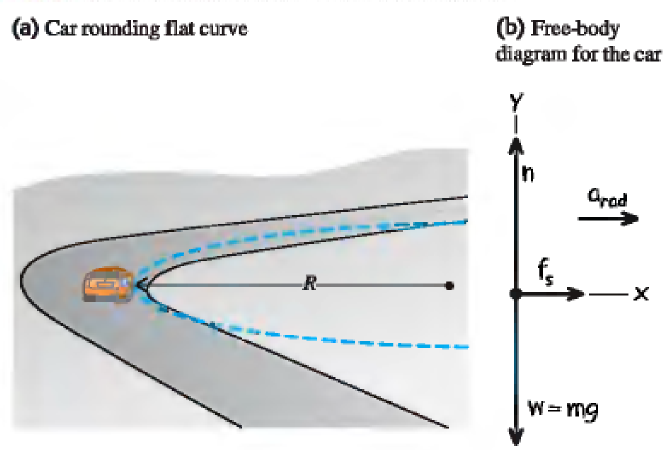
The sports car in Example 3.11 (Section 3.4) is rounding a flat, unbanked curve with radius R (Fig. 5.33a). If the coefficient of static friction between tires and road is μ_s, what is the maximum speed v_{max} at which the driver can take the curve without sliding?

SOLUTION

IDENTIFY: The car's acceleration as it rounds the curve has magnitude $a_{rad} = v^2/R$. Hence the maximum speed v_{max} (our target variable) corresponds to the maximum acceleration a_{rad} and to the maximum horizontal force on the car toward the center of its circular path. The only horizontal force acting on the car is the friction force exerted by the road. So we'll need Newton's second law and our knowledge of the friction force from Section 5.3.

SET UP: The free-body diagram in Fig. 5.33b includes the car's weight $w = mg$ and the two forces exerted by the road, the normal force n and the horizontal friction force f. The friction force must

5.33 (a) The situation. (b) Our free-body diagram.



point toward the center of the circular path in order to cause the radial acceleration. Since the car doesn't move in the radial direction (it doesn't slide toward or away from the center of the circle), the friction force is *static* friction with a maximum magnitude $f_{max} = \mu_s n$ [see Eq. (5.6)].

EXECUTE: The acceleration toward the center of the circular path is $a_{rad} = v^2/R$ and there is no vertical acceleration. Thus we have

$$\sum F_x = f = ma_{rad} = m\frac{v^2}{R}$$
$$\sum F_y = n + (-mg) = 0$$

The second equation shows that $n = mg$. The first equation shows that the friction force *needed* to keep the car moving in its circular path increases with the car's speed. But the maximum friction force *available* is $f_{max} = \mu_s n = \mu_s mg$, and this determines the car's maximum speed. Substituting f_{max} for f and v_{max} for v in the $\sum F_x$ equation, we find

$$\mu_s mg = m\frac{v_{max}{}^2}{R}$$

so the maximum speed is

$$v_{max} = \sqrt{\mu_s gR}$$

As an example, if $\mu_s = 0.96$ and $R = 230$ m, then

$$v_{max} = \sqrt{(0.96)(9.8\ \mathrm{m/s^2})(230\ \mathrm{m})} = 47\ \mathrm{m/s}$$

or about 170 km/h (100 mi/h). This is the maximum speed for this radius.

EVALUATE: If the car's speed is slower than $\sqrt{\mu_s gR}$, the required friction force is less than the maximum possible value $f_{max} = \mu_s mg$ and the car can easily make the curve. If we try to take the curve going *faster* than the maximum speed, the car can still go in a circle without skidding, but the radius will be larger and the car will run off the road.

Note that the maximum centripetal acceleration (called the "lateral acceleration" in Example 3.11) is equal to $\mu_s g$. If the coefficient of friction is reduced, the maximum centripetal acceleration and v_{max} are also reduced. That's why it's best to take curves at a lower speed if the road is wet or icy (either of which can reduce the value of μ_s).

Example 5.23 Rounding a banked curve

For a car traveling at a certain speed, it is possible to bank a curve at just the right angle so that no friction at all is needed to maintain the car's turning radius. Then a car can safely round the curve even on wet ice. (Bobsled racing depends on this same idea.) Your engineering firm plans to rebuild the curve in Example 5.22 so that a car moving at speed v can safely make the turn even with no friction (Fig. 5.34a). At what angle β should the curve be banked?

SOLUTION

IDENTIFY: With no friction, the only two forces acting on the car are its weight and the normal force. Because the road is banked, the normal force (which acts perpendicular to the road surface) has a horizontal component. This component causes the car's horizontal acceleration toward the center of the car's circular path. Since forces and acceleration are involved, we'll use Newton's second law to find the target variable β.

SET UP: Our free-body diagram (Fig. 5.34b) is very similar to the diagram for the conical pendulum in Example 5.21 (Fig. 5.32b). The normal force acting on the car plays the role of the tension acting on the pendulum bob.

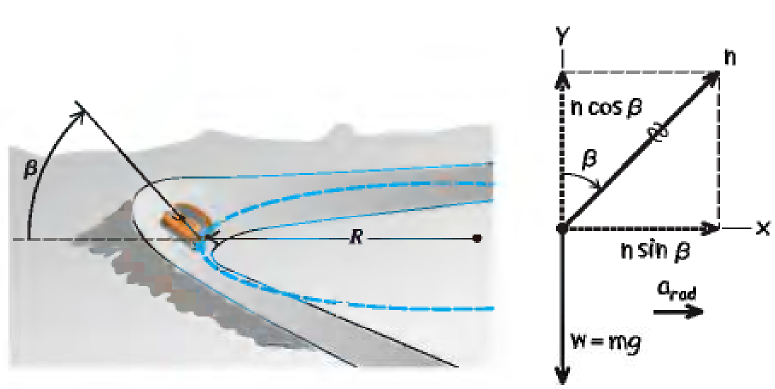
EXECUTE: The normal force $\vec{n}$ is perpendicular to the roadway at an angle β with the vertical. Thus it has a vertical component $n\cos\beta$ and a horizontal component $n\sin\beta$, as Fig. 5.34b shows. The acceleration in the x-direction is the centripetal acceleration, $a_{rad} = v^2/R$; there is no acceleration in the y-direction. Thus the equations of Newton's second law are

$$\sum F_x = n\sin\beta = ma_{rad}$$
$$\sum F_y = n\cos\beta + (-mg) = 0$$

5.34 (a) The situation. (b) Our free-body diagram.

(a) Car rounding banked curve

(b) Free-body diagram for the car



Continued

From the $\sum F_y$ equation, $n = mg/\cos\beta$. Substituting this into the $\sum F_x$ equation gives an expression for the bank angle:

$$\tan\beta = \frac{a_{rad}}{g}$$

This is the same expression we found in Example 5.21. Finally, substituting the expression $a_{rad} = v^2/R$, we have

$$\tan\beta = \frac{v^2}{gR}$$

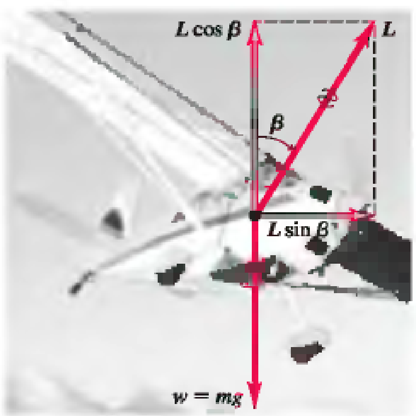
EVALUATE: The bank angle depends on the speed and the radius. For a given radius, no one angle is correct for all speeds. In the design of highways and railroads, curves are often banked for the average speed of the traffic over them. If $R = 230\text{ m}$ and $v = 25\text{ m/s}$ (equal to a highway speed of 88 km/h, or 55 mi/h), then

$$\beta = \arctan\frac{(25\text{ m/s})^2}{(9.8\text{ m/s}^2)(230\text{ m})} = 15°$$

This is within the range of banking angles actually used in highways. With the same radius and $v = 47\text{ m/s}$, as in Example 5.22, $\beta = 44°$; such steeply banked curves are found at automobile raceways.

Banked Curves and the Flight of Airplanes

5.35 An airplane banks to one side in order to turn in that direction. The vertical component of the lift force $\vec{L}$ balances the force of gravity; the horizontal component of $\vec{L}$ causes the acceleration v^2/R.



The results of Example 5.23 also apply to an airplane when it makes a turn in level flight (Fig. 5.35). When an airplane is flying in a straight line at a constant speed and at a steady altitude, the airplane's weight is exactly balanced by the lift force $\vec{L}$ exerted by the air. (The upward lift force that the air exerts on the wings is a reaction to the downward push the wings exert on the air as they move through it.) To make the airplane turn, the pilot banks the airplane to one side so that the lift force has a horizontal component as Fig. 5.35 shows. (The pilot also changes the angle at which the wings "bite" into the air so that the vertical component of lift continues to balance the weight.) The bank angle is related to the airplane's speed v and the radius R of the turn by the same expression as in Example 5.23: $\tan\beta = v^2/gR$. For an airplane to make a tight turn (small R) at high speed (large v), $\tan\beta$ must be large and the required bank angle β must approach 90°.

We can also apply the results of Example 5.23 to the *pilot* of an airplane. The free-body diagram for the pilot of the airplane is exactly as shown in Fig. 5.34b; the normal force $n = mg/\cos\beta$ is exerted on the pilot by the seat. As in Example 5.9, n is equal to the apparent weight of the pilot, which is greater than the pilot's true weight mg. In a tight turn with a large bank angle β, the pilot's apparent weight can be tremendous: $n = 5.8mg$ at $\beta = 80°$ and $n = 9.6mg$ at $\beta = 84°$. Pilots black out in such tight turns because the apparent weight of their blood increases by the same factor, and the human heart isn't strong enough to pump such apparently "heavy" blood to the brain.

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4.2 Circular Motion Problem Solving
4.3 Cart Goes Over Circular Path
4.4 Ball Swings on a String
4.5 Car Circles a Track

Motion in a Vertical Circle

In Examples 5.20, 5.21, 5.22, and 5.23 the body moved in a horizontal circle. Motion in a *vertical* circle is no different in principle, but the weight of the body has to be treated carefully. The following example shows what we mean.

Example 5.24 Uniform circular motion in a vertical circle

A passenger on a carnival Ferris wheel moves in a vertical circle of radius R with constant speed v. The seat remains upright during the motion. Find expressions for the force the seat exerts on the passenger at the top of the circle and at the bottom.

SOLUTION

IDENTIFY: At both the top and bottom of the circle, the target variable is the magnitude n of the normal force that the seat exerts on the passenger. We'll find this force at each position using Newton's second law and the equations of uniform circular motion.

SET UP: Figure 5.36a shows the passenger's velocity and acceleration at the two positions. Note that the acceleration points *downward* at the *top* of the circle but *upward* at the *bottom* of the circle. At each position the only forces acting are vertical: the upward normal force and the downward force of gravity. Hence we need only the vertical component of Newton's second law.

EXECUTE: Figures 5.36b and 5.36c show free-body diagrams for the two positions. We take the positive y-direction as upward in both cases. Let n_T be the upward normal force the seat applies to the passenger at the top of the circle, and let n_B be the normal force at the bottom. At the top the acceleration has magnitude v^2/R, but its vertical component is negative because its direction is downward. Hence $a_y = -v^2/R$, and Newton's second law tells us that

$$\text{Top:} \qquad \sum F_y = n_T + (-mg) = -m\frac{v^2}{R} \qquad \text{or}$$

$$n_T = m\left(g - \frac{v^2}{R}\right)$$

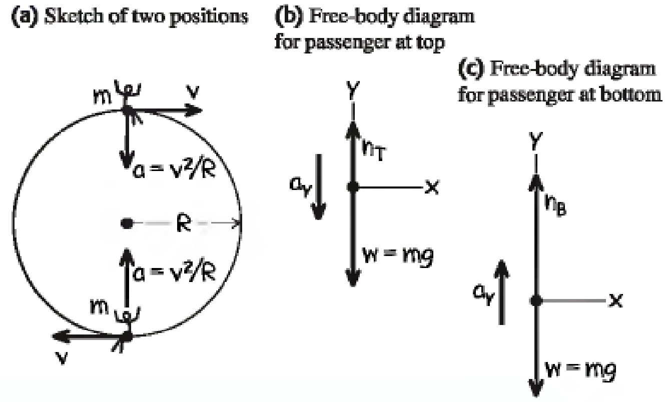
At the bottom the acceleration is upward, so $a_y = +v^2/R$ and Newton's second law is

$$\text{Bottom:} \qquad \sum F_y = n_B + (-mg) = +m\frac{v^2}{R} \qquad \text{or}$$

$$n_B = m\left(g + \frac{v^2}{R}\right)$$

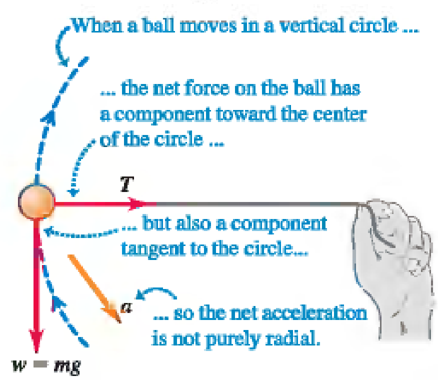
EVALUATE: Our result for n_T tells us that at the top of the Ferris wheel, the upward force the seat applies to the passenger is *smaller* in magnitude than the passenger's weight $w = mg$. If the ride goes fast enough that $g - v^2/R$ becomes zero, the seat applies *no* force, and the passenger is about to become airborne. If v becomes still larger, n_T becomes negative; this means that a *downward* force (such as from a seat belt) is needed to keep the passenger in the seat. By contrast, the normal force n_B at the bottom is always *greater than* the passenger's weight. You feel the seat pushing up on you more firmly than when you are at rest. You can see that n_T and n_B are the values of the passenger's *apparent weight* at the top and bottom of the circle (see Section 5.2).

5.36 Our sketches for this problem.



When we tie a string to an object and whirl it in a vertical circle, the analysis in Example 5.24 isn't directly applicable. The reason is that v is *not* constant in this case; except at the top and bottom of the circle, the net force (and hence the acceleration) does *not* point toward the center of the circle (Fig. 5.37). So both $\sum\vec{F}$ and $\vec{a}$ have a component tangent to the circle, which means that the speed changes. Hence this is a case of *nonuniform* circular motion (see Section 3.4). Even worse, we can't use the constant-acceleration formulas to relate the speeds at various points because *neither* the magnitude nor the direction of the acceleration is constant. The speed relationships we need are best obtained by using the concept of energy. We'll consider such problems in Chapter 7.

5.37 A ball moving in a vertical circle.



Test Your Understanding of Section 5.4 Satellites are held in orbit by the force of our planet's gravitational attraction. A satellite in a small-radius orbit moves at a higher speed than a satellite in an orbit of large radius. Based on this information, what you can conclude about the earth's gravitational attraction for the satellite? (i) It increases with increasing distance from the earth. (ii) It is the same at all distances from the earth. (iii) It decreases with increasing distance from the earth. (iv) This information by itself isn't enough to answer the question.

*5.5 The Fundamental Forces of Nature

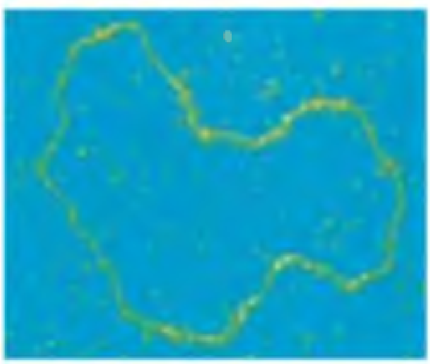
We have discussed several kinds of forces—including weight, tension, friction, fluid resistance, and the normal force—and we will encounter others as we continue our study of physics. But just how many kinds of forces are there? Our current understanding is that all forces are expressions of just four distinct classes of *fundamental* forces, or interactions between particles (Fig. 5.38). Two are familiar in everyday experience. The other two involve interactions between subatomic particles that we cannot observe with the unaided senses.

Gravitational interactions include the familiar force of your *weight,* which results from the earth's gravitational attraction acting on you. The mutual gravitational attraction of various parts of the earth for each other holds our planet

5.38 Examples of the fundamental interactions in nature. **(a)** The moon and the earth are held together and held in orbit by gravitational forces. **(b)** This molecule of bacterial plasmid DNA is held together by electromagnetic forces between its atoms. **(c)** The sun shines because in its core, strong forces between nuclear particles cause the release of energy. **(d)** When a massive star explodes into a supernova, a flood of energy is released by weak interactions between the star's nuclear particles.

(a) Gravitational forces hold planets together.

(b) Electromagnetic forces hold molecules together.



(c) Strong forces release energy to power the sun.

(d) Weak forces play a role in exploding stars.

together (Fig. 5.38a). Newton recognized that the sun's gravitational attraction for the earth keeps the earth in its nearly circular orbit around the sun. In Chapter 12 we will study gravitational interactions in greater detail, and we will analyze their vital role in the motions of planets and satellites.

The second familiar class of forces, **electromagnetic interactions,** includes electric and magnetic forces. If you run a comb through your hair, the comb ends up with an electric charge; you can use the electric force exerted by this charge to pick up bits of paper. All atoms contain positive and negative electric charge, so atoms and molecules can exert electric forces on each other (Fig. 5.38b). Contact forces, including the normal force, friction, and fluid resistance, are the combination of all such forces exerted on the atoms of a body by atoms in its surroundings. *Magnetic* forces, such as those between magnets or between a magnet and a piece of iron, are actually the result of electric charges in motion. For example, an electromagnet causes magnetic interactions because electric charges move through its wires. We will study electromagnetic interactions in detail in the second half of this book.

On the atomic or molecular scale, gravitational forces play no role because electric forces are enormously stronger: The electrical repulsion between two protons is stronger than their gravitational attraction by a factor of about 10^{35}. But in bodies of astronomical size, positive and negative charges are usually present in nearly equal amounts, and the resulting electrical interactions nearly cancel out. Gravitational interactions are thus the dominant influence in the motion of planets and in the internal structure of stars.

The other two classes of interactions are less familiar. One, the **strong interaction,** is responsible for holding the nucleus of an atom together. Nuclei contain electrically neutral neutrons and positively charged protons. The electric force between charged protons tries to push them apart; the strong attractive force between nuclear particles counteracts this repulsion and makes the nucleus stable. In this context the strong interaction is also called the *strong nuclear force*. It has much shorter range than electrical interactions, but within its range it is much stronger. The strong interaction plays a key role in thermonuclear reactions that take place at the sun's core and generate the sun's heat and light (Fig. 5.38c).

Finally, there is the **weak interaction.** Its range is so short that it plays a role only on the scale of the nucleus or smaller. The weak interaction is responsible for a common form of radioactivity called beta decay, in which a neutron in a radioactive nucleus is transformed into a proton while ejecting an electron and a nearly massless particle called an antineutrino. The weak interaction between the antineutrino and ordinary matter is so feeble that an antineutrino could easily penetrate a wall of lead a million kilometers thick! Yet when a giant star undergoes a cataclysmic explosion called a supernova, most of the energy is released by way of the weak interaction (Fig. 5.38d).

In the 1960s physicists developed a theory that described the electromagnetic and weak interactions as aspects of a single *electroweak* interaction. This theory has passed every experimental test to which it has been put. Encouraged by this success, physicists have made similar attempts to describe the strong, electromagnetic, and weak interactions in terms of a single *grand unified theory* (GUT), and have taken steps toward a possible unification of all interactions into a *theory of everything* (TOE). Such theories are still speculative, and there are many unanswered questions in this very active field of current research.

CHAPTER 5 SUMMARY

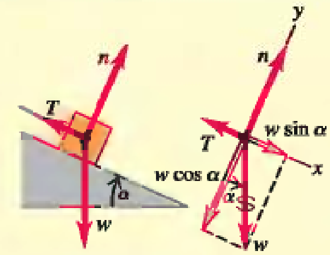
Using Newton's first law: When a body is in equilibrium in an inertial frame of reference—that is, either at rest or moving with constant velocity—the vector sum of forces acting on it must be zero (Newton's first law). Free-body diagrams are essential in identifying the forces that act on the body being considered.

Newton's third law (action and reaction) is also frequently needed in equilibrium problems. The two forces in an action–reaction pair *never* act on the same body. (See Examples 5.1–5.5.)

The normal force exerted on a body by a surface is *not* always equal to the body's weight. (See Example 5.3.)

$$\sum \vec{F} = 0 \quad \text{(vector form)} \qquad (5.1)$$

$$\sum F_x = 0 \qquad \sum F_y = 0 \quad \text{(component form)} \qquad (5.2)$$



Using Newton's second law: If the vector sum of forces on a body is *not* zero, the body accelerates. The acceleration is related to the net force by Newton's second law.

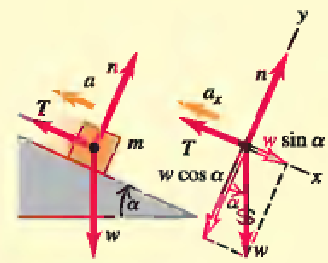
Just as for equilibrium problems, free-body diagrams are essential for solving problems involving Newton's second law, and the normal force exerted on a body is not always equal to its weight. (See Examples 5.6–5.12.)

Vector form:

$$\sum \vec{F} = m\vec{a} \qquad (5.3)$$

Component form:

$$\sum F_x = ma_x \qquad \sum F_y = ma_y \qquad (5.4)$$



Friction and fluid resistance: The contact force between two bodies can always be represented in terms of a normal force $\vec{n}$ perpendicular to the surface of contact and a friction force $\vec{f}$ parallel to the surface.

When a body is sliding over the surface, the friction force is called *kinetic* friction. Its magnitude f_k is approximately equal to the normal force magnitude n multiplied by the coefficient of kinetic friction μ_k. When a body is *not* moving relative to a surface, the friction force is called *static* friction. The *maximum* possible static friction force is approximately equal to the magnitude n of the normal force multiplied by the coefficient of static friction μ_s. The *actual* static friction force may be anything from zero to this maximum value, depending on the situation. Usually μ_s is greater than μ_k for a given pair of surfaces in contact. (See Examples 5.13–5.17.)

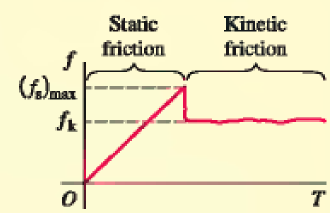
Rolling friction is similar to kinetic friction, but the force of fluid resistance depends on the speed of an object through a fluid. (See Examples 5.18 and 5.19.)

Magnitude of kinetic friction force:

$$f_k = \mu_k n \qquad (5.5)$$

Magnitude of static friction force:

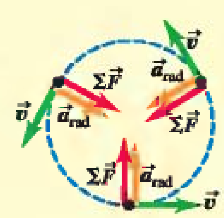
$$f_s \leq \mu_s n \qquad (5.6)$$



Forces in circular motion: In uniform circular motion, the acceleration vector is directed toward the center of the circle. The motion is governed by Newton's second law, $\sum \vec{F} = m\vec{a}$. (See Examples 5.20–5.24.)

Acceleration in uniform circular motion:

$$a_{rad} = \frac{v^2}{R} = \frac{4\pi^2 R}{T^2} \qquad (5.14), (5.16)$$



Key Terms

apparent weight, *145*
friction force, *149*
kinetic friction force, *150*
coefficient of kinetic friction, *150*
static friction force, *151*
coefficient of static friction, *151*
coefficient of rolling friction, *155*
fluid resistance, *155*
air drag, *155*
terminal speed, *156*
gravitational interaction, *163*
electromagnetic interaction, *164*
strong interaction, *164*
weak interaction, *164*

Answer to Chapter Opening Question ?

Neither; the upward force of the air has the *same* magnitude as the force of gravity. Although the bird is ascending, its vertical velocity is constant and so its vertical acceleration is zero. Hence the net vertical force on the bird must also be zero, and the individual vertical forces must balance.

Answers to Test Your Understanding Questions

5.1 **Answer: (ii)** The two cables are arranged symmetrically, so the tension in either cable has the same magnitude T. The vertical component of the tension from each cable is $T\sin 45°$ (or, equivalently, $T\cos 45°$), so Newton's first law applied to the vertical forces tells us that $2T\sin 45° - w = 0$. Hence $T = w/(2\sin 45°) = w/\sqrt{2} = 0.71w$. Each cable supports half of the weight of the traffic light, but the tension is greater than $w/2$ because only the vertical component of the tension counteracts the weight.

5.2 **Answer: (ii)** No matter what the instantaneous velocity of the glider, its acceleration is constant and has the value found in Example 5.12. In the same way, the acceleration of a body in free fall is the same whether it is ascending, descending, or at the high point of its motion (see Section 2.5).

5.3 **Answers to (a): (i), (iii); answers to (b): (ii), (iv); answer to (c): (v)** In situations (i) and (iii) the box is not accelerating (so the net force on it must be zero) and there is no other force acting parallel to the horizontal surface; hence no friction force is needed to prevent sliding. In situations (ii) and (iv) the box would start to slide over the surface if no friction were present, so a static friction force must act to prevent this. In situation (v) the box is sliding over a rough surface, so a kinetic friction force acts on it.

5.4 **Answer: (iii)** A satellite of mass m orbiting the earth at speed v in an orbit of radius r has an acceleration of magnitude v^2/r, so the net force acting on it from the earth's gravity has magnitude $F = mv^2/r$. The farther the satellite is from earth, the greater the value of r, the smaller the value of v, and hence the smaller the values of v^2/r and of F. In other words, the earth's gravitational force decreases with increasing distance.

PROBLEMS

For instructor-assigned homework, go to www.masteringphysics.com

Discussion Questions

Q5.1. A man sits in a seat that is suspended from a rope. The rope passes over a pulley suspended from the ceiling, and the man holds the other end of the rope in his hands. What is the tension in the rope, and what force does the seat exert on the man? Draw a free-body force diagram for the man.

Q5.2. "In general, the normal force is not equal to the weight." Give an example where the two forces are equal in magnitude, and at least two examples where they are not.

Q5.3. A clothesline hangs between two poles. No matter how tightly the line is stretched, it always sags a little at the center. Explain why.

Q5.4. A car is driven up a steep hill at constant speed. Discuss all the forces acting on the car. What pushes it up the hill?

Q5.5. For medical reasons it is important for astronauts in outer space to determine their body mass at regular intervals. Devise a scheme for measuring body mass in an apparently weightless environment.

Q5.6. To push a box up a ramp, is the force required smaller if you push horizontally or if you push parallel to the ramp? Why?

Q5.7. A woman in an elevator lets go of her briefcase but it does not fall to the floor. How is the elevator moving?

Q5.8. You can classify scales for weighing objects as those that use springs and those that use standard masses to balance unknown masses. Which group would be more accurate when you use it in an accelerating spaceship? When you use it on the moon?

Q5.9. When you tighten a nut on a bolt, how are you increasing the frictional force? How does a lock washer work?

Q5.10. A block rests on an inclined plane with enough friction to prevent it from sliding down. To start the block moving, is it easier to push it up the plane or down the plane? Why?

Q5.11. A crate of books rests on a level floor. To move it along the floor at a constant velocity, why do you exert a smaller force if you pull it at an angle θ above the horizontal than if you push it at the same angle below the horizontal?

Q5.12. In a world without friction, which of the following activities could you do (or not do)? Explain your reasoning. (a) drive around an unbanked highway curve; (b) jump into the air; (c) start walking on a horizontal sidewalk; (d) climb a vertical ladder; (e) change lanes on the freeway.

Q5.13. Walking on horizontal slippery ice can be much more tiring than walking on ordinary pavement. Why?

Q5.14. When you stand with bare feet in a wet bathtub, the grip feels fairly secure, and yet a catastrophic slip is quite possible. Explain this in terms of the two coefficients of friction.

Q5.15. You are pushing a large crate from the back of a freight elevator to the front as the elevator is moving to the next floor. In which situation is the force you must apply to move the crate the smallest and in which is it the largest: when the elevator is accelerating upward, when it is accelerating downward, or when it is traveling at constant speed? Explain.

Q5.16. The moon is accelerating toward the earth. Why isn't it getting closer to us?

Q5.17. An automotive magazine calls decreasing-radius curves "the bane of the Sunday driver." Explain.

Q5.10. You often hear people say that "friction always opposes motion." Give at least one example where (a) static friction *causes* motion, and (b) kinetic friction *causes* motion.

Q5.19. If there is a net force on a particle in uniform circular motion, why doesn't the particle's speed change?

Q5.20. A curve in a road has the banking angle calculated and posted for 80 km/h. However, the road is covered with ice so you cautiously plan to drive slower than this limit. What may happen to your car? Why?

Q5.21. You swing a ball on the end of a lightweight string in a horizontal circle at constant speed. Can the string ever be truly horizontal? If not, would it slope above the horizontal or below the horizontal? Why?

Q5.22. The centrifugal force is not included in the free-body diagrams of Figs. 5.34b and 5.35. Explain why not.

Q5.23. A professor swings a rubber stopper in a horizontal circle on the end of a string in front of his class. He tells Caroline, in the first row, that he is going to let the string go when the stopper is directly in front of her face. Should Caroline worry?

Q5.24. To keep the forces on the riders within allowable limits, loop-the-loop roller coaster rides are often designed so that the loop, rather than being a perfect circle, has a larger radius of curvature at the bottom than at the top. Explain.

Q5.25. A tennis ball drops from rest at the top of a tall glass cylinder, first with the air pumped out of the cylinder so there is no air resistance, and then a second time after the air has been readmitted to the cylinder. You examine multiflash photographs of the two drops. From these photos how can you tell which one is which, or can you?

Q5.26. If you throw a baseball straight upward with speed v_0, how does its speed, when it returns to the point from where you threw it, compare to v_0 (a) in the absence of air resistance and (b) in the presence of air resistance? Explain.

Q5.27. You throw a baseball straight upward. If air resistance is *not* ignored, how does the time required for the ball to go from the height at which it was thrown up to its maximum height compare to the time required for it to fall from its maximum height back down to the height from which it was thrown? Explain your answer.

Q5.26. You take two identical tennis balls and fill one with water. You release both balls simultaneously from the top of a tall building. If air resistance is negligible, which ball strikes the ground first? Explain. What is the answer if air resistance is *not* negligible?

Q5.26. A ball is dropped from rest and feels air resistance as it falls. Which of the graphs in Fig. 5.39 best represents its acceleration as a function of time?

Q5.30. A ball is dropped from rest and feels air resistance as it falls. Which of the graphs in Fig. 5.40 best represents its vertical velocity component as a function of time?

Q5.31. When does a baseball in flight have an acceleration with a positive upward component? Explain in terms of the forces on the ball and also in terms of the velocity components compared to the terminal speed. Do *not* ignore air resistance.

Q5.32. When a batted baseball moves with air drag, does it travel a greater horizontal distance while climbing to its maximum height or while descending from its maximum height back to the ground? Or is the horizontal distance traveled the same for both? Explain in terms of the forces acting on the ball.

Figure 5.39 Question Q5.29.

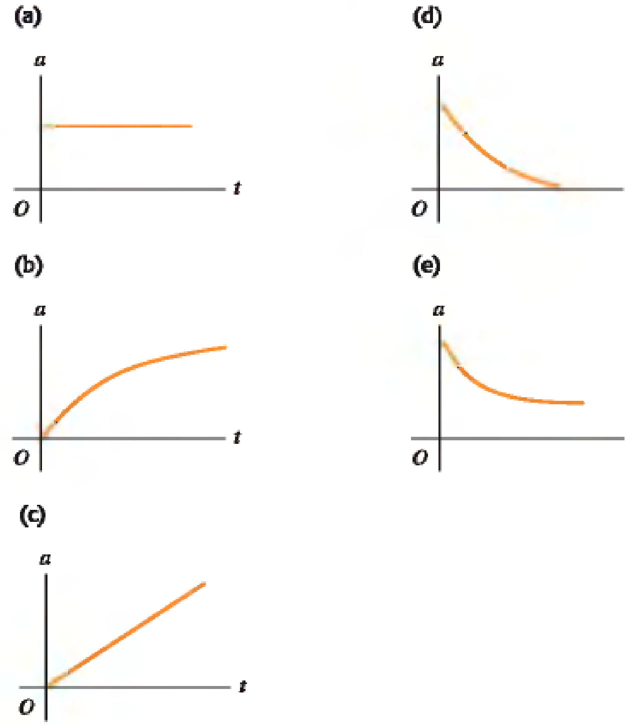
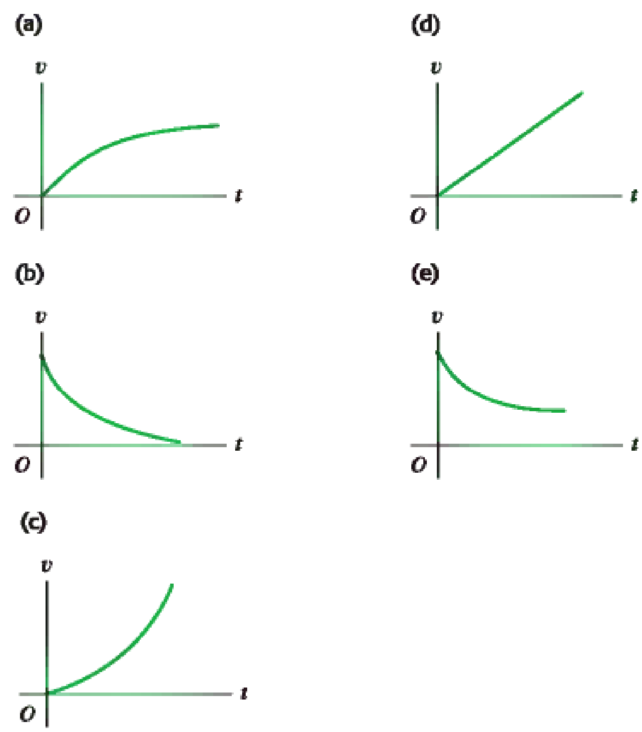


Figure 5.40 Question Q5.30.



Q5.33. "A ball is thrown from the edge of a high cliff. No matter what the angle at which it is thrown, due to air resistance, the ball will eventually end up moving vertically downward." Justify this statement.

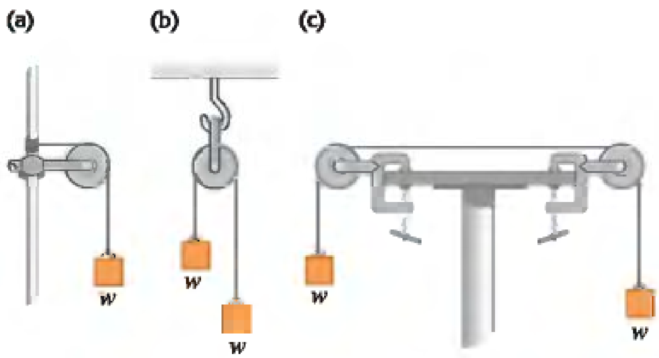
Exercises

Section 5.1 Using Newton's First Law: Particles in Equilibrium

5.1. Two 25.0-N weights are suspended at opposite ends of a rope that passes over a light, frictionless pulley. The pulley is attached to a chain that goes to the ceiling. (a) What is the tension in the rope? (b) What is the tension in the chain?

5.2. In Fig. 5.41 each of the suspended blocks has weight w. The pulleys are frictionless and the ropes have negligible weight. Calculate, in each case, the tension T in the rope in terms of the weight w. In each case, include the free-body diagram or diagrams you used to determine the answer.

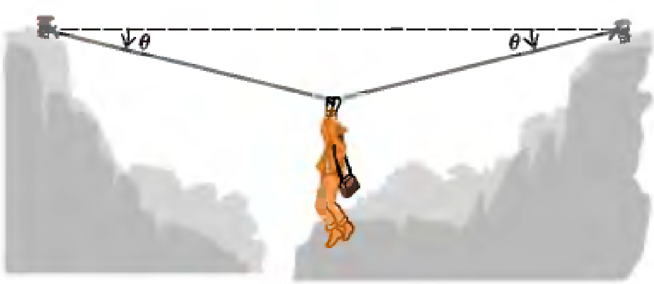
Figure 5.41 Exercise 5.2.



5.3. A 75.0-kg wrecking ball hangs from a uniform heavy-duty chain having a mass of 26.0 kg. (a) Find the maximum and minimum tension in the chain. (b) What is the tension at a point three-fourths of the way up from the bottom of the chain?

5.4. An adventurous archaeologist crosses between two rock cliffs by slowly going hand over hand along a rope stretched between the cliffs. He stops to rest at the middle of the rope (Fig. 5.42). The rope will break if the tension in it exceeds 2.50×10^4 N, and our hero's mass is 90.0 kg. (a) If the angle θ is 10.0°, find the tension in the rope. (b) What is the smallest value the angle θ can have if the rope is not to break?

Figure 5.42 Exercise 5.4.



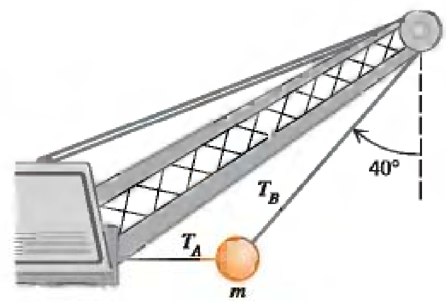
5.5. A picture frame hung against a wall is suspended by two wires attached to its upper corners. If the two wires make the same angle with the vertical, what must this angle be if the tension in each wire is equal to 0.75 of the weight of the frame? (Ignore any friction between the wall and the picture frame.)

5.6. Solve the problem in Example 5.5 using coordinate axes where the y-axis is vertical and the x-axis is horizontal. Do you get the same answers using this different set of axes?

5.7. Certain streets in San Francisco make an angle of 17.5° with the horizontal. What force parallel to the street surface is required to keep a loaded 1967 Corvette of mass 1390 kg from rolling down such a street?

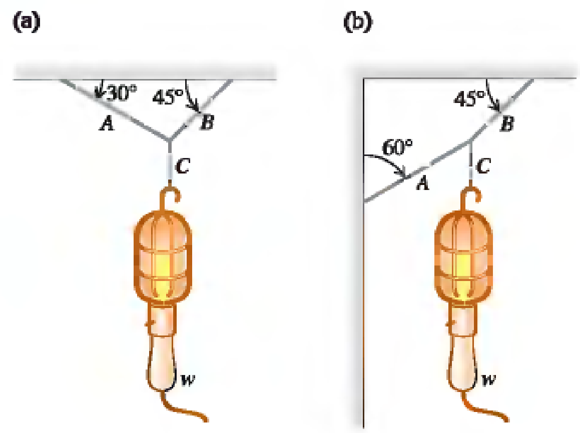
5.8. A large wrecking ball is held in place by two light steel cables (Fig. 5.43). If the mass m of the wrecking ball is 4090 kg, what are (a) the tension T_B in the cable that makes an angle of 40° with the vertical and (b) the tension T_A in the horizontal cable?

Figure 5.43 Exercise 5.8.



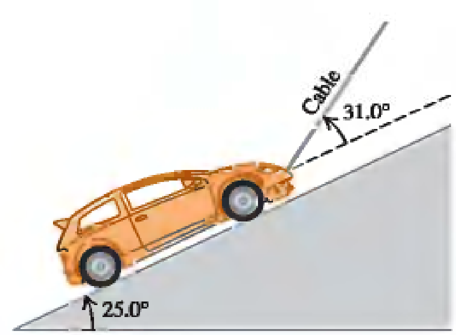
5.9. Find the tension in each cord in Fig. 5.44 if the weight of the suspended object is w.

Figure 5.44 Exercise 5.9.



5.10. A 1130-kg car is held in place by a light cable on a very smooth (frictionless) ramp, as shown in Fig. 5.45. The cable makes an angle of 31.0° above the surface of the ramp, and the ramp itself rises at 25.0° above the horizontal. (a) Draw a free-body diagram for the car. (b) Find the tension in the cable. (c) How hard does the surface of the ramp push on the car?

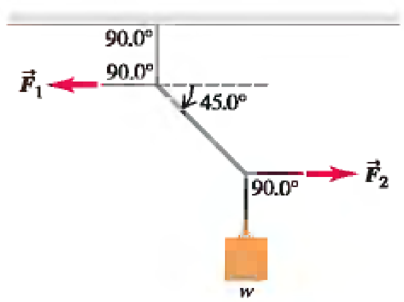
Figure 5.45 Exercise 5.10.



5.11. A man pushes on a piano with mass 180 kg so that it slides at constant velocity down a ramp that is inclined at 11.0° above the horizontal floor. Neglect any friction acting on the piano. Calculate the magnitude of the force applied by the man if he pushes (a) parallel to the incline and (b) parallel to the floor.

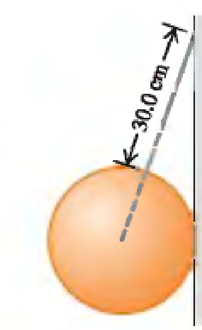
5.12. In Fig. 5.46 the weight w is 60.0 N. (a) What is the tension in the diagonal string? (b) Find the magnitudes of the horizontal forces $\vec{F}_1$ and $\vec{F}_2$ that must be applied to hold the system in the position shown.

Figure 5.46 Exercise 5.12.



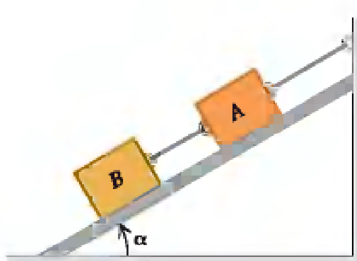
5.13. A solid uniform 45.0-kg ball of diameter 32.0 cm is supported against a vertical frictionless wall using a thin 30.0-cm wire of negligible mass, as shown in Fig. 5.47. (a) Make a free-body diagram for the ball and use it to find the tension in the wire. (b) How hard does the ball push against the wall?

Figure 5.47 Exercise 5.13.



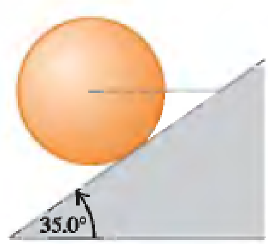
5.14. Two blocks, each with weight w, are held in place on a frictionless incline (Fig. 5.48). In terms of w and the angle α of the incline, calculate the tension in (a) the rope connecting the blocks and (b) the rope that connects block A to the wall. (c) Calculate the magnitude of the force that the incline exerts on each block. (d) Interpret your answers for the cases $\alpha = 0$ and $\alpha = 90°$.

Figure 5.48 Exercise 5.14.



5.15. A horizontal wire holds a solid uniform ball of mass m in place on a tilted ramp that rises 35.0° above the horizontal. The surface of this ramp is perfectly smooth, and the wire is directed away from the center of the ball (Fig. 5.49). (a) Draw a free-body diagram for the ball. (b) How hard does the surface of the ramp push on the ball? (c) What is the tension in the wire?

Figure 5.49 Exercise 5.15.



Section 5.2 Using Newton's Second Law: Dynamics of Particles

5.16. A 125-kg (including all the contents) rocket has an engine that produces a constant vertical force (the *thrust*) of 1720 N. Inside this rocket, a 15.5-N electrical power supply rests on the floor. (a) Find the acceleration of the rocket. (b) When it has reached an altitude of 120 m, how hard does the floor push on the power supply? (*Hint:* Start with a free-body diagram for the power supply.)

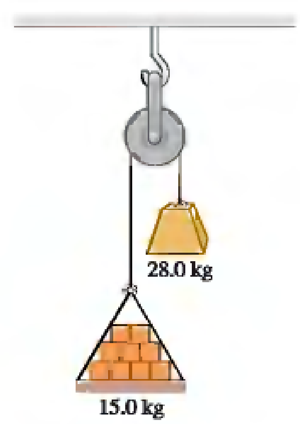
5.17. ***Genesis* Crash.** On September 8, 2004, the *Genesis* spacecraft crashed in the Utah desert because its parachute did not open. The 210-kg capsule hit the ground at 311 km/h and penetrated the soil to a depth of 81.0 cm. (a) Assuming it to be constant, what was its acceleration (in m/s^2 and in g's) during the crash? (b) What force did the ground exert on the capsule during the crash? Express the force in newtons and as a multiple of the capsule's weight. (c) For how long did this force last?

5.18. Three sleds are being pulled horizontally on frictionless horizontal ice using horizontal ropes (Fig. 5.50). The pull is horizontal and of magnitude 125 N. Find (a) the acceleration of the system and (b) the tension in ropes A and B.

Figure 5.50 Exercise 5.18.

5.19. Atwood's Machine. A 15.0-kg load of bricks hangs from one end of a rope that passes over a small, frictionless pulley. A 28.0-kg counterweight is suspended from the other end of the rope, as shown in Fig. 5.51. The system is released from rest. (a) Draw two free-body diagrams, one for the load of bricks and one for the counterweight. (b) What is the magnitude of the upward acceleration of the load of bricks? (c) What is the tension in the rope while the load is moving? How does the tension compare to the weight of the load of bricks? To the weight of the counterweight?

Figure 5.51 Exercise 5.19.



5.20. A 8.00-kg block of ice, released from rest at the top of a 1.50-m-long frictionless ramp, slides downhill, reaching a speed of 2.50 m/s at the bottom. (a) What is the angle between the ramp and the horizontal? (b) What would be the speed of the ice at the bottom if the motion were opposed by a constant friction force of 10.0 N parallel to the surface of the ramp?

5.21. A light rope is attached to a block with mass 4.00 kg that rests on a frictionless, horizontal surface. The horizontal rope passes over a frictionless, massless pulley, and a block with mass m is suspended from the other end. When the blocks are released, the tension in the rope is 10.0 N. (a) Draw two free-body diagrams, one for the 4.00-kg block and one for the block with mass m. (b) What is the acceleration of either block? (c) Find the mass m of the hanging block. (d) How does the tension compare to the weight of the hanging block?

5.22. Runway Design. A transport plane takes off from a level landing field with two gliders in tow, one behind the other. The mass of each glider is 700 kg, and the total resistance (air drag plus friction with the runway) on each may be assumed constant and equal to 2500 N. The tension in the towrope between the transport plane and the first glider is not to exceed 12,000 N. (a) If a speed of 40 m/s is required for takeoff, what minimum length of runway is needed? (b) What is the tension in the towrope between the two gliders while they are accelerating for the takeoff?

5.23. A 750.0-kg boulder is raised from a quarry 125 m deep by a long uniform chain having a mass of 575 kg. This chain is of uniform strength, but at any point it can support a maximum tension no greater than 2.50 times its weight without breaking. (a) What is the maximum acceleration the boulder can have and still get out of the quarry, and (b) how long does it take to be lifted out at maximum acceleration if it started from rest?

5.24. Apparent Weight. A 550-N physics student stands on a bathroom scale in an 850-kg (including the student) elevator that is supported by a cable. As the elevator starts moving, the scale reads 450 N. (a) Find the acceleration of the elevator (magnitude and direction). (b) What is the acceleration if the scale reads 670 N? (c) If the scale reads zero, should the student worry? Explain. (d) What is the tension in the cable in parts (a) and (c)?

5.25. A physics student playing with an air hockey table (a frictionless surface) finds that if she gives the puck a velocity of 3.80 m/s along the length (1.75 m) of the table at one end, by the time it has reached the other end the puck has drifted 2.50 cm to the right but still has a velocity component along the length of 3.80 m/s. She correctly concludes that the table is not level and correctly calculates its inclination from the given information. What is the angle of inclination?

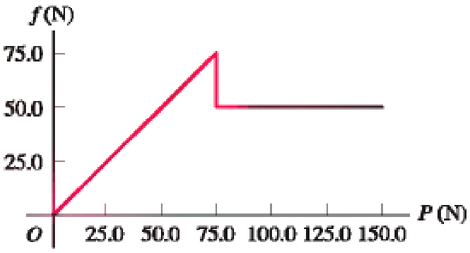
5.26. A 2540-kg test rocket is launched vertically from the launch pad. Its fuel (of negligible mass) provides a thrust force so that its vertical velocity as a function of time is given by $v(t) = At + Bt^2$, where A and B are constants and time is measured from the instant the fuel is ignited. At the instant of ignition, the rocket has an upward acceleration of 1.50 m/s^2 and 1.00 s later an upward velocity of 2.00 m/s. (a) Determine A and B, including their SI units. (b) At 4.00 s after fuel ignition, what is the acceleration of the rocket, and (c) what thrust force does the burning fuel exert on it, assume no air resistance? Express the thrust in newtons and as a multiple of the rocket's weight. (d) What was the initial thrust due to the fuel?

Section 5.3 Frictional Forces

5.27. Free-Body Diagrams. The first two steps in the solution of Newton's second-law problems are to select an object for analysis and then to draw free-body diagrams for that object. Draw free-body diagrams for the following situations: (a) a mass M sliding down a frictionless inclined plane of angle α, and (b) a mass M sliding up a frictionless inclined plane of angle α; (c) a mass M sliding up an inclined plane of angle α with kinetic friction present.

5.20. In a laboratory experiment on friction, a 135-N block resting on a rough horizontal table is pulled by a horizontal wire. The pull gradually increases until the block begins to move and continues to increase thereafter. Figure 5.52 shows a graph of the friction force on this block as a function of the pull. (a) Identify the regions of the graph where static and kinetic friction occur. (b) Find the coefficients of static and kinetic friction between the block and the table. (c) Why does the graph slant upward in the first part but then level out? (d) What would the graph look like if a 135-N brick were placed on the box, and what would be the coefficients of friction be in that case?

Figure 5.52 Exercise 5.28.



5.29. A stockroom worker pushes a box with mass 11.2 kg on a horizontal surface with a constant speed of 3.50 m/s. The coefficient of kinetic friction between the box and the surface is 0.20. (a) What horizontal force must the worker apply to maintain the motion? (b) If the force calculated in part (a) is removed, how far does the box slide before coming to rest?

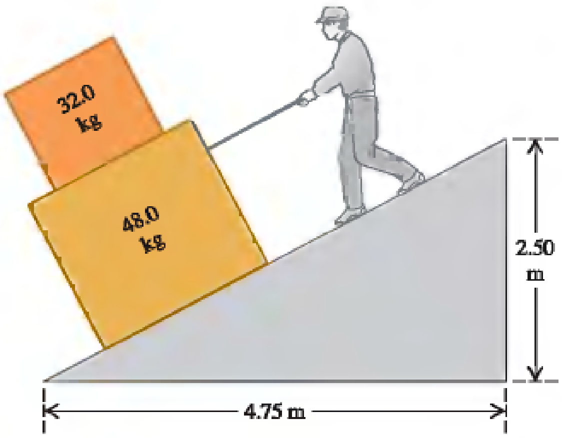
5.30. A box of bananas weighing 40.0 N rests on a horizontal surface. The coefficient of static friction between the box and the surface is 0.40, and the coefficient of kinetic friction is 0.20. (a) If no horizontal force is applied to the box and the box is at rest, how large is the friction force exerted on the box? (b) What is the magnitude of the friction force if a monkey applies a horizontal force of 6.0 N to the box and the box is initially at rest? (c) What minimum horizontal force must the monkey apply to start the box in motion? (d) What minimum horizontal force must the monkey apply to keep the box moving at constant velocity once it has been started? (e) If the monkey applies a horizontal force of 18.0 N, what is the magnitude of the friction force and what is the box's acceleration?

5.31. A crate of 45.0-kg tools rests on a horizontal floor. You exert a gradually increasing horizontal push on it and observe that the crate just begins to move when your force exceeds 313 N. After that you must reduce your push to 208 N to keep it moving at a steady 25.0 cm/s. (a) What are the coefficients of static and kinetic friction between the crate and the floor? (b) What push must you exert to give it an acceleration of 1.10 m/s^2? (c) Suppose you were performing the same experiment on this crate but were doing it on the moon instead, where the acceleration due to gravity is 1.62 m/s^2. (i) What magnitude push would cause it to move? (ii) What would its acceleration be if you maintained the push in part (b)?

5.32. An 85-N box of oranges is being pushed across a horizontal floor. As it moves, it is slowing at a constant rate of 0.90 m/s each second. The push force has a horizontal component of 20 N and a vertical component of 25 N downward. Calculate the coefficient of kinetic friction between the box and floor.

5.33. You are lowering two boxes, one on top of the other, down the ramp shown in Figure 5.53 by pulling on a rope parallel to the surface of the ramp. Both boxes move together at a constant speed of 15.0 cm/s. The coefficient of kinetic friction between the ramp and the lower box is 0.444, and the coefficient of static friction between the two boxes is 0.800. (a) What force do you need to

Figure 5.53 Exercise 5.33.



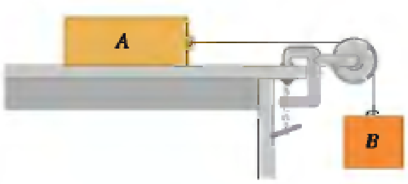
exert to accomplish this? (b) What are the magnitude and direction of the friction force on the upper box?

5.34. Stopping Distance. (a) If the coefficient of kinetic friction between tires and dry pavement is 0.80, what is the shortest distance in which you can stop an automobile by locking the brakes when traveling at 28.7 m/s (about 65 mi/h)? (b) On wet pavement the coefficient of kinetic friction may be only 0.25. How fast should you drive on wet pavement in order to be able to stop in the same distance as in part (a)? (*Note:* Locking the brakes is *not* the safest way to stop.)

5.35. Coefficient of Friction. A clean brass washer slides along a horizontal clean steel surface until it stops. Using the values from Table 5.1, how many times farther would it slide with the same initial speed if the washer were Teflon-coated?

5.38. Consider the system shown in Fig. 5.54. Block A weighs 45.0 N and block B weighs 25.0 N. Once block B is set into downward motion, it descends at a constant speed. (a) Calculate the coefficient of kinetic friction between block A and the tabletop. (b) A cat, also of weight 45.0 N, falls asleep on top of block A. If block B is now set into downward motion, what is its acceleration (magnitude and direction)?

Figure 5.54 Exercises 5.36 and 5.41; Problem 5.77.



5.37. Two crates connected by a rope lie on a horizontal surface (Fig. 5.55). Crate A has mass m_A and crate B has mass m_B. The coefficient of kinetic friction between each crate and the surface is μ_k. The crates are pulled to the right at constant velocity by a horizontal force $\vec{F}$. In terms of m_A, m_B, and μ_k, calculate (a) the magnitude of the force $\vec{F}$ and (b) the tension in the rope connecting the blocks. Include the free-body diagram or diagrams you used to determine each answer.

Figure 5.55 Exercise 5.37.

5.38. Rolling Friction. Two bicycle tires are set rolling with the same initial speed of 3.50 m/s on a long, straight road, and the distance each travels before its speed is reduced by half is measured. One tire is inflated to a pressure of 40 psi and goes 18.1 m; the other is at 105 psi and goes 92.9 m. What is the coefficient of rolling friction μ_r for each? Assume that the net horizontal force is due to rolling friction only.

5.39. Wheels. You find that it takes a horizontal force of 160 N to slide a box along the surface of a level floor at constant speed. The coefficient of static friction is 0.52, and the coefficient of kinetic friction is 0.47. If you place the box on a dolly of mass 5.3 kg and with coefficient of rolling friction 0.018, what horizontal acceleration would that 160-N force provide?

5.40. You find it takes 200 N of horizontal force to move an empty pickup truck along a level road at a speed of 2.4 m/s. You then load the pickup and pump up its tires so that its total weight increases by 42% while the coefficient of rolling friction decreases by 19%. Now what horizontal force will you need to move the pickup along the same road at the same speed? The speed is low enough that you can ignore air resistance.

5.41. As shown in Fig. 5.54, block A (mass 2.25 kg) rests on a tabletop. It is connected by a horizontal cord passing over a light, frictionless pulley to a hanging block B (mass 1.30 kg). The coefficient of kinetic friction between block A and the tabletop is 0.450. After the blocks are released from rest, find (a) the speed of each block after moving 3.00 cm and (b) the tension in the cord. Include the free-body diagram or diagrams you used to determine the answers.

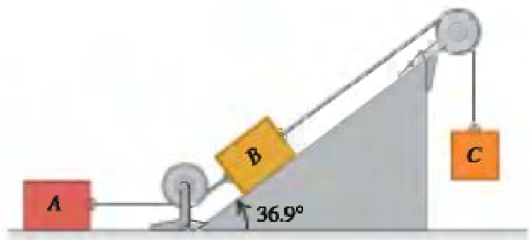
5.42. A 25.0-kg box of textbooks rests on a loading ramp that makes an angle α with the horizontal. The coefficient of kinetic friction is 0.25, and the coefficient of static friction is 0.35. (a) As the angle α is increased, find the minimum angle at which the box starts to slip. (b) At this angle, find the acceleration once the box has begun to move. (c) At this angle, how fast will the box be moving after it has slid 5.0 m along the loading ramp?

5.43. A large crate with mass m rests on a horizontal floor. The coefficients of friction between the crate and the floor are μ_s and μ_k. A woman pushes downward at an angle θ below the horizontal on the crate with a force $\vec{F}$. (a) What magnitude of force $\vec{F}$ is required to keep the crate moving at constant velocity? (b) If μ_s is greater than some critical value, the woman cannot start the crate moving no matter how hard she pushes. Calculate this critical value of μ_s.

5.44. A box with mass m is dragged across a level floor having a coefficient of kinetic friction μ_k by a rope that is pulled upward at an angle θ above the horizontal with a force of magnitude F. (a) In terms of m, μ_k, θ, and g, obtain an expression for the magnitude of force required to move the box with constant speed. (b) Knowing that you are studying physics, a CPR instructor asks you how much force it would take to slide a 90-kg patient across a floor at constant speed by pulling on him at an angle of 25° above the horizontal. By dragging some weights wrapped in an old pair of pants down the hall with a spring balance, you find that $\mu_k = 0.35$. Use the result of part (a) to answer the instructor's question.

5.45. Blocks A, B, and C are placed as in Fig. 5.56 and connected by ropes of negligible mass. Both A and B weigh 25.0 N each, and the coefficient of kinetic friction between each block and the surface is 0.35. Block C descends with constant velocity. (a) Draw two separate free-body diagrams showing the forces acting on A and on B. (b) Find the tension in the rope connecting blocks A and B. (c) What is the weight of block C? (d) If the rope connecting A and B were cut, what would be the acceleration of C?

Figure 5.56 Exercise 5.45.



5.46. Starting from Eq. (5.10), derive Eqs. (5.11) and (5.12).

5.47. (a) In Example 5.19 (Section 5.3), what value of D is required to make $v_t = 42\ \text{m/s}$ for the skydiver? (b) If the skydiver's daughter, whose mass is 45 kg, is falling through the air and has the same D $(0.25\ \text{kg/m})$ as her father, what is the daughter's terminal speed?

5.48. You throw a baseball straight up. The drag force is proportional to v^2. In terms of g, what is the y-component of the ball's acceleration when its speed is half its terminal speed and (a) it is moving up? (b) It is moving back down?

Section 5.4 Dynamics of Circular Motion

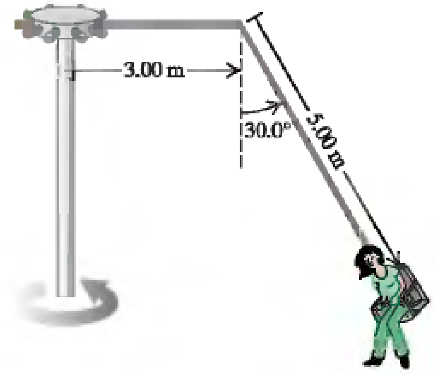
5.48. A machine part consists of a thin 40.0-cm-long bar with small 1.15-kg masses fastened by screws to its ends. The screws can support a maximum force of 75.0 N without pulling out. This bar rotates about an axis perpendicular to it at its center. (a) As the bar is turning at a constant rate on a horizontal frictionless surface, what is the maximum speed the masses can have without pulling out the screws? (b) Suppose the machine is redesigned so that the bar turns at a constant rate in a vertical circle. Will one of the screws be more likely to pull out when the mass is at the top of the circle or at the bottom? Use a free-body diagram to see why. (c) Using the result of part (b), what is the greatest speed the masses can have without pulling a screw?

5.50. A flat (unbanked) curve on a highway has a radius of 220.0 m. A car rounds the curve at a speed of 25.0 m/s. (a) What is the minimum coefficient of friction that will prevent sliding? (b) Suppose the highway is icy and the coefficient of friction between the tires and pavement is only one-third what you found in part (a). What should be the maximum speed of the car so it can round the curve safely?

5.51. A 1125-kg car and a 2250-kg pickup truck aproach a curve on the expressway that has a radius of 225 m. (a) At what angle should the highway engineer bank this curve so that vehicles traveling at 65.0 mi/h can safely round it regardless of the condition of their tires? Should the heavy truck go slower than the lighter car? (b) As the car and truck round the curve at 65.0 mi/h, find the normal force on each one due to the highway surface.

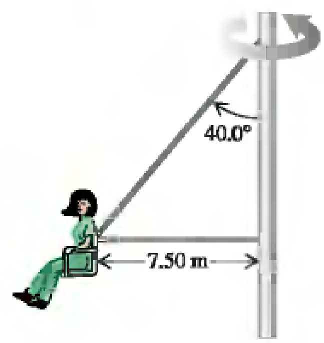
5.52. The "Giant Swing" at a county fair consists of a vertical central shaft with a number of horizontal arms attached at its upper end (Fig. 5.57). Each arm supports a seat suspended from a cable 5.00 m long, the upper end of the cable being fastened to the arm at a point 3.00 m from the central shaft. (a) Find the time of one revolution of the swing if the cable supporting a seat makes an angle of 30.0° with the vertical. (b) Does the angle depend on the weight of the passenger for a given rate of revolution?

Figure 5.57 Exercise 5.52.



5.53. In another version of the "Giant Swing" (see Exercise 5.52), the seat is connected to two cables as shown in Fig. 5.58, one of which is horizontal. The seat swings in a horizontal circle at a rate of 32.0 rpm (rev/min). If the seat weighs 255 N and a 825-N person is sitting in it, find the tension in each cable.

Figure 5.56 Exercise 5.53.



5.54. A small button placed on a horizontal rotating platform with diameter 0.320 m will revolve with the platform when it is brought up to a speed of 40.0 rev/min, provided the button is no more than 0.150 m from the axis. (a) What is the coefficient of static friction between the button and the platform? (b) How far from the axis can the button be placed, without slipping, if the platform rotates at 60.0 rev/min?

5.55. Rotating Space Stations. One problem for humans living in outer space is that they are apparently weightless. One way around this problem is to design a space station that spins about its center at a constant rate. This creates "artificial gravity" at the outside rim of the station. (a) If the diameter of the space station is 800 m, how many revolutions per minute are needed for the "artificial gravity" acceleration to be $9.80\ \text{m/s}^2$? (b) If the space station is a waiting area for travelers going to Mars, it might be desirable to simulate the acceleration due to gravity on the Martian surface $(3.70\ \text{m/s}^2)$. How many revolutions per minute are needed in this case?

5.56. The Cosmoclock 21 Ferris wheel in Yokohama City, Japan, has a diameter of 100 m. Its name comes from its 60 arms, each of which can function as a second hand (so that it makes one revolution every 60.0 s). (a) Find the speed of the passengers when the Ferris wheel is rotating at this rate. (b) A passenger weighs 882 N at the weight-guessing booth on the ground. What is his apparent weight at the highest and at the lowest point on the Ferris wheel? (c) What would be the time for one revolution if the passenger's apparent weight at the highest point were zero? (d) What then would be the passenger's apparent weight at the lowest point?

5.57. An airplane flies in a loop (a circular path in a vertical plane) of radius 150 m. The pilot's head always points toward the center of the loop. The speed of the airplane is not constant; the airplane goes slowest at the top of the loop and fastest at the bottom. (a) At the top of the loop, the pilot feels weightless. What is the speed of the airplane at this point? (b) At the bottom of the loop, the speed of the airplane is 280 km/h. What is the apparent weight of the pilot at this point? His true weight is 700 N.

5.50. A 50.0-kg stunt pilot who has been diving her airplane vertically pulls out of the dive by changing her course to a circle in a vertical plane. (a) If the plane's speed at the lowest point of the circle is 95.0 m/s, what is the minimum radius of the circle for the acceleration at this point not to exceed 4.00g? (b) What is the apparent weight of the pilot at the lowest point of the pullout?

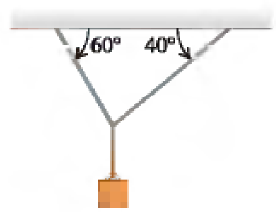
5.59. Stay Dry! You tie a cord to a pail of water, and you swing the pail in a vertical circle of radius 0.600 m. What minimum speed must you give the pail at the highest point of the circle if no water is to spill from it?

5.60. A bowling ball weighing 71.2 N (16.0 lb) is attached to the ceiling by a 3.80-m rope. The ball is pulled to one side and released; it then swings back and forth as a pendulum. As the rope swings through the vertical, the speed of the bowling ball is 4.20 m/s. (a) What is the acceleration of the bowling ball, in magnitude and direction, at this instant? (b) What is the tension in the rope at this instant?

Problems

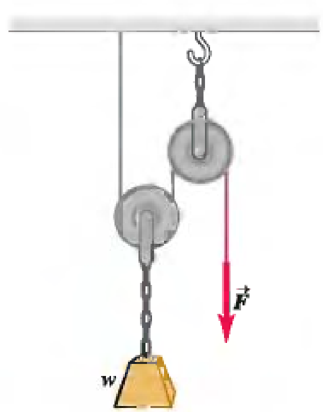
5.61. Two ropes are connected to a steel cable that supports a hanging weight as shown in Fig. 5.59. (a) Draw a free-body diagram showing all of the forces acting at the knot that connects the two ropes to the steel cable. Based on your force diagram, which of the two ropes will have the greater tension? (b) If the maximum tension either rope can sustain without breaking is 5000 N, determine the maximum value of the hanging weight that these ropes can safely support. You can ignore the weight of the ropes and the steel cable.

Figure 5.59 Problem 5.61.



5.62. In Fig. 5.60 a worker lifts a weight w by pulling down on a rope with a force $\vec{F}$. The upper pulley is attached to the ceiling by a chain, and the lower pulley is attached to the weight by another chain. In terms of w, find the tension in each chain and the magnitude of the force $\vec{F}$ if the weight is lifted at constant speed. Include the free-body diagram or diagrams you used to determine your answers. Assume that the rope, pulleys, and chains all have negligible weights.

Figure 5.60 Problem 5.62.



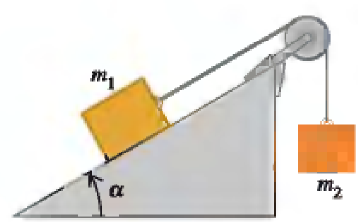
5.63. A Rope with Mass. In most problems in this book, the ropes, cords, or cables have so little mass compared to other objects in the problem that you can safely ignore their mass. But if the rope is the *only* object in the problem, then clearly you cannot ignore its mass. For example, suppose we have a clothesline attached to two poles (Fig. 5.61). The clothesline has a mass M, and each end makes an angle θ with the horizontal. What are (a) the tension at the ends of the clothesline and (b) the tension at the lowest point? (c) Why can't we have $\theta = 0$? (See Discussion Question Q5.3.) (d) Discuss your results for parts (a) and (b) in the limit that $\theta \to 90°$. The curve of the clothesline, or of any flexible cable hanging under its own weight, is called a *catenary*. [For a more advanced treatment of this curve, see K. R. Symon, *Mechanics*, 3rd ed. (Reading, MA: Addison-Wesley, 1971), pp. 237–241.]

Figure 5.61 Problem 5.63.

5.64. Another Rope with Mass. A block with mass M is attached to the lower end of a vertical, uniform rope with mass m and length L. A constant upward force $\vec{F}$ is applied to the top of the rope, causing the rope and block to accelerate upward. Find the tension in the rope at a distance x from the top end of the rope, where x can have any value from 0 to L.

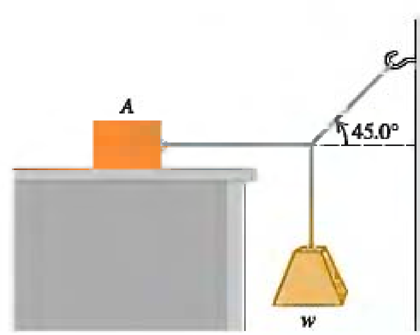
5.65. A block with mass m_1 is placed on an inclined plane with slope angle α and is connected to a second hanging block with mass m_2 by a cord passing over a small, frictionless pulley (Fig. 5.62). The coefficient of static friction is μ_s and the coefficient of kinetic friction is μ_k. (a) Find the mass m_2 for which block m_1 moves up the plane at constant speed once it is set in motion. (b) Find the mass m_2 for which block m_1 moves down the plane at constant speed once it is set in motion. (c) For what range of values of m_2 will the blocks remain at rest if they are released from rest?

Figure 5.62 Problem 5.65.



5.66. (a) Block A in Fig. 5.63 weighs 60.0 N. The coefficient of static friction between the block and the surface on which it rests is

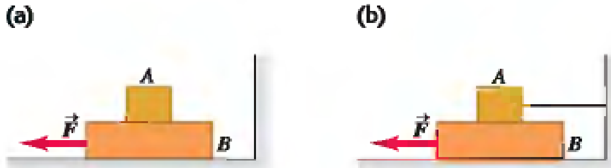
Figure 5.63 Problem 5.66.



0.25. The weight w is 12.0 N and the system is in equilibrium. Find the friction force exerted on block A. (b) Find the maximum weight w for which the system will remain in equilibrium.

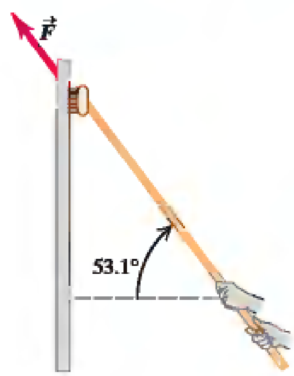
5.67. Block A in Fig. 5.64 weighs 1.20 N and block B weighs 3.60 N. The coefficient of kinetic friction between all surfaces is 0.300. Find the magnitude of the horizontal force $\vec{F}$ necessary to drag block B to the left at constant speed (a) if A rests on B and moves with it (Fig. 5.64a) and (b) if A is held at rest (Fig. 5.64b).

Figure 5.64 Problem 5.67.



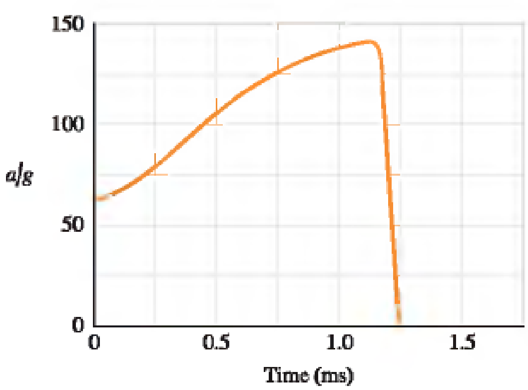
5.68. A window washer pushes his scrub brush up a vertical window at constant speed by applying a force $\vec{F}$ as shown in Fig. 5.65. The brush weighs 12.0 N and the coefficient of kinetic friction is $\mu_k = 0.150$. Calculate (a) the magnitude of the force $\vec{F}$ and (b) the normal force exerted by the window on the brush.

Figure 5.65 Problem 5.68.



5.69. The Flying Leap of a Flea. High-speed motion pictures $(3500\ \text{frames/second})$ of a jumping 210-μg flea yielded the data to plot the flea's acceleration as a function of time as shown in Fig. 5.66. (See "The Flying Leap of the Flea," by M. Rothschild et al. in the November 1973 *Scientific American*.) This flea was about 2 mm long and jumped at a nearly vertical takeoff angle. Use the measurements shown on the graph to answer the questions. (a) Find the *initial* net external force on the flea. How does it compare to the flea's weight? (b) Find the *maximum* net external force on this jumping flea. When does this maximum force occur? (c) Use the graph to find the flea's maximum speed.

Figure 5.66 Problem 5.69.



5.70. A 25,000-kg rocket blasts off vertically from the earth's surface with a constant acceleration. During the motion considered in the problem, assume that g remains constant (see Chapter 12). Inside the rocket, a 15.0-N instrument hangs from a wire that can support a maximum tension of 35.0 N. (a) Find the minimum time for this rocket to reach the sound barrier $(330\ \text{m/s})$ without breaking the inside wire and the maximum vertical thrust of the rocket engines under these conditions. (b) How far is the rocket above the earth's surface when it breaks the sound barrier?

5.71. You are standing on a bathroom scale in an elevator in a tall building. Your mass is 72 kg. The elevator starts from rest and travels upward with a speed that varies with time according to $v(t) = (3.0\ \text{m/s}^2)t + (0.20\ \text{m/s}^3)t^2$. When $t = 4.0$ s, what is the reading of the bathroom scale?

5.72. Elevator Design. You are designing an elevator for a hospital. The force exerted on a passenger by the floor of the elevator is not to exceed 1.60 times the passenger's weight. The elevator accelerates upward with constant acceleration for a distance of 3.0 m and then starts to slow down. What is the maximum speed of the elevator?

5.73. You are working for a shipping company. Your job is to stand at the bottom of a 8.0-m-long ramp that is inclined at 37° above the horizontal. You grab packages off a conveyor belt and propel them up the ramp. The coefficient of kinetic friction between the packages and the ramp is $\mu_k = 0.30$. (a) What speed do you need to give a package at the bottom of the ramp so that it has zero speed at the top of the ramp? (b) Your coworker is supposed to grab the packages as they arrive at the top of the ramp, but she misses one and it slides back down. What is its speed when it returns to you?

5.74. A hammer is hanging by a light rope from the ceiling of a bus. The ceiling of the bus is parallel to the roadway. The bus is traveling in a straight line on a horizontal street. You observe that the hammer hangs at rest with respect to the bus when the angle between the rope and the ceiling of the bus is 74°. What is the acceleration of the bus?

5.75. A steel washer is suspended inside an empty shipping crate from a light string attached to the top of the crate. The crate slides down a long ramp that is inclined at an angle of 37° above the horizontal. The crate has mass 180 kg. You are sitting inside the crate (with a flashlight); your mass is 55 kg. As the crate is sliding down the ramp, you find the washer is at rest with respect to the crate when the string makes an angle of 68° with the top of the crate. What is the coefficient of kinetic friction between the ramp and the crate?

5.76. Lunch Time! You are riding your motorcycle one day down a wet street that slopes downward at an angle of 20° below the horizontal. As you start to ride down the hill, you notice a construction crew has dug a deep hole in the street at the bottom of the hill. A Siberian tiger, escaped from the City Zoo, has taken up residence in the hole. You apply the brakes and lock your wheels at the top of the hill, where you are moving with a speed of 20 m/s. The inclined street in front of you is 40 m long. (a) Will you plunge into the hole and become the tiger's lunch, or do you skid to a stop before you reach the hole? (The coefficients of friction between your motorcycle tires and the wet pavement are $\mu_s = 0.90$ and $\mu_k = 0.70$.) (b) What must your initial speed be if you are to stop just before reaching the hole?

5.77. In the system shown in Fig. 5.54, block A has mass m_A, block B has mass m_B, and the rope connecting them has a *nonzero* mass m_{rope}. The rope has a total length L, and the pulley has a very small radius. You can ignore any sag in the horizontal part of the rope. (a) If there is no friction between block A and the tabletop, find the acceleration of the blocks at an instant when a length d of rope hangs vertically between the pulley and block B. As block B falls, will the magnitude of the acceleration of the system increase, decrease, or remain constant? Explain. (b) Let $m_A = 2.00$ kg, $m_B = 0.400$ kg, $m_{rope} = 0.160$ kg, and $L = 1.00$ m. If there is friction between block A and the tabletop, with $\mu_k = 0.200$ and $\mu_s = 0.250$, find the minimum value of the distance d such that the blocks will start to move if they are initially at rest. (c) Repeat part (b) for the case $m_{rope} = 0.040$ kg. Will the blocks move in this case?

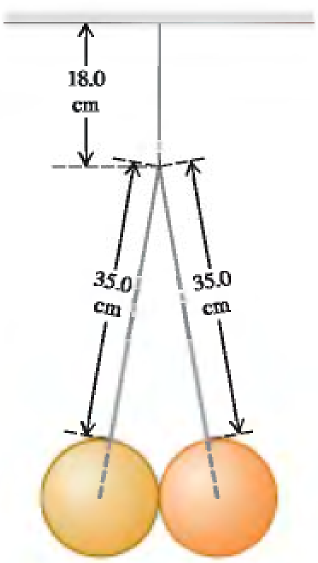
5.78. If the coefficient of static friction between a table and a uniform massive rope is μ_s, what fraction of the rope can hang over the edge of the table without the rope sliding?

5.79. A 30.0-kg packing case is initially at rest on the floor of a 1500-kg pickup truck. The coefficient of static friction between the case and the truck floor is 0.30, and the coefficient of kinetic friction is 0.20. Before each acceleration given below, the truck is traveling due north at constant speed. Find the magnitude and direction of the friction force acting on the case (a) when the truck accelerates at 2.20 m/s^2 northward and (b) when it accelerates at 3.40 m/s^2 southward.

5.80. Traffic Court. You are called as an expert witness in the trial of a traffic violation. The facts are these: A driver slammed on his brakes and came to a stop with constant acceleration. Measurements of his tires and the skid marks on the pavement indicate that he locked his car's wheels, the car traveled 192 ft before stopping, and the coefficient of kinetic friction between the road and his tires was 0.750. The charge is that he was speeding in a 45-mi/h zone. He pleads innocent. What is your conclusion, guilty or innocent? How fast was he going when he hit his brakes?

5.81. Two identical 15.0-kg balls, each 25.0 cm in diameter, are suspended by two 35.0-cm wires as shown in Fig. 5.67. The entire apparatus is supported by a single 18.0-cm wire, and the surfaces of the balls are perfectly smooth. (a) Find the tension in each of the three wires. (b) How hard does each ball push on the other one?

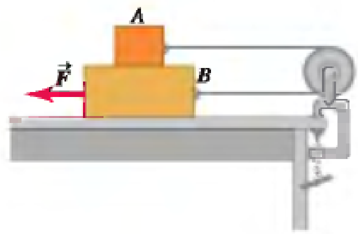
Figure 5.67 Problem 5.81.



5.82. Losing Cargo. A 12.0-kg box rests on the flat floor of a truck. The coefficients of friction between the box and floor are $\mu_s = 0.19$ and $\mu_k = 0.15$. The truck stops at a stop sign and then starts to move with an acceleration of 2.20 m/s^2. If the box is 1.80 m from the rear of the truck when the truck starts, how much time elapses before the box falls off the truck? How far does the truck travel in this time?

5.83. Block A in Fig. 5.68 weighs 1.40 N, and block B weighs 4.20 N. The coefficient of kinetic friction between all surfaces is 0.30. Find the magnitude of the horizontal force $\vec{F}$ necessary to drag block B to the left at constant speed if A and B are connected by a light, flexible cord passing around a fixed, frictionless pulley.

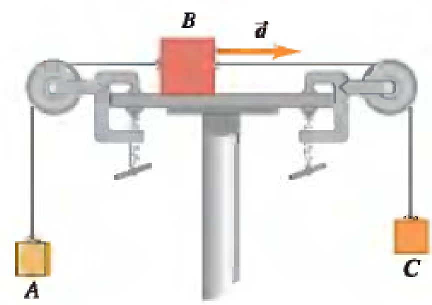
Figure 5.68 Problem 5.83.



5.84. You are part of a design team for future exploration of the planet Mars, where $g = 3.7$ m/s^2. An explorer is to step out of a survey vehicle traveling horizontally at 33 m/s when it is 1200 m above the surface and then fall freely for 20 s. At that time, a portable advanced propulsion system (PAPS) is to exert a constant force that will decrease the explorer's speed to zero at the instant she touches the surface. The total mass (explorer, suit, equipment, and PAPS) is 150 kg. Assume the change in mass of the PAPS to be negligible. Find the horizontal and vertical components of the force the PAPS must exert, and for what interval of time the PAPS must exert it. You can ignore air resistance.

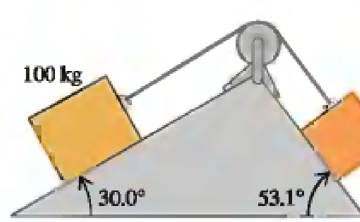
5.85. Block A in Fig. 5.69 has a mass of 4.00 kg, and block B has mass 12.0 kg. The coefficient of kinetic friction between block B and the horizontal surface is 0.25. (a) What is the mass of block C if block B is moving to the right and speeding up with an acceleration 2.00 m/s^2? (b) What is the tension in each cord when block B has this acceleration?

Figure 5.69 Problem 5.85.



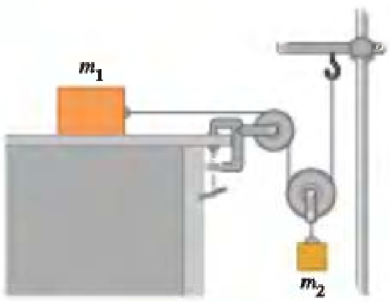
5.86. Two blocks connected by a cord passing over a small, frictionless pulley rest on frictionless planes (Fig. 5.70). (a) Which way will the system move when the blocks are released from rest? (b) What is the acceleration of the blocks? (c) What is the tension in the cord?

Figure 5.70 Problem 5.86.



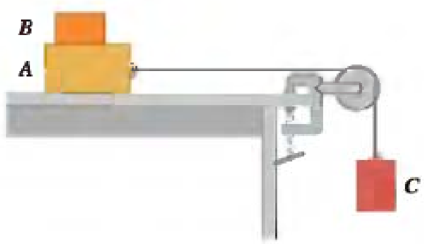
5.67. In terms of m_1, m_2, and g, find the accelerations of each block in Fig. 5.71. There is no friction anywhere in the system.

Figure 5.71 Problem 5.87.



5.68. Block B, with mass 5.00 kg, rests on block A, with mass 8.00 kg, which in turn is on a horizontal tabletop (Fig. 5.72). There is no friction between block A and the tabletop, but the coefficient of static friction between block A and block B is 0.750. A light string attached to block A passes over a frictionless, massless pulley, and block C is suspended from the other end of the string. What is the largest mass that block C can have so that blocks A and B still slide together when the system is released from rest?

Figure 5.72 Problem 5.88.



5.68. Two objects with masses 5.00 kg and 2.00 kg hang 0.600 m above the floor from the ends of a cord 6.00 m long passing over a frictionless pulley. Both objects start from rest. Find the maximum height reached by the 2.00-kg object.

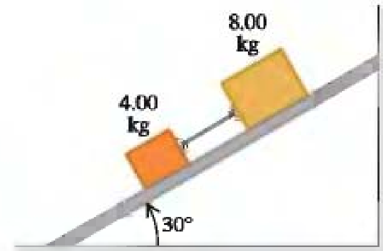
5.90. Friction in an Elevator. You are riding in an elevator on the way to the 18th floor of your dormitory. The elevator is accelerating upward with $a = 1.90\ \text{m/s}^2$. Beside you is the box containing your new computer; the box and its contents have a total mass of 28.0 kg. While the elevator is accelerating upward, you push horizontally on the box to slide it at constant speed toward the elevator door. If the coefficient of kinetic friction between the box and the elevator floor is $\mu_k = 0.32$, what magnitude of force must you apply?

5.91. A block is placed against the vertical front of a cart as shown in Fig. 5.73. What acceleration must the cart have so that block A does not fall? The coefficient of static friction between the block and the cart is μ_s. How would an observer on the cart describe the behavior of the block?

Figure 5.73 Problem 5.91.

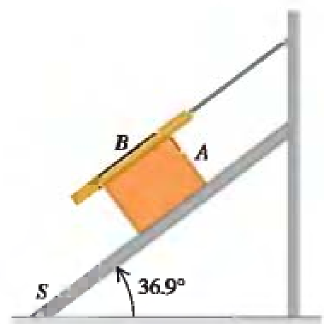
5.92. Two blocks with masses 4.00 kg and 8.00 kg are connected by a string and slide down a 30.0° inclined plane (Fig. 5.74). The coefficient of kinetic friction between the 4.00-kg block and the plane is 0.25; that between the 8.00-kg block and the plane is 0.35. (a) Calculate the acceleration of each block. (b) Calculate the tension in the string. (c) What happens if the positions of the blocks are reversed, so the 4.00-kg block is above the 8.00-kg block?

Figure 5.74 Problem 5.92.



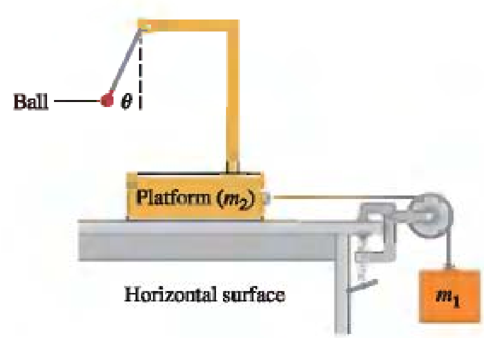
5.93. Block A, with weight $3w$, slides down an inclined plane S of slope angle 36.9° at a constant speed while plank B, with weight w, rests on top of A. The plank is attached by a cord to the wall (Fig. 5.75). (a) Draw a diagram of all the forces acting on block A. (b) If the coefficient of kinetic friction is the same between A and B and between S and A, determine its value.

Figure 5.75 Problem 5.93.



5.94. Accelerometer. The system shown in Fig. 5.76 can be used to measure the acceleration of the system. An observer riding on the platform measures the angle θ that the thread supporting the light ball makes with the vertical. There is no friction anywhere. (a) How is θ related to the acceleration of the system? (b) If $m_1 = 250$ kg and $m_2 = 1250$ kg, what is θ? (c) If you can vary m_1 and m_2, what is the largest angle θ you could achieve? Explain how you need to adjust m_1 and m_2 to do this.

Figure 5.76 Problem 5.94.



5.95. Banked Curve I. A curve with a 120-m radius on a level road is banked at the correct angle for a speed of 20 m/s. If an automobile rounds this curve at 30 m/s, what is the minimum coefficient of static friction needed between tires and road to prevent skidding?

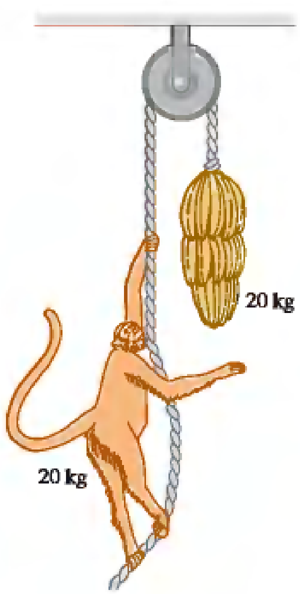
5.96. Banked Curve II. Consider a wet roadway banked as in Example 5.23 (Section 5.4), where there is a coefficient of static friction of 0.30 and a coefficient of kinetic friction of 0.25 between the tires and the roadway. The radius of the curve is $R = 50$ m. (a) If the banking angle is $\beta = 25°$, what is the *maximum* speed the automobile can have before sliding *up* the banking? (b) What is the *minimum* speed the automobile can have before sliding *down* the banking?

5.97. Maximum Safe Speed. As you travel every day to campus, the road makes a large turn that is approximately an arc of a circle. You notice the warning sign at the start of the turn, asking for a maximum speed of 55 mi/h. You also notice that in the curved portion the road is level—that is, not banked at all. On a dry day with very little traffic, you enter the turn at a constant speed of 80 mi/h and feel that the car may skid if you do not slow down quickly. You conclude that your speed is at the limit of safety for this curve and you slow down. However, you remember reading that on dry pavement new tires have an average coefficient of static friction of about 0.76, while under the worst winter driving conditions, you may encounter wet ice for which the coefficient of static friction can be as low as 0.20. Wet ice is not unheard of on this road, so you ask yourself whether the speed limit for the turn on the roadside warning sign is for the worst-case scenario. (a) Estimate the radius of the curve from your 80-mi/h experience in the dry turn. (b) Use this estimate to find the maximum speed limit in the turn under the worst wet-ice conditions. How does this compare with the speed limit on the sign? Is the sign misleading drivers? (c) On a rainy day, the coefficient of static friction would be about 0.37. What is the maximum safe speed for the turn when the road is wet? Does your answer help you understand the maximum-speed sign?

5.98. You are riding in a school bus. As the bus rounds a flat curve at constant speed, a lunch box with mass 0.500 kg, suspended from the ceiling of the bus by a string 1.80 m long, is found to hang at rest relative to the bus when the string makes an angle of 30.0° with the vertical. In this position the lunch box is 50.0 m from the center of curvature of the curve. What is the speed v of the bus?

5.99. The Monkey and Bananas Problem. A 20-kg monkey has a firm hold on a light rope that passes over a frictionless pulley and is attached to a 20-kg bunch of bananas (Fig. 5.77). The monkey looks up, sees the bananas, and starts to climb the rope to get them. (a) As the monkey climbs, do the bananas move up, down, or remain at rest? (b) As the monkey climbs, does the distance between the monkey and the bananas decrease, increase, or remain constant? (c) The monkey releases her hold on the rope. What happens to the distance between the monkey and the bananas while she is falling? (d) Before reaching the ground, the monkey grabs the rope to stop her fall. What do the bananas do?

Figure 5.77 Problem 5.99.



5.100. You throw a rock downward into water with a speed of $3mg/k$, where k is the coefficient in Eq. (5.7). Assume that the relationship between fluid resistance and speed is as given in Eq. (5.7), and calculate the speed of the rock as a function of time.

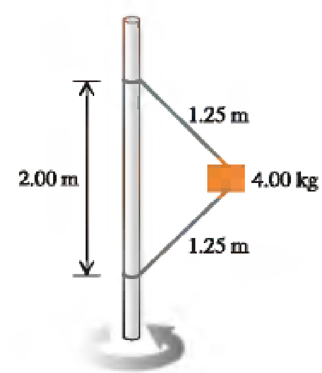
5.101. A rock with mass $m = 3.00$ kg falls from rest in a viscous medium. The rock is acted on by a net constant downward force of 18.0 N (a combination of gravity and the buoyant force exerted by the medium) and by a fluid resistance force $f = kv$, where v is the speed in m/s and $k = 2.20\ \text{N} \cdot \text{s/m}$ (see Section 5.3). (a) Find the initial acceleration a_0. (b) Find the acceleration when the speed is 3.00 m/s. (c) Find the speed when the acceleration equals $0.1a_0$. (d) Find the terminal speed v_t. (e) Find the coordinate, speed, and acceleration 2.00 s after the start of the motion. (f) Find the time required to reach a speed $0.9v_t$.

5.102. A rock with mass m slides with initial velocity v_0 on a horizontal surface. A retarding force F_R that the surface exerts on the rock is proportional to the square root of the instantaneous velocity of the rock $(F_R = -kv^{1/2})$. (a) Find expressions for the velocity and position of the rock as a function of time. (b) In terms of m, k, and v_0, at what time will the rock come to rest? (c) In terms of m, k, and v_0, what is the distance of the rock from its starting point when it comes to rest?

5.103. A fluid exerts an upward buoyancy force on an object immersed in it. In the derivation of Eq. (5.9) the buoyancy force exerted on an object by the fluid was ignored. But in some situations, where the density of the object is not much greater than the density of the fluid, you cannot ignore the buoyancy force. For a plastic sphere falling in water, you calculate the terminal speed to be 0.36 m/s when you ignore buoyancy, but you measure it to be 0.24 m/s. The buoyancy force is what fraction of the weight?

5.104. The 4.00-kg block in Fig. 5.78 is attached to a vertical rod by means of two strings. When the system rotates about the axis of the rod, the strings are extended as shown in the diagram and the tension in the upper string is 80.0 N. (a) What is the tension in the lower cord? (b) How many revolutions per minute does the system make? (c) Find the number of revolutions per minute at which the lower cord just goes slack. (d) Explain what happens if the number of revolutions per minute is less than in part (c).

Figure 5.78 Problem 5.104.



5.105. Equation (5.10) applies to the case where the initial velocity is zero. (a) Derive the corresponding equation for $v_y(t)$ when the falling object has an initial downward velocity with magnitude v_0. (b) For the case where $v_0 < v_t$, sketch a graph of v_y as a function of t and label v_t on your graph. (c) Repeat part (b) for the case where $v_0 > v_t$. (d) Discuss what your result says about $v_y(t)$ when $v_0 = v_t$.

5.106. A small rock moves in water, and the force exerted on it by the water is given by Eq. (5.7). The terminal speed of the rock is measured and found to be 2.0 m/s. The rock is projected *upward* at an initial speed of 6.0 m/s. You can ignore the buoyancy force on the rock. (a) In the absence of fluid resistance, how high will the rock rise and how long will it take to reach this maximum height?

(b) When the effects of fluid resistance are included, what are the answers to the questions in part (a)?

5.107. You observe a 1350-kg sports car rolling along flat pavement in a straight line. The only horizontal forces acting on it are a constant rolling friction and air resistance (proportional to the square of its speed). You take the following data during a time interval of 25 s: When its speed is 32 m/s, the car slows down at a rate of $-0.42\ \text{m/s}^2$, and when its speed is decreased to 24 m/s, it slows down at $-0.30\ \text{m/s}^2$. (a) Find the coefficient of rolling friction and the air drag constant D. (b) At what constant speed will this car move down an incline that makes a 2.2° angle with the horizontal? (c) How is the constant speed for an incline of angle β related to the terminal speed of this sports car if the car drops off a high cliff? Assume that in both cases the air resistance force is proportional to the square of the speed, and the air drag constant is the same.

5.108. A 70-kg person rides in a 30-kg cart moving at 12 m/s at the top of a hill that is in the shape of an arc of a circle with a radius of 40 m. (a) What is the apparent weight of the person as the cart passes over the top of the hill? (b) Determine the maximum speed that the cart may travel at the top of the hill without losing contact with the surface. Does your answer depend on the mass of the cart or the mass of the person? Explain.

5.109. Merry-Go-Round. One December identical twins Jena and Jackie are playing on a large merry-go-round (a disk mounted parallel to the ground, on a vertical axle through its center) in their school playground in northern Minnesota. Each twin has mass 30.0 kg. The icy coating on the merry-go-round surface makes it frictionless. The merry-go-round revolves at a constant rate as the twins ride on it. Jena, sitting 1.80 m from the center of the merry-go-round, must hold on to one of the metal posts attached to the merry-go-round with a horizontal force of 60.0 N to keep from sliding off. Jackie is sitting at the edge, 3.60 m from the center. (a) With what horizontal force must Jackie hold on to keep from falling off? (b) If Jackie falls off, what will be her horizontal velocity when she becomes airborne?

5.110. A passenger with mass 85 kg rides in a Ferris wheel like that in Example 5.24 (Section 5.4). The seats travel in a circle of radius 35 m. The Ferris wheel rotates at constant speed and makes one complete revolution every 25 s. Calculate the magnitude and direction of the net force exerted on the passenger by the seat when she is (a) one-quarter revolution past her lowest point and (b) one-quarter revolution past her highest point.

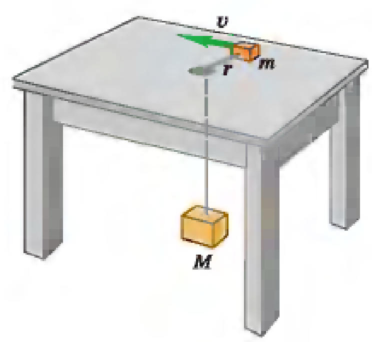
5.111. On the ride "Spindletop" at the amusement park Six Flags Over Texas, people stood against the inner wall of a hollow vertical cylinder with radius 2.5 m. The cylinder started to rotate, and when it reached a constant rotation rate of 0.60 rev/s, the floor on which people were standing dropped about 0.5 m. The people remained pinned against the wall. (a) Draw a force diagram for a person on this ride, after the floor has dropped. (b) What minimum coefficient of static friction is required if the person on the ride is not to slide downward to the new position of the floor? (c) Does your answer in part (b) depend on the mass of the passenger? (*Note:* When the ride is over, the cylinder is slowly brought to rest. As it slows down, people slide down the walls to the floor.)

5.112. A physics major is working to pay his college tuition by performing in a traveling carnival. He rides a motorcycle inside a hollow transparent plastic sphere. After gaining sufficient speed, he travels in a vertical circle with a radius of 13.0 m. The physics major has mass 70.0 kg, and his motorcycle has mass 40.0 kg. (a) What minimum speed must he have at the top of the circle if the tires of the motorcycle are not to lose contact with the sphere? (b) At the bottom of the circle, his speed is twice the value calculated in part (a). What is the magnitude of the normal force exerted on the motorcycle by the sphere at this point?

5.113. Ulterior Motives. You are driving a classic 1954 Nash Ambassador with a friend who is sitting to your right on the passenger side of the front seat. The Ambassador has flat bench seats. You would like to be closer to your friend and decide to use physics to achieve your romantic goal by making a quick turn. (a) Which way (to the left or to the right) should you turn the car to get your friend to slide closer to you? (b) If the coefficient of static friction between your friend and the car seat is 0.35, and you keep driving at a constant speed of 20 m/s, what is the maximum radius you could make your turn and still have your friend slide your way?

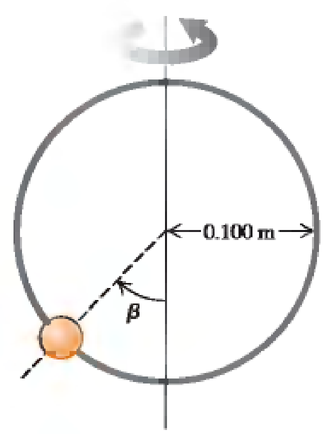
5.114. A small block with mass m rests on a frictionless horizontal tabletop a distance r from a hole in the center of the table (Fig. 5.79). A string tied to the small block passes down through the hole, and a larger block with mass M is suspended from the free end of the string. The small block is set into uniform circular motion with radius r and speed v. What must v be if the large block is to remain motionless when released?

Figure 5.79 Problem 5.114.



5.115. A small bead can slide without friction on a circular hoop that is in a vertical plane and has a radius of 0.100 m. The hoop rotates at a constant rate of 4.00 rev/s about a vertical diameter (Fig. 5.80). (a) Find the angle β at which the bead is in vertical equilibrium. (Of course, it has a radial acceleration toward the axis.) (b) Is it possible for the bead to "ride" at the same elevation as the center of the hoop? (c) What will happen if the hoop rotates at 1.00 rev/s?

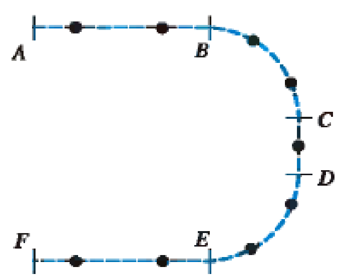
Figure 5.80 Problem 5.115.



5.116. A model airplane with mass 2.20 kg moves in the xy-plane such that its x- and y-coordinates vary in time according to $x(t) = \alpha - \beta t^3$ and $y(t) = \gamma t - \delta t^2$, where $\alpha = 1.50$ m, $\beta = 0.120$ m/s^3, $\gamma = 3.00$ m/s, and $\delta = 1.00$ m/s^2. (a) Calculate the x- and y-components of the net force on the plane as functions of time. (b) Sketch the trajectory of the airplane between $t = 0$ and $t = 3.00$ s, and draw on your sketch vectors showing the net force on the airplane at $t = 0$, $t = 1.00$ s, $t = 2.00$ s, and $t = 3.00$ s. For each of these times, relate the direction of the net force to the direction that the airplane is turning, and to whether the airplane is speeding up or slowing down (or neither). (c) What are the magnitude and direction of the net force at $t = 3.00$ s?

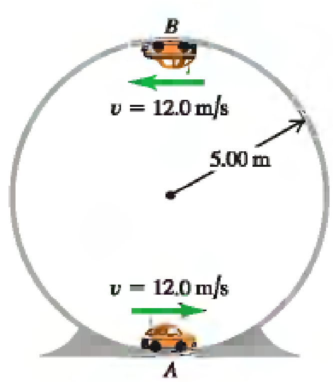
5.117. A particle moves on a frictionless surface along a path as shown in Fig. 5.81. (The figure gives a view looking down on the surface.) The particle is initially at rest at point A and then begins to move toward B as it gains speed at a constant rate. From B to C, the particle moves along a circular path at a constant speed. The speed remains constant along the straight-line path from C to D. From D to E, the particle moves along a circular path, but now its speed is decreasing at a constant rate. The speed continues to decrease at a constant rate as the particle moves from E to F; the particle comes to a halt at F. (The time intervals between the marked points are not equal.) At each point marked with a dot, draw arrows to represent the velocity, the acceleration, and the net force acting on the particle. Draw longer or shorter arrows to represent vectors of larger or smaller magnitude.

Figure 5.81 Problem 5.117.



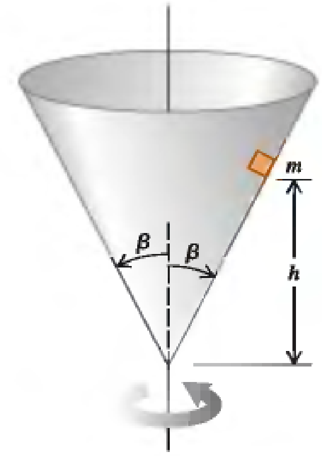
5.119. A small remote-control car with mass 1.60 kg moves at a constant speed of $v = 12.0$ m/s in a vertical circle inside a hollow metal cylinder that has a radius of 5.00 m (Fig 5.82). What is the magnitude of the normal force exerted on the car by the walls of the cylinder at (a) point A (at the bottom of the vertical circle) and (b) point B (at the top of the vertical circle)?

Figure 5.82 Problem 5.118.



5.119. A small block with mass m is placed inside an inverted cone that is rotating about a vertical axis such that the time for one revolution of the cone is T (Fig. 5.83). The walls of the cone make an angle β with the vertical. The coefficient of static friction between the block and the cone is μ_s. If the block is to remain at a constant height h above the apex of the cone, what are the maximum and minimum values of T?

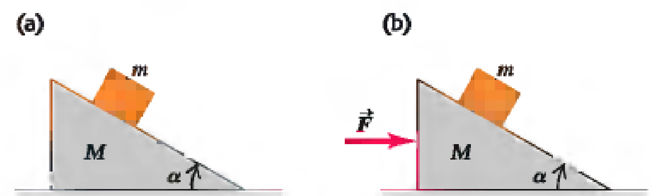
Figure 5.83 Problem 5.119.



Challenge Problems

5.120. Moving Wedge. A wedge with mass M rests on a frictionless, horizontal tabletop. A block with mass m is placed on the wedge (Fig. 5.84a). There is no friction between the block and the wedge. The system is released from rest. (a) Calculate the acceleration of the wedge and the horizontal and vertical components of the acceleration of the block. (b) Do your answers to part (a) reduce to the correct results when M is very large? (c) As seen by a stationary observer, what is the shape of the trajectory of the block?

Figure 5.84 Challenge Problems 5.120 and 5.121.



5.121. A wedge with mass M rests on a frictionless horizontal tabletop. A block with mass m is placed on the wedge and a horizontal force $\vec{F}$ is applied to the wedge (Fig. 5.84b). What must the magnitude of $\vec{F}$ be if the block is to remain at a constant height above the tabletop?

5.122. A box of weight w is accelerated up a ramp by a rope that exerts a tension T. The ramp makes an angle α with the horizontal, and the rope makes an angle θ above the ramp. The coefficient of kinetic friction between the box and the ramp is μ_k. Show that no matter what the value of α, the acceleration is maximum if $\theta = \arctan \mu_k$ (as long as the box remains in contact with the ramp).

5.123. Angle for Minimum Force. A box with weight w is pulled at constant speed along a level floor by a force $\vec{F}$ that is at an angle θ above the horizontal. The coefficient of kinetic friction between the floor and box is μ_k. (a) In terms of θ, μ_k, and w, calculate F. (b) For $w = 400$ N and $\mu_k = 0.25$, calculate F for θ ranging from 0° to 90° in increments of 10°. Graph F versus θ. (c) From the general expression in part (a), calculate the value of θ for which the value of F, required to maintain constant speed, is a minimum. (*Hint:* At a point where a function is minimum, what are the first and second derivatives of the function? Here F is a function of θ.) For the special case of $w = 400$ N and $\mu_k = 0.25$, evaluate this optimal θ and compare your result to the graph you constructed in part (b).

5.124. Falling Baseball. You drop a baseball from the roof of a tall building. As the ball falls, the air exerts a drag force proportional to the square of the ball's speed $(f = Dv^2)$. (a) In a diagram, show the direction of motion and indicate, with the aid of vectors, all the forces acting on the ball. (b) Apply Newton's second law and infer from the resulting equation the general properties of the motion. (c) Show that the ball acquires a terminal speed that is as given in Eq. (5.13). (d) Derive the equation for the speed at any time. (*Note:*

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a}\operatorname{arctanh}\left(\frac{x}{a}\right)$$

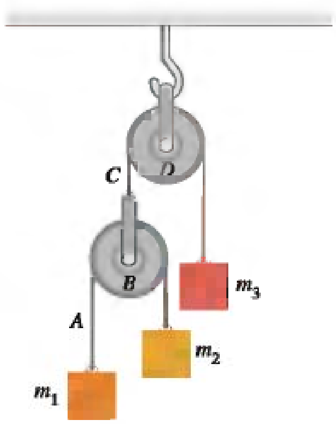
where

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

defines the hyperbolic tangent.)

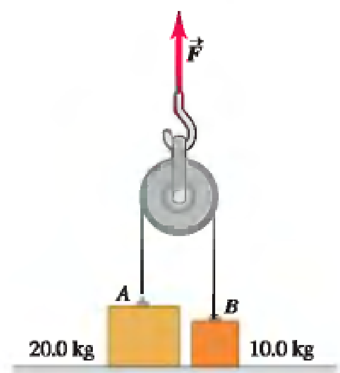
5.125. Double Atwood's Machine. In Fig. 5.85 masses m_1 and m_2 are connected by a light string A over a light, frictionless pulley B. The axle of pulley B is connected by a second light string C over a second light, frictionless pulley D to a mass m_3. Pulley D is suspended from the ceiling by an attachment to its axle. The system is released from rest. In terms of m_1, m_2, m_3, and g, what are (a) the acceleration of block m_3; (b) the acceleration of pulley B; (c) the acceleration of block m_1; (d) the acceleration of block m_2; (e) the tension in string A; (f) the tension in string C? (g) What do your expressions give for the special case of $m_1 = m_2$ and $m_3 = m_1 + m_2$? Is this sensible?

Figure 5.85 Challenge Problem 5.125.



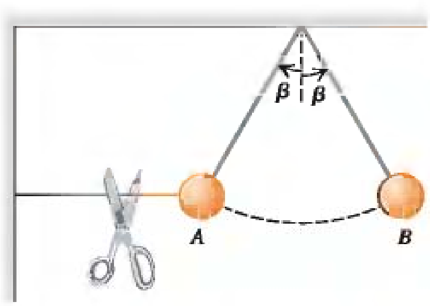
5.126. The masses of blocks A and B in Fig. 5.86 are 20.0 kg and 10.0 kg, respectively. The blocks are initially at rest on the floor and are connected by a massless string passing over a massless and frictionless pulley. An upward force $\vec{F}$ is applied to the pulley. Find the accelerations $\vec{a}_A$ of block A and $\vec{a}_B$ of block B when F is (a) 124 N; (b) 294 N; (c) 424 N.

Figure 5.86 Challenge Problem 5.126.



5.127. A ball is held at rest at position A in Fig. 5.87 by two light strings. The horizontal string is cut and the ball starts swinging as a pendulum. Point B is the farthest to the right the ball goes as it swings back and forth. What is the ratio of the tension in the supporting string in position B to its value at A before the horizontal string was cut?

Figure 5.87 Challenge Problem 5.127.



6 WORK AND KINETIC ENERGY

?When a shotgun fires, the expanding gases in the barrel push the shell out. According to Newton's third law, the shell exerts as much force on the gases as the gases exert on the shell. Would it be correct to say that the *shell* does work on the *gases?*

LEARNING GOALS

By studying this chapter, you will learn:

- What it means for a force to do work on a body, and how to calculate the amount of work done.
- The definition of the kinetic energy (energy of motion) of a body, and what it means physically.
- How the total work done on a body changes the body's kinetic energy, and how to use this principle to solve problems in mechanics.
- How to use the relationship between total work and change in kinetic energy when the forces are not constant, the body follows a curved path, or both.
- How to solve problems involving power (the rate of doing work).

Suppose you try to find the speed of an arrow that has been shot from a bow. You apply Newton's laws and all the problem-solving techniques that we've learned, but you run across a major stumbling block: After the archer releases the arrow, the bow string exerts a *varying* force that depends on the arrow's position. As a result, the simple methods that we've learned aren't enough to calculate the speed. Never fear; we aren't by any means finished with mechanics, and there are other methods for dealing with such problems.

The new method that we're about to introduce uses the ideas of *work* and *energy*. The importance of the energy idea stems from the *principle of conservation of energy*: Energy is a quantity that can be converted from one form to another but cannot be created or destroyed. In an automobile engine, chemical energy stored in the fuel is converted partially to the energy of the automobile's motion and partially to thermal energy. In a microwave oven, electromagnetic energy obtained from your power company is converted to thermal energy of the food being cooked. In these and all other processes, the *total* energy—the sum of all energy present in all different forms—remains the same. No exception has ever been found.

We'll use the energy idea throughout the rest of this book to study a tremendous range of physical phenomena. This idea will help you understand why a sweater keeps you warm, how a camera's flash unit can produce a short burst of light, and the meaning of Einstein's famous equation $E = mc^2$.

In this chapter, though, our concentration will be on mechanics. We'll learn about one important form of energy called *kinetic energy*, or energy of motion, and how it relates to the concept of *work*. We'll also consider *power*, which is the time rate of doing work. In Chapter 7 we'll expand the ideas of work and kinetic energy into a deeper understanding of the concepts of energy and the conservation of energy.

6.1 Work

You'd probably agree that it's hard work to pull a heavy sofa across the room, to lift a stack of encyclopedias from the floor to a high shelf, or to push a stalled car off the road. Indeed, all of these examples agree with the everyday meaning of *work*—any activity that requires muscular or mental effort.

In physics, work has a much more precise definition. By making use of this definition we'll find that in any motion, no matter how complicated, the total work done on a particle by all forces that act on it equals the change in its *kinetic energy*—a quantity that's related to the particle's speed. This relationship holds even when the forces acting on the particle aren't constant, a situation that can be difficult or impossible to handle with the techniques you learned in Chapters 4 and 5. The ideas of work and kinetic energy enable us to solve problems in mechanics that we could not have attempted before.

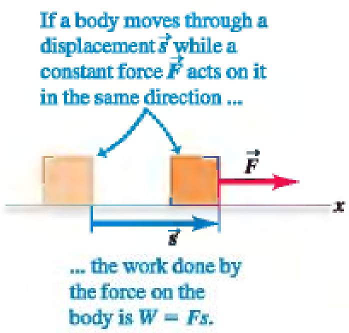
In this section we'll see how work is defined and how to calculate work in a variety of situations involving *constant* forces. Even though we already know how to solve problems in which the forces are constant, the idea of work is still useful in such problems. Later in this chapter we'll relate work and kinetic energy, and then apply these ideas to problems in which the forces are *not* constant.

6.1 These people are doing work as they push on the stalled car because they exert a force on the car as it moves.

The three examples of work described above—pulling a sofa, lifting encyclopedias, and pushing a car—have something in common. In each case you do work by exerting a *force* on a body while that body *moves* from one place to another—that is, undergoes a *displacement* (Fig. 6.1). You do more work if the force is greater (you push harder on the car) or if the displacement is greater (you push the car farther down the road).

The physicist's definition of work is based on these observations. Consider a body that undergoes a displacement of magnitude s along a straight line. (For now, we'll assume that any body we discuss can be treated as a particle so that we can ignore any rotation or changes in shape of the body.) While the body moves, a constant force $\vec{F}$ acts on it in the same direction as the displacement $\vec{s}$ (Fig. 6.2). We define the **work** W done by this constant force under these circumstances as the product of the force magnitude F and the displacement magnitude s:

6.2 The work done by a constant force acting in the same direction as the displacement.



$$W = Fs \qquad \text{(constant force in direction of straight-line displacement)} \tag{6.1}$$

The work done on the body is greater if either the force F or the displacement s is greater, in agreement with our observations above.

CAUTION **Work = W, weight = w** Don't confuse W (work) with w (weight). Though the symbols are similar, work and weight are different quantities.

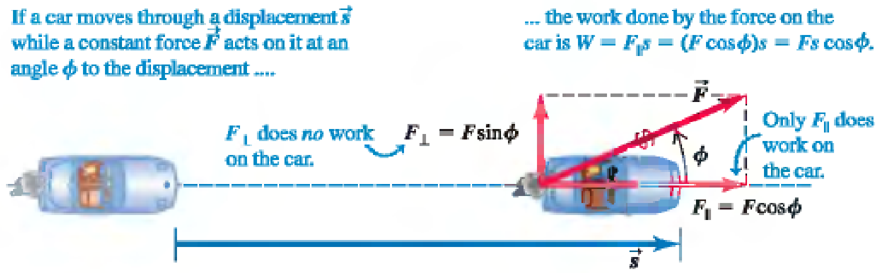
The SI unit of work is the **joule** (abbreviated J, pronounced "jewel," and named in honor of the 19th-century English physicist James Prescott Joule). From Eq. (6.1) we see that in any system of units, the unit of work is the unit of force multiplied by the unit of distance. In SI units the unit of force is the newton and the unit of distance is the meter, so 1 joule is equivalent to 1 *newton-meter* $(\text{N} \cdot \text{m})$:

$$1 \text{ joule} = (1 \text{ newton})(1 \text{ meter}) \quad \text{or} \quad 1 \text{ J} = 1 \text{ N} \cdot \text{m}$$

In the British system the unit of force is the pound (lb), the unit of distance is the foot (ft), and the unit of work is the *foot-pound* $(\text{ft} \cdot \text{lb})$. The following conversions are useful:

$$1 \text{ J} = 0.7376 \text{ ft} \cdot \text{lb} \qquad 1 \text{ ft} \cdot \text{lb} = 1.356 \text{ J}$$

As an illustration of Eq. (6.1), think of a person pushing a stalled car. If he pushes the car through a displacement $\vec{s}$ with a constant force $\vec{F}$ in the direction of motion, the amount of work he does on the car is given by Eq. (6.1): $W = Fs$. But what if the person pushes at an angle ϕ with the car's displacement (Fig. 6.3)? Then $\vec{F}$ has a component $F_{\parallel} = F\cos\phi$ in the direction of the displacement and a component $F_{\perp} = F\sin\phi$ that acts perpendicular to the displacement. (Other



6.3 The work done by a constant force acting at an angle to the displacement.

forces must act on the car so that it moves along $\vec{s}$, not in the direction of $\vec{F}$. We're interested only in the work that the person does, however, so we'll consider only the force he exerts.) In this case only the parallel component $F_{\parallel}$ is effective in moving the car, so we define the work as the product of this force component and the magnitude of the displacement. Hence $W = F_{\parallel}s = (F\cos\phi)s$, or

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5.1 Work Calculations

$$W = Fs\cos\phi \qquad \text{(constant force, straight-line displacement)} \qquad (6.2)$$

We are assuming that F and ϕ are constant during the displacement. If $\phi = 0$, so that $\vec{F}$ and $\vec{s}$ are in the same direction, then $\cos\phi = 1$ and we are back to Eq. (6.1).

Equation (6.2) has the form of the *scalar product* of two vectors, which we introduced in Section 1.10: $\vec{A}\cdot\vec{B} = AB\cos\phi$. You may want to review that definition. Hence we can write Eq. (6.2) more compactly as

$$W = \vec{F}\cdot\vec{s} \qquad \text{(constant force, straight-line displacement)} \qquad (6.3)$$

CAUTION **Work is a scalar** Here's an essential point: Work is a *scalar* quantity, even though it's calculated by using two vector quantities (force and displacement). A 5-N force toward the east acting on a body that moves 6 m to the east does exactly the same amount of work as a 5-N force toward the north acting on a body that moves 6 m to the north.

Example 6.1 Work done by a constant force

(a) Steve exerts a steady force of magnitude 210 N (about 47 lb) on the stalled car in Fig. 6.3 as he pushes it a distance of 18 m. The car also has a flat tire, so to make the car track straight Steve must push at an angle of 30° to the direction of motion. How much work does Steve do? (b) In a helpful mood, Steve pushes a second stalled car with a steady force $\vec{F} = (160\,\text{N})\hat{\imath} - (40\,\text{N})\hat{\jmath}$. The displacement of the car is $\vec{s} = (14\,\text{m})\hat{\imath} + (11\,\text{m})\hat{\jmath}$. How much work does Steve do in this case?

SOLUTION

IDENTIFY: In both parts (a) and (b), the target variable is the work W done by Steve. In each case the force is constant and the displacement is along a straight line, so we can use Eq. (6.2) or (6.3).

SET UP: The angle between $\vec{F}$ and $\vec{s}$ is given explicitly in part (a), so we can apply Eq. (6.2) directly. In part (b) the angle isn't given, so we're better off calculating the scalar product in Eq. (6.3) from the components of $\vec{F}$ and $\vec{s}$, as in Eq. (1.21): $\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$.

EXECUTE: (a) From Eq. (6.2),

$$W = Fs\cos\phi = (210\,\text{N})(18\,\text{m})\cos 30° = 3.3\times10^3\,\text{J}$$

(b) The components of $\vec{F}$ are $F_x = 160$ N and $F_y = -40$ N, and the components of $\vec{s}$ are $x = 14$ m and $y = 11$ m. (There are no z-components for either vector.) Hence, using Eqs. (1.21) and (6.3),

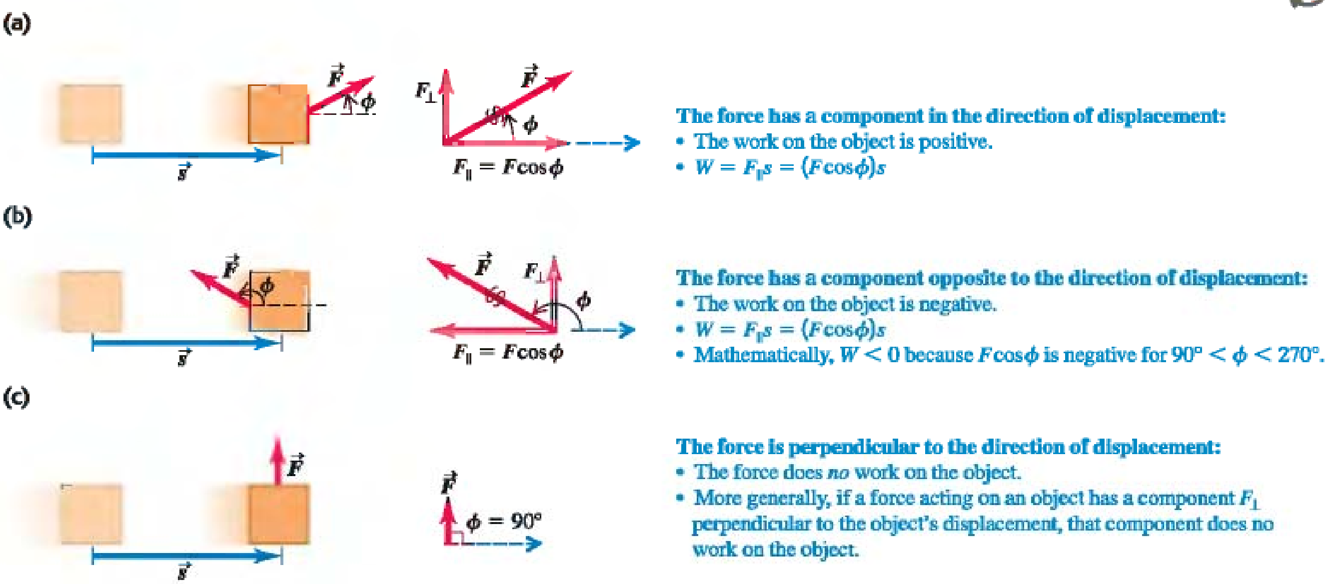
$$\begin{aligned} W &= \vec{F}\cdot\vec{s} = F_x x + F_y y \\ &= (160\,\text{N})(14\,\text{m}) + (-40\,\text{N})(11\,\text{m}) \\ &= 1.8\times10^3\,\text{J} \end{aligned}$$

EVALUATE: In each case the work that Steve does is more than 1000 J. This shows that 1 joule is a rather small amount of work.

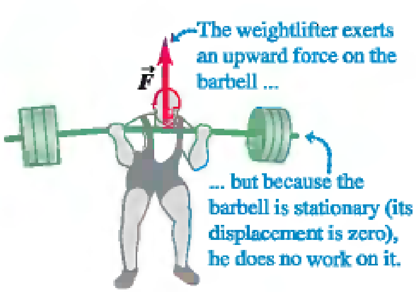
Work: Positive, Negative, or Zero

In Example 6.1 the work done in pushing the cars was positive. But it's important to understand that work can also be negative or zero. This is the essential way in which work as defined in physics differs from the "everyday" definition of work. When the force has a component in the *same direction* as the displacement

6.4 A constant force $\vec{F}$ can do positive, negative, or zero work depending on the angle between $\vec{F}$ and the displacement $\vec{s}$.



6.5 A weightlifter does no work on a barbell as long as he holds it stationary.



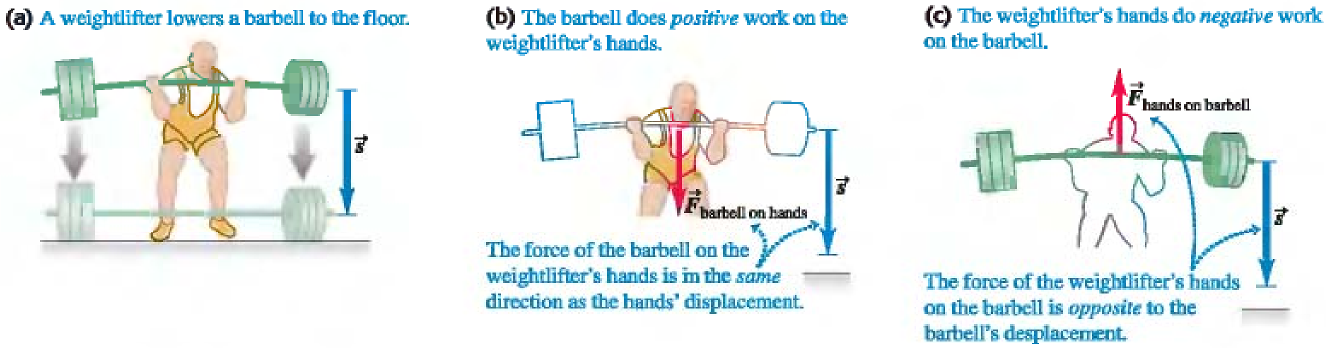
(ϕ between zero and 90°), $\cos\phi$ in Eq. (6.2) is positive and the work W is *positive* (Fig. 6.4a). When the force has a component *opposite* to the displacement (ϕ between 90° and 180°), $\cos\phi$ is negative and the work is *negative* (Fig. 6.4b). When the force is *perpendicular* to the displacement, $\phi = 90°$ and the work done by the force is *zero* (Fig. 6.4c). The cases of zero work and negative work bear closer examination, so let's look at some examples.

There are many situations in which forces act but do zero work. You might think it's "hard work" to hold a barbell motionless in the air for 5 minutes (Fig. 6.5). But in fact, you aren't doing any work at all on the barbell because there is no displacement. You get tired because the components of muscle fibers in your arm do work as they continually contract and relax. This is work done by one part of the arm exerting force on another part, however, *not* on the barbell. (We'll say more in Section 6.2 about work done by one part of a body on another part.) Even when you walk with constant velocity on a level floor while carrying a book, you still do no work on it. The book has a displacement, but the (vertical) supporting force that you exert on the book has no component in the direction of the (horizontal) motion. Then $\phi = 90°$ in Eq. (6.2), and $\cos\phi = 0$. When a body slides along a surface, the work done on the body by the normal force is zero; and when a ball on a string moves in uniform circular motion, the work done on the ball by the tension in the string is also zero. In both cases the work is zero because the force has no component in the direction of motion.

What does it really mean to do *negative* work? The answer comes from Newton's third law of motion. When a weightlifter lowers a barbell as in Fig. 6.6a, his hands and the barbell move together with the same displacement $\vec{s}$. The barbell exerts a force $\vec{F}_{\text{barbell on hands}}$ on his hands in the same direction as the hands' displacement, so the work done by the *barbell* on his *hands* is positive. (Fig. 6.6b). But by Newton's third law the weightlifter's hands exert an equal and opposite force $\vec{F}_{\text{hands on barbell}} = -\vec{F}_{\text{barbell on hands}}$ on the barbell (Fig. 6.6c). This force, which keeps the barbell from crashing to the floor, acts opposite to the barbell's displacement. Thus the work done by his *hands* on the *barbell* is negative. Because the weightlifter's hands and the barbell have the same displacement, the work that his hands do on the barbell is just the negative of the work that the barbell does on his hands. In general, when one body does negative work on a second body, the second body does an equal amount of *positive* work on the first body.

CAUTION **Keep track of who's doing the work** We always speak of work done *on* a particular body *by* a specific force. Always be sure to specify exactly what force is doing the

6.6 This weightlifter's hands do negative work on a barbell as the barbell does positive work on his hands.



work you are talking about. When you lift a book, you exert an upward force on the book and the book's displacement is upward, so the work done by the lifting force on the book is positive. But the work done by the *gravitational* force (weight) on a book being lifted is *negative* because the downward gravitational force is opposite to the upward displacement.

Total Work

How do we calculate work when *several* forces act on a body? One way is to use Eq. (6.2) or (6.3) to compute the work done by each separate force. Then, becanse work is a scalar quantity, the *total* work W_{tot} done on the body by all the forces is the algebraic sum of the quantities of work done by the individual forces. An alternative way to find the total work W_{tot} is to compute the vector sum of the forces (that is, the net force) and then use this vector sum as $\vec{F}$ in Eq. (6.2) or (6.3). The following example illustrates both of these techniques.

Example 6.2 Work done by several forces

A farmer hitches her tractor to a sled loaded with firewood and pulls it a distance of 20 m along level ground (Fig. 6.7a). The total weight of sled and load is 14,700 N. The tractor exerts a constant 5000-N force at an angle of 36.9° above the horizontal, as shown in Fig. 6.7b. There is a 3500-N friction force opposing the sled's motion. Find the work done by each force acting on the sled and the total work done by all the forces.

SOLUTION

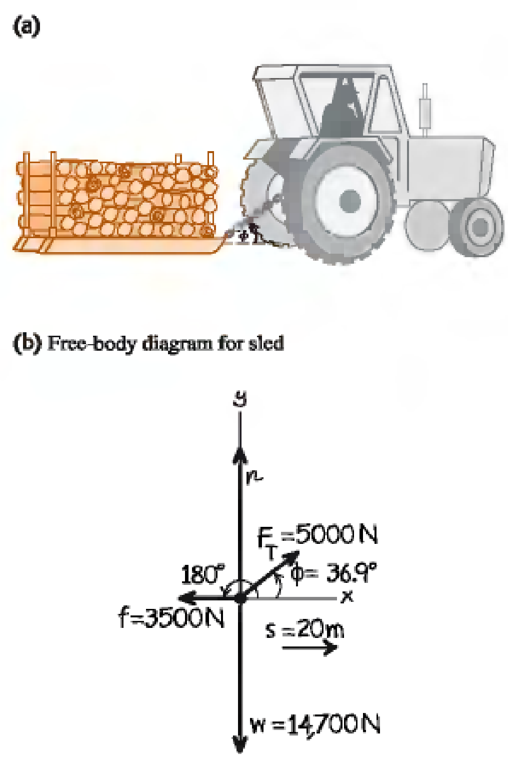
IDENTIFY: Each force is constant and the displacement is along a straight line, so we can calculate the work using the ideas of this section. We'll find the total work in two ways: (1) by adding together the work done on the sled by each force and (2) by finding the amount of work done by the net force on the sled.

SET UP: Since we're working with forces, we first draw a free-body diagram showing all of the forces acting on the sled and we choose a coordinate system (Fig. 6.7b). For each force—weight, normal force, force of the tractor, and friction force—we know the angle between the displacement (in the positive *x*-direction) and the force. Hence we can calculate the work each force does using Eq. (6.2).

As we did in Chapter 5, we'll find the net force by adding the components of the four forces. Newton's second law tells us that because the sled's motion is purely horizontal, the net force has only a horizontal component.

EXECUTE: The work W_w done by the weight is zero because its direction is perpendicular to the displacement (compare Fig. 6.4c). For the same reason, the work W_n done by the normal force is

6.7 Calculating the work done on a sled of firewood being pulled by a tractor.



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also zero. So $W_w = W_n = 0$. (Incidentally, can you see that the magnitude of the normal force is less than the weight? Compare Example 5.15 in Section 5.3, which has a very similar free-body diagram.)

That leaves the force F_T exerted by the tractor and the friction force f. From Eq. (6.2) the work W_T done by the tractor is

$$W_T = F_T s\cos\phi = (5000\,\text{N})(20\,\text{m})(0.800) = 80{,}000\,\text{N}\cdot\text{m}$$
$$= 80\,\text{kJ}$$

The friction force $\vec{f}$ is opposite to the displacement, so for this force $\phi = 180°$ and $\cos\phi = -1$. The work W_f done by the friction force is

$$W_f = fs\cos 180° = (3500\,\text{N})(20\,\text{m})(-1) = -70{,}000\,\text{N}\cdot\text{m}$$
$$= -70\,\text{kJ}$$

The total work W_{tot} done on the sled by all forces is the *algebraic* sum of the work done by the individual forces:

$$W_{tot} = W_w + W_n + W_T + W_f = 0 + 0 + 80\,\text{kJ} + (-70\,\text{kJ})$$
$$= 10\,\text{kJ}$$

In the alternative approach, we first find the *vector* sum of all the forces (the net force) and then use it to compute the total work. The vector sum is best found by using components. From Fig. 6.7b,

$$\sum F_x = F_T\cos\phi + (-f) = (5000\,\text{N})\cos 36.9° - 3500\,\text{N}$$
$$= 500\,\text{N}$$
$$\sum F_y = F_T\sin\phi + n + (-w)$$
$$= (5000\,\text{N})\sin 36.9° + n - 14{,}700\,\text{N}$$

We don't really need the second equation; we know that the y-component of force is perpendicular to the displacement, so it does no work. Besides, there is no y-component of acceleration, so $\sum F_y$ has to be zero anyway. The total work is therefore the work done by the total x-component:

$$W_{tot} = \left(\sum \vec{F}\right)\cdot\vec{s} = \left(\sum F_x\right)s = (500\,\text{N})(20\,\text{m}) = 10{,}000\,\text{J}$$
$$= 10\,\text{kJ}$$

EVALUATE: We get the same result for W_{tot} with either method, as we should.

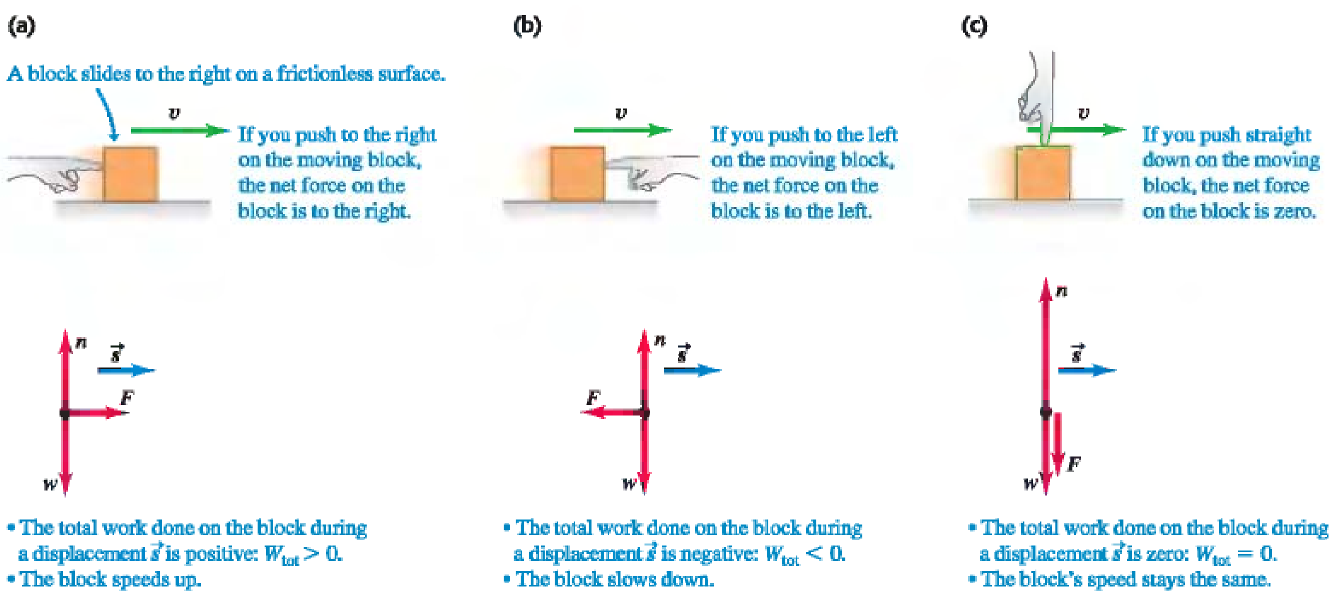
Note that the net force in the x-direction is *not* zero, and so the sled must accelerate as it moves. In Section 6.2 we'll return to this example and see how to use the concept of work to explore the sled's motion.

Test Your Understanding of Section 6.1 An electron moves in a straight line toward the east with a constant speed of 8×10^7 m/s. It has electric, magnetic, and gravitational forces acting on it. During a 1-m displacement, the total work done on the electron is (i) positive; (ii) negative; (iii) zero; (iv) not enough information given to decide.

6.2 Kinetic Energy and the Work–Energy Theorem

The total work done on a body by external forces is related to the body's displacement—that is, to changes in its position. But the total work is also related to changes in the *speed* of the body. To see this, consider Fig. 6.8, which shows

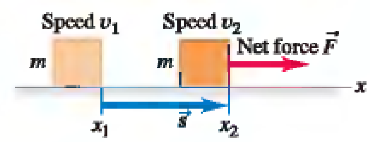
6.8 The relationship between the total work done on a body and how the body's speed changes.



three examples of a block sliding on a frictionless table. The forces acting on the block are its weight $\vec{w}$, the normal force $\vec{n}$, and the force $\vec{F}$ exerted on it by the hand.

In Fig. 6.8a the net force on the block is in the direction of its motion. From Newton's second law, this means that the block speeds up; from Eq. (6.1), this also means that the total work W_{tot} done on the block is positive. The total work is *negative* in Fig. 6.8b because the net force opposes the displacement; in this case the block slows down. The net force is zero in Fig. 6.8c, so the speed of the block stays the same and the total work done on the block is zero. We can conclude that *when a particle undergoes a displacement, it speeds up if* $W_{tot} > 0$, *slows down if* $W_{tot} < 0$, *and maintains the same speed if* $W_{tot} = 0$.

Let's make these observations more quantitative. Consider a particle with mass m moving along the x-axis under the action of a constant net force with magnitude F directed along the positive x-axis (Fig. 6.9). The particle's acceleration is constant and given by Newton's second law, $F = ma_x$. Suppose the speed changes from v_1 to v_2 while the particle undergoes a displacement $s = x_2 - x_1$ from point x_1 to x_2. Using a constant-acceleration equation, Eq. (2.13), and replacing v_{0x} by v_1, v_x by v_2, and $(x - x_0)$ by s, we have



6.9 A constant net force $\vec{F}$ does work on a moving body.

$$v_2^2 = v_1^2 + 2a_x s$$
$$a_x = \frac{v_2^2 - v_1^2}{2s}$$

When we multiply this equation by m and equate ma_x to the net force F, we find

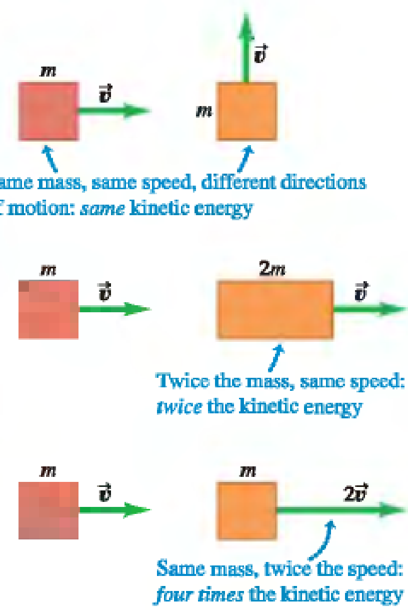
$$F = ma_x = m\frac{v_2^2 - v_1^2}{2s} \quad \text{and}$$
$$Fs = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \tag{6.4}$$

The product Fs is the work done by the net force F and thus is equal to the total work W_{tot} done by all the forces acting on the particle. The quantity $\frac{1}{2}mv^2$ is called the **kinetic energy** K of the particle:

$$K = \frac{1}{2}mv^2 \qquad \text{(definition of kinetic energy)} \tag{6.5}$$

Like work, the kinetic energy of a particle is a scalar quantity; it depends on only the particle's mass and speed, not its direction of motion (Fig. 6.10). A car (viewed as a particle) has the same kinetic energy when going north at 10 m/s as when going east at 10 m/s. Kinetic energy can never be negative, and it is zero only when the particle is at rest.

6.10 Comparing the kinetic energy $K = \frac{1}{2}mv^2$ of different bodies.



We can now interpret Eq. (6.4) in terms of work and kinetic energy. The first term on the right side of Eq. (6.4) is $K_2 = \frac{1}{2}mv_2^2$, the final kinetic energy of the particle (that is, after the displacement). The second term is the initial kinetic energy, $K_1 = \frac{1}{2}mv_1^2$, and the difference between these terms is the *change* in kinetic energy. So Eq. (6.4) says:

The work done by the net force on a particle equals the change in the particle's kinetic energy:

$$W_{tot} = K_2 - K_1 = \Delta K \qquad \text{(work–energy theorem)} \tag{6.6}$$

This result is the **work–energy theorem.**

The work–energy theorem agrees with our observations about the block in Fig. 6.8. When W_{tot} is *positive*, the kinetic energy *increases* (the final kinetic energy K_2 is greater than the initial kinetic energy K_1) and the particle is going faster at the end of the displacement than at the beginning. When W_{tot} is *negative*, the kinetic energy *decreases* $(K_2$ is less than $K_1)$ and the speed is less after the displacement. When $W_{tot} = 0$, the kinetic energy stays the same $(K_1 = K_2)$ and the speed is unchanged. Note that the work–energy theorem by itself tells us only about changes in *speed*, not velocity, since the kinetic energy doesn't depend on the direction of motion.

From Eq. (6.4) or (6.6), kinetic energy and work must have the same units. Hence the joule is the SI unit of both work and kinetic energy (and, as we will see later, of all kinds of energy). To verify this, note that in SI units the quantity $K = \frac{1}{2}mv^2$ has units $\text{kg} \cdot (\text{m/s})^2$ or $\text{kg} \cdot \text{m}^2/\text{s}^2$; we recall that $1\ \text{N} = 1\ \text{kg} \cdot \text{m/s}^2$, so

$$1\ \text{J} = 1\ \text{N} \cdot \text{m} = 1\ (\text{kg} \cdot \text{m/s}^2) \cdot \text{m} = 1\ \text{kg} \cdot \text{m}^2/\text{s}^2$$

In the British system the unit of kinetic energy and of work is

$$1\ \text{ft} \cdot \text{lb} = 1\ \text{ft} \cdot \text{slug} \cdot \text{ft/s}^2 = 1\ \text{slug} \cdot \text{ft}^2/\text{s}^2$$

Because we used Newton's laws in deriving the work–energy theorem, we can use this theorem only in an inertial frame of reference. Note also that the work–energy theorem is valid in *any* inertial frame, but the values of W_{tot} and $K_2 - K_1$ may differ from one inertial frame to another (because the displacement and speed of a body may be different in different frames).

We have derived the work–energy theorem for the special case of straight-line motion with constant forces, and in the following examples we'll apply it to this special case only. We'll find in the next section that the theorem is valid in general, even when the forces are not constant and the particle's trajectory is curved.

Problem-Solving Strategy 6.1 Work and Kinetic Energy

MP

IDENTIFY *the relevant concepts:* The work–energy theorem, $W_{tot} = K_2 - K_1$, is extremely useful when you want to relate a body's speed v_1 at one point in its motion to its speed v_2 at a different point. (It's less useful for problems that involve the *time* it takes a body to go from point 1 to point 2, because the work–energy theorem doesn't involve time at all. For such problems it's usually best to use the relationships among time, position, velocity, and acceleration described in Chapters 2 and 3.)

SET UP *the problem* using the following steps:

1. Choose the initial and final positions of the body, and draw a free-body diagram showing all the forces that act on the body.
2. Choose a coordinate system. (If the motion is along a straight line, it's usually easiest to have both the initial and final positions lie along the x-axis.)
3. List the unknown and known quantities, and decide which unknowns are your target variables. The target variable may be the body's initial or final speed, the magnitude of one of the forces acting on the body, or the body's displacement.

EXECUTE *the solution:* Calculate the work W done by each force. If the force is constant and the displacement is a straight line, you can use Eq. (6.2) or (6.3). (Later in this chapter we'll see how to handle varying forces and curved trajectories.) Be sure to check signs; W must be positive if the force has a component in the direction of the displacement, negative if the force has a component opposite to the displacement, and zero if the force and displacement are perpendicular.

Add the amounts of work done by each force to find the total work W_{tot}. Sometimes it's easier to calculate the vector sum of the forces (the net force) and then find the work done by the net force; this value is also equal to W_{tot}.

Write expressions for the initial and final kinetic energies, K_1 and K_2. Note that kinetic energy involves *mass*, not *weight;* if you are given the body's weight, you'll need to use the relationship $w = mg$ to find the mass.

Finally, use $W_{tot} = K_2 - K_1$ to solve for the target variable. Remember that the right-hand side of this equation is the *final* kinetic energy minus the *initial* kinetic energy, never the other way around.

EVALUATE *your answer:* Check whether your answer makes physical sense. A key point to remember is that kinetic energy $K = \frac{1}{2}mv^2$ can never be negative. If you come up with a negative value of K, perhaps you interchanged the initial and final kinetic energies in $W_{tot} = K_2 - K_1$ or made a sign error in one of the work calculations.

Example 6.3 Using work and energy to calculate speed

Let's look again at the sled in Fig. 6.7 and the numbers at the end of Example 6.2. Suppose the initial speed v_1 is 2.0 m/s. What is the speed of the sled after it moves 20 m?

SOLUTION

IDENTIFY: We'll use the work–energy theorem, Eq. (6.6) ($W_{tot} = K_2 - K_1$), since we are given the initial speed $v_1 = 2.0\text{ m/s}$ and want to find the final speed.

SET UP: Figure 6.11 shows our sketch of the situation. The motion is in the positive x-direction.

EXECUTE: In Example 6.2 we calculated the total work done by all the forces: $W_{tot} = 10\text{ kJ}$. Hence the kinetic energy of the sled and its load must increase by 10 kJ.

To write expressions for the initial and final kinetic energies, we need the mass of the sled and load. We are given that the *weight* is 14,700 N, so the mass is

$$m = \frac{w}{g} = \frac{14{,}700\text{ N}}{9.8\text{ m/s}^2} = 1500\text{ kg}$$

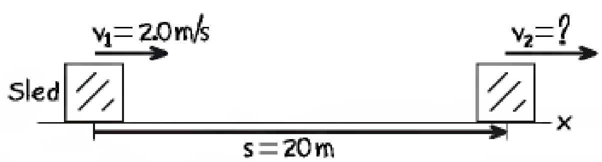
Then the initial kinetic energy K_1 is

$$K_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}(1500\text{ kg})(2.0\text{ m/s})^2 = 3000\text{ kg}\cdot\text{m}^2/\text{s}^2$$
$$= 3000\text{ J}$$

The final kinetic energy K_2 is

$$K_2 = \frac{1}{2}mv_2^2 = \frac{1}{2}(1500\text{ kg})v_2^2$$

6.11 Our sketch for this problem.



where v_2 is the unknown speed we want to find. Equation (6.6) gives

$$K_2 = K_1 + W_{tot} = 3000\text{ J} + 10{,}000\text{ J} = 13{,}000\text{ J}$$

Setting these two expressions for K_2 equal, substituting $1\text{ J} = 1\text{ kg}\cdot\text{m}^2/\text{s}^2$, and solving for v_2, we find

$$v_2 = 4.2\text{ m/s}$$

EVALUATE: The total work is positive, so the kinetic energy increases ($K_2 > K_1$) and the speed increases ($v_2 > v_1$).

This problem can also be done without the work–energy theorem. We can find the acceleration from $\sum\vec{F} = m\vec{a}$ and then use the equations of motion for constant acceleration to find v_2. Since the acceleration is along the x-axis,

$$a = a_x = \frac{\sum F_x}{m} = \frac{(5000\text{ N})\cos 36.9^\circ - 3500\text{ N}}{1500\text{ kg}}$$
$$= 0.333\text{ m/s}^2$$

Then, using Eq. (2.13),

$$v_2^2 = v_1^2 + 2as = (2.0\text{ m/s})^2 + 2(0.333\text{ m/s}^2)(20\text{ m})$$
$$= 17.3\text{ m}^2/\text{s}^2$$
$$v_2 = 4.2\text{ m/s}$$

This is the same result we obtained with the work–energy approach, but there we avoided the intermediate step of finding the acceleration. You will find several other examples in this chapter and the next that *can* be done without using energy considerations but that are easier when energy methods are used. When a problem can be done by two different methods, doing it by both methods (as we did in this example) is a very good way to check your work.

Example 6.4 Forces on a hammerhead

In a pile driver, a steel hammerhead with mass 200 kg is lifted 3.00 m above the top of a vertical I-beam being driven into the ground (Fig. 6.12a). The hammer is then dropped, driving the I-beam 7.4 cm farther into the ground. The vertical rails that guide the hammerhead exert a constant 60-N friction force on the hammerhead. Use the work–energy theorem to find (a) the speed of the hammerhead just as it hits the I-beam and (b) the average force the hammerhead exerts on the I-beam. Ignore the effects of the air.

SOLUTION

IDENTIFY: We'll use the work–energy theorem to relate the hammerhead's speed at different locations and the forces acting on it. There are *three* locations of interest: point 1, where the hammerhead starts from rest; point 2, where it first contacts the I-beam; and point 3, where the hammerhead comes to a halt (see Fig. 6.12a). The two unknowns are the hammerhead's speed at point 2 and the force the hammerhead exerts between points 2 and 3. Hence we'll apply the work–energy theorem twice: once for the motion from 1 to 2, and once for the motion from 2 to 3.

SET UP: Figure 6.12b shows the vertical forces on the hammerhead as it falls from point 1 to point 2. (We can ignore any horizontal forces that may be present because they do no work as the hammerhead moves vertically.) For this part of the motion, our target variable is the hammerhead's speed v_2.

Figure 6.12c shows the vertical forces on the hammerhead during the motion from point 2 to point 3. In addition to the forces shown in Fig. 6.12b, the I-beam exerts an upward normal force of magnitude n on the hammerhead. This force actually varies as the hammerhead comes to a halt, but for simplicity we'll treat n as a

Continued

constant. Hence n represents the *average* value of this upward force during the motion. Our target variable for this part of the motion is the force that the *hammerhead* exerts on the I-beam; it is the reaction force to the normal force exerted by the I-beam, so by Newton's third law its magnitude is also n.

EXECUTE: (a) From point 1 to point 2, the vertical forces are the downward weight $w = mg = (200\text{ kg})(9.8\text{ m/s}^2) = 1960\text{ N}$ and the upward friction force $f = 60\text{ N}$. Thus the net downward force is $w - f = 1900\text{ N}$. The displacement of the hammerhead from point 1 to point 2 is downward and equal to $s_{12} = 3.00\text{ m}$. The total work done on the hammerhead as it moves from point 1 to point 2 is then

$$W_{\text{tot}} = (w - f)s_{12} = (1900\text{ N})(3.00\text{ m}) = 5700\text{ J}$$

At point 1 the hammerhead is at rest, so its initial kinetic energy K_1 is zero. Hence the kinetic energy K_2 at point 2 equals the total work done on the hammerhead between points 1 and 2:

$$W_{\text{tot}} = K_2 - K_1 = K_2 - 0 = \frac{1}{2}mv_2^2 - 0$$

$$v_2 = \sqrt{\frac{2W_{\text{tot}}}{m}} = \sqrt{\frac{2(5700\text{ J})}{200\text{ kg}}} = 7.55\text{ m/s}$$

This is the hammerhead's speed at point 2, just as it hits the I-beam.

(b) As the hammerhead moves downward between points 2 and 3, the net downward force acting on it is $w - f - n$ (see Fig. 6.12c). The total work done on the hammerhead during this displacement is

$$W_{\text{tot}} = (w - f - n)s_{23}$$

The initial kinetic energy for this part of the motion is K_2, which from part (a) equals 5700 J. The final kinetic energy is $K_3 = 0$, since the hammerhead ends at rest. Then, from the work–energy theorem,

$$W_{\text{tot}} = (w - f - n)s_{23} = K_3 - K_2$$

$$n = w - f - \frac{K_3 - K_2}{s_{23}}$$

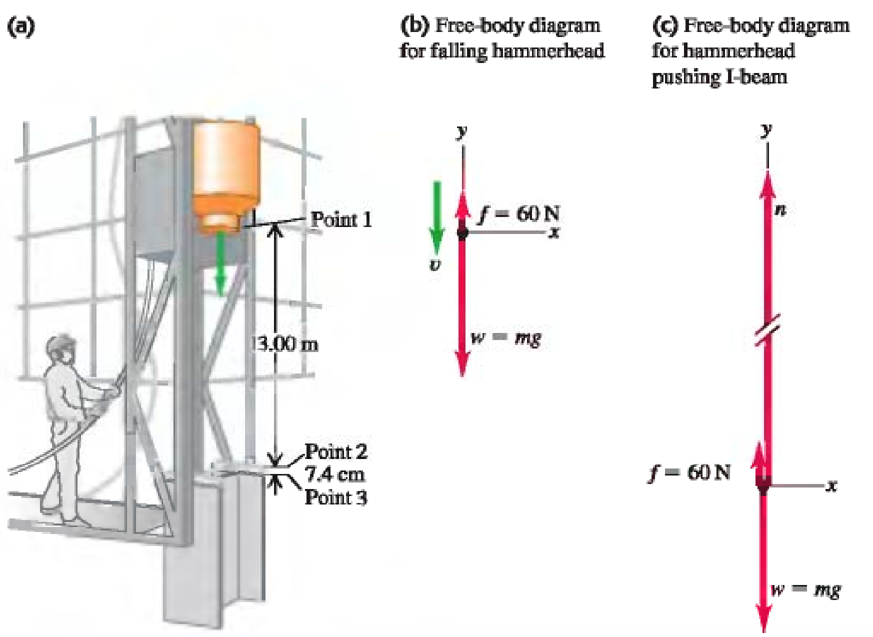
$$= 1960\text{ N} - 60\text{ N} - \frac{0\text{ J} - 5700\text{ J}}{0.074\text{ m}}$$

$$= 79{,}000\text{ N}$$

The downward force that the hammerhead exerts on the I-beam has this same magnitude, 79,000 N (about 9 tons)—more than 40 times the weight of the hammerhead.

EVALUATE: The net change in the hammerhead's kinetic energy from point 1 to point 3 is zero; a relatively small net force does positive work over a large distance, and then a much larger net force does negative work over a much smaller distance. The same thing happens if you speed up your car gradually and then drive it into a brick wall. The very large force needed to reduce the kinetic energy to zero over a short distance is what does the damage to your car—and possibly to you.

6.12 (a) A pile driver pounds an I-beam into the ground. (b), (c) Free-body diagrams. Vector lengths are not to scale.



The Meaning of Kinetic Energy

Example 6.4 gives insight into the physical meaning of kinetic energy. The hammerhead is dropped from rest, and its kinetic energy when it hits the I-beam equals the total work done on it up to that point by the net force. This result is true in general: To accelerate a particle of mass m from rest (zero kinetic energy) up to a speed v, the total work done on it must equal the change in kinetic energy from zero to $K = \frac{1}{2}mv^2$:

$$W_{\text{tot}} = K - 0 = K$$

So *the kinetic energy of a particle is equal to the total work that was done to accelerate it from rest to its present speed* (Fig. 6.13). The definition $K = \frac{1}{2}mv^2$, Eq. (6.5), wasn't chosen at random; it's the *only* definition that agrees with this interpretation of kinetic energy.

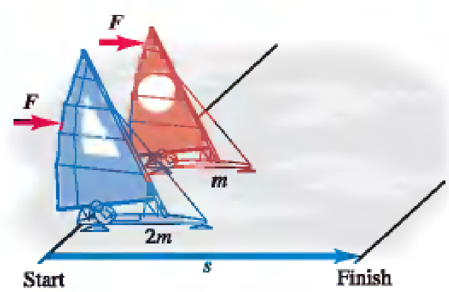
In the second part of Example 6.4 the kinetic energy of the hammerhead did work on the I-beam and drove it into the ground. This gives us another interpretation of kinetic energy: *The kinetic energy of a particle is equal to the total work that particle can do in the process of being brought to rest.* This is why you pull your hand and arm backward when you catch a ball. As the ball comes to rest, it does an amount of work (force times distance) on your hand equal to the ball's initial kinetic energy. By pulling your hand back, you maximize the distance over which the force acts and so minimize the force on your hand.

6.13 When a billiards player hits a cue ball at rest, the ball's kinetic energy after being hit is equal to the work that was done on it by the cue. The greater the force exerted by the cue and the greater the distance the ball moves while in contact with it, the greater the ball's kinetic energy.

Conceptual Example 6.5 Comparing kinetic energies

Two iceboats like the one in Example 5.6 (Section 5.2) hold a race on a frictionless horizontal lake (Fig. 6.14). The two iceboats have masses m and $2m$. Each iceboat has an identical sail, so the wind exerts the same constant force $\vec{F}$ on each iceboat. The two iceboats start from rest and cross the finish line a distance s away. Which iceboat crosses the finish line with greater kinetic energy?

6.14 A race between iceboats.



SOLUTION

If you use the mathematical definition of kinetic energy, $K = \frac{1}{2}mv^2$, Eq. (6.5), the answer to this problem isn't immediately obvious. The iceboat of mass $2m$ has greater mass, so you might guess that the larger iceboat attains a greater kinetic energy at the finish line. But the smaller iceboat, of mass m, crosses the finish line with a greater speed, and you might guess that *this* iceboat has the greater kinetic energy. How can we decide?

The correct way to approach this problem is to remember that *the kinetic energy of a particle is equal to the total work done to accelerate it from rest.* Both iceboats travel the same distance s, and only the horizontal force F in the direction of motion does work on either iceboat. Hence the total work done between the starting line and the finish line is the *same* for each iceboat, $W_{\text{tot}} = Fs$. At the finish line, each iceboat has a kinetic energy equal to the work W_{tot} done on it, because each iceboat started from rest. So both iceboats have the *same* kinetic energy at the finish line!

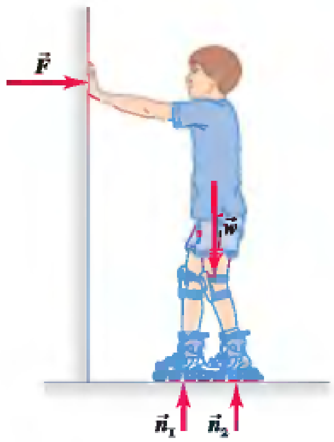
You might think this is a "trick" question, but it isn't. If you really understand the physical meanings of quantities such as kinetic energy, you can solve problems more easily and with better insight into the physics.

Notice that we didn't need to say anything about how much time each iceboat took to reach the finish line. This is because the work–energy theorem makes no direct reference to time, only to displacement. In fact, the iceboat of mass m takes less time to reach the finish line than does the larger iceboat of mass $2m$ because it has a greater acceleration.

Work and Kinetic Energy in Composite Systems

In this section we've been careful to apply the work–energy theorem only to bodies that we can represent as *particles*—that is, as moving point masses. New subtleties appear for more complex systems that have to be represented as many particles with different motions. We can't go into these subtleties in detail in this chapter, but here's an example.

6.15 The external forces acting on a skater pushing off a wall. The work done by these forces is zero, but the skater's kinetic energy changes nonetheless.

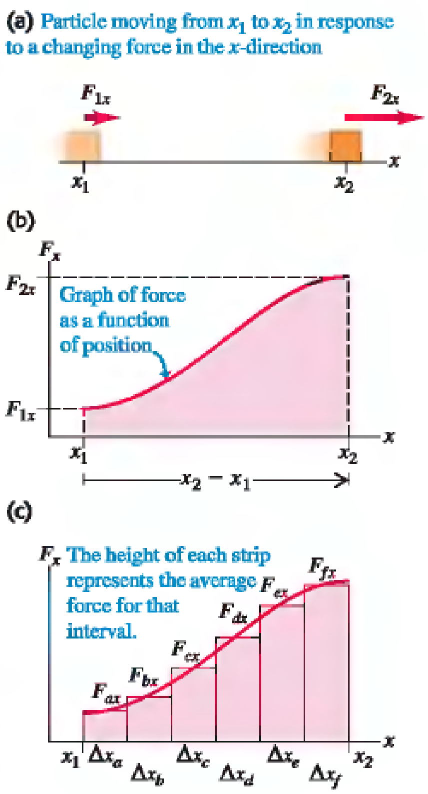


Suppose a boy stands on frictionless roller skates on a level surface, facing a rigid wall (Fig. 6.15). He pushes against the wall, which makes him move to the right. The forces acting on him are his weight $\vec{w}$, the upward normal forces $\vec{n}_1$ and $\vec{n}_2$ exerted by the ground on his skates, and the horizontal force $\vec{F}$ exerted on him by the wall. There is no vertical displacement, so $\vec{w}, \vec{n}_1$, and $\vec{n}_2$ do no work. Force $\vec{F}$ accelerates him to the right, but the parts of his body where that force is applied (the man's hands) do not move while the force acts. Thus the force $\vec{F}$ also does no work. Where, then, does the boy's kinetic energy come from?

The explanation is that it's not adequate to represent the boy as a single point mass. Different parts of the boy's body have different motions; his hands remain stationary against the wall while his torso is moving away from the wall. The various parts of his body interact with each other, and one part can exert forces and do work on another part. Therefore the *total* kinetic energy of this *composite* system of body parts can change, even though no work is done by forces applied by bodies (such as the wall) that are outside the system. In Chapter 8 we'll consider further the motion of a collection of particles that interact with each other. We'll discover that just as for the boy in this example, the total kinetic energy of such a system can change even when no work is done on any part of the system by anything outside it.

Test Your Understanding of Section 6.2 Rank the following bodies in order of their kinetic energy, from least to greatest. (i) a 2.0-kg body moving at 5.0 m/s; (ii) a 1.0 kg body that initially was at rest and then had 30 J of work done on it; (iii) a 1.0-kg body that initially was moving at 4.0 m/s and then had 20 J of work done on it; (iv) a 2.0 kg body that initially was moving at 10 m/s and then did 80 J of work on another body.

6.16 Calculating the work done by a varying force F_x in the x-direction as a particle moves from x_1 to x_2.



6.3 Work and Energy with Varying Forces

So far in this chapter we've considered work done by *constant forces* only. But what happens when you stretch a spring? The more you stretch it, the harder you have to pull, so the force you exert is *not* constant as the spring is stretched. We've also restricted our discussion to *straight-line* motion. There are many situations in which a body moves along a curved path and is acted on by a force that varies in magnitude, direction, or both. We need to be able to compute the work done by the force in these more general cases. Fortunately, we'll find that the work–energy theorem holds true even when varying forces are considered and when the body's path is not straight.

Work Done by a Varying Force, Straight-Line Motion

To add only one complication at a time, let's consider straight-line motion along the x-axis with a force whose x-component F_x may change as the body moves. (A real-life example is driving a car along a straight road with stop signs, so the driver has to alternately step on the gas and apply the brakes.) Suppose a particle moves along the x-axis from point x_1 to x_2 (Fig. 6.16a). Figure 6.16b is a graph of the x-component of force as a function of the particle's coordinate x. To find the work done by this force, we divide the total displacement into small segments Δx_a, Δx_b, and so on (Fig. 6.16c). We approximate the work done by the force during segment Δx_a as the average x-component of force F_{ax} in that segment multiplied by the x-displacement Δx_a. We do this for each segment and then add the results for all the segments. The work done by the force in the total displacement from x_1 to x_2 is approximately

$$W = F_{ax}\Delta x_a + F_{bx}\Delta x_b + \cdots$$

In the limit that the number of segments becomes very large and the width of each becomes very small, this sum becomes the *integral* of F_x from x_1 to x_2:

$$W = \int_{x_1}^{x_2} F_x\, dx \qquad \text{(varying } x\text{-component of force, straight-line displacement)} \tag{6.7}$$

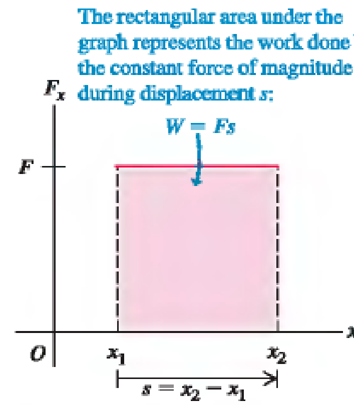
Note that $F_{ax}\Delta x_a$ represents the *area* of the first vertical strip in Fig. 6.16c and that the integral in Eq. (6.7) represents the area under the curve of Fig. 6.16b between x_1 and x_2. *On a graph of force as a function of position, the total work done by the force is represented by the area under the curve between the initial and final positions.* An alternative interpretation of Eq. (6.7) is that the work W equals the average force that acts over the entire displacement, multiplied by the displacement.

In the special case that F_x, the x-component of the force, is constant, it may be taken outside the integral in Eq. (6.7):

$$W = \int_{x_1}^{x_2} F_x\, dx = F_x \int_{x_1}^{x_2} dx = F_x(x_2 - x_1) \qquad \text{(constant force)}$$

But $x_2 - x_1 = s$, the total displacement of the particle. So in the case of a constant force F, Eq. (6.7) says that $W = Fs$, in agreement with Eq. (6.1). The interpretation of work as the area under the curve of F_x as a function of x also holds for a constant force; $W = Fs$ is the area of a rectangle of height F and width s (Fig. 6.17).

6.17 The work done by a constant force F in the x-direction as a particle moves from x_1 to x_2.

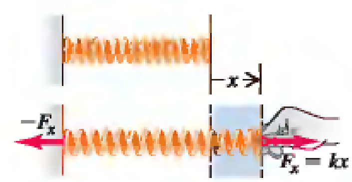


Now let's apply these ideas to the stretched spring. To keep a spring stretched beyond its unstretched length by an amount x, we have to apply a force of equal magnitude at each end (Fig. 6.18). If the elongation x is not too great, the force we apply to the right-hand end has an x-component directly proportional to x:

$$F_x = kx \qquad \text{(force required to stretch a spring)} \tag{6.8}$$

where k is a constant called the **force constant** (or spring constant) of the spring. The units of k are force divided by distance: N/m in SI units and lb/ft in British units. A floppy toy spring such as a Slinky™ has a force constant of about 1 N/m; for the much stiffer springs in an automobile's suspension, k is about 10^5 N/m. The observation that force is directly proportional to elongation for elongations that are not too great was made by Robert Hooke in 1678 and is known as **Hooke's law.** It really shouldn't be called a "law," since it's a statement about a specific device and not a fundamental law of nature. Real springs don't always obey Eq. (6.8) precisely, but it's still a useful idealized model. We'll discuss Hooke's law more fully in Chapter 11.

6.18 The force needed to stretch an ideal spring is proportional to the spring's elongation: $F_x = kx$.



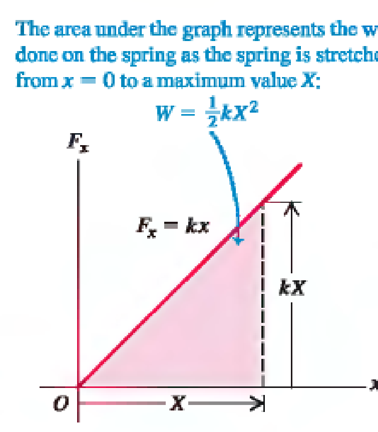
To stretch a spring, we must do work. We apply equal and opposite forces to the ends of the spring and gradually increase the forces. We hold the left end stationary, so the force we apply at this end does no work. The force at the moving end *does* do work. Figure 6.19 is a graph of F_x as a function of x, the elongation of the spring. The work done by this force when the elongation goes from zero to a maximum value X is

$$W = \int_0^X F_x\, dx = \int_0^X kx\, dx = \frac{1}{2}kX^2 \tag{6.9}$$

We can also obtain this result graphically. The area of the shaded triangle in Fig. 6.19, representing the total work done by the force, is equal to half the product of the base and altitude, or

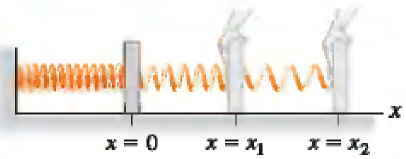
$$W = \frac{1}{2}(X)(kX) = \frac{1}{2}kX^2$$

6.19 Calculating the work done to stretch a spring by a length X.

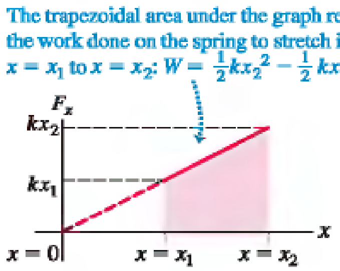


6.20 Calculating the work done to stretch a spring from one extension to a greater one.

(a) Stretching a spring from elongation x_1 to elongation x_2



(b) Force-versus-distance graph



This equation also says that the work is the *average* force $kX/2$ multiplied by the total displacement X. We see that the total work is proportional to the *square* of the final elongation X. To stretch an ideal spring by 2 cm, you must do four times as much work as is needed to stretch it by 1 cm.

Equation (6.9) assumes that the spring was originally unstretched. If initially the spring is already stretched a distance x_1, the work we must do to stretch it to a greater elongation x_2 (Fig. 6.20a) is

$$W = \int_{x_1}^{x_2} F_x\,dx = \int_{x_1}^{x_2} kx\,dx = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2 \qquad (6.10)$$

You should use your knowledge of geometry to convince yourself that the trapezoidal area under the graph in Fig. 6.20b is given by the expression in Eq. (6.10).

If the spring has spaces between the coils when it is unstretched, then it can also be compressed, and Hooke's law holds for compression as well as stretching. In this case the force and displacement are in the opposite directions from those shown in Fig. 6.18, and so F_x and x in Eq. (6.8) are both negative. Since both F_x and x are reversed, the force again is in the same direction as the displacement, and the work done by F_x is again positive. So the total work is still given by Eq. (6.9) or (6.10), even when X is negative or either or both of x_1 and x_2 are negative.

CAUTION **Work done *on* a spring vs. work done *by* a spring** Note that Eq. (6.10) gives the work that *you* must do *on* a spring to change its length. For example, if you stretch a spring that's originally relaxed, then $x_1 = 0$, $x_2 > 0$, and $W > 0$: The force you apply to one end of the spring is in the same direction as the displacement, and the work you do is positive. By contrast, the work that the *spring* does on whatever it's attached to is given by the *negative* of Eq. (6.10). Thus, as you pull on the spring, the spring does negative work on you. Paying careful attention to the sign of work will eliminate confusion later on!

Example 6.6 Work done on a spring scale

A woman weighing 600 N steps on a bathroom scale containing a stiff spring (Fig. 6.21). In equilibrium the spring is compressed 1.0 cm under her weight. Find the force constant of the spring and the total work done on it during the compression.

SOLUTION

IDENTIFY: In equilibrium the upward force exerted by the spring balances the downward force of the woman's weight. We'll use this principle and Eq. (6.8) to determine the force constant k, and we'll use Eq. (6.10) to calculate the work W that the woman does on the spring to compress it.

SET UP: We take positive values of x to correspond to elongation (upward in Fig. 6.21), so that the displacement of the spring (x) and the x-component of the force that the woman exerts on it (F_x) are both negative.

EXECUTE: The top of the spring is displaced by $x = -1.0\text{ cm} = -0.010\text{ m}$, and the woman exerts a force $F_x = -600\text{ N}$ on the spring. From Eq. (6.8) the force constant is

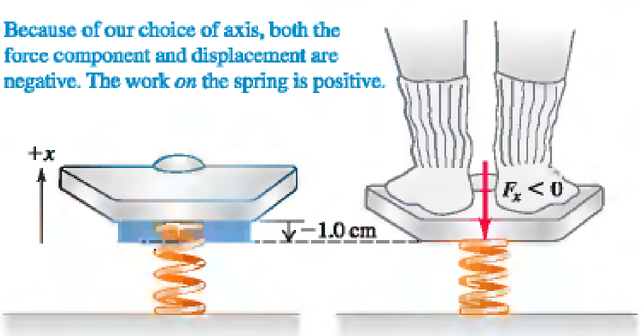
$$k = \frac{F_x}{x} = \frac{-600\text{ N}}{-0.010\text{ m}} = 6.0 \times 10^4\text{ N/m}$$

Then, using $x_1 = 0$ and $x_2 = -0.010\text{ m}$ in Eq. (6.10),

$$W = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$$
$$= \frac{1}{2}(6.0 \times 10^4\text{ N/m})(-0.010\text{ m})^2 - 0 = 3.0\text{ J}$$

EVALUATE: The applied force and the displacement of the end of the spring were in the same direction, so the work done must have been positive—just as we found. Our arbitrary choice of the positive direction has no effect on the answer for W. (You can test this by taking the positive x-direction to be downward, corresponding to compression. You'll get the same values for k and W.)

6.21 Compressing a spring in a bathroom scale.



Work–Energy Theorem for Straight-Line Motion, Varying Forces

In Section 6.2 we derived the work–energy theorem, $W_{tot} = K_2 - K_1$, for the special case of straight-line motion with a constant net force. We can now prove that this theorem is true even when the force varies with position. As in Section 6.2, let's consider a particle that undergoes a displacement x while being acted on by a net force with x-component F_x, which we now allow to vary. Just as in Fig. 6.16, we divide the total displacement x into a large number of small segments Δx. We can apply the work–energy theorem, Eq. (6.6), to each segment because the value of F_x in each small segment is approximately constant. The change in kinetic energy in segment Δx_a is equal to the work $F_{ax}\Delta x_a$, and so on. The total change of kinetic energy is the sum of the changes in the individual segments, and thus is equal to the total work done on the particle during the entire displacement. So $W_{tot} = \Delta K$ holds for varying forces as well as for constant ones.

Here's an alternative derivation of the work–energy theorem for a force that may vary with position. It involves making a change of variable from x to v_x in the work integral. As a preliminary, we note that the acceleration a of the particle can be expressed in various ways, using $a_x = dv_x/dt$, $v_x = dx/dt$, and the chain rule for derivatives:

$$a_x = \frac{dv_x}{dt} = \frac{dv_x}{dx}\frac{dx}{dt} = v_x\frac{dv_x}{dx} \tag{6.11}$$

From this result, Eq. (6.7) tells us that the total work done by the *net* force F_x is

$$W_{tot} = \int_{x_1}^{x_2} F_x\,dx = \int_{x_1}^{x_2} ma_x\,dx = \int_{x_1}^{x_2} mv_x\frac{dv_x}{dx}\,dx \tag{6.12}$$

Now $(dv_x/dx)\,dx$ is the change in velocity dv_x during the displacement dx, so in Eq. (6.12) we can substitute dv_x for $(dv_x/dx)\,dx$. This changes the integration variable from x to v_x, so we change the limits from x_1 and x_2 to the corresponding x-velocities v_1 and v_2 at these points. This gives us

$$W_{tot} = \int_{v_1}^{v_2} mv_x\,dv_x$$

The integral of $v_x\,dv_x$ is just $v_x^2/2$. Substituting the upper and lower limits, we finally find

$$W_{tot} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \tag{6.13}$$

This is the same as Eq. (6.6), so the work–energy theorem is valid even without the assumption that the net force is constant.

Example 6.7 Motion with a varying force

An air-track glider of mass 0.100 kg is attached to the end of a horizontal air track by a spring with force constant 20.0 N/m (Fig. 6.22a). Initially the spring is unstretched and the glider is moving at 1.50 m/s to the right. Find the maximum distance d that the glider moves to the right (a) if the air track is turned on so that there is no friction, and (b) if the air is turned off so that there is kinetic friction with coefficient $\mu_k = 0.47$.

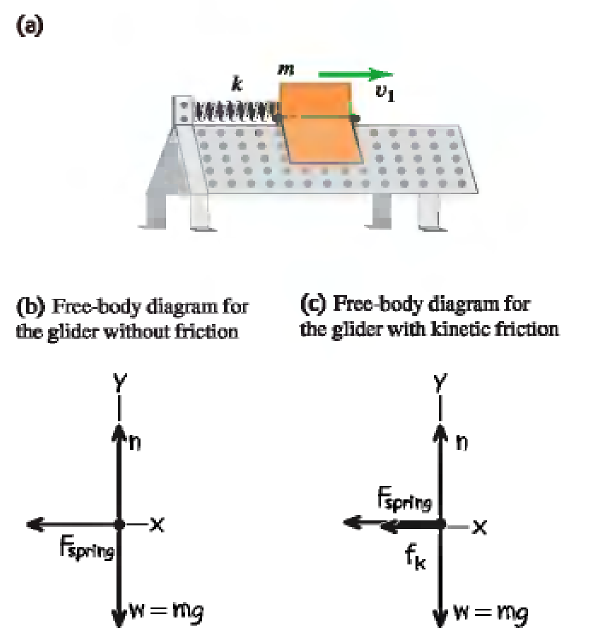
SOLUTION

IDENTIFY: The force exerted by the spring is not constant, so we *cannot* use the constant-acceleration formulas of Chapter 2 to solve this problem. Instead, we'll use the work–energy theorem, which involves the distance moved (our target variable) through the formula for work.

SET UP: In Figs. 6.22b and 6.22c we chose the positive x-direction to be to the right (in the direction of the glider's motion). We take $x = 0$ at the glider's initial position (where the spring is unstretched) and $x = d$ (the target variable) at the position where the glider stops. The motion is purely horizontal, so only the horizontal forces do work. Note that Eq. (6.10) gives the work done *on* the spring as it stretches, but to use the work–energy theorem we

Continued

6.22 (a) A glider attached to an air track by a spring. (b), (c) Our free-body diagrams.



need the work done *by* the spring *on* the glider—which is the negative of Eq. (6.10).

EXECUTE: (a) As the glider moves from $x_1 = 0$ to $x_2 = d$, it does an amount of work on the spring given by Eq. (6.10): $W = \frac{1}{2}kd^2 - \frac{1}{2}k(0)^2 = \frac{1}{2}kd^2$. The amount of work that the spring does on the glider is the negative of this value, or $-\frac{1}{2}kd^2$. The spring stretches until the glider comes instantaneously to rest, so the final kinetic energy K_2 is zero. The initial kinetic energy is $\frac{1}{2}mv_1^2$, where $v_1 = 1.50\ \text{m/s}$ is the glider's initial speed. Using the work–energy theorem, we find

$$-\frac{1}{2}kd^2 = 0 - \frac{1}{2}mv_1^2$$

We solve for the distance d the glider moves:

$$d = v_1\sqrt{\frac{m}{k}} = (1.50\ \text{m/s})\sqrt{\frac{0.100\ \text{kg}}{20.0\ \text{N/m}}}$$
$$= 0.106\ \text{m} = 10.6\ \text{cm}$$

The stretched spring subsequently pulls the glider back to the left, so the glider is at rest only instantaneously.

(b) If the air is turned off, we must also include the work done by the constant force of kinetic friction. The normal force n is equal in magnitude to the weight of the glider, since the track is horizontal and there are no other vertical forces. Hence the magnitude of the kinetic friction force is $f_k = \mu_k n = \mu_k mg$. The friction force is directed opposite to the displacement, so the work done by friction is

$$W_{fric} = f_k d\cos 180° = -f_k d = -\mu_k mgd$$

The total work is the sum of W_{fric} and the work done by the spring, $-\frac{1}{2}kd^2$. The work–energy theorem then says that

$$-\mu_k mgd - \frac{1}{2}kd^2 = 0 - \frac{1}{2}mv_1^2$$

$$-(0.47)(0.100\ \text{kg})(9.8\ \text{m/s}^2)d - \frac{1}{2}(20.0\ \text{N/m})d^2$$
$$= -\frac{1}{2}(0.100\ \text{kg})(1.50\ \text{m/s})^2$$
$$(10.0\ \text{N/m})d^2 + (0.461\ \text{N})d - (0.113\ \text{N}\cdot\text{m}) = 0$$

This is a quadratic equation for d. The solutions are

$$d = \frac{-(0.461\ \text{N}) \pm \sqrt{(0.461\ \text{N})^2 - 4(10.0\ \text{N/m})(-0.113\ \text{N}\cdot\text{m})}}{2(10.0\ \text{N/m})}$$
$$= 0.086\ \text{m} \quad \text{or} \quad -0.132\ \text{m}$$

We have used d as the symbol for a positive displacement, so only the positive value of d makes sense. Thus with friction the glider moves a distance

$$d = 0.086\ \text{m} = 8.6\ \text{cm}$$

EVALUATE: With friction present, the glider goes a shorter distance and the spring stretches less, as you might expect. Again the glider stops instantaneously, and again the spring force pulls the glider to the left; whether it moves or not depends on how great the *static* friction force is. How large would the coefficient of static friction μ_s have to be to keep the glider from springing back to the left?

Work–Energy Theorem for Motion Along a Curve

We can generalize our definition of work further to include a force that varies in direction as well as magnitude, and a displacement that lies along a curved path. Suppose a particle moves from point P_1 to P_2 along a curve, as shown in Fig. 6.23a. We divide the portion of the curve between these points into many infinitesimal vector displacements, and we call a typical one of these $d\vec{l}$. Each $d\vec{l}$ is tangent to the path at its position. Let $\vec{F}$ be the force at a typical point along the path, and let ϕ be the angle between $\vec{F}$ and $d\vec{l}$ at this point. Then the small element of work dW done on the particle during the displacement $d\vec{l}$ may be written as

$$dW = F\cos\phi\, dl = F_{\parallel}\, dl = \vec{F}\cdot d\vec{l}$$

where $F_{\parallel} = F\cos\phi$ is the component of $\vec{F}$ in the direction parallel to $d\vec{l}$ (Fig. 6.23b). The total work done by $\vec{F}$ on the particle as it moves from P_1 to P_2 is then

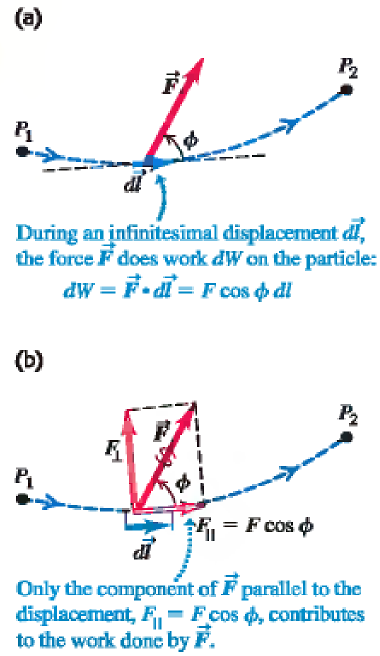
$$W = \int_{P_1}^{P_2} F\cos\phi\, dl = \int_{P_1}^{P_2} F_{\parallel}\, dl = \int_{P_1}^{P_2} \vec{F}\cdot d\vec{l} \qquad \text{(work done on a curved path)} \qquad (6.14)$$

We can now show that the work–energy theorem, Eq. (6.6), holds true even with varying forces and a displacement along a curved path. The force $\vec{F}$ is essentially constant over any given infinitesimal segment $d\vec{l}$ of the path, so we can apply the work–energy theorem for straight-line motion to that segment. Thus the change in the particle's kinetic energy K over that segment equals the work $dW = F_{\parallel}\, dl = \vec{F}\cdot d\vec{l}$ done on the particle. Adding up these infinitesimal quantities of work from all the segments along the whole path gives the total work done, Eq. (6.14), which equals the total change in kinetic energy over the whole path. So $W_{tot} = \Delta K = K_2 - K_1$ is true *in general,* no matter what the path and no matter what the character of the forces. This can be proved more rigorously by using steps like those in Eqs. (6.11) through (6.13) (see Challenge Problem 6.104).

Note that only the component of the net force parallel to the path, $F_{\parallel}$, does work on the particle, so only this component can change the speed and kinetic energy of the particle. The component perpendicular to the path, $F_{\perp} = F\sin\phi$, has no effect on the particle's speed; it acts only to change the particle's direction.

The integral in Eq. (6.14) is called a *line integral.* To evaluate this integral in a specific problem, we need some sort of detailed description of the path and of the way in which $\vec{F}$ varies along the path. We usually express the line integral in terms of some scalar variable, as in the following example.

6.23 A particle moves along a curved path from point P_1 to P_2, acted on by a force $\vec{F}$ that varies in magnitude and direction.



Example 6.8 Motion on a curved path I

At a family picnic you are appointed to push your obnoxious cousin Throckmorton in a swing (Fig. 6.24a). His weight is w, the length of the chains is R, and you push Throcky until the chains make an angle θ_0 with the vertical. To do this, you exert a varying horizontal force $\vec{F}$ that starts at zero and gradually increases just enough so that Throcky and the swing move very slowly and remain very nearly in equilibrium. What is the total work done on Throcky by all forces? What is the work done by the tension T in the chains? What is the work you do by exerting the force $\vec{F}$? (Neglect the weight of the chains and seat.)

SOLUTION

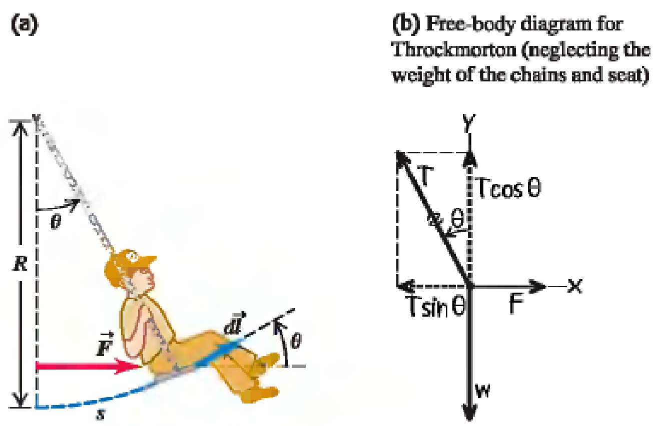
IDENTIFY: The motion is along a curve, so we will use Eq. (6.14) to calculate the work done by the net force, by the tension force, and by the force $\vec{F}$.

SET UP: Figure 6.24b shows our free-body diagram and coordinate system. We have replaced the tensions in the two chains with a single tension T.

EXECUTE: There are two ways to find the total work done during the motion: (1) by calculating the work done by each force and then adding the quantities of work together, and (2) by calculating the work done by the net force. The second approach is far easier in this situation because Throcky is in equilibrium at every point. Hence the net force on him is zero, the integral of the net force in Eq. (6.14) is zero, and the total work done on him by all forces is zero.

It's also easy to find the work done by the chain tension on Throcky because this force is perpendicular to the direction of motion at all points along the path. Hence at all points the angle between the chain tension and the displacement vector $d\vec{l}$ is 90° and the scalar product in Eq. (6.14) is zero. Thus the chain tension does zero work.

6.24 (a) Pushing cousin Throckmorton in a swing. (b) Our free-body diagram.



Continued

To compute the work done by $\vec{F}$, we need to know how this force varies with the angle θ. The net force on Throcky is zero, so $\sum F_x = 0$ and $\sum F_y = 0$. From Fig. 6.24b, we get

$$\sum F_x = F + (-T\sin\theta) = 0$$
$$\sum F_y = T\cos\theta + (-w) = 0$$

By eliminating T from these two equations, we obtain

$$F = w\tan\theta$$

The point where $\vec{F}$ is applied swings through the arc s. The arc length s equals the radius R of the circular path multiplied by the length θ (in radians), so $s = R\theta$. Therefore the displacement $d\vec{l}$ corresponding to a small change of angle $d\theta$ has a magnitude $dl = ds = R\,d\theta$. The work done by $\vec{F}$ is

$$W = \int \vec{F}\cdot d\vec{l} = \int F\cos\theta\,ds$$

Now we express everything in terms of the angle θ, whose value increases from 0 to θ_0:

$$W = \int_0^{\theta_0} (w\tan\theta)\cos\theta\,(R\,d\theta) = wR\int_0^{\theta_0} \sin\theta\,d\theta$$
$$= wR(1 - \cos\theta_0)$$

EVALUATE: If $\theta_0 = 0$, there is no displacement; then $\cos\theta_0 = 1$ and $W = 0$, as we should expect. If $\theta_0 = 90°$, then $\cos\theta_0 = 0$ and $W = wR$. In that case the work you do is the same as if you had lifted Throcky straight up a distance R with a force equal to his weight w. In fact, the quantity $R(1 - \cos\theta_0)$ is the increase in his height above the ground during the displacement, so for any value of θ_0 the work done by the force $\vec{F}$ is the change in height multiplied by the weight. This is an example of a more general result that we'll prove in Section 7.1.

Example 6.9 Motion on a curved path II

In Example 6.8 the infinitesimal displacement $d\vec{l}$ (Fig. 6.24a) has a magnitude of ds, an x-component of $ds\cos\theta$, and a y-component of $ds\sin\theta$. Hence $d\vec{l} = \hat{\imath}\,ds\cos\theta + \hat{\jmath}\,ds\sin\theta$. Use this expression and Eq. (6.14) to calculate the work done during the motion by the chain tension, by the force of gravity, and by the force $\vec{F}$.

SOLUTION

IDENTIFY: We again use Eq. (6.14), but now we'll use Eq. (1.21) to find the scalar product in terms of components.

SET UP: We use the same free-body diagram, Fig. 6.24b, as in Example 6.8.

EXECUTE: From Fig. 6.24b, we can write the three forces in terms of unit vectors:

$$\vec{T} = \hat{\imath}(-T\sin\theta) + \hat{\jmath}T\cos\theta$$
$$\vec{w} = \hat{\jmath}(-w)$$
$$\vec{F} = \hat{\imath}F$$

To use Eq. (6.14), we must calculate the scalar product of each of these forces with $d\vec{l}$. Using Eq. (1.21),

$$\vec{T}\cdot d\vec{l} = (-T\sin\theta)(ds\cos\theta) + (T\cos\theta)(ds\sin\theta) = 0$$
$$\vec{w}\cdot d\vec{l} = (-w)(ds\sin\theta) = -w\sin\theta\,ds$$
$$\vec{F}\cdot d\vec{l} = F(ds\cos\theta) = F\cos\theta\,ds$$

Since $\vec{T}\cdot d\vec{l} = 0$, the integral of this quantity is zero and the work done by the chain tension is zero (just as we found in Example 6.8). Using $ds = R\,d\theta$ as in Example 6.8, we find the work done by the force of gravity is

$$\int \vec{w}\cdot d\vec{l} = \int (-w\sin\theta)\,R\,d\theta = -wR\int_0^{\theta_0} \sin\theta\,d\theta$$
$$= -wR(1 - \cos\theta_0)$$

The work done by gravity is negative because gravity pulls down while Throcky moves upward. Finally, the work done by the force $\vec{F}$ is the integral $\int \vec{F}\cdot d\vec{l} = \int F\cos\theta\,ds$, which we calculated in Example 6.8; the answer is $+wR(1 - \cos\theta_0)$.

EVALUATE: As a check on our answers, note that the sum of all three quantities of work is zero. This is just what we concluded in Example 6.8 using the work–energy theorem.

The method of components is often the most convenient way to calculate scalar products. Use it when it makes your life easier!

Test Your Understanding of Section 6.3 In Example 5.21 (Section 5.4) we examined a conical pendulum. The speed of the pendulum bob remains constant as it travels around the circle shown in Fig. 5.32a. (a) Over one complete circle, how much work does the tension force F do on the bob? (i) a positive amount; (ii) a negative amount; (iii) zero. (b) Over one complete circle, how much work does the weight do on the bob? (i) a positive amount; (ii) a negative amount; (iii) zero.

6.4 Power

The definition of work makes no reference to the passage of time. If you lift a barbell weighing 100 N through a vertical distance of 1.0 m at constant velocity, you do $(100\text{ N})(1.0\text{ m}) = 100\text{ J}$ of work whether it takes you 1 second, 1 hour, or 1 year to do it. But often we need to know how quickly work is done. We describe this in terms of *power*. In ordinary conversation the word "power" is often synonymous with "energy" or "force." In physics we use a much more precise definition: **Power** is the time *rate* at which work is done. Like work and energy, power is a scalar quantity.

When a quantity of work ΔW is done during a time interval Δt, the average work done per unit time or **average power** P_{av} is defined to be

$$P_{av} = \frac{\Delta W}{\Delta t} \qquad \text{(average power)} \tag{6.15}$$

The rate at which work is done might not be constant. We can define **instantaneous power** P as the quotient in Eq. (6.15) as Δt approaches zero:

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt} \qquad \text{(instantaneous power)} \tag{6.16}$$

The SI unit of power is the **watt** (W), named for the English inventor James Watt. One watt equals 1 joule per second: $1\text{ W} = 1\text{ J/s}$ (Fig. 6.25). The kilowatt $(1\text{ kW} = 10^3\text{ W})$ and the megawatt $(1\text{ MW} = 10^6\text{ W})$ are also commonly used. In the British system, work is expressed in foot-pounds, and the unit of power is the foot-pound per second. A larger unit called the *horsepower* (hp) is also used (Fig. 6.26):

$$1\text{ hp} = 550\text{ ft}\cdot\text{lb/s} = 33{,}000\text{ ft}\cdot\text{lb/min}$$

That is, a 1-hp motor running at full load does $33{,}000\text{ ft}\cdot\text{lb}$ of work every minute. A useful conversion factor is

$$1\text{ hp} = 746\text{ W} = 0.746\text{ kW}$$

The watt is a familiar unit of *electrical* power; a 100-W light bulb converts 100 J of electrical energy into light and heat each second. But there's nothing inherently electrical about a watt. A light bulb could be rated in horsepower, and an engine can be rated in kilowatts.

The *kilowatt-hour* $(\text{kW}\cdot\text{h})$ is the usual commercial unit of electrical energy. One kilowatt-hour is the total work done in 1 hour (3600 s) when the power is 1 kilowatt (10^3 J/s), so

$$1\text{ kW}\cdot\text{h} = (10^3\text{ J/s})(3600\text{ s}) = 3.6\times10^6\text{ J} = 3.6\text{ MJ}$$

The kilowatt-hour is a unit of *work* or *energy*, not power.

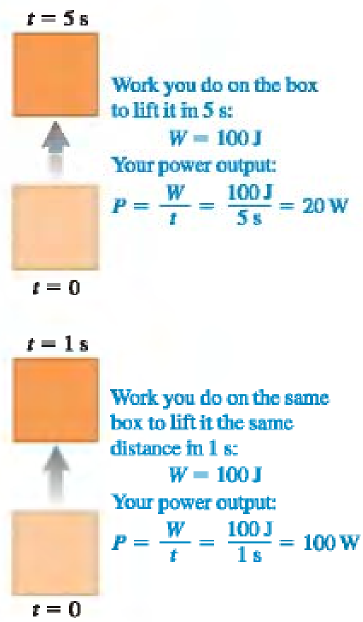
In mechanics we can also express power in terms of force and velocity. Suppose that a force $\vec{F}$ acts on a body while it undergoes a vector displacement $\Delta\vec{s}$. If $F_\parallel$ is the component of $\vec{F}$ tangent to the path (parallel to $\Delta\vec{s}$), then the work done by the force is $\Delta W = F_\parallel \Delta s$. The average power is

$$P_{av} = \frac{F_\parallel \Delta s}{\Delta t} = F_\parallel \frac{\Delta s}{\Delta t} = F_\parallel v_{av} \tag{6.17}$$

Instantaneous power P is the limit of this expression as $\Delta t \to 0$:

$$P = F_\parallel v \tag{6.18}$$

6.25 The same amount of work is done in both of these situations, but the power (the rate at which work is done) is different.



6.26 The value of the horsepower derives from experiments by James Watt, who measured that a horse could do 33,000 foot-pounds of work per minute in lifting coal from a coal pit.

where v is the magnitude of the instantaneous velocity. We can also express Eq. (6.18) in terms of the scalar product:

$$P = \vec{F} \cdot \vec{v} \quad \text{(instantaneous rate at which force } \vec{F} \text{ does work on a particle)} \tag{6.19}$$

Example 6.10 Force and power

Each of the two jet engines in a Boeing 767 airliner develops a thrust (a forward force on the airplane) of 197,000 N (44,300 lb). When the airplane is flying at 250 m/s (900 km/h, or roughly 560 mi/h), what horsepower does each engine develop?

SOLUTION

IDENTIFY: Our target variable is the instantaneous power P, which is the rate at which the thrust does work.

SET UP: We use Eq. (6.18). The thrust is in the direction of motion, so $F_{\parallel}$ is just equal to the thrust.

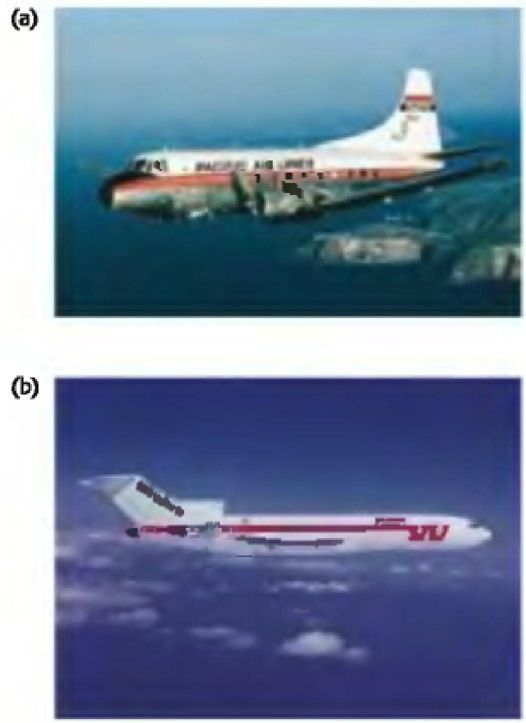
EXECUTE: At $v = 250\text{ m/s}$, the power developed by each engine is

$$P = F_{\parallel}v = (1.97 \times 10^5\text{ N})(250\text{ m/s}) = 4.93 \times 10^7\text{ W}$$
$$= (4.93 \times 10^7\text{ W})\frac{1\text{ hp}}{746\text{ W}} = 66{,}000\text{ hp}$$

EVALUATE: The speed of modern airliners is directly related to the power of their engines (Fig. 6.27). The largest propeller-driven airliners of the 1950s had engines that developed about 3400 hp $(2.5 \times 10^6\text{ W})$, giving them maximum speeds of about 600 km/h (370 mi/h). Each engine in a Boeing 767 develops nearly 20 times more power, enabling it to fly at about 900 km/h (560 mi/h) and to carry a much heavier load.

If the engines are at maximum thrust while the airliner is at rest on the ground so that $v = 0$, the engines develop *zero* power. Force and power are not the same thing!

6.27 (a) Propeller-driven and (b) jet airliners.



Example 6.11 A "power climb"

A 50.0-kg marathon runner runs up the stairs to the top of Chicago's 443-m-tall Sears Tower, the tallest building in the United States (Fig. 6.28). To lift herself to the top in 15.0 minutes, what must be her average power output in watts? In kilowatts? In horsepower?

SOLUTION

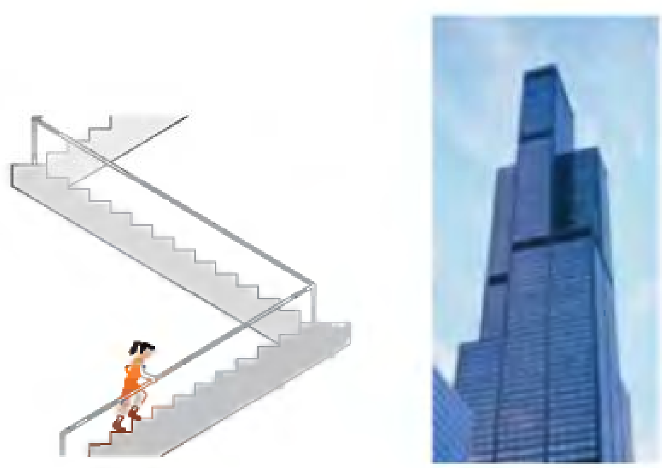
IDENTIFY: We'll treat the runner as a particle of mass m. Her average power output P_{av} must be enough to lift her at constant speed against gravity.

SET UP: We can find P_{av} in two ways: (1) by first determining how much work she must do and then dividing it by the elapsed time, as in Eq. (6.15), or (2) by calculating the average upward force she must exert (in the direction of the climb) and then multiplying it by her upward velocity, as in Eq. (6.17).

EXECUTE: As in Example 6.8, lifting a mass m against gravity requires an amount of work equal to the weight mg multiplied by the height h it is lifted. Hence the work she must do is

$$W = mgh = (50.0\text{ kg})(9.80\text{ m/s}^2)(443\text{ m})$$
$$= 2.17 \times 10^5\text{J}$$

6.28 How much power is required to run up the stairs of Chicago's Sears Tower in 15 minutes?



The time is 15.0 min = 900 s, so from Eq. (6.15) the average power is

$$P_{av} = \frac{2.17 \times 10^5\,\text{J}}{900\,\text{s}} = 241\,\text{W} = 0.241\,\text{kW} = 0.323\,\text{hp}$$

Let's try the calculation again using Eq. (6.17). The force exerted is vertical, and the average vertical component of velocity is $(443\,\text{m})/(900\,\text{s}) = 0.492\,\text{m/s}$, so the average power is

$$\begin{aligned} P_{av} &= F_{\parallel}v_{av} = (mg)v_{av} \\ &= (50.0\,\text{kg})(9.80\,\text{m/s}^2)(0.492\,\text{m/s}) = 241\,\text{W} \end{aligned}$$

which is the same result as before.

EVALUATE: The runner's *total* power output will be several times greater than 241 W. The reason is that the runner isn't really a particle but a collection of parts that exert forces on each other and do work, such as the work done to inhale and exhale and to make her arms and legs swing. What we've calculated is only the part of her power output that lifts her to the top of the building.

Test Your Understanding of Section 6.4 The air surrounding an airplane in flight exerts a drag force that acts opposite to the airplane's motion. When the Boeing 767 in Example 6.10 is flying in a straight line at a constant altitude at a constant 250 m/s, what is the rate at which the drag force does work on it? (i) 132,000 hp; (ii) 66,000 hp; (iii) 0; (iv) −66,000 hp; (v) −132,000 hp.

CHAPTER 6 SUMMARY

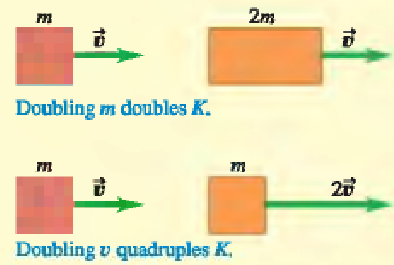
Work done by a force: When a constant force $\vec{F}$ acts on a particle that undergoes a straight-line displacement $\vec{s}$, the work done by the force on the particle is defined to be the scalar product of $\vec{F}$ and $\vec{s}$. The unit of work in SI units is 1 joule = 1 newton-meter $(1\ \text{J} = 1\ \text{N}\cdot\text{m})$. Work is a scalar quantity; it can be positive or negative, but it has no direction in space. (See Examples 6.1 and 6.2.)

$$W = \vec{F}\cdot\vec{s} = Fs\cos\phi \qquad (6.2), (6.3)$$

ϕ = angle between $\vec{F}$ and $\vec{s}$

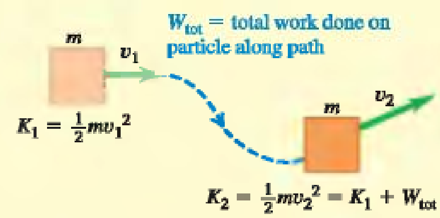
Kinetic energy: The kinetic energy K of a particle equals the amount of work required to accelerate the particle from rest to speed v. It is also equal to the amount of work the particle can do in the process of being brought to rest. Kinetic energy is a scalar that has no direction in space; it is always positive or zero. Its units are the same as the units of work: $1\ \text{J} = 1\ \text{N}\cdot\text{m} = 1\ \text{kg}\cdot\text{m}^2/\text{s}^2$.

$$K = \frac{1}{2}mv^2 \qquad (6.5)$$



The work–energy theorem: When forces act on a particle while it undergoes a displacement, the particle's kinetic energy changes by an amount equal to the total work done on the particle by all the forces. This relationship, called the work–energy theorem, is valid whether the forces are constant or varying and whether the particle moves along a straight or curved path. It is applicable only to bodies that can be treated as a particle. (See Examples 6.3–6.5)

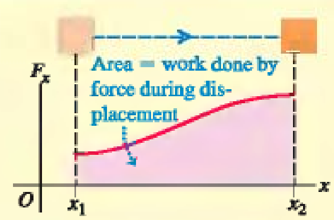
$$W_{\text{tot}} = K_2 - K_1 = \Delta K. \qquad (6.6)$$



Work done by a varying force or on a curved path: When a force varies during a straight-line displacement, the work done by the force is given by an integral, Eq. (6.7). (See Examples 6.6 and 6.7.) When a particle follows a curved path, the work done on it by a force $\vec{F}$ is given by an integral that involves the angle ϕ between the force and the displacement. This expression is valid even if the force magnitude and the angle ϕ vary during the displacement. (See Examples 6.8 and 6.9.)

$$W = \int_{x_1}^{x_2} F_x\,dx \qquad (6.7)$$

$$W = \int_{P_1}^{P_2} F\cos\phi\,dl = \int_{P_1}^{P_2} F_\parallel\,dl = \int_{P_1}^{P_2} \vec{F}\cdot d\vec{l} \qquad (6.14)$$

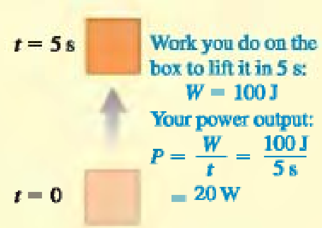


Power: Power is the time rate of doing work. The average power P_{av} is the amount of work ΔW done in time Δt divided by that time. The instantaneous power is the limit of the average power as Δt goes to zero. When a force $\vec{F}$ acts on a particle moving with velocity $\vec{v}$, the instantaneous power (the rate at which the force does work) is the scalar product of $\vec{F}$ and $\vec{v}$. Like work and kinetic energy, power is a scalar quantity. The SI unit of power is 1 watt = 1 joule/second $(1\ \text{W} = 1\ \text{J/s})$. (See Examples 6.10 and 6.11.)

$$P_{\text{av}} = \frac{\Delta W}{\Delta t} \qquad (6.15)$$

$$P = \lim_{\Delta t \to 0}\frac{\Delta W}{\Delta t} = \frac{dW}{dt} \qquad (6.16)$$

$$P = \vec{F}\cdot\vec{v} \qquad (6.19)$$



Key Terms

work, *182*
joule, *182*
kinetic energy, *187*
work–energy theorem, *187*
force constant, *193*
Hooke's law, *193*
power, *199*
average power, *199*
instantaneous power, *199*
watt, *199*

Answer to Chapter Opening Question ?

It is indeed true that the shell does work on the gases. However, because the shell exerts a backward force on the gases as the gases and shell move forward through the barrel, the work done by the shell is *negative* (see Section 6.1).

Answers to Test Your Understanding Questions

6.1 Answer: (iii) The electron has constant velocity, so its acceleration is zero and (by Newton's second law) the net force on the electron is also zero. Therefore the total work done by all the forces (equal to the work done by the net force) must be zero as well. The individual forces may do nonzero work, but that's not what the question asks.

6.2 Answer: (iv), (i), (iii), (ii) Body (i) has kinetic energy $K = \frac{1}{2}mv^2 = \frac{1}{2}(2.0\,\text{kg})(5.0\,\text{m/s})^2 = 25\,\text{J}$. Body (ii) had zero kinetic energy initially and then had 30 J of work done it, so its final kinetic energy is $K_2 = K_1 + W = 0 + 30\,\text{J} = 30\,\text{J}$. Body (iii) had initial kinetic energy $K_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}(1.0\,\text{kg})(4.0\,\text{m/s})^2 = 8.0\,\text{J}$ and then had 20 J of work done on it, so its final kinetic energy is $K_2 = K_1 + W = 8.0\,\text{J} + 20\,\text{J} = 28\,\text{J}$. Body (iv) had initial kinetic energy $K_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}(2.0\,\text{kg})(10\,\text{m/s})^2 = 100\,\text{J}$; when it did 80 J of work on another body, the other body did -80 J of work on body (iv), so the final kinetic energy of body (iv) is $K_2 = K_1 + W = 100\,\text{J} + (-80\,\text{J}) = 20\,\text{J}$.

6.3 Answers:(a) (iii), (b) (iii) At any point during the pendulum bob's motion, the tension force and the weight both act perpendicular to the motion—that is, perpendicular to an infinitesimal displacement $d\vec{l}$ of the bob. (In Fig. 5.32b, the displacement $d\vec{l}$ would be directed outward from the plane of the free-body diagram.) Hence for either force the scalar product inside the integral in Eq. (6.14) is $\vec{F}\cdot d\vec{l} = 0$, and the work done along any part of the circular path (including a complete circle) is $W = \int \vec{F}\cdot d\vec{l} = 0$.

6.4 Answer: (v) The airliner has a constant horizontal velocity, so the net horizontal force on it must be zero. Hence the backward drag force must have the same magnitude as the forward force due to the combined thrust of the two engines. This means that the drag force must do *negative* work on the airplane at the same rate that the combined thrust force does *positive* work. The combined thrust does work at a rate of $2(66{,}000\,\text{hp}) = 132{,}000\,\text{hp}$, so the drag force must do work at a rate of $-132{,}000$ hp.

PROBLEMS

For instructor-assigned homework, go to www.masteringphysics.com

Discussion Questions

Q6.1. The sign of many physical quantities depends on the choice of coordinates. For example, g can be negative or positive, depending on whether we choose upward or downward as positive. Is the same thing true of work? In other words, can we make positive work negative by a different choice of coordinates? Explain.

Q6.2. An elevator is hoisted by its cables at constant speed. Is the total work done on the elevator positive, negative, or zero? Explain.

Q6.3. A rope tied to a body is pulled, causing the body to accelerate. But according to Newton's third law, the body pulls back on the rope with an equal and opposite force. Is the total work done then zero? If so, how can the body's kinetic energy change? Explain.

Q6.4. If it takes total work W to give an object a speed v and kinetic energy K, starting from rest, what will be the object's speed (in terms of v) and kinetic energy (in terms of K) if we do twice as much work on it, again starting from rest?

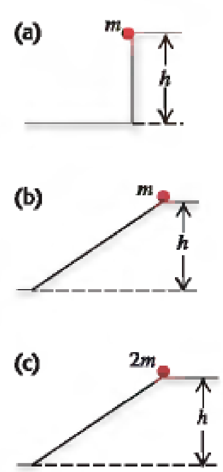
Q6.5. If there is a net nonzero force on a moving object, is it possible for the total work done on the object to be zero? Explain, with an example that illustrates your answer.

Q6.6. In Example 5.5 (Section 5.1), how does the work done on the bucket by the tension in the cable compare to the work done on the cart by the tension in the cable?

Q6.7. In the conical pendulum in Example 5.21 (Section 5.4), which of the forces do work on the bob while it is swinging?

Q6.8. For the cases shown in Fig. 6.29, the object is released from rest at the top and feels no friction or air resistance. In which (if any) cases will the mass have (i) the greatest speed at the bottom and (ii) the most work done on it by the time it reaches the bottom?

Figure 6.29 Question Q6.8.



Q6.9. A force $\vec{F}$ is in the x-direction and has a magnitude that depends on x. Sketch a possible graph of F versus x such that the force does zero work on an object that moves from x_1 to x_2, even though the force magnitude is not zero at all x in this range.

Q6.10. Does the kinetic energy of a car change more when it speeds up from 10 to 15 m/s or from 15 to 20 m/s? Explain.

Q6.11. A falling brick has a mass of 1.5 kg and is moving straight downward with a speed of 5.0 m/s. A 1.5-kg physics book is sliding across the floor with a speed of 5.0 m/s. A 1.5-kg melon is traveling with a horizontal velocity component 3.0 m/s to the right and a vertical component 4.0 m/s upward. Do these objects all have the same velocity? Do these objects all have the same kinetic energy? For each question, give the reasoning behind your answer.

Q6.12. Can the *total* work done on an object during a displacement be negative? Explain. If the total work is negative, can its magnitude be larger than the initial kinetic energy of the object? Explain.

Q6.13. A net force acts on an object and accelerates it from rest to a speed v_1. In doing so, the force does an amount of work W_1. By what factor must the work done on the object be increased to produce three times the final speed, with the object again starting from rest?

Q6.14. A truck speeding down the highway has a lot of kinetic energy relative to a stopped state trooper, but no kinetic energy relative to the truck driver. In these two frames of reference, is the same amount of work required to stop the truck? Explain.

Q6.15. You are holding a briefcase by the handle, with your arm straight down by your side. Does the force your hand exerts do work on the briefcase when (a) you walk at a constant speed down a horizontal hallway and (b) you ride an escalator from the first to second floor of a building? In each case justify your answer.

Q6.16. When a book slides along a tabletop, the force of friction does negative work on it. Can friction ever do *positive* work? Explain. (*Hint:* Think of a box in the back of an accelerating truck.)

Q6.17. Time yourself while running up a flight of steps, and compute the average rate at which you do work against the force of gravity. Express your answer in watts and in horsepower.

Q6.10. Fractured Physics. Many terms from physics are badly misused in everyday language. In each case, explain the errors involved. (a) A *strong* person is called *powerful.* What is wrong with this use of *power*? (b) When a worker carries a bag of concrete along a level construction site, people say he did a lot of *work*. Did he?

Q6.19. An advertisement for a portable electrical generating unit claims that the unit's diesel engine produces 28,000 hp to drive an electrical generator that produces 30 MW of electrical power. Is this possible? Explain.

Q6.20. A car speeds up while the engine delivers constant power. Is the acceleration greater at the beginning of this process or at the end? Explain.

Q6.21. Consider a graph of instantaneous power versus time, with the vertical P axis starting at $P = 0$. What is the physical significance of the area under the P versus t curve between vertical lines at t_1 and t_2? How could you find the average power from the graph? Draw a P versus t curve that consists of two straight-line sections and for which the peak power is equal to twice the average power.

Q6.22. A nonzero net force acts on an object. Is it possible for any of the following quantities to be constant: (a) the particle's speed; (b) the particle's velocity; (c) the particle's kinetic energy.

Q6.23. When a certain force is applied to an ideal spring, the spring stretches a distance x from its unstretched length and does work W. If instead twice the force is applied, what distance (in terms of x) does the spring stretch from its unstretched length, and how much work (in terms of W) is required to stretch it this distance?

Q6.24. If work W is required to stretch a spring a distance x from its unstretched length, what work (in terms of W) is required to stretch the spring an *additional* distance x?

Exercises

Section 6.1 Work

6.1. An old oaken bucket of mass 6.75 kg hangs in a well at the end of a rope. The rope passes over a frictionless pulley at the top of the well, and you pull horizontally on the end of the rope to raise the bucket slowly a distance of 4.00 m. (a) How much work do you do on the bucket in pulling it up? (b) How much work does gravity do on the bucket? (c) What is the total work done on the bucket?

6.2. A tow truck pulls a car 5.00 km along a horizontal roadway using a cable having a tension of 850 N. (a) How much work does the cable do on the car if it pulls horizontally? If it pulls at 35.0° above the horizontal? (b) How much work does the cable do on the tow truck in both cases of part (a)? (c) How much work does gravity do on the car in part (a)?

6.3. A factory worker pushes a 30.0-kg crate a distance of 4.5 m along a level floor at constant velocity by pushing horizontally on it. The coefficient of kinetic friction between the crate and the floor is 0.25. (a) What magnitude of force must the worker apply? (b) How much work is done on the crate by this force? (c) How much work is done on the crate by friction? (d) How much work is done on the crate by the normal force? By gravity? (e) What is the total work done on the crate?

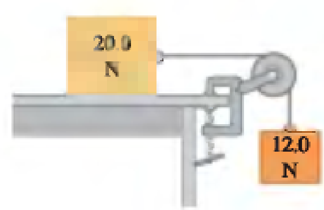
6.4. Suppose the worker in Exercise 6.3 pushes downward at an angle of 30° below the horizontal. (a) What magnitude of force must the worker apply to move the crate at constant velocity? (b) How much work is done on the crate by this force when the crate is pushed a distance of 4.5 m? (c) How much work is done on the crate by friction during this displacement? (d) How much work is done on the crate by the normal force? By gravity? (e) What is the total work done on the crate?

6.5. A 75.0-kg painter climbs a ladder that is 2.75 m long leaning against a vertical wall. The ladder makes an 30.0° angle with the wall. (a) How much work does gravity do on the painter? (b) Does the answer to part (a) depend on whether the painter climbs at constant speed or accelerates up the ladder?

6.6. Two tugboats pull a disabled supertanker. Each tug exerts a constant force of 1.80×10^6 N, one 14° west of north and the other 14° east of north, as they pull the tanker 0.75 km toward the north. What is the total work they do on the supertanker?

6.7. Two blocks are connected by a very light string passing over a massless and frictionless pulley (Figure 6.30). Traveling at constant speed, the 20.0-N block moves 75.0 cm to the right and the 12.0-N block moves 75.0 cm downward. During this process, how much work is done (a) on the 12.0-N block by (i) gravity and (ii) the tension in the string? (b) On the 20.0-N block by (i) gravity, (ii) the tension in the string, (iii) friction, and (iv) the normal force? (c) Find the total work done on each block.

Figure 6.30 Exercise 6.7.



6.6. A loaded grocery cart is rolling across a parking lot in a strong wind. You apply a constant force $\vec{F} = (30\,\text{N})\hat{\imath} - (40\,\text{N})\hat{\jmath}$ to the cart as it undergoes a displacement $\vec{s} = (-9.0\,\text{m})\hat{\imath} - (3.0\,\text{m})\hat{\jmath}$. How much work does the force you apply do on the grocery cart?

6.9. A 0.800-kg ball is tied to the end of a string 1.60 m long and swung in a vertical circle. (a) During one complete circle, starting anywhere, calculate the total work done on the ball by (i) the tension in the string and (ii) gravity. (b) Repeat part (a) for motion along the semicircle from the lowest to the highest point on the path.

Section 6.2 Kinetic Energy and the Work–Energy Theorem

6.10. (a) How many joules of kinetic energy does a 750-kg automobile traveling at a typical highway speed of 65 mi/h have? (b) By what factor would its kinetic energy decrease if the car traveled half as fast? (c) How fast (in mi/h) would the car have to travel to have half as much kinetic energy as in part (a)?

6.11. Meteor Crater. About 50,000 years ago, a meteor crashed into the earth near present-day Flagstaff, Arizona. Recent (2005) measurements estimate that this meteor had a mass of about 1.4×10^8 kg (around 150,000 tons) and hit the ground at 12 km/s. (a) How much kinetic energy did this meteor deliver to the ground? (b) How does this energy compare to the energy released by a 1.0-megaton nuclear bomb? (A megaton bomb releases the same energy as a million tons of TNT, and 1.0 ton of TNT releases 4.184×10^9 J of energy.)

6.12. Some Typical Kinetic Energies. (a) How many joules of kinetic energy does a 75-kg person have when walking and when running? (b) In the Bohr model of the atom, the ground-state electron in hydrogen has an orbital speed of 2190 km/s. What is its kinetic energy? (Consult Appendix F.) (c) If you drop a 1.0-kg weight (about 2 lb) from shoulder height, how many joules of kinetic energy will it have when it reaches the ground? (d) Is it reasonable that a 30-kg child could run fast enough to have 100 J of kinetic energy?

6.13. The mass of a proton is 1836 times the mass of an electron. (a) A proton is traveling at speed V. At what speed (in terms of V) would an electron have the same kinetic energy as the proton? (b) An electron has kinetic energy K. If a proton has the same speed as the electron, what is its kinetic energy (in terms of K)?

6.14. A 4.80-kg watermelon is dropped from rest from the roof of a 25.0-m-tall building and feels no appreciable air resistance. (a) Calculate the work done by gravity on the watermelon during its displacement from the roof to the ground. (b) Just before it strikes the ground, what is the watermelon's (i) kinetic energy and (ii) speed? (c) Which of the answers in parts (a) and (b) would be *different* if there were appreciable air resistance?

6.15. Use the work–energy theorem to solve each of these problems. You can use Newton's laws to check your answers. Neglect air resistance in all cases. (a) A branch falls from the top of a 95.0-m-tall redwood tree, starting from rest. How fast is it moving when it reaches the ground? (b) A volcano ejects a boulder directly upward 525 m into the air. How fast was the boulder moving just as it left the volcano? (c) A skier moving at 5.00 m/s encounters a long, rough horizontal patch of snow having coefficient of kinetic friction 0.220 with her skis. How far does she travel on this patch before stopping? (d) Suppose the rough patch in part (c) was only 2.90 m long? How fast would the skier be moving when she reached the end of the patch? (e) At the base of a frictionless icy hill that rises at 25.0° above the horizontal, a toboggan has a speed of 12.0 m/s toward the hill. How high vertically above the base will it go before stopping?

6.16. You throw a 20-N rock vertically into the air from ground level. You observe that when it is 15.0 m above the ground, it is traveling at 25.0 m/s upward. Use the work–energy theorem to find (a) the rock's speed just as it left the ground and (b) its maximum height.

6.17. You are a member of an Alpine Rescue Team. You must project a box of supplies up an incline of constant slope angle α so that it reaches a stranded skier who is a vertical distance h above the bottom of the incline. The incline is slippery, but there is some friction present, with kinetic friction coefficient μ_k. Use the work–energy theorem to calculate the minimum speed you must give the box at the bottom of the incline so that it will reach the skier. Express your answer in terms of g, h, μ_k, and α.

6.18. A mass m slides down a smooth inclined plane from an initial vertical height h, making an angle α with the horizontal. (a) The work done by a force is the sum of the work done by the components of the force. Consider the components of gravity parallel and perpendicular to the surface of the plane. Calculate the work done on the mass by each of the components, and use these results to show that the work done by gravity is exactly the same as if the mass had fallen straight down through the air from a height h. (b) Use the work–energy theorem to prove that the speed of the mass at the bottom of the incline is the same as if it had been dropped from height h, independent of the angle α of the incline. Explain how this speed can be independent of the slope angle. (c) Use the results of part (b) to find the speed of a rock that slides down an icy frictionless hill, starting from rest 15.0 m above the bottom.

6.19. A car is stopped in a distance D by a constant friction force that is independent of the car's speed. What is the stopping distance (in terms of D) (a) if the car's initial speed is tripled, and (b) if the speed is the same as it originally was but the friction force is tripled? (Solve using the work–energy theorem.)

6.20. A moving electron has kinetic energy K_1. After a net amount of work W has been done on it, the electron is moving one-quarter as fast in the opposite direction. (a) Find W in terms of K_1. (b) Does your answer depend on the final direction of the electron's motion?

6.21. A sled with mass 8.00 kg moves in a straight line on a frictionless horizontal surface. At one point in its path, its speed is 4.00 m/s; after it has traveled 2.50 m beyond this point, its speed is 6.00 m/s. Use the work–energy theorem to find the force acting on the sled, assuming that this force is constant and that it acts in the direction of the sled's motion.

6.22. A soccer ball with mass 0.420 kg is initially moving with speed 2.00 m/s. A soccer player kicks the ball, exerting a constant force of magnitude 40.0 N in the same direction as the ball's motion. Over what distance must the player's foot be in contact with the ball to increase the ball's speed to 6.00 m/s?

6.23. A 12-pack of Omni-Cola (mass 4.30 kg) is initially at rest on a horizontal floor. It is then pushed in a straight line for 1.20 m by a trained dog that exerts a horizontal force with magnitude 36.0 N. Use the work–energy theorem to find the final speed of the 12-pack if (a) there is no friction between the 12-pack and the floor, and (b) the coefficient of kinetic friction between the 12-pack and the floor is 0.30.

6.24. A batter hits a baseball with mass 0.145 kg straight upward with an initial speed of 25.0 m/s. (a) How much work has gravity done on the baseball when it reaches a height of 20.0 m above the bat? (b) Use the work–energy theorem to calculate the speed of the baseball at a height of 20.0 m above the bat. You can ignore air resistance. (c) Does the answer to part (b) depend on whether the baseball is moving upward or downward at a height of 20.0 m? Explain.

6.25. A little red wagon with mass 7.00 kg moves in a straight line on a frictionless horizontal surface. It has an initial speed of 4.00 m/s and then is pushed 3.0 m in the direction of the initial velocity by a force with a magnitude of 10.0 N. (a) Use the work–energy theorem to calculate the wagon's final speed. (b) Calculate the acceleration produced by the force. Use this acceleration in the kinematic relationships of Chapter 2 to calculate the wagon's final speed. Compare this result to that calculated in part (a).

6.26. A block of ice with mass 2.00 kg slides 0.750 m down an inclined plane that slopes downward at an angle of 36.9° below the horizontal. If the block of ice starts from rest, what is its final speed? You can ignore friction.

6.27. Stopping Distance. A car is traveling on a level road with speed v_0 at the instant when the brakes lock, so that the tires slide rather than roll. (a) Use the work–energy theorem to calculate the minimum stopping distance of the car in terms of v_0, g, and the coefficient of kinetic friction μ_k between the tires and the road. (b) By what factor would the minimum stopping distance change if (i) the coefficient of kinetic friction were doubled, or (ii) the initial speed were doubled, or (iii) both the coefficient of kinetic friction and the initial speed were doubled?

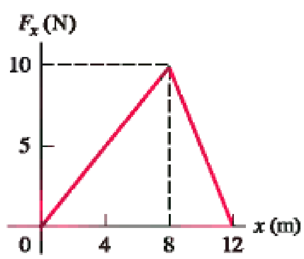
Section 6.3 Work and Energy with Varying Forces

6.26. To stretch a spring 3.00 cm from its unstretched length, 12.0 J of work must be done. (a) What is the force constant of this spring? (b) What magnitude force is needed to stretch the spring 3.00 cm from its unstretched length? (c) How much work must be done to compress this spring 4.00 cm from its unstretched length, and what force is needed to stretch it this distance?

6.29. A force of 160 N stretches a spring 0.050 m beyond its unstretched length. (a) What magnitude of force is required to stretch the spring 0.015 m beyond its unstretched length? To compress the spring 0.020 m? (b) How much work must be done to stretch the spring 0.015 m beyond its unstretched length? To compress the spring 0.020 m from its unstretched length?

6.30. A child applies a force $\vec{F}$ parallel to the x-axis to a 10.0-kg sled moving on the frozen surface of a small pond. As the child controls the speed of the sled, the x-component of the force she applies varies with the x-coordinate of the sled as shown in Fig. 6.31. Calculate the work done by the force $\vec{F}$ when the sled moves (a) from $x = 0$ to $x = 8.0$ m; (b) from $x = 8.0$ m to $x = 12.0$ m; (c) from $x = 0$ to 12.0 m.

Figure 6.31 Exercises 6.30 and 6.31.



6.31. Suppose the sled in Exercise 6.30 is initially at rest at $x = 0$. Use the work–energy theorem to find the speed of the sled at (a) $x = 8.0$ m and (b) $x = 12.0$ m. You can ignore friction between the sled and the surface of the pond.

6.32. A balky cow is leaving the barn as you try harder and harder to push her back in. In coordinates with the origin at the barn door, the cow walks from $x = 0$ to $x = 6.9$ m as you apply a force with x-component $F_x = -[20.0\text{ N} + (3.0\text{ N/m})x]$. How much work does the force you apply do on the cow during this displacement?

6.33. A 6.0-kg box moving at 3.0 m/s on a horizontal, frictionless surface runs into a light spring of force constant 75 N/cm. Use the work–energy theorem to find the maximum compression of the spring.

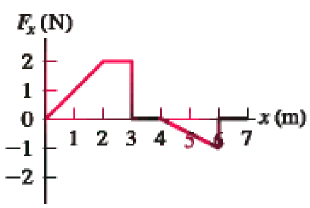
6.34. Leg Presses. As part of your daily workout, you lie on your back and push with your feet against a platform attached to two stiff springs arranged side by side so that they are parallel to each other. When you push the platform, you compress the springs. You do 80.0 J of work when you compress the springs 0.200 m from their uncompressed length. (a) What magnitude of force must you apply to hold the platform in this position? (b) How much *additional* work must you do to move the platform 0.200 m *farther*, and what maximum force must you apply?

6.35. (a) In Example 6.7 (Section 6.3) it was calculated that with the air track turned off, the glider travels 8.6 cm before it stops instantaneously. How large would the coefficient of static friction μ_s have to be to keep the glider from springing back to the left? (b) If the coefficient of static friction between the glider and the track is $\mu_s = 0.60$, what is the maximum initial speed v_1 that the glider can be given and still remain at rest after it stops instantaneously? With the air track turned off, the coefficient of kinetic friction is $\mu_k = 0.47$.

6.36. A 4.00-kg block of ice is placed against a horizontal spring that has force constant $k = 200$ N/m and is compressed 0.025 m. The spring is released and accelerates the block along a horizontal surface. You can ignore friction and the mass of the spring. (a) Calculate the work done on the block by the spring during the motion of the block from its initial position to where the spring has returned to its uncompressed length. (b) What is the speed of the block after it leaves the spring?

6.37. A force $\vec{F}$ is applied to a 2.0-kg radio-controlled model car parallel to the x-axis as it moves along a straight track. The x-component of the force varies with the x-coordinate of the car as shown in Fig. 6.32. Calculate the work done by the force $\vec{F}$ when the car moves from (a) $x = 0$ to $x = 3.0$ m; (b) $x = 3.0$ m to $x = 4.0$ m; (c) $x = 4.0$ m to $x = 7.0$ m; (d) $x = 0$ to $x = 7.0$ m; (e) $x = 7.0$ m to $x = 2.0$ m.

Figure 6.32 Exercises 6.37 and 6.38.



6.30. Suppose the 2.0-kg model car in Exercise 6.37 is initially at rest at $x = 0$ and $\vec{F}$ is the net force acting on it. Use the work–energy theorem to find the speed of the car at (a) $x = 3.0$ m; (b) $x = 4.0$ m; (c) $x = 7.0$ m.

6.39. At a waterpark, sleds with riders are sent along a slippery, horizontal surface by the release of a large compressed spring. The spring with force constant $k = 40.0$ N/cm and negligible mass rests on the frictionless horizontal surface. One end is in contact with a stationary wall. A sled and rider with total mass 70.0 kg are pushed against the other end, compressing the spring 0.375 m. The sled is then released with zero initial velocity. What is the sled's speed when the spring (a) returns to its uncompressed length and (b) is still compressed 0.200 m?

6.40. Half of a Spring. (a) Suppose you cut a massless ideal spring in half. If the full spring had a force constant k, what is the force constant of each half, in terms of k? (*Hint:* Think of the original spring as two equal halves, each producing the same force as the entire spring. Do you see why the forces must be equal?) (b) If you cut the spring into three equal segments instead, what is the force constant of each one, in terms of k?

6.41. A small glider is placed against a compressed spring at the bottom of an air track that slopes upward at an angle of 40.0° above the horizontal. The glider has mass 0.0900 kg. The spring has $k = 640$ N/m and negligible mass. When the spring is released, the glider travels a maximum distance of 1.80 m along the air track before sliding back down. Before reaching this maximum distance, the glider loses contact with the spring. (a) What distance was the spring originally compressed? (b) When the glider has traveled along the air track 0.80 m from its initial position against the compressed spring, is it still in contact with the spring? What is the kinetic energy of the glider at this point?

6.42. An ingenious bricklayer builds a device for shooting bricks up to the top of the wall where he is working. He places a brick on a vertical compressed spring with force constant $k = 450\ \text{N/m}$ and negligible mass. When the spring is released, the brick is propelled upward. If the brick has mass 1.80 kg and is to reach a maximum height of 3.6 m above its initial position on the compressed spring, what distance must the bricklayer compress the spring initially? (The brick loses contact with the spring when the spring returns to its uncompressed length. Why?)

Section 6.4 Power

6.43. How many joules of energy does a 100-watt light bulb use per hour? How fast would a 70-kg person have to run to have that amount of kinetic energy?

6.44. The total consumption of electrical energy in the United States is about 1.0×10^{19} J per year. (a) What is the average rate of electrical energy consumption in watts? (b) The population of the United States is about 300 million people. What is the average rate of electrical energy consumption per person? (c) The sun transfers energy to the earth by radiation at a rate of approximately 1.0 kW per square meter of surface. If this energy could be collected and converted to electrical energy with 40% efficiency, how great an area (in square kilometers) would be required to collect the electrical energy used in the United States?

6.45. Magnetar. On December 27, 2004, astronomers observed the greatest flash of light ever recorded from outside the solar system. It came from the highly magnetic neutron star SGR 1806-20 (a *magnetar*). During 0.20 s, this star released as much energy as our sun does in 250,000 years. If P is the average power output of our sun, what was the average power output (in terms of P) of this magnetar?

6.46. A 20.0-kg rock is sliding on a rough, horizontal surface at 8.00 m/s and eventually stops due to friction. The coefficient of kinetic friction between the rock and the surface is 0.200. What average power is produced by friction as the rock stops?

6.47. A tandem (two-person) bicycle team must overcome a force of 165 N to maintain a speed of 9.00 m/s. Find the power required per rider, assuming that each contributes equally. Express your answer in watts and in horsepower.

6.46. When its 75-kW (100-hp) engine is generating full power, a small single-engine airplane with mass 700 kg gains altitude at a rate of 2.5 m/s (150 m/min, or 500 ft/min). What fraction of the engine power is being used to make the airplane climb? (The remainder is used to overcome the effects of air resistance and of inefficiencies in the propeller and engine.)

6.49. Working Like a Horse. Your job is to lift 30-kg crates a vertical distance of 0.90 m from the ground onto the bed of a truck. (a) How many crates would you have to load onto the truck in 1 minute for the average power output you use to lift the crates to equal 0.50 hp? (b) How many crates for an average power output of 100 W?

6.58. An elevator has mass 600 kg, not including passengers. The elevator is designed to ascend, at constant speed, a vertical distance of 20.0 m (five floors) in 16.0 s, and it is driven by a motor that can provide up to 40 hp to the elevator. What is the maximum number of passengers that can ride in the elevator? Assume that an average passenger has mass 65.0 kg.

6.51. Automotive Power. It is not unusual for a 1000-kg car to get 30 mi/gal when traveling at 60 mi/h on a level road. If this car makes a 200-km trip, (a) how many joules of energy does it consume, and (b) what is the average rate of energy consumption during the trip? Note that 1.0 gal of gasoline yields 1.3×10^9 J (although this can vary). Consult Appendix E.

6.52. The aircraft carrier *John F. Kennedy* has mass 7.4×10^7 kg. When its engines are developing their full power of 280,000 hp, the *John F. Kennedy* travels at its top speed of 35 knots (65 km/h). If 70% of the power output of the engines is applied to pushing the ship through the water, what is the magnitude of the force of water resistance that opposes the carrier's motion at this speed?

6.53. A ski tow operates on a 15.0° slope of length 300 m. The rope moves at 12.0 km/h and provides power for 50 riders at one time, with an average mass per rider of 70.0 kg. Estimate the power required to operate the tow.

6.54. A typical flying insect applies an average force equal to twice its weight during each downward stroke while hovering. Take the mass of the insect to be 10 g, and assume the wings move an average downward distance of 1.0 cm during each stroke. Assuming 100 downward strokes per second, estimate the average power output of the insect.

Problems

6.55. Rotating Bar. A thin, uniform 12.0-kg bar that is 2.00 m long rotates uniformly about a pivot at one end, making 5.00 complete revolutions every 3.00 seconds. What is the kinetic energy of this bar? (*Hint*: Different points in the bar have different speeds. Break the bar up into infinitesimal segments of mass dm and integrate to add up the kinetic energy of all these segments.)

6.56. A Near-Earth Asteroid. On April 13, 2029 (Friday the 13th!), the asteroid 99942 Apophis will pass within 18,600 mi of the earth—about 1/13 the distance to the moon! It has a density of $2600\ \text{kg/m}^3$, can be modeled as a sphere 320 m in diameter, and will be traveling at 12.6 km/s. (a) If, due to a small disturbance in its orbit, the asteroid were to hit the earth, how much kinetic energy would it deliver? (b) The largest nuclear bomb ever tested by the United States was the "Castle/Bravo" bomb, having a yield of 15 megatons of TNT. (A megaton of TNT releases 4.184×10^{15} J of energy.) How many Castle/Bravo bombs would be equivalent to the energy of Apophis?

6.57. A luggage handler pulls a 20.0-kg suitcase up a ramp inclined at 25.0° above the horizontal by a force $\vec{F}$ of magnitude 140 N that acts parallel to the ramp. The coefficient of kinetic friction between the ramp and the incline is $\mu_k = 0.300$. If the suitcase travels 3.80 m along the ramp, calculate (a) the work done on the suitcase by the force $\vec{F}$; (b) the work done on the suitcase by the gravitational force; (c) the work done on the suitcase by the normal force; (d) the work done on the suitcase by the friction force; (e) the total work done on the suitcase. (f) If the speed of the suitcase is zero at the bottom of the ramp, what is its speed after it has traveled 3.80 m along the ramp?

6.58. Chin-Ups. While doing a chin-up, a man lifts his body 0.40 m. (a) How much work must the man do per kilogram of body mass? (b) The muscles involved in doing a chin-up can generate about 70 J of work per kilogram of muscle mass. If the man can just barely do a 0.40-m chin-up, what percentage of his body's mass do these muscles constitute? (For comparison, the *total* percentage of muscle in a typical 70-kg man with 14% body fat is about 43%.) (c) Repeat part (b) for the man's young son, who has arms half as long as his father's but whose muscles can also generate 70 J of work per kilogram of muscle mass. (d) Adults and children have about the same percentage of muscle in their bodies. Explain why children can commonly do chin-ups more easily than their fathers.

6.59. Simple Machines. Ramps for the disabled are used because a large weight w can be raised by a relatively small force equal to $w \sin \alpha$ plus the small friction force. Such inclined planes are an example of a class of devices called *simple machines*. An input force F_{in} is applied to the system and results in an output force F_{out} applied to the object that is moved. For a simple machine the ratio of these forces, F_{out}/F_{in}, is called the actual mechanical advantage (AMA). The inverse ratio of the distances that the points of application of these forces move through during the motion of the object, s_{in}/s_{out}, is called the ideal mechanical advantage (IMA). (a) Find the IMA for an inclined plane. (b) What can we say about the relationship between the work supplied to the machine, W_{in}, and the work output of the machine, W_{out}, if AMA = IMA? (c) Sketch a single pulley arranged to give IMA = 2. (d) We define the efficiency e of a simple machine to equal the ratio of the output work to the input work, $e = W_{out}/W_{in}$. Show that $e = \text{AMA}/\text{IMA}$.

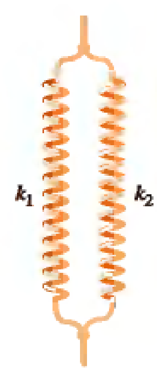
6.60. Consider the blocks in Exercise 6.7 as they move 75.0 cm. Find the total work done on each one (a) if there is no friction between the table and the 20.0-N block, and (b) if $\mu_s = 0.500$ and $\mu_k = 0.325$ between the table and the 20.0-N block.

6.61. The space shuttle *Endeavour*, with mass 86,400 kg, is in a circular orbit of radius 6.66×10^6 m around the earth. It takes 90.1 min for the shuttle to complete each orbit. On a repair mission, the shuttle is cautiously moving 1.00 m closer to a disabled satellite every 3.00 s. Calculate the shuttle's kinetic energy (a) relative to the earth and (b) relative to the satellite.

6.62. A 5.00-kg package slides 1.50 m down a long ramp that is inclined at 12.0° below the horizontal. The coefficient of kinetic friction between the package and the ramp is $\mu_k = 0.310$. Calculate (a) the work done on the package by friction; (b) the work done on the package by gravity; (c) the work done on the package by the normal force; (d) the total work done on the package. (e) If the package has a speed of 2.20 m/s at the top of the ramp, what is its speed after sliding 1.50 m down the ramp?

6.63. Springs in Parallel. Two springs are in *parallel* if they are parallel to each other and are connected at their ends (Figure 6.33). We can think of this combination as being equivalent to a single spring. The force constant of the equivalent single spring is called the *effective* force constant, k_{eff}, of the combination. (a) Show that the effective force constant of this combination is $k_{eff} = k_1 + k_2$. (b) Generalize this result for N springs in parallel.

Figure 6.33 Problem 6.63.



6.64. Springs in Series. Two massless springs are connected in series when they are attached one after the other, head to tail. (a) Show that the effective force constant (see Problem 6.63) of a series combination is given by $\dfrac{1}{k_{eff}} = \dfrac{1}{k_1} + \dfrac{1}{k_2}$. (*Hint:* For a given force, the total distance stretched by the equivalent single spring is the sum of the distances stretched by the springs in combination. Also, each spring must exert the same force. Do you see why?) (b) Generalize this result for N springs in series.

6.65. An object is attracted toward the origin with a force given by $F_x = -k/x^2$. (Gravitational and electrical forces have this distance dependence.) (a) Calculate the work done by the force F_x when the object moves in the x-direction from x_1 to x_2. If $x_2 > x_1$, is the work done by F_x positive or negative? (b) The only other force acting on the object is a force that you exert with your hand to move the object slowly from x_1 to x_2. How much work do you do? If $x_2 > x_1$, is the work you do positive or negative? (c) Explain the similarities and differences between your answers to parts (a) and (b).

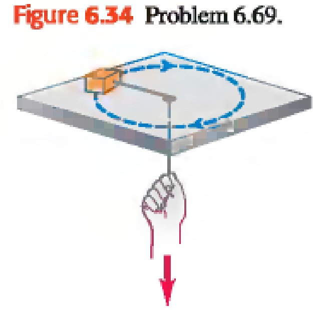
6.66. The gravitational pull of the earth on an object is inversely proportional to the square of the distance of the object from the center of the earth. At the earth's surface this force is equal to the object's normal weight mg, where $g = 9.8\ \text{m/s}^2$, and at large distances, the force is zero. If a 20,000-kg asteroid falls to earth from a very great distance away, what will be its minimum speed as it strikes the earth's surface, and how much kinetic energy will it impart to our planet? You can ignore the effects of the earth's atmosphere.

6.67. Varying Coefficient of Friction. A box is sliding with a speed of 4.50 m/s on a horizontal surface when, at point P, it encounters a rough section. On the rough section, the coefficient of friction is not constant, but starts at 0.100 at P and increases linearly with distance past P, reaching a value of 0.600 at 12.5 m past point P. (a) Use the work–energy theorem to find how far this box slides before stopping. (b) What is the coefficient of friction at the stopping point? (c) How far would the box have slid if the friction coefficient didn't increase but instead had the constant value of 0.100?

6.68. Consider a spring that does not obey Hooke's law very faithfully. One end of the spring is fixed. To keep the spring stretched or compressed an amount x, a force along the x-axis with x-component $F_x = kx - bx^2 + cx^3$ must be applied to the free end. Here $k = 100\ \text{N/m}$, $b = 700\ \text{N/m}^2$, and $c = 12{,}000\ \text{N/m}^3$. Note that $x > 0$ when the spring is stretched and $x < 0$ when it is compressed. (a) How much work must be done to stretch this spring by 0.050 m from its unstretched length? (b) How much work must be done to *compress* this spring by 0.050 m from its unstretched length? (c) Is it easier to stretch or compress this spring? Explain why in terms of the dependence of F_x on x. (Many real springs behave qualitatively in the same way.)

6.68. A small block with a mass of 0.120 kg is attached to a cord passing through a hole in a frictionless, horizontal surface (Fig. 6.34). The block is originally revolving at a distance of 0.40 m from the hole with a speed of 0.70 m/s. The cord is then pulled from below, shortening the radius of the circle in which the block revolves to 0.10 m. At this new distance, the speed of the block is observed to be 2.80 m/s. (a) What is the tension in the cord in the original situation when the block has speed $v = 0.70$ m/s? (b) What is the tension in the cord in the final situation when the block has speed $v = 2.80$ m/s? (c) How much work was done by the person who pulled on the cord?

Figure 6.34 Problem 6.69.



6.70. Proton Bombardment. A proton with mass 1.67×10^{-27} kg is propelled at an initial speed of 3.00×10^5 m/s directly toward a uranium nucleus 5.00 m away. The proton is repelled by the uranium nucleus with a force of magnitude $F = \alpha/x^2$, where x is the separation between the two objects and $\alpha = 2.12 \times 10^{-26}\ \text{N} \cdot \text{m}^2$. Assume that the uranium nucleus remains at rest.

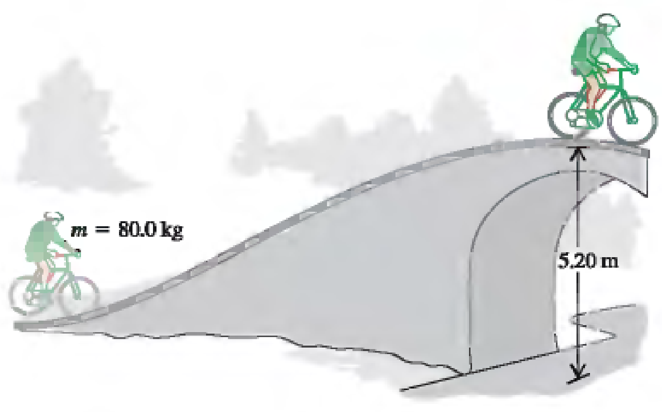
(a) What is the speed of the proton when it is 8.00×10^{-10} m from the uranium nucleus? (b) As the proton approaches the uranium nucleus, the repulsive force slows down the proton until it comes momentarily to rest, after which the proton moves away from the uranium nucleus. How close to the uranium nucleus does the proton get? (c) What is the speed of the proton when it is again 5.00 m away from the uranium nucleus?

6.71. A block of ice with mass 6.00 kg is initially at rest on a frictionless, horizontal surface. A worker then applies a horizontal force $\vec{F}$ to it. As a result, the block moves along the x-axis such that its position as a function of time is given by $x(t) = \alpha t^2 + \beta t^3$, where $\alpha = 0.200\ \text{m/s}^2$ and $\beta = 0.0200\ \text{m/s}^3$. (a) Calculate the velocity of the object when $t = 4.00$ s. (b) Calculate the magnitude of $\vec{F}$ when $t = 4.00$ s. (c) Calculate the work done by the force $\vec{F}$ during the first 4.00 s of the motion.

6.72. The Genesis Crash. When the 210-kg Genesis Mission capsule crashed (see Exercise 5.17 in Chapter 5) with a speed of 311 km/h, it buried itself 81.0 cm deep in the desert floor. Assuming constant acceleration during the crash, at what average rate did the capsule do work on the desert?

6.73. You and your bicycle have combined mass 80.0 kg. When you reach the base of a bridge, you are traveling along the road at 5.00 m/s (Fig. 6.35). At the top of the bridge, you have climbed a vertical distance of 5.20 m and have slowed to 1.50 m/s. You can ignore work done by friction and any inefficiency in the bike or your legs. (a) What is the total work done on you and your bicycle when you go from the base to the top of the bridge? (b) How much work have you done with the force you apply to the pedals?

Figure 6.35 Problem 6.73.



6.74. A force in the $+x$-direction has magnitude $F = b/x^n$, where b and n are constants. (a) For $n > 1$, calculate the work done on a particle by this force when the particle moves along the x-axis from $x = x_0$ to infinity. (b) Show that for $0 < n < 1$, even though F becomes zero as x becomes very large, an infinite amount of work is done by F when the particle moves from $x = x_0$ to infinity.

6.75. You are asked to design spring bumpers for the walls of a parking garage. A freely rolling 1200-kg car moving at 0.65 m/s is to compress the spring no more than 0.070 m before stopping. What should be the force constant of the spring? Assume that the spring has negligible mass.

6.76. The spring of a spring gun has force constant $k = 400$ N/m and negligible mass. The spring is compressed 6.00 cm, and a ball with mass 0.0300 kg is placed in the horizontal barrel against the compressed spring. The spring is then released, and the ball is propelled out the barrel of the gun. The barrel is 6.00 cm long, so the ball leaves the barrel at the same point that it loses contact with the spring. The gun is held so the barrel is horizontal. (a) Calculate the speed with which the ball leaves the barrel if you can ignore friction. (b) Calculate the speed of the ball as it leaves the barrel if a constant resisting force of 6.00 N acts on the ball as it moves along the barrel. (c) For the situation in part (b), at what position along the barrel does the ball have the greatest speed, and what is that speed? (In this case, the maximum speed does not occur at the end of the barrel.)

6.77. A 2.50-kg textbook is forced against a horizontal spring of negligible mass and force constant 250 N/m, compressing the spring a distance of 0.250 m. When released, the textbook slides on a horizontal tabletop with coefficient of kinetic friction $\mu_k = 0.30$. Use the work–energy theorem to find how far the textbook moves from its initial position before coming to rest.

6.78. Pushing a Cat. Your cat "Ms." (mass 7.00 kg) is trying to make it to the top of a frictionless ramp 2.00 m long and inclined upward at 30.0° above the horizontal. Since the poor cat can't get any traction on the ramp, you push her up the entire length of the ramp by exerting a constant 100-N force parallel to the ramp. If Ms. takes a running start so that she is moving at 2.40 m/s at the bottom of the ramp, what is her speed when she reaches the top of the incline? Use the work–energy theorem.

6.79. Crash Barrier. A student proposes a design for an automobile crash barrier in which a 1700-kg sport utility vehicle moving at 20.0 m/s crashes into a spring of negligible mass that slows it to a stop. So that the passengers are not injured, the acceleration of the vehicle as it slows can be no greater than 5.00g. (a) Find the required spring constant k, and find the distance the spring will compress in slowing the vehicle to a stop. In your calculation, disregard any deformation or crumpling of the vehicle and the friction between the vehicle and the ground. (b) What disadvantages are there to this design?

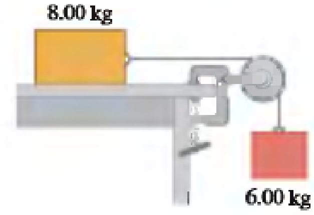
6.80. A physics professor is pushed up a ramp inclined upward at 30.0° above the horizontal as he sits in his desk chair that slides on frictionless rollers. The combined mass of the professor and chair is 85.0 kg. He is pushed 2.50 m along the incline by a group of students who together exert a constant horizontal force of 600 N. The professor's speed at the bottom of the ramp is 2.00 m/s. Use the work–energy theorem to find his speed at the top of the ramp.

6.81. A 5.00-kg block is moving at $v_0 = 6.00$ m/s along a frictionless, horizontal surface toward a spring with force constant $k = 500$ N/m that is attached to a wall (Fig. 6.36). The spring has negligible mass. (a) Find the maximum distance the spring will be compressed. (b) If the spring is to compress by no more than 0.150 m, what should be the maximum value of v_0?

Figure 6.36 Problem 6.81.

6.82. Consider the system shown in Fig. 6.37. The rope and pulley have negligible mass, and the pulley is frictionless. The coefficient of kinetic friction between the 8.00-kg block and the tabletop is $\mu_k = 0.250$. The blocks are released from rest. Use energy methods to calculate the speed of the 6.00-kg block after it has descended 1.50 m.

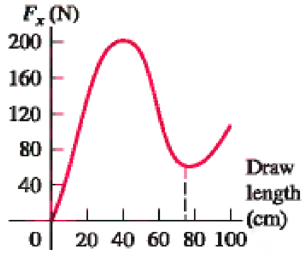
Figure 6.37 Problems 6.82 and 6.83.



6.83. Consider the system shown in Fig. 6.37. The rope and pulley have negligible mass, and the pulley is frictionless. Initially the 6.00-kg block is moving downward and the 8.00-kg block is moving to the right, both with a speed of 0.900 m/s. The blocks come to rest after moving 2.00 m. Use the work–energy theorem to calculate the coefficient of kinetic friction between the 8.00-kg block and the tabletop.

6.84. Bow and Arrow. Figure 6.38 shows how the force exerted by the string of a compound bow on an arrow varies as a function of how far back the arrow is pulled (the draw length). Assume that the same force is exerted on the arrow as it moves forward after being released. Full draw for this bow is at a draw length of 75.0 cm. If the bow shoots a 0.0250-kg arrow from full draw, what is the speed of the arrow as it leaves the bow?

Figure 6.38 Problem 6.84.



6.65. On an essentially frictionless, horizontal ice rink, a skater moving at 3.0 m/s encounters a rough patch that reduces her speed by 45% due to a friction force that is 25% of her weight. Use the work–energy theorem to find the length of this rough patch.

6.66. Rescue. Your friend (mass 65.0 kg) is standing on the ice in the middle of a frozen pond. There is very little friction between her feet and the ice, so she is unable to walk. Fortunately, a light rope is tied around her waist and you stand on the bank holding the other end. You pull on the rope for 3.00 s and accelerate your friend from rest to a speed of 6.00 m/s while you remain at rest. What is the average power supplied by the force you applied?

6.67. A pump is required to lift 800 kg of water (about 210 gallons) per minute from a well 14.0 m deep and eject it with a speed of 18.0 m/s. (a) How much work is done per minute in lifting the water? (b) How much work is done in giving the water the kinetic energy it has when ejected? (c) What must be the power output of the pump?

6.80. Find the power output of the worker in Problem 6.71 as a function of time. What is the numerical value of the power (in watts) at $t = 4.00$ s?

6.80. A physics student spends part of her day walking between classes or for recreation, during which time she expends energy at an average rate of 280 W. The remainder of the day she is sitting in class, studying, or resting; during these activities, she expends energy at an average rate of 100 W. If she expends a total of 1.1×10^7 J of energy in a 24-hour day, how much of the day did she spend walking?

6.90. All birds, independent of their size, must maintain a power output of 10–25 watts per kilogram of body mass in order to fly by flapping their wings. (a) The Andean giant hummingbird (*Patagona gigas*) has mass 70 g and flaps its wings 10 times per second while hovering. Estimate the amount of work done by such a hummingbird in each wingbeat. (b) A 70-kg athlete can maintain a power output of 1.4 kW for no more than a few seconds; the *steady* power output of a typical athlete is only 500 W or so. Is it possible for a human-powered aircraft to fly for extended periods by flapping its wings? Explain.

6.91. The Grand Coulee Dam is 1270 m long and 170 m high. The electrical power output from generators at its base is approximately 2000 MW. How many cubic meters of water must flow from the top of the dam per second to produce this amount of power if 92% of the work done on the water by gravity is converted to electrical energy? (Each cubic meter of water has a mass of 1000 kg.)

6.92. The engine of a car with mass m supplies a constant power P to the wheels to accelerate the car. You can ignore rolling friction and air resistance. The car is initially at rest. (a) Show that the speed of the car is given as a function of time by $v = (2Pt/m)^{1/2}$. (b) Show that the acceleration of the car is not constant but is given as a function of time by $a = (P/2mt)^{1/2}$. (c) Show that the displacement as a function of time is given by $x - x_0 = (8P/9m)^{1/2} t^{3/2}$.

6.93. Power of the Human Heart. The human heart is a powerful and extremely reliable pump. Each day it takes in and discharges about 7500 L of blood. Assume that the work done by the heart is equal to the work required to lift this amount of blood a height equal to that of the average American woman (1.63 m). The density (mass per unit volume) of blood is $1.05 \times 10^3\ \text{kg/m}^3$. (a) How much work does the heart do in a day? (b) What is the heart's power output in watts?

6.94. Six diesel units in series can provide 13.4 MW of power to the lead car of a freight train. The diesel units have total mass 1.10×10^6 kg. The average car in the train has mass 8.2×10^4 kg and requires a horizontal pull of 2.8 kN to move at a constant 27 m/s on level tracks. (a) How many cars can be in the train under these conditions? (b) This would leave no power for accelerating or climbing hills. Show that the extra force needed to accelerate the train is about the same for a 0.10-m/s² acceleration or a 1.0% slope (slope angle $\alpha = \arctan 0.010$). (c) With the 1.0% slope, show that an extra 2.9 MW of power is needed to maintain the 27-m/s speed of the diesel units. (d) With 2.9 MW less power available, how many cars can the six diesel units pull up a 1.0% slope at a constant 27-m/s?

6.95. It takes a force of 53 kN on the lead car of a 16-car passenger train with mass 9.1×10^5 kg to pull it at a constant 45 m/s (101 mi/h) on level tracks. (a) What power must the locomotive provide to the lead car? (b) How much more power to the lead car than calculated in part (a) would be needed to give the train an acceleration of 1.5 m/s², at the instant that the train has a speed of 45 m/s on level tracks? (c) How much more power to the lead car than that calculated in part (a) would be needed to move the train up a 1.5% grade (slope angle $\alpha = \arctan 0.015$) at a constant 45 m/s?

6.96. An object has several forces acting on it. One of these forces is $\vec{F} = \alpha xy\hat{\imath}$, a force in the x-direction whose magnitude depends on the position of the object, with $\alpha = 2.50\ \text{N/m}^2$. Calculate the work done on the object by this force for the following displacements of the object: (a) The object starts at the point $x = 0$, $y = 3.00$ m and moves parallel to the x-axis to the point $x = 2.00$ m, $y = 3.00$ m. (b) The object starts at the point $x = 2.00$ m, $y = 0$ and moves in the y-direction to the point $x = 2.00$ m, $y = 3.00$ m. (c) The object starts at the origin and moves on the line $y = 1.5x$ to the point $x = 2.00$ m, $y = 3.00$ m.

6.97. Cycling. For a touring bicyclist the drag coefficient $C\left(f_{\text{air}} = \frac{1}{2}CA\rho v^2\right)$ is 1.00, the frontal area A is 0.463 m², and the coefficient of rolling friction is 0.0045. The rider has mass 50.0 kg, and her bike has mass 12.0 kg. (a) To maintain a speed of 12.0 m/s (about 27 mi/h) on a level road, what must the rider's power output to the rear wheel be? (b) For racing, the same rider uses a different bike with coefficient of rolling friction 0.0030 and mass 9.00 kg. She also crouches down, reducing her drag coefficient to 0.88 and reducing her frontal area to $0.366 \mathrm{~m}^2$. What must her power output to the rear wheel be then to maintain a speed of $12.0 \mathrm{~m/s}$? (c) For the situation in part (b), what power output is required to maintain a speed of $6.0 \mathrm{~m/s}$? Note the great drop in power requirement when the speed is only halved. (For more on aerodynamic speed limitations for a wide variety of human-powered vehicles, see "The Aerodynamics of Human-Powered Land Vehicles," *Scientific American*, December 1983.)

6.98. Automotive Power I. A truck engine transmits 28.0 kW (37.5 hp) to the driving wheels when the truck is traveling at a constant velocity of magnitude $60.0 \mathrm{~km/h}$ $(37.3 \mathrm{~mi/h})$ on a level road. (a) What is the resisting force acting on the truck? (b) Assume that 65% of the resisting force is due to rolling friction and the remainder is due to air resistance. If the force of rolling friction is independent of speed, and the force of air resistance is proportional to the square of the speed, what power will drive the truck at $30.0 \mathrm{~km/h}$? At $120.0 \mathrm{~km/h}$? Give your answers in kilowatts and in horsepower.

6.99. Automotive Power II. (a) If 8.00 hp are required to drive a 1800-kg automobile at $60.0 \mathrm{~km/h}$ on a level road, what is the total retarding force due to friction, air resistance, and so on? (b) What power is necessary to drive the car at $60.0 \mathrm{~km/h}$ up a 10.0% grade (a hill rising 10.0 m vertically in 100.0 m horizontally)? (c) What power is necessary to drive the car at $60.0 \mathrm{~km/h}$ *down* a 1.00% grade? (d) Down what percent grade would the car coast at $60.0 \mathrm{~km/h}$?

Challenge Problems

6.100. On a winter's day in Maine, a warehouse worker is shoving boxes up a rough plank inclined at an angle α above the horizontal. The plank is partially covered with ice, with more ice near the bottom of the plank than near the top, so that the coefficient of friction increases with the distance x along the plank: $\mu = Ax$, where A is a positive constant and the bottom of the plank is at $x = 0$. (For this plank the coefficients of kinetic and static friction are equal: $\mu_k = \mu_s = \mu$.) The worker shoves a box up the plank so that it leaves the bottom of the plank moving at speed v_0. Show that when the box first comes to rest, it will remain at rest if

$$v_0^2 \geq \frac{3g\sin^2\alpha}{A\cos\alpha}$$

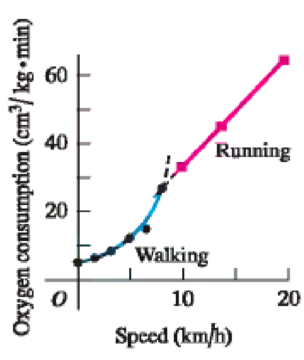
6.101. A Spring with Mass. We usually ignore the kinetic energy of the moving coils of a spring, but let's try to get a reasonable approximation to this. Consider a spring of mass M, equilibrium length L_0, and spring constant k. The work done to stretch or compress the spring by a distance L is $\frac{1}{2}kX^2$, where $X = L - L_0$. (a) Consider a spring, as described above, that has one end fixed and the other end moving with speed v. Assume that the speed of points along the length of the spring varies linearly with distance l from the fixed end. Assume also that the mass M of the spring is distributed uniformly along the length of the spring. Calculate the kinetic energy of the spring in terms of M and v. (*Hint:* Divide the spring into pieces of length dl; find the speed of each piece in terms of l, v, and L; find the mass of each piece in terms of dl, M, and L; and integrate from 0 to L. The result is *not* $\frac{1}{2}Mv^2$, since not all of the spring moves with the same speed.) In a spring gun, a spring of mass 0.243 kg and force constant $3200 \mathrm{~N/m}$ is compressed 2.50 cm from its unstretched length. When the trigger is pulled, the spring pushes horizontally on a 0.053-kg ball. The work done by friction is negligible. Calculate the ball's speed when the spring reaches its uncompressed length (b) ignoring the mass of the spring and (c) including, using the results of part (a), the mass of the spring. (d) In part (c), what is the final kinetic energy of the ball and of the spring?

6.102. An airplane in flight is subject to an air resistance force proportional to the square of its speed v. But there is an additional resistive force because the airplane has wings. Air flowing over the wings is pushed down and slightly forward, so from Newton's third law the air exerts a force on the wings and airplane that is up and slightly backward (Fig. 6.39). The upward force is the lift force that keeps the airplane aloft, and the backward force is called *induced drag*. At flying speeds, induced drag is inversely proportional to v^2, so that the total air resistance force can be expressed by $F_{\mathrm{air}} = \alpha v^2 + \beta/v^2$, where α and β are positive constants that depend on the shape and size of the airplane and the density of the air. For a Cessna 150, a small single-engine airplane, $\alpha = 0.30 \mathrm{~N \cdot s^2/m^2}$ and $\beta = 3.5 \times 10^5 \mathrm{~N \cdot m^2/s^2}$. In steady flight, the engine must provide a forward force that exactly balances the air resistance force. (a) Calculate the speed (in km/h) at which this airplane will have the maximum *range* (that is, travel the greatest distance) for a given quantity of fuel. (b) Calculate the speed (in km/h) for which the airplane will have the maximum *endurance* (that is, remain in the air the longest time).

Figure 6.39 Challenge Problem 6.102.

6.103. Figure 6.40 shows the oxygen consumption rate of men walking and running at different speeds. The vertical axis shows the volume of oxygen (in cm^3) that a man consumes per kilogram

Figure 6.40 Challenge Problem 6.103.



of body mass per minute. Note the transition from walking to running that occurs naturally at about 9 km/h. The metabolism of 1 cm^3 of oxygen releases about 20 J of energy. Using the data in the graph, calculate the energy required for a 70-kg man to travel 1 km on foot at (a) 5 km/h (walking); (b) 10 km/h (running); (c) 15 km/h (running). (d) Which speed is the most efficient—that is, requires the least energy to travel 1 km?

6.104. General Proof of the Work–Energy Theorem. Consider a particle that moves along a curved path in space from (x_1, y_1, z_1) to (x_2, y_2, z_2). At the initial point, the particle has velocity $\vec{v} = v_{1x}\hat{\imath} + v_{1y}\hat{\jmath} + v_{1z}\hat{k}$. The path that the particle follows may be divided into infinitesimal segments $d\vec{l} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$. As the particle moves, it is acted on by a net force $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$. The force components F_x, F_y, and F_z are in general functions of position. By the same sequence of steps used in Eqs. (6.11) through (6.13), prove the work–energy theorem for this general case. That is, prove that

$$W_{tot} = K_2 - K_1$$

where

$$W_{tot} = \int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} \vec{F} \cdot d\vec{l} = \int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} (F_x\,dx + F_y\,dy + F_z\,dz)$$

7 POTENTIAL ENERGY AND ENERGY CONSERVATION

? As this diver enters the water, is the force of gravity doing positive or negative work on him? Is the water doing positive or negative work on him?

LEARNING GOALS

By studying this chapter, you will learn:

- How to use the concept of gravitational potential energy in problems that involve vertical motion.
- How to use the concept of elastic potential energy in problems that involve a moving body attached to a stretched or compressed spring.
- The distinction between conservative and nonconservative forces, and how to solve problems in which both kinds of forces act on a moving body.
- How to calculate the properties of a conservative force if you know the corresponding potential-energy function.
- How to use energy diagrams to understand the motion of an object moving in a straight line under the influence of a conservative force.

When a diver jumps off a high board into a swimming pool, he hits the water moving pretty fast, with a lot of kinetic energy. Where does that energy come from? The answer we learned in Chapter 6 was that the gravitational force (his weight) does work on the diver as he falls. The diver's kinetic energy—energy associated with his *motion*—increases by an amount equal to the work done.

However, there is a very useful alternative way to think about work and kinetic energy. This new approach is based on the concept of *potential energy,* which is energy associated with the *position* of a system rather than its motion. In this approach, there is *gravitational potential energy* even while the diver is standing on the high board. Energy is not added to the earth–diver system as the diver falls, but rather a storehouse of energy is *transformed* from one form (potential energy) to another (kinetic energy) as he falls. In this chapter we'll see how the work–energy theorem explains this transformation.

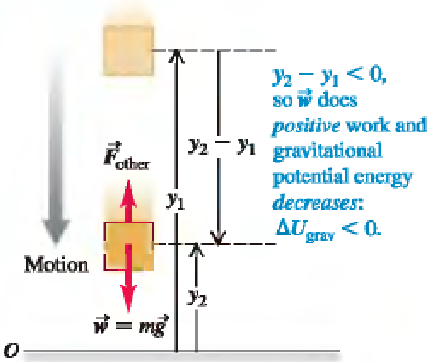
If the diver bounces on the end of the board before he jumps, the bent board stores a second kind of potential energy called *elastic potential energy.* We'll discuss elastic potential energy of simple systems such as a stretched or compressed spring. (An important third kind of potential energy is associated with the positions of electrically charged particles relative to each other. We'll encounter this potential energy in Chapter 23.)

We will prove that in some cases the sum of a system's kinetic and potential energy, called the *total mechanical energy* of the system, is constant during the motion of the system. This will lead us to the general statement of the *law of conservation of energy,* one of the most fundamental and far-reaching principles in all of science.

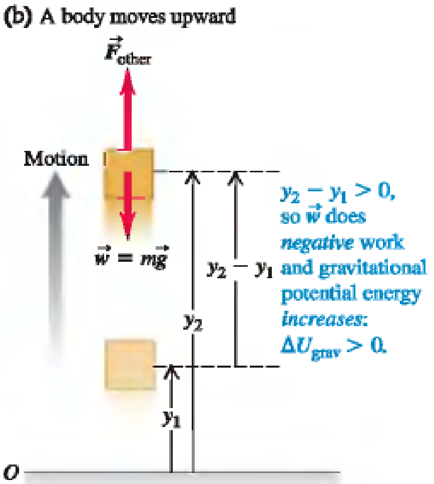
7.1 As a basketball descends, gravitational potential energy is converted to kinetic energy and the basketball's speed increases.

7.2 When a body moves vertically from an initial height y_1 to a final height y_2, the gravitational force $\vec{w}$ does work and the gravitational potential energy changes.

(a) A body moves downward



(b) A body moves upward



7.1 Gravitational Potential Energy

We learned in Chapter 6 that a particle gains or loses kinetic energy because it interacts with other objects that exert forces on it. During any interaction, the change in a particle's kinetic energy is equal to the total work done on the particle by the forces that act on it.

In many situations it seems as though energy has been stored in a system, to be recovered later. For example, you must do work to lift a heavy stone over your head. It seems reasonable that in hoisting the stone into the air you are storing energy in the system, energy that is later converted into kinetic energy when you let the stone fall.

This example points to the idea of an energy associated with the *position* of bodies in a system. This kind of energy is a measure of the *potential* or *possibility* for work to be done; when a stone is raised into the air, there is a potential for work to be done on it by the gravitational force, but only if the stone is allowed to fall to the ground. For this reason, energy associated with position is called **potential energy.** Our discussion suggests that there is potential energy associated with a body's weight and its height above the ground. We call this *gravitational potential energy* (Fig. 7.1).

We now have *two* ways to describe what happens when a body falls without air resistance. One way is to say that gravitational potential energy decreases and the falling body's kinetic energy increases. The other way, which we learned in Chapter 6, is that a falling body's kinetic energy increases because the force of the earth's gravity (the body's weight) does work on the body. Later in this section we'll use the work–energy theorem to show that these two descriptions are equivalent.

To begin with, however, let's derive the expression for gravitational potential energy. Suppose a body with mass m moves along the (vertical) y-axis, as in Fig. 7.2. The forces acting on it are its weight, with magnitude $w = mg$, and possibly some other forces; we call the vector sum (resultant) of all the other forces $\vec{F}_{other}$. We'll assume that the body stays close enough to the earth's surface that the weight is constant. (We'll find in Chapter 12 that weight decreases with altitude.) We want to find the work done by the weight when the body moves downward from a height y_1 above the origin to a lower height y_2 (Fig. 7.2a). The weight and displacement are in the same direction, so the work W_{grav} done on the body by its weight is positive;

$$W_{grav} = Fs = w(y_1 - y_2) = mgy_1 - mgy_2 \tag{7.1}$$

This expression also gives the correct work when the body moves *upward* and y_2 is greater than y_1 (Fig. 7.2b). In that case the quantity $(y_1 - y_2)$ is negative, and W_{grav} is negative because the weight and displacement are opposite in direction.

Equation (7.1) shows that we can express W_{grav} in terms of the values of the quantity mgy at the beginning and end of the displacement. This quantity, the product of the weight mg and the height y above the origin of coordinates, is called the **gravitational potential energy,** U_{grav}:

$$U_{grav} = mgy \qquad \text{(gravitational potential energy)} \tag{7.2}$$

Its initial value is $U_{grav,1} = mgy_1$ and its final value is $U_{grav,2} = mgy_2$. The change in U_{grav} is the final value minus the initial value, or $\Delta U_{grav} = U_{grav,2} - U_{grav,1}$. We can express the work W_{grav} done by the gravitational force during the displacement from y_1 to y_2 as

$$W_{grav} = U_{grav,1} - U_{grav,2} = -(U_{grav,2} - U_{grav,1}) = -\Delta U_{grav} \tag{7.3}$$

The negative sign in front of ΔU_{grav} is *essential*. When the body moves up, y increases, the work done by the gravitational force is negative, and the gravitational potential energy increases $(\Delta U_{grav} > 0)$. When the body moves down, y decreases, the gravitational force does positive work, and the gravitational potential energy decreases $(\Delta U_{grav} < 0)$. It's like drawing money out of the bank (decreasing U_{grav}) and spending it (doing positive work). As Eq. (7.3) shows, the unit of potential energy is the joule (J), the same unit as is used for work.

CAUTION **To what body does gravitational potential energy "belong"?** It is *not* correct to call $U_{grav} = mgy$ the "gravitational potential energy of the body." The reason is that gravitational potential energy U_{grav} is a *shared* property of the body and the earth. The value of U_{grav} increases if the earth stays fixed and the body moves upward, away from the earth; it also increases if the body stays fixed and the earth is moved away from it. Notice that the formula $U_{grav} = mgy$ involves characteristics of both the body (its mass m) and the earth (the value of g).

Conservation of Mechanical Energy (Gravitational Forces Only)

To see what gravitational potential energy is good for, suppose the body's weight is the *only* force acting on it, so $\vec{F}_{other} = \mathbf{0}$. The body is then falling freely with no air resistance, and can be moving either up or down. Let its speed at point y_1 be v_1 and let its speed at y_2 be v_2. The work–energy theorem, Eq. (6.6), says that the total work done on the body equals the change in the body's kinetic energy: $W_{tot} = \Delta K = K_2 - K_1$. If gravity is the only force that acts, then from Eq. (7.3), $W_{tot} = W_{grav} = -\Delta U_{grav} = U_{grav,1} - U_{grav,2}$. Putting these together, we get

$$\Delta K = -\Delta U_{grav} \qquad \text{or} \qquad K_2 - K_1 = U_{grav,1} - U_{grav,2}$$

which we can rewrite as

$$K_1 + U_{grav,1} = K_2 + U_{grav,2} \qquad \text{(if only gravity does work)} \tag{7.4}$$

or

$$\frac{1}{2}mv_1^2 + mgy_1 = \frac{1}{2}mv_2^2 + mgy_2 \qquad \text{(if only gravity does work)} \tag{7.5}$$

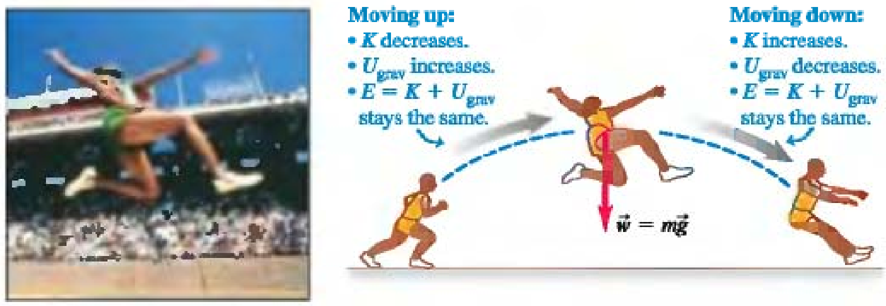
The sum $K + U_{grav}$ of kinetic and potential energy is called E, the **total mechanical energy of the system.** By "system" we mean the body of mass m and the earth considered together, because gravitational potential energy U is a shared property of both bodies. Then $E_1 = K_1 + U_{grav,1}$ is the total mechanical energy at y_1 and $E_2 = K_2 + U_{grav,2}$ is the total mechanical energy at y_2. Equation (7.4) says that when the body's weight is the only force doing work on it, $E_1 = E_2$. That is, E is constant; it has the same value at y_1 and y_2. But since the positions y_1 and y_2 are arbitrary points in the motion of the body, the total mechanical energy E has the same value at *all* points during the motion:

$$E = K + U_{grav} = \text{constant} \qquad \text{(if only gravity does work)}$$

A quantity that always has the same value is called a *conserved* quantity. *When only the force of gravity does work, the total mechanical energy is constant—that is, is conserved* (Fig. 7.3). This is our first example of the **conservation of mechanical energy.**

When we throw a ball into the air, its speed decreases on the way up as kinetic energy is converted to potential energy; $\Delta K < 0$ and $\Delta U_{grav} > 0$. On the way back down, potential energy is converted back to kinetic energy and the ball's speed increases; $\Delta K > 0$ and $\Delta U_{grav} < 0$. But the *total* mechanical energy (kinetic plus potential) is the same at every point in the motion, provided that no force other than gravity does work on the ball (that is, air resistance must be

7.3 While this athlete is in midair, only gravity does work on him (if we neglect the minor effects of air resistance). Mechanical energy E—the sum of kinetic and gravitational potential energy—is conserved.



negligible). It's still true that the gravitational force does work on the body as it moves up or down, but we no longer have to calculate work directly; keeping track of changes in the value of U_{grav} takes care of this completely.

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5.2 Upward-Moving Elevator Stops
5.3 Stopping a Downward-Moving Elevator
5.6 Skier Speed

CAUTION **Choose "zero height" to be wherever you like** When working with gravitational potential energy, we may choose any height to be $y = 0$. If we shift the origin for y, the values of y_1 and y_2 change, as do the values of $U_{grav,1}$ and $U_{grav,2}$. But this shift has no effect on the *difference* in height $y_2 - y_1$ or on the *difference* in gravitational potential energy $U_{grav,2} - U_{grav,1} = mg(y_2 - y_1)$. As the following example shows, the physically significant quantity is not the value of U_{grav} at a particular point, but only the *difference* in U_{grav} between two points. So we can define U_{grav} to be zero at whatever point we choose without affecting the physics.

Example 7.1 Height of a baseball from energy conservation

You throw a 0.145-kg baseball straight up in the air, giving it an initial upward velocity of magnitude 20.0 m/s. Find how high it goes, ignoring air resistance.

SOLUTION

IDENTIFY: After the ball leaves your hand, the only force doing work on the ball is gravity. Hence we can use conservation of mechanical energy.

SET UP: We'll use Eqs. (7.4) and (7.5), taking point 1 to be where the ball leaves your hand and point 2 to be where it reaches its maximum height. As in Fig. 7.2, we take the positive y-direction to be upward. The ball's speed at point 1 is $v_1 = 20.0$ m/s; at its maximum height the ball is instantaneously at rest, so $v_2 = 0$.

We want to know how far the ball moves vertically between the two points, so our target variable is the displacement $y_2 - y_1$. If we take the origin to be where the ball leaves your hand (point 1), then $y_1 = 0$ (Fig. 7.4) and the target variable is just y_2.

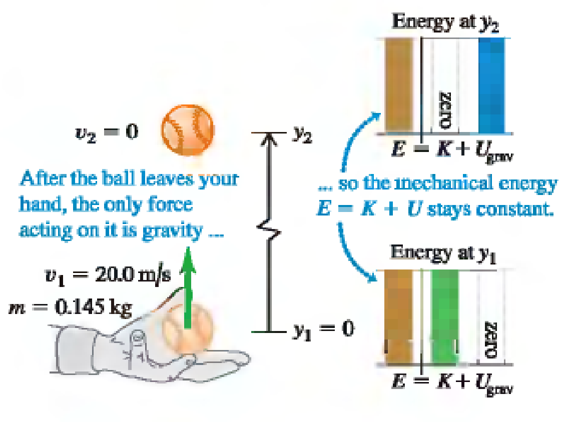
EXECUTE: Since $y_1 = 0$, the potential energy at point 1 is $U_{grav,1} = mgy_1 = 0$. Furthermore, since the ball is at rest at point 2, the kinetic energy at that point is $K_2 = \frac{1}{2}mv_2^2 = 0$. Hence Eq. (7.4), which says that $K_1 + U_{grav,1} = K_2 + U_{grav,2}$, becomes

$$K_1 = U_{grav,2}$$

As the energy bar graphs in Fig. 7.4 show, the kinetic energy of the ball at point 1 is completely converted to gravitational potential energy at point 2. At point 1 the kinetic energy is

$$K_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}(0.145\text{ kg})(20.0\text{ m/s})^2 = 29.0\text{ J}$$

7.4 After a baseball leaves your hand, mechanical energy $E = K + U$ is conserved.



This equals the gravitational potential energy $U_{grav,2} = mgy_2$ at point 2, so

$$y_2 = \frac{U_{grav,2}}{mg} = \frac{29.0\text{ J}}{(0.145\text{ kg})(9.80\text{ m/s}^2)} = 20.4\text{ m}$$

We can also solve the equation $K_1 = U_{grav,2}$ algebraically for y_2:

$$\frac{1}{2}mv_1^2 = mgy_2$$

$$y_2 = \frac{v_1^2}{2g} = \frac{(20.0\text{ m/s})^2}{2(9.80\text{ m/s}^2)} = 20.4\text{ m}$$

EVALUATE: The mass divides out, as we should expect; we learned in Chapter 2 that the motion of a body in free fall doesn't depend on its mass. Indeed, we could have derived the result $y_2 = v_1^2/2g$ using Eq. (2.13).

In our calculation we chose the origin to be at point 1, so $y_1 = 0$ and $U_{grav,1} = 0$. What happens if we make a different choice? As an example, suppose we choose the origin to be 5.0 m below point 1, so $y_1 = 5.0$ m. Then the total mechanical energy at point 1 is part kinetic and part potential, while at point 2 it's purely potential energy. If you work through the calculation again with this choice of origin, you'll find $y_2 = 25.4$ m; this is 20.4 m above point 1, just as with the first choice of origin. In problems like this, the choice of height at which $U_{grav} = 0$ is up to you; don't agonize over the choice, though, because the physics of the answer doesn't depend on your choice.

When Forces Other Than Gravity Do Work

If other forces act on the body in addition to its weight, then $\vec{F}_{other}$ in Fig. 7.2 is *not* zero. For the pile driver described in Example 6.4 (Section 6.2), the force applied by the hoisting cable and the friction with the vertical guide rails are examples of forces that might be included in $\vec{F}_{other}$. The gravitational work W_{grav} is still given by Eq. (7.3), but the total work W_{tot} is then the sum of W_{grav} and the work done by $\vec{F}_{other}$. We will call this additional work W_{other}, so the total work done by all forces is $W_{tot} = W_{grav} + W_{other}$. Equating this to the change in kinetic energy, we have

$$W_{other} + W_{grav} = K_2 - K_1 \tag{7.6}$$

Also, from Eq. (7.3), $W_{grav} = U_{grav,1} - U_{grav,2}$, so

$$W_{other} + U_{grav,1} - U_{grav,2} = K_2 - K_1$$

which we can rearrange in the form

$$K_1 + U_{grav,1} + W_{other} = K_2 + U_{grav,2} \quad \text{(if forces other than gravity do work)} \tag{7.7}$$

Finally, using the appropriate expressions for the various energy terms, we obtain

$$\frac{1}{2}mv_1^2 + mgy_1 + W_{other} = \frac{1}{2}mv_2^2 + mgy_2 \quad \text{(if forces other than gravity do work)} \tag{7.8}$$

The meaning of Eqs. (7.7) and (7.8) is this: *The work done by all forces* other than the gravitational force *equals the change in the total mechanical energy* $E = K + U_{grav}$ *of the system, where* U_{grav} *is the gravitational potential energy.* When W_{other} is positive, E increases, and $K_2 + U_{grav,2}$ is greater than $K_1 + U_{grav,1}$. When W_{other} is negative, E decreases (Fig. 7.5). In the special case in which no forces other than the body's weight do work, $W_{other} = 0$. The total mechanical energy is then constant, and we are back to Eq. (7.4) or (7.5).

7.5 As this skydiver moves downward, the upward force of air resistance does negative work W_{other} on him. Hence the total mechanical energy $E = K + U$ decreases: The skydiver's speed and kinetic energy K stay the same, while the gravitational potential energy U goes down.

Problem-Solving Strategy 7.1 Problems Using Mechanical Energy I

IDENFITY *the relevant concepts:* Decide whether the problem should be solved by energy methods, by using $\sum\vec{F} = m\vec{a}$ directly, or by a combination of these. The energy approach is best when the problem involves varying forces, motion along a curved path (discussed later in this section), or both. If the problem involves elapsed time, the energy approach is usually *not* the best choice, because it doesn't involve time directly.

SET UP *the problem* using the following steps:

1. When using the energy approach, first decide what the initial and final states (the positions and velocities) of the system are. Use the subscript 1 for the initial state and the subscript 2 for the final state. It helps to draw sketches showing the initial and final states.
2. Define your coordinate system, particularly the level at which $y = 0$. You will use it to compute gravitational potential energies. We suggest that you always choose the positive y-direction to be upward because this is what Eq. (7.2) assumes.
3. Identify all forces that do work that can't be described in terms of potential energy. (So far this means any forces other than gravity. But later in this chapter we'll see that the work done by an ideal spring can also be expressed as a change in potential energy.) A free-body diagram is always helpful.

Continued

4. List the unknown and known quantities, including the coordinates and velocities at each point. Decide which unknowns are your target variables.

EXECUTE *the solution:* Write expressions for the initial and final kinetic and potential energies—that is, K_1, K_2, $U_{grav,1}$, and $U_{grav,2}$. Then relate the kinetic and potential energies and the work done by other forces, W_{other}, using Eq. (7.7). (You will have to calculate W_{other} in terms of these forces.) If no other forces do work, this expression becomes Eq. (7.4). It's helpful to draw bar graphs showing the initial and final values of K, U_{grav}, and $E = K + U_{grav}$. Then solve to find whatever unknown quantity is required.

EVALUATE *your answer:* Check whether your answer makes physical sense. Keep in mind, here and in later sections, that the work done by each force must be represented either in $U_{grav,1} - U_{grav,2} = -\Delta U_{grav}$ or as W_{other}, but *never* in both places. The gravitational work is included in ΔU_{grav}, so make sure you did not include it again in W_{other}.

Example 7.2 Work and energy in throwing a baseball

In Example 7.1, suppose your hand moves up 0.50 m while you are throwing the ball, which leaves your hand with an upward velocity of 20.0 m/s. Again ignore air resistance. (a) Assuming that your hand exerts a constant upward force on the ball, find the magnitude of that force. (b) Find the speed of the ball at a point 15.0 m above the point where it leaves your hand.

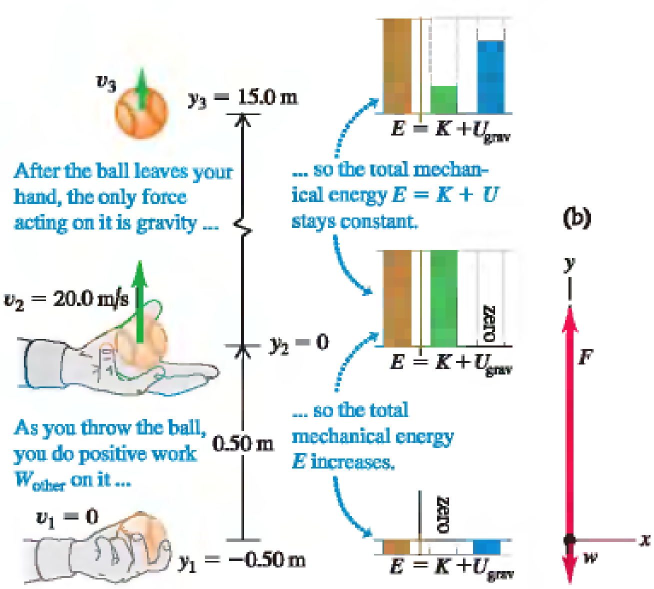
SOLUTION

IDENTIFY: In Example 7.1 we used conservation of mechanical energy because only gravity did work. In this example, however, we must also include the nongravitational work done by your hand.

SET UP: Figure 7.6 shows a diagram of the situation, including a free-body diagram for the ball while it is being thrown. We let point 1 be where your hand first starts to move, point 2 be where the ball leaves your hand, and point 3 be where the ball is 15.0 m above point 2. The nongravitational force $\vec{F}$ of your hand acts only between points 1 and 2. Using the same coordinate system as in Example 7.1, we have $y_1 = -0.50$ m, $y_2 = 0$, and $y_3 = 15.0$ m. The ball starts at rest at point 1, so $v_1 = 0$, and we are given that the ball's speed as it leaves your hand is $v_2 = 20.0$ m/s. Our target variables are (a) the magnitude F of the force of your hand and (b) the speed v_3 at point 3.

7.6 (a) Applying energy ideas to a ball thrown vertically upward. (b) Free-body diagram for the ball as you throw it.

(a)



EXECUTE: (a) To determine the magnitude of $\vec{F}$, we'll first use Eq. (7.7) to calculate the work W_{other} done by this force. We have

$$K_1 = 0$$
$$U_{grav,1} = mgy_1 = (0.145\text{ kg})(9.80\text{ m/s}^2)(-0.50\text{ m}) = -0.71\text{ J}$$
$$K_2 = \frac{1}{2}mv_2^2 = \frac{1}{2}(0.145\text{ kg})(20.0\text{ m/s})^2 = 29.0\text{ J}$$
$$U_{grav,2} = mgy_2 = (0.145\text{ kg})(9.80\text{ m/s}^2)(0) = 0$$

The initial potential energy $U_{grav,1}$ is *negative* because the ball was initially below the origin. (Don't worry about having a potential energy that's less than zero. Remember, all that matters is the *difference* in potential energy from one point to another.) According to Eq. (7.7), $K_1 + U_{grav,1} + W_{other} = K_2 + U_{grav,2}$, so

$$W_{other} = (K_2 - K_1) + (U_{grav,2} - U_{grav,1})$$
$$= (29.0\text{ J} - 0) + (0 - (-0.71\text{ J})) = 29.7\text{ J}$$

The kinetic energy of the ball increases by $K_2 - K_1 = 29.0$ J, and the potential energy increases by $U_{grav,2} - U_{grav,1} = 0.71$ J; the sum is $E_2 - E_1$, the change in total mechanical energy, which is equal to W_{other}.

Assuming the upward force $\vec{F}$ that your hand applies is constant, the work W_{other} done by this force is equal to the magnitude F of the force multiplied by the upward displacement $y_2 - y_1$ over which it acts:

$$W_{other} = F(y_2 - y_1)$$
$$F = \frac{W_{other}}{y_2 - y_1} = \frac{29.7\text{ J}}{0.50\text{ m}} = 59\text{ N}$$

This is about 40 times greater than the weight of the ball.

(b) To find the speed at point 3, note that between points 2 and 3, total mechanical energy is conserved; the force of your hand no longer acts, so $W_{other} = 0$. We can then find the kinetic energy at point 3 using Eq. (7.4):

$$K_2 + U_{grav,2} = K_3 + U_{grav,3}$$
$$U_{grav,3} = mgy_3 = (0.145\text{ kg})(9.80\text{ m/s}^2)(15.0\text{ m}) = 21.3\text{ J}$$
$$K_3 = (K_2 + U_{grav,2}) - U_{grav,3}$$

$$= (29.0\text{ J} + 0\text{ J}) - 21.3\text{ J} = 7.7\text{ J}$$

Since $K_3 = \frac{1}{2}mv_{3y}^2$, where v_{3y} is the y-component of the ball's velocity at point 3, we have

$$v_{3y} = \pm\sqrt{\frac{2K_3}{m}} = \pm\sqrt{\frac{2(7.7\text{ J})}{0.145\text{ kg}}} = \pm 10\text{ m/s}$$

The significance of the plus-or-minus sign is that the ball passes point 3 *twice*, once on the way up and again on the way down. The total mechanical energy E is constant and equal to 29.0 J while the ball is in free fall, and the potential energy at point 3 is $U_{\text{grav},3} = 21.3$ J whether the ball is moving up or down. So at point 3, the ball's kinetic energy K_3 and *speed* don't depend on the direction the ball is moving. The velocity v_{3y} is positive $(+10\text{ m/s})$ when the ball is moving up and negative (-10 m/s) when it is moving down; the speed v_3 is 10 m/s in either case.

EVALUATE: As a check on our result, recall from Example 7.1 that the ball reaches a maximum height $y = 20.4$ m. At that point all of the kinetic energy that the ball had when it left your hand at $y = 0$ has been converted to gravitational potential energy. At $y = 15.0$ m, the ball is about three-fourths of the way to its maximum height, so about three-fourths of its mechanical energy should be in the form of potential energy. (This is shown in the energy bar graphs in Fig. 7.6a.) Can you show that this is true from our results for K_3 and $U_{\text{grav},3}$?

Gravitational Potential Energy for Motion Along a Curved Path

In our first two examples the body moved along a straight vertical line. What happens when the path is slanted or curved (Fig. 7.7a)? The body is acted on by the gravitational force $\vec{w} = m\vec{g}$ and possibly by other forces whose resultant we call $\vec{F}_{\text{other}}$. To find the work done by the gravitational force during this displacement, we divide the path into small segments $\Delta\vec{s}$; Fig. 7.7b shows a typical segment. The work done by the gravitational force over this segment is the scalar product of the force and the displacement. In terms of unit vectors, the force is $\vec{w} = m\vec{g} = -mg\hat{\jmath}$ and the displacement is $\Delta\vec{s} = \Delta x\hat{\imath} + \Delta y\hat{\jmath}$, so the work done by the gravitational force is

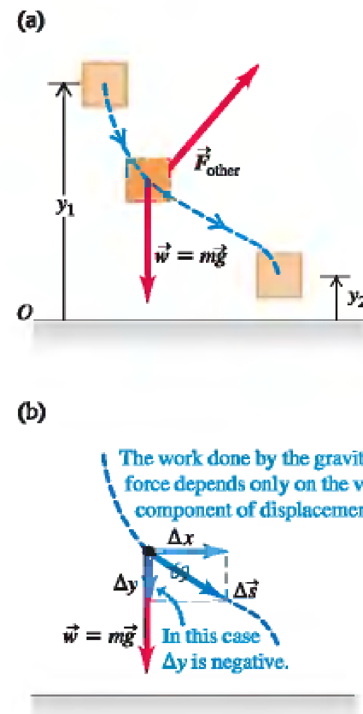
$$\vec{w}\cdot\Delta\vec{s} = -mg\hat{\jmath}\cdot(\Delta x\hat{\imath} + \Delta y\hat{\jmath}) = -mg\Delta y$$

The work done by gravity is the same as though the body had been displaced vertically a distance Δy, with no horizontal displacement. This is true for every segment, so the *total* work done by the gravitational force is $-mg$ multiplied by the *total* vertical displacement $(y_2 - y_1)$:

$$W_{\text{grav}} = -mg(y_2 - y_1) = mgy_1 - mgy_2 = U_{\text{grav},1} - U_{\text{grav},2}$$

This is the same as Eq. (7.1) or (7.3), in which we assumed a purely vertical path. So even if the path a body follows between two points is curved, the total work done by the gravitational force depends only on the difference in height between the two points of the path. This work is unaffected by any horizontal motion that may occur. So *we can use the same expression for gravitational potential energy whether the body's path is curved or straight.*

7.7 Calculating the change in gravitational potential energy for a displacement along a curved path.



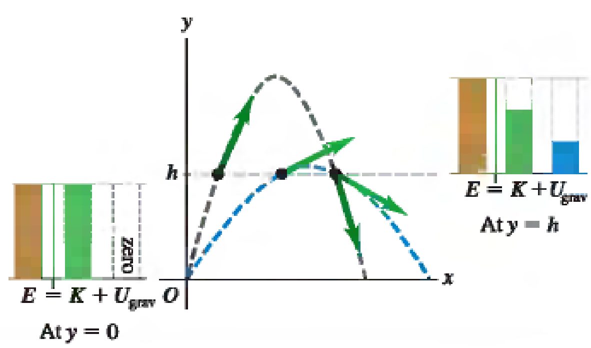
Conceptual Example 7.3 Energy in projectile motion

A batter hits two identical baseballs with the same initial speed and height but different initial angles. Prove that at a given height h, both balls have the same speed if air resistance can be neglected.

SOLUTION

If there is no air resistance, the only force acting on each ball after it is hit is its weight. Hence the total mechanical energy for each ball is constant. Figure 7.8 shows the trajectories of two balls batted at the same height with the same initial speed, and thus the same total mechanical energy, but with different initial angles. At all points at the same height the potential energy is the same. Thus the kinetic energy at this height must be the same for both balls, and the speeds are the same.

7.8 For the same initial speed and initial height, the speed of a projectile at a given elevation h is always the same, neglecting air resistance.



Example 7.4 Calculating speed along a vertical circle

Your cousin Throckmorton skateboards down a curved playground ramp. If we treat Throcky and his skateboard as a particle, he moves through a quarter-circle with radius $R = 3.00$ m (Fig. 7.9). The total mass of Throcky and his skateboard is 25.0 kg. He starts from rest and there is no friction. (a) Find his speed at the bottom of the ramp. (b) Find the normal force that acts on him at the bottom of the curve.

SOLUTION

IDENTIFY: We can't use the constant-acceleration equations because Throcky's acceleration isn't constant; the slope decreases as he descends. Instead, we'll use the energy approach. Since Throcky moves along a circular arc, we'll also use what we learned about circular motion in Section 5.4.

SET UP: Since there is no friction, the only force other than Throcky's weight is the normal force $\vec{n}$ exerted by the ramp (Fig. 7.9b). Although this force acts all along the path, it does *zero* work because $\vec{n}$ is perpendicular to Throcky's displacement at every point. Hence $W_{\text{other}} = 0$ and mechanical energy is conserved.

We take point 1 at the starting point and point 2 at the bottom of the curved ramp, and we let $y = 0$ be at the bottom of the ramp (Fig. 7.9a). Then $y_1 = R$ and $y_2 = 0$. (We are treating Throcky as if his entire mass were concentrated at his center.) Throcky starts at rest at the top, so $v_1 = 0$. Our target variable in part (a) is his speed at the bottom, v_2. In part (b) we want to find the magnitude n of the normal force at point 2. Because this force does no work, it doesn't appear in the energy equation, so we'll use Newton's second law instead.

EXECUTE: (a) The various energy quantities are

$$K_1 = 0 \qquad U_{\text{grav},1} = mgR$$
$$K_2 = \frac{1}{2}mv_2^2 \qquad U_{\text{grav},2} = 0$$

From conservation of mechanical energy,

$$\begin{aligned} K_1 + U_{\text{grav},1} &= K_2 + U_{\text{grav},2} \\ 0 + mgR &= \frac{1}{2}mv_2^2 + 0 \\ v_2 &= \sqrt{2gR} \\ &= \sqrt{2(9.80\ \text{m/s}^2)(3.00\ \text{m})} = 7.67\ \text{m/s} \end{aligned}$$

Notice that this answer doesn't depend on the ramp being circular; no matter what the shape of the ramp, Throcky will have the same speed $v_2 = \sqrt{2gR}$ at the bottom. This would be true even if the wheels of his skateboard lost contact with the ramp during the ride, because only the gravitational force would still do work. In fact, the speed is the same as if Throcky had fallen vertically through a height R. The answer is also independent of his mass.

(b) To find n at point 2 using Newton's second law, we need the free-body diagram at that point (Fig. 7.9b). At point 2, Throcky is moving at speed $v_2 = \sqrt{2gR}$ in a circle of radius R; his acceleration is toward the center of the circle and has magnitude

$$a_{\text{rad}} = \frac{v_2^2}{R} = \frac{2gR}{R} = 2g$$

If we take the positive y-direction to be upward, the y-component of Newton's second law is

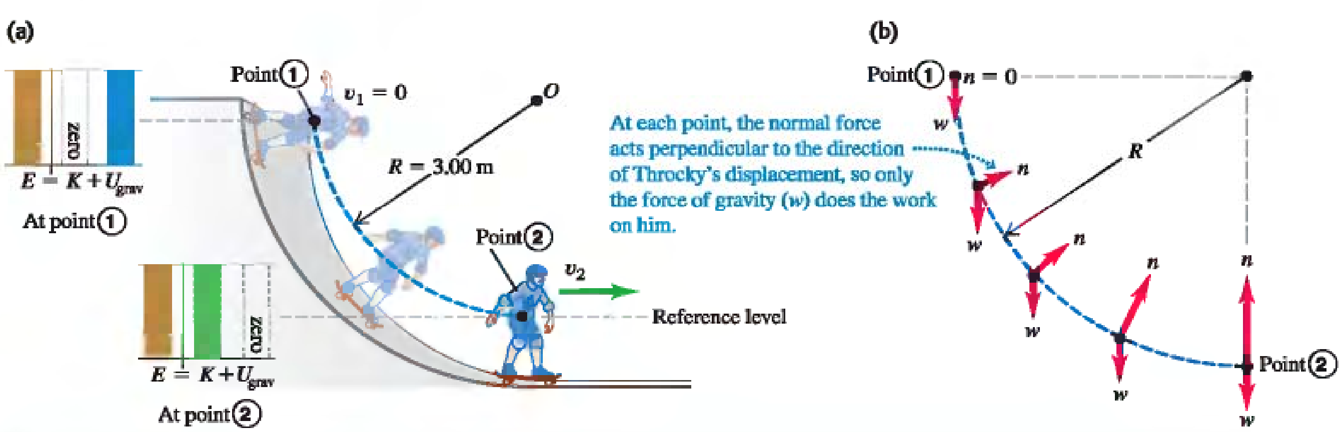
$$\begin{aligned} \sum F_y &= n + (-w) = ma_{\text{rad}} = 2mg \\ n &= w + 2mg = 3mg \\ &= 3(25.0\ \text{kg})(9.80\ \text{m/s}^2) = 735\ \text{N} \end{aligned}$$

At point 2 the normal force is three times Throcky's weight. This result is independent of the radius of the circular ramp. We learned in Example 5.9 (Section 5.2) and Example 5.24 (Section 5.4) that the magnitude of n is the *apparent weight*, so Throcky feels as though he weighs three times his true weight mg. But as soon as he reaches the horizontal part of the ramp to the right of point 2, the normal force decreases to $w = mg$ and Throcky feels normal again. Can you see why?

EVALUATE: This example shows a general rule about the role of forces in problems in which we use energy techniques: What matters is not simply whether a force *acts*, but whether that force *does work*. If the force does no work, like the normal force $\vec{n}$ in this example, then it does not appear at all in Eq. (7.7), $K_1 + U_{\text{grav},1} + W_{\text{other}} = K_2 + U_{\text{grav},2}$.

Notice we had to use *both* the energy approach and Newton's second law to solve this problem; energy conservation gave us the speed and $\sum\vec{F} = m\vec{a}$ gave us the normal force. For each part of the problem we used the technique that most easily led us to the answer.

7.9 (a) Throcky skateboarding down a frictionless circular ramp. The total mechanical energy is constant. (b) Free-body diagrams for Throcky and his skateboard at various points on the ramp.



Example 7.5 A vertical circle with friction

In Example 7.4, suppose that the ramp is not frictionless and that Throcky's speed at the bottom is only 6.00 m/s. What work was done by the friction force acting on him?

SOLUTION

IDENTIFY: Figure 7.10 shows that again the normal force does no work, but now there is a friction force $\vec{f}$ that *does* do work. Hence the nongravitational work done on Throcky between points 1 and 2, W_{other}, is not zero.

SET UP: We use the same coordinate system and the same initial and final points as in Example 7.4 (see Fig. 7.10). Our target variable is the work done by friction, W_f; since friction is the only force other than gravity that does work, this is just equal to W_{other}. We'll find W_f using Eq. (7.7).

EXECUTE: The energy quantities are

$$K_1 = 0$$

$$U_{\text{grav},1} = mgR = (25.0\ \text{kg})(9.80\ \text{m/s}^2)(3.00\ \text{m}) = 735\ \text{J}$$

$$K_2 = \frac{1}{2}mv_2^2 = \frac{1}{2}(25.0\ \text{kg})(6.00\ \text{m/s})^2 = 450\ \text{J}$$

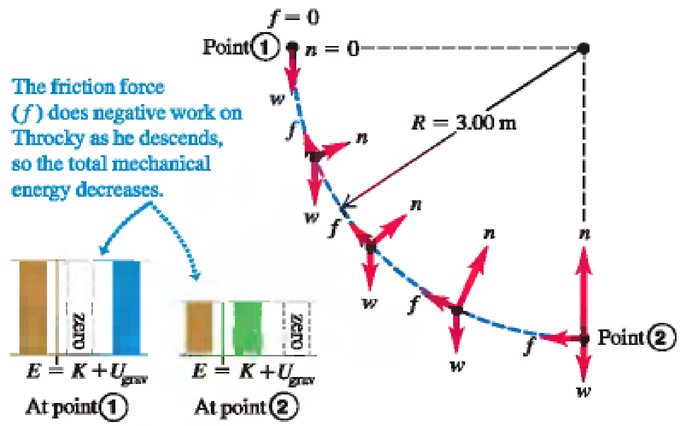
$$U_{\text{grav},2} = 0$$

From Eq. (7.7),

$$\begin{aligned} W_f &= K_2 + U_{\text{grav},2} - K_1 - U_{\text{grav},1} \\ &= 450\ \text{J} + 0 - 0 - 735\ \text{J} = -285\ \text{J} \end{aligned}$$

The work done by the friction force is -285 J, and the total mechanical energy *decreases* by 285 J. Do you see why W_f has to be negative?

7.10 Free-body diagram and energy bar graphs for Throcky skateboarding down a ramp with friction.



EVALUATE: Throcky's motion is determined by Newton's second law, $\sum\vec{F} = m\vec{a}$. But it would be very difficult to apply the second law directly to this problem because the normal and friction forces and the acceleration are continuously changing in both magnitude and direction as Throcky moves. The energy approach, by contrast, relates the motions at the top and bottom of the ramp without involving the details of what happens in between. Many problems are easy if energy considerations are used but very complex if we try to use Newton's laws directly.

Example 7.6 An inclined plane with friction

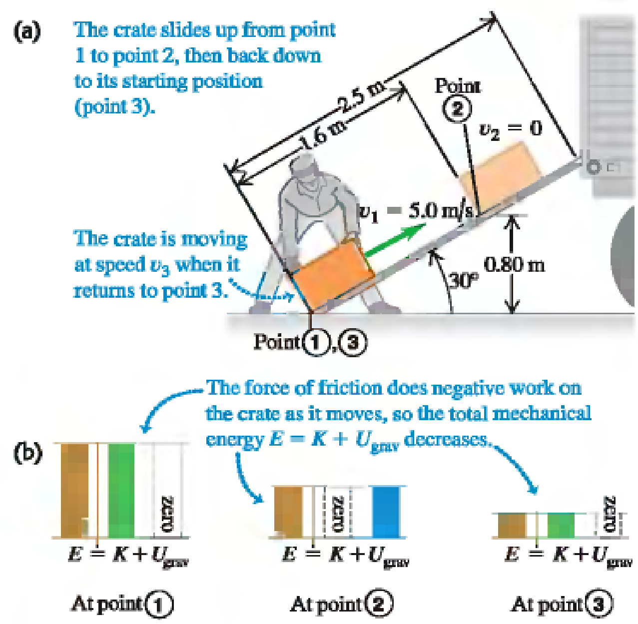
We want to load a 12-kg crate into a truck by sliding it up a ramp 2.5 m long, inclined at 30°. A worker, giving no thought to friction, calculates that he can get the crate up the ramp by giving it an initial speed of 5.0 m/s at the bottom and letting it go. But friction is *not* negligible; the crate slides 1.6 m up the ramp, stops, and slides back down (Fig. 7.11). (a) Assuming that the friction force acting on the crate is constant, find its magnitude. (b) How fast is the crate moving when it reaches the bottom of the ramp?

SOLUTION

IDENTIFY: The friction force does work on the crate as it slides. As in Example 7.2, we'll use the energy approach in part (a) to find the magnitude of the nongravitational force that does work (in this case, friction). In part (b) we'll calculate how much nongravitational work this force does as the crate slides back down and then use the energy approach to find the crate's speed at the bottom of the ramp.

SET UP: The first part of the motion is from point 1, at the bottom of the ramp, to point 2, where the crate stops instantaneously. In the second part of the motion, the crate returns to the bottom of the ramp, which we'll also call point 3 (Fig. 7.11a). We take $y = 0$ (and hence $U_{\text{grav}} = 0$) to be at ground level, so $y_1 = 0$,

7.11 (a) A crate slides partway up the ramp, stops, and slides back down. (b) Energy bar graphs for points 1, 2, and 3.



Continued

$y_2 = (1.6\text{ m})\sin 30° = 0.80\text{ m}$, and $y_3 = 0$. We are given that $v_1 = 5.0\text{ m/s}$ and $v_2 = 0$ (the crate is instantaneously at rest at point 2). Our target variable in part (a) is f, the magnitude of the friction force. In part (b) our target variable is v_3, the speed at the bottom of the ramp.

EXECUTE: (a) The energy quantities are

$$\begin{aligned} K_1 &= \frac{1}{2}(12\text{ kg})(5.0\text{ m/s})^2 = 150\text{ J} \\ U_{\text{grav},1} &= 0 \\ K_2 &= 0 \\ U_{\text{grav},2} &= (12\text{ kg})(9.8\text{ m/s}^2)(0.80\text{ m}) = 94\text{ J} \\ W_{\text{other}} &= -fs \end{aligned}$$

Here f is the unknown magnitude of the friction force and $s = 1.6\text{ m}$. Using Eq. (7.7), we find

$$\begin{aligned} K_1 + U_{\text{grav},1} + W_{\text{other}} &= K_2 + U_{\text{grav},2} \\ W_{\text{other}} = -fs &= (K_2 + U_{\text{grav},2}) - (K_1 + U_{\text{grav},1}) \\ f &= -\frac{(K_2 + U_{\text{grav},2}) - (K_1 + U_{\text{grav},1})}{s} \\ &= -\frac{(0 + 94\text{ J}) - (150\text{ J} + 0)}{1.6\text{ m}} = 35\text{ N} \end{aligned}$$

The friction force of 35 N, acting over 1.6 m, causes the mechanical energy of the crate to decrease from 150 J to 94 J (Fig. 7.11b).

(b) On the way down from point 2 to point 3 at the bottom of the ramp, the friction force and the displacement both reverse direction but have the same magnitudes, so the frictional work has the same negative value as from point 1 to point 2. The total work done by friction between points 1 and 3 is

$$W_{\text{other}} = W_{\text{fric}} = -2fs = -2(35\text{ N})(1.6\text{ m}) = -112\text{ J}$$

From part (a), $K_1 = 150\text{ J}$ and $U_{\text{grav},1} = 0$. Equation (7.7) then gives

$$\begin{aligned} K_1 + U_{\text{grav},1} + W_{\text{other}} &= K_3 + U_{\text{grav},3} \\ K_3 &= K_1 + U_{\text{grav},1} - U_{\text{grav},3} + W_{\text{other}} \\ &= 150\text{ J} + 0 - 0 + (-112\text{ J}) = 38\text{ J} \end{aligned}$$

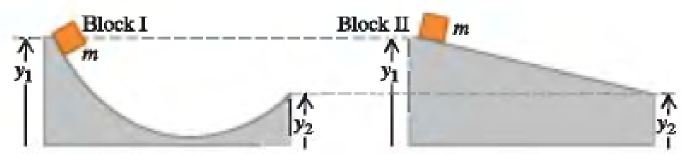
The crate returns to the bottom of the ramp with only 38 J of the original 150 J of mechanical energy (Fig. 7.11b). Using $K_3 = \frac{1}{2}mv_3^2$, we get

$$v_3 = \sqrt{\frac{2K_3}{m}} = \sqrt{\frac{2(38\text{ J})}{12\text{ kg}}} = 2.5\text{ m/s}$$

EVALUATE: The crate's speed when it returns to the bottom of the ramp, $v_3 = 2.5\text{ m/s}$, is less than the speed $v_1 = 5.0\text{ m/s}$ at which it left that point. That's good—energy was lost due to friction.

In part (b) we applied Eq. (7.7) to points 1 and 3, considering the entire round trip as a whole. Alternatively, we could have considered the second part of the motion by itself and applied Eq. (7.7) to points 2 and 3. Try it and see whether you get the same result for v_3.

Test Your Understanding of Section 7.1 The figure shows two different frictionless ramps. The heights y_1 and y_2 are the same for both ramps. If a block of mass m is released from rest at the left-hand end of each ramp, which block arrives at the right-hand end with the greater speed? (i) block I; (ii) block II; (iii) the speed is the same for both blocks.



7.12 The Achilles tendon, which runs along the back of the ankle to the heel bone, acts like a natural spring. When it stretches and then relaxes, this tendon stores and then releases elastic potential energy. This spring action reduces the amount of work your leg muscles must do as you run.

7.2 Elastic Potential Energy

There are many situations in which we encounter potential energy that is not gravitational in nature. One example is a rubber-band slingshot. Work is done on the rubber band by the force that stretches it, and that work is stored in the rubber band until you let it go. Then the rubber band gives kinetic energy to the projectile.

This is the same pattern we saw with the pile driver in Section 7.1: Do work on the system to store energy, which can later be converted to kinetic energy. We'll describe the process of storing energy in a deformable body such as a spring or rubber band in terms of *elastic potential energy* (Fig. 7.12). A body is called *elastic* if it returns to its original shape and size after being deformed.

To be specific, we'll consider storing energy in an ideal spring, like the ones we discussed in Section 6.3. To keep such an ideal spring stretched by a distance x, we must exert a force $F = kx$, where k is the force constant of the spring. The ideal spring is a useful idealization because many elastic bodies show this same direct proportionality between force $\vec{F}$ and displacement x, provided that x is sufficiently small.

We proceed just as we did for gravitational potential energy. We begin with the work done by the elastic (spring) force and then combine this with the work–energy theorem. The difference is that gravitational potential energy is a shared property of a body and the earth, but elastic potential energy is stored just in the spring (or other deformable body).

Figure 7.13 shows the ideal spring from Fig. 6.18, with its left end held stationary and its right end attached to a block with mass m that can move along the x-axis. In Fig. 7.13a the body is at $x = 0$ when the spring is neither stretched nor compressed. We move the block to one side, thereby stretching or compressing the spring, and then let it go. As the block moves from one position x_1 to another position x_2, how much work does the elastic (spring) force do on the block?

We found in Section 6.3 that the work we must do *on* the spring to move one end from an elongation x_1 to a different elongation x_2 is

$$W = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2 \qquad \text{(work done } on \text{ a spring)}$$

where k is the force constant of the spring. If we stretch the spring farther, we do positive work on the spring; if we let the spring relax while holding one end, we do negative work on it. We also saw that this expression for work is still correct if the spring is compressed, not stretched, so that x_1 or x_2 or both are negative. Now we need to find the work done *by* the spring. From Newton's third law the two quantities of work are just negatives of each other. Changing the signs in this equation, we find that in a displacement from x_1 to x_2 the spring does an amount of work W_{el} given by

$$W_{el} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 \qquad \text{(work done } by \text{ a spring)}$$

The subscript "el" stands for *elastic*. When x_1 and x_2 are both positive and $x_2 > x_1$ (Fig. 7.13b), the spring does negative work on the block, which moves in the $+x$-direction while the spring pulls on it in the $-x$-direction. The spring stretches farther, and the block slows down. When x_1 and x_2 are both positive and $x_2 < x_1$ (Fig. 7.13c), the spring does positive work as it relaxes and the block speeds up. If the spring can be compressed as well as stretched, x_1 or x_2 or both may be negative, but the expression for W_{el} is still valid. In Fig. 7.13d, both x_1 and x_2 are negative, but x_2 is less negative than x_1; the compressed spring does positive work as it relaxes, speeding the block up.

Just as for gravitational work, we can express the work done by the spring in terms of a given quantity at the beginning and end of the displacement. This quantity is $\frac{1}{2}kx^2$, and we define it to be the **elastic potential energy:**

$$U_{el} = \frac{1}{2}kx^2 \qquad \text{(elastic potential energy)} \tag{7.9}$$

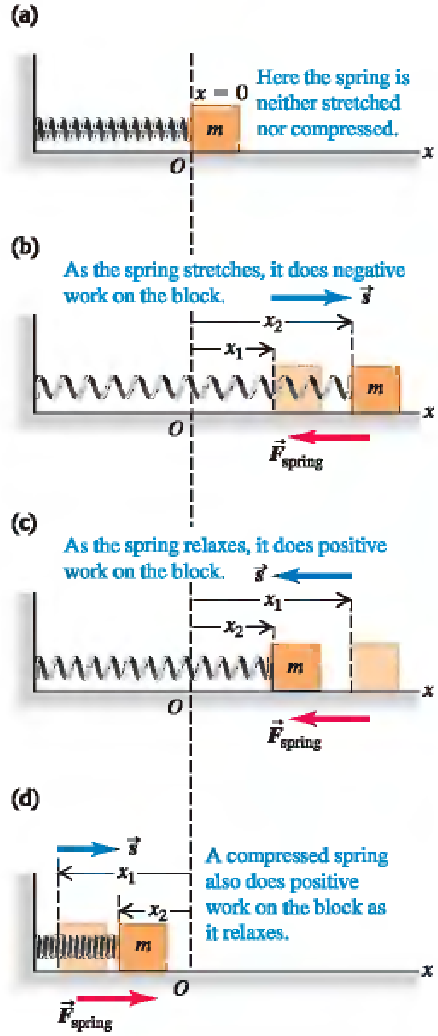
Figure 7.14 is a graph of Eq. (7.9). The unit of U_{el} is the joule (J), the unit used for *all* energy and work quantities; to see this from Eq. (7.9), recall that the units of k are N/m and that $1\ \text{N} \cdot \text{m} = 1\ \text{J}$.

We can use Eq. (7.9) to express the work W_{el} done on the block by the elastic force in terms of the change in elastic potential energy:

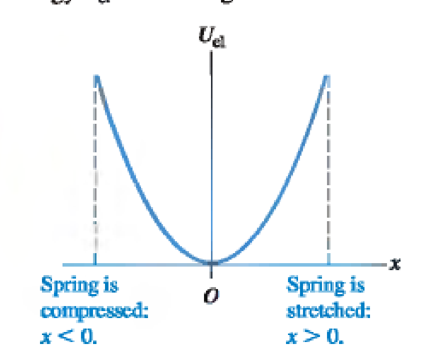
$$W_{el} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 = U_{el,1} - U_{el,2} = -\Delta U_{el} \tag{7.10}$$

When a stretched spring is stretched farther, as in Fig. 7.13b, W_{el} is negative and U_{el} *increases;* a greater amount of elastic potential energy is stored in the spring. When a stretched spring relaxes, as in Fig. 7.13c, x decreases, W_{el} is positive, and U_{el} *decreases;* the spring loses elastic potential energy. Negative values of x refer

7.13 Calculating the work done by a spring attached to a block on a horizontal surface. The quantity x is the extension or compression of the spring.



7.14 The graph of elastic potential energy for an ideal spring is a parabola: $U_{el} = \frac{1}{2}kx^2$, where x is the extension or compression of the spring. Elastic potential energy U_{el} is never negative.



to a compressed spring. But, as Fig. 7.14 shows, U_{el} is positive for both positive and negative x, and Eqs. (7.9) and (7.10) are valid for both cases. The more a spring is compressed *or* stretched, the greater its elastic potential energy.

CAUTION **Gravitational potential energy vs. elastic potential energy** An important difference between gravitational potential energy $U_{grav} = mgy$ and elastic potential energy $U_{el} = \frac{1}{2}kx^2$ is that we do *not* have the freedom to choose $x = 0$ to be wherever we wish. To be consistent with Eq. (7.9), $x = 0$ *must* be the position at which the spring is neither stretched nor compressed. At that position, its elastic potential energy and the force that it exerts are both zero.

The work–energy theorem says that $W_{tot} = K_2 - K_1$, no matter what kind of forces are acting on a body. If the elastic force is the *only* force that does work on the body, then

$$W_{tot} = W_{el} = U_{el,1} - U_{el,2}$$

The work–energy theorem $W_{tot} = K_2 - K_1$ then gives us

$$K_1 + U_{el,1} = K_2 + U_{el,2} \qquad \text{(if only the elastic force does work)} \qquad (7.11)$$

Here U_{el} is given by Eq. (7.9), so

$$\frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2 = \frac{1}{2}mv_2^2 + \frac{1}{2}kx_2^2 \qquad \text{(if only the elastic force does work)} \qquad (7.12)$$

In this case the total mechanical energy $E = K + U_{el}$—the sum of kinetic and *elastic* potential energy—is *conserved.* An example of this is the motion of the block in Fig. 7.13, provided the horizontal surface is frictionless so that no force does work other than that exerted by the spring.

For Eq. (7.12) to be strictly correct, the ideal spring that we've been discussing must also be *massless.* If the spring has a mass, it also has kinetic energy as the coils of the spring move back and forth. We can neglect the kinetic energy of the spring if its mass is much less than the mass m of the body attached to the spring. For instance, a typical automobile has a mass of 1200 kg or more. The springs in its suspension have masses of only a few kilograms, so their mass can be neglected if we want to study how a car bounces on its suspension.

Situations with Both Gravitational and Elastic Potential Energy

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5.4 Inverse Bungee Jumper

5.5 Spring-Launched Bowler

Equations (7.11) and (7.12) are valid when the only potential energy in the system is elastic potential energy. What happens when we have *both* gravitational and elastic forces, such as a block attached to the lower end of a vertically hanging spring? And what if work is also done by other forces that *cannot* be described in terms of potential energy, such as the force of air resistance on a moving block? Then the total work is the sum of the work done by the gravitational force (W_{grav}), the work done by the elastic force (W_{el}), and the work done by other forces (W_{other}): $W_{tot} = W_{grav} + W_{el} + W_{other}$. Then the work–energy theorem gives

$$W_{grav} + W_{el} + W_{other} = K_2 - K_1$$

The work done by the gravitational force is $W_{grav} = U_{grav,1} - U_{grav,2}$ and the work done by the spring is $W_{el} = U_{el,1} - U_{el,2}$. Hence we can rewrite the work–energy theorem for this most general case as

$$K_1 + U_{grav,1} + U_{el,1} + W_{other} = K_2 + U_{grav,2} + U_{el,2} \qquad \text{(valid in general)} \qquad (7.13)$$

or, equivalently,

$$K_1 + U_1 + W_{\text{other}} = K_2 + U_2 \qquad \text{(valid in general)} \qquad (7.14)$$

where $U = U_{\text{grav}} + U_{\text{el}} = mgy + \frac{1}{2}kx^2$ is the *sum* of gravitational potential energy and elastic potential energy. For short, we call U simply "the potential energy."

Equation (7.14) is *the most general statement* of the relationship among kinetic energy, potential energy, and work done by other forces. It says:

The work done by all forces other than the gravitational force or elastic force equals the change in the total mechanical energy $E = K + U$ of the system, where $U = U_{\text{grav}} + U_{\text{el}}$ is the sum of the gravitational potential energy and the elastic potential energy.

The "system" is made up of the body of mass m, the earth with which it interacts through the gravitational force, and the spring of force constant k.

If W_{other} is positive, $E = K + U$ increases; if W_{other} is negative, E decreases. If the gravitational and elastic forces are the *only* forces that do work on the body, then $W_{\text{other}} = 0$ and the total mechanical energy (including both gravitational and elastic potential energy) is conserved. (You should compare Eq. (7.14) to Eqs. (7.7) and (7.8), which describe situations in which there is gravitational potential energy but no elastic potential energy.)

Bungee jumping (Fig. 7.15) is an example of transformations among kinetic energy, elastic potential energy, and gravitational potential energy. As the jumper falls, gravitational potential energy decreases and is converted into the kinetic energy of the jumper and the elastic potential energy of the bungee cord. Beyond a certain point in the fall, the jumper's speed decreases so that both gravitational potential energy and kinetic energy are converted into elastic potential energy.

7.15 The fall of a bungee jumper involves an interplay among kinetic energy, gravitational potential energy, and elastic potential energy. Due to air resistance and frictional forces within the bungee cord, mechanical energy is not conserved. (If mechanical energy *were* conserved, the bungee jumper would keep bouncing up and down forever!)

Problem-Solving Strategy 7.2 Problems Using Mechanical Energy II

MP

Problem-Solving Strategy 7.1 (Section 7.1) is equally useful in solving problems that involve elastic forces as well as gravitational forces. The only new wrinkle is that the potential energy U now includes the elastic potential energy $U_{\text{el}} = \frac{1}{2}kx^2$, where x is the displacement of the spring *from its unstretched length.* The work done by the gravitational and elastic forces is accounted for by their potential energies; the work of the other forces, W_{other}, has to be included separately.

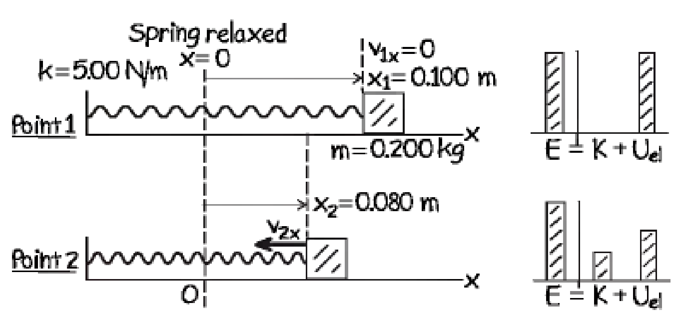
Example 7.7 Motion with elastic potential energy

A glider with mass $m = 0.200$ kg sits on a frictionless horizontal air track, connected to a spring with force constant $k = 5.00$ N/m. You pull on the glider, stretching the spring 0.100 m, and then release it with no initial velocity. The glider begins to move back toward its equilibrium position ($x = 0$). What is its x-velocity when $x = 0.080$ m?

SOLUTION

IDENTIFY: Because the spring force varies with position, this problem can't be solved with the equations for motion with constant acceleration. Instead, we'll use the idea that as the glider starts to move, elastic potential energy is converted into kinetic energy. (The glider remains at the same height throughout the motion, so gravitational potential energy is not a factor. Hence $U = U_{\text{el}} = \frac{1}{2}kx^2$.)

7.16 Our sketches and energy bar graphs for this problem.



SET UP: Figure 7.16 shows our sketches. The spring force is the only force doing work on the glider, so $W_{\text{other}} = 0$ and we may use

Continued

Eq. (7.11). We designate the point where the glider is released as point 1 and $x = 0.080$ m as point 2. We know the velocity at point 1 ($v_{1x} = 0$); our target variable is the x-velocity at point 2, v_{2x}.

EXECUTE: The energy quantities are

$$K_1 = \frac{1}{2}mv_{1x}{}^2 = \frac{1}{2}(0.200\text{ kg})(0)^2 = 0$$

$$U_1 = \frac{1}{2}kx_1{}^2 = \frac{1}{2}(5.00\text{ n/m})(0.100\text{ m})^2 = 0.0250\text{ J}$$

$$K_2 = \frac{1}{2}mv_{2x}{}^2$$

$$U_2 = \frac{1}{2}kx_2{}^2 = \frac{1}{2}(5.00\text{ N/m})(0.080\text{ m})^2 = 0.0160\text{ J}$$

Then from Eq. (7.11),

$$K_2 = K_1 + U_1 - U_2 = 0 + 0.0250\text{ J} - 0.0160\text{ J} = 0.0090\text{ J}$$

$$v_{2x} = \pm\sqrt{\frac{2K_2}{m}} = \pm\sqrt{\frac{2(0.0090\text{ J})}{0.200\text{ kg}}} = \pm 0.30\text{ m/s}$$

We choose the negative root because the glider is moving in the $-x$-direction; the answer we want is $v_{2x} = -0.30$ m/s.

EVALUATE: What is the meaning of the second solution, $v_{2x} = +0.30$ m/s? Eventually the spring will compress and push the glider back to the right in the positive x-direction (see Fig. 7.13d). The second solution tells us that when the glider passes through $x = 0.080$ m while moving to the right, its speed will be 0.30 m/s—the same speed as when it passed through this point while moving to the left.

When the glider passes through the point $x = 0$, the spring is relaxed and all of the mechanical energy is in the form of kinetic energy. Can you show that the speed of the glider at this point is 0.50 m/s?

Example 7.8 Motion with elastic potential energy and work done by other forces

For the system of Example 7.7, suppose the glider is initially at rest at $x = 0$, with the spring unstretched. You then apply a constant force $\vec{F}$ in the $+x$-direction with magnitude 0.610 N to the glider. What is the glider's velocity when it has moved to $x = 0.100$ m?

SOLUTION

IDENTIFY: Although the force $\vec{F}$ you apply is constant, the spring force isn't, so the acceleration of the glider won't be constant. Total mechanical energy is not conserved because of the work done by the force $\vec{F}$, so we must use the generalized energy relationship given by Eq. (7.13). (As in Example 7.7, we ignore gravitational potential energy because the glider's height doesn't change. Hence we have only elastic potential energy, and so $U = U_{el} = \frac{1}{2}kx^2$.)

SET UP: Let point 1 be at $x = 0$, where the velocity is $v_{1x} = 0$, and let point 2 be at $x = 0.100$ m. (These points are different from the ones labeled in Fig. 7.16.) Our target variable is v_{2x}, the velocity at point 2.

EXECUTE: The energy quantities are

$$K_1 = 0$$

$$U_1 = \frac{1}{2}kx_1^2 = 0$$

$$K_2 = \frac{1}{2}mv_{2x}{}^2$$

$$U_2 = \frac{1}{2}kx_2^2 = \frac{1}{2}(5.00\text{ N/m})(0.100\text{ m})^2 = 0.0250\text{ J}$$

$$W_{other} = (0.610\text{ N})(0.100\text{ m}) = 0.0610\text{ J}$$

(To calculate W_{other} we multiplied the magnitude of the force by the displacement, since both are in the $+x$-direction.) Initially, the total mechanical energy is zero; the work done by the force $\vec{F}$ increases the total mechanical energy to 0.0610 J, of which 0.0250 J is elastic potential energy. The remainder is kinetic energy. From Eq. (7.13),

$$K_1 + U_1 + W_{other} = K_2 + U_2$$

$$K_2 = K_1 + U_1 + W_{other} - U_2$$

$$= 0 + 0 + 0.0610\text{ J} - 0.0250\text{ J} = 0.0360\text{ J}$$

$$v_{2x} = \sqrt{\frac{2K_2}{m}} = \sqrt{\frac{2(0.0360\text{ J})}{0.200\text{ kg}}} = 0.60\text{ m/s}$$

We choose the positive square root because the glider is moving in the $+x$-direction.

EVALUATE: To test our answer, think what would be different if we disconnected the glider from the spring. Then $\vec{F}$ would be the only force doing work, there would be zero potential energy at all times, and Eq. (7.13) would give us

$$K_2 = K_1 + W_{other} = 0 + 0.0610\text{ J}$$

$$v_{2x} = \sqrt{\frac{2K_2}{m}} = \sqrt{\frac{2(0.0610\text{ J})}{0.200\text{ kg}}} = 0.78\text{ m/s}$$

We found a lower velocity than this value because the spring does negative work on the glider as it stretches (see Fig. 7.13b).

If you stop pushing on the glider when it reaches the point $x = 0.100$ m, beyond that point the only force that does work on the glider is the spring force. Hence for $x > 0.100$ m, the total mechanical energy $E = K + U$ is conserved and maintains the same value of 0.0610 J. The glider will slow down as the spring continues to stretch, so the kinetic energy K will decrease as the potential energy increases. The glider will come to rest at a point $x = x_3$; at this point the kinetic energy is zero and the potential energy $U = U_{el} = \frac{1}{2}kx_3^2$ is equal to the total mechanical energy 0.0610 J. You should be able to show that the glider comes to rest at $x_3 = 0.156$ m, which means that it moves an additional 0.056 m after the force $\vec{F}$ is removed at $x_2 = 0.100$ m. (Since there's no friction, the glider will not remain at rest but will start moving back toward $x = 0$ due to the force of the stretched spring.)

Example 7.9 Motion with gravitational, elastic, and friction forces

In a "worst-case" design scenario, a 2000-kg elevator with broken cables is falling at 4.00 m/s when it first contacts a cushioning spring at the bottom of the shaft. The spring is supposed to stop the elevator, compressing 2.00 m as it does so (Fig. 7.17). During the motion a safety clamp applies a constant 17,000-N frictional force to the elevator. As a design consultant, you are asked to determine what the force constant of the spring should be.

SOLUTION

IDENTIFY: We'll use the energy approach to determine the force constant, which appears in the expression for elastic potential energy. Note that this problem involves *both* gravitational and elastic potential energy. Furthermore, total mechanical energy is not conserved because the friction force does negative work W_{other} on the elevator.

SET UP: Since mechanical energy isn't conserved and more than one kind of potential energy is involved, we'll use the most general form of the energy relationship, Eq. (7.13). We take point 1 as the position of the bottom of the elevator when it initially contacts the spring, and take point 2 as its position when it is at rest. We choose the origin to be at point 1, so $y_1 = 0$ and $y_2 = -2.00$ m. With this choice the coordinate of the upper end of the spring is the same as the coordinate of the elevator, so the elastic potential energy at any point between point 1 and point 2 is $U_{el} = \frac{1}{2}ky^2$. (The gravitational potential energy is $U_{grav} = mgy$ as usual.) We know the initial and final speeds of the elevator and the magnitude of the friction force, so the only unknown is the force constant k (our target variable).

EXECUTE: The elevator's initial speed is $v_1 = 4.00$ m/s, so the initial kinetic energy is

$$K_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}(2000\text{ kg})(4.00\text{ m/s})^2 = 16{,}000\text{ J}$$

The elevator stops at point 2, so $K_2 = 0$. The potential energy at point 1, U_1, is zero; U_{grav} is zero because $y_1 = 0$, and $U_{el} = 0$ because the spring is not yet compressed. At point 2 there is both gravitational and elastic potential energy, so

$$U_2 = mgy_2 + \frac{1}{2}ky_2^2$$

The gravitational potential energy at point 2 is

$$mgy_2 = (2000\text{ kg})(9.80\text{ m/s}^2)(-2.00\text{ m}) = -39{,}200\text{ J}$$

The other force is the 17,000-N friction force, acting opposite to the 2.00-m displacement, so

$$W_{other} = -(17{,}000\text{ N})(2.00\text{ m}) = -34{,}000\text{ J}$$

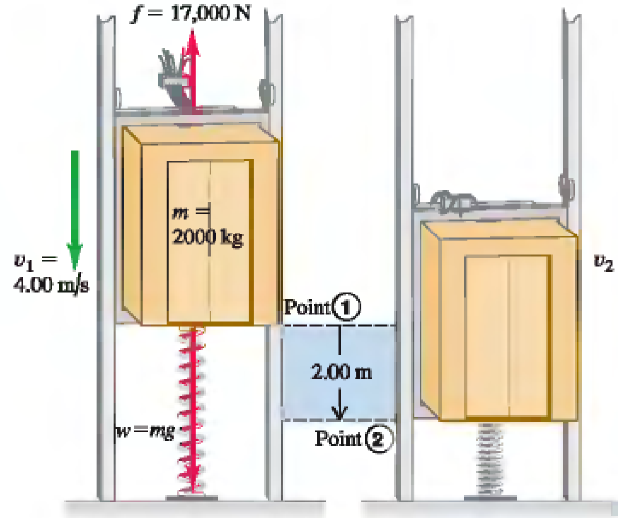
Putting these terms into $K_1 + U_1 + W_{other} = K_2 + U_2$, we have

$$K_1 + 0 + W_{other} = 0 + \left(mgy_2 + \frac{1}{2}ky_2^2\right)$$

so the force constant of the spring is

$$\begin{aligned} k &= \frac{2(K_1 + W_{other} - mgy_2)}{y_2^2} \\ &= \frac{2[16{,}000\text{ J} + (-34{,}000\text{ J}) - (-39{,}200\text{ J})]}{(-2.00\text{ m})^2} \\ &= 1.06 \times 10^4\text{ N/m} \end{aligned}$$

7.17 The fall of an elevator is stopped by a spring and by a constant friction force.



This is about one-tenth the force constant of a spring in an automobile suspension.

EVALUATE: Let's note what might seem to be a paradox in this problem. The elastic potential energy in the spring at point 2 is

$$\frac{1}{2}ky_2^2 = \frac{1}{2}(1.06 \times 10^4\text{ N/m})(-2.00\text{ m})^2 = 21{,}200\text{ J}$$

This is *more* than the total mechanical energy at point 1:

$$E_1 = K_1 + U_1 = 16{,}000\text{ J} + 0 = 16{,}000\text{ J}$$

But the friction force caused the mechanical energy of the system to *decrease* by 34,000 J between point 1 and point 2. Does this mean that energy appeared from nowhere? Don't panic; there is no paradox. At point 2 there is also *negative* gravitational potential energy, $mgy_2 = -39{,}200$ J, because point 2 is below the origin. The total mechanical energy at point 2 is

$$\begin{aligned} E_2 &= K_2 + U_2 = 0 + \frac{1}{2}ky_2^2 + mgy_2 \\ &= 0 + 21{,}200\text{ J} + (-39{,}200\text{ J}) = -18{,}000\text{ J} \end{aligned}$$

This is just the initial mechanical energy of 16,000 J, minus 34,000 J lost to friction.

Will the elevator stay at the bottom of the shaft? At point 2 the compressed spring exerts an upward force of magnitude $F_{spring} = (1.06 \times 10^4\text{ N/m})(2.00\text{ m}) = 21{,}200$ N, while the downward force of gravity on the elevator is only $w = mg = (2000\text{ kg})(9.80\text{ m/s}^2) = 19{,}600$ N. So if there were no friction, there would be a net upward force of $21{,}200\text{ N} - 19{,}600\text{ N} = 1600$ N and the elevator would bounce back upward. However, there *is* friction in the safety clamp, which can exert a force of as much as 17,000 N; hence the clamp can keep the elevator from rebounding.

Test Your Understanding of Section 7.2 Consider the situation in Example 7.9 at the instant when the elevator is still moving downward and the spring is compressed by 1.00 m. Which of the energy bar graphs in the figure most accurately shows the kinetic energy K, gravitational potential energy U_{grav}, and elastic potential energy U_{el} at this instant?

7.3 Conservative and Nonconservative Forces

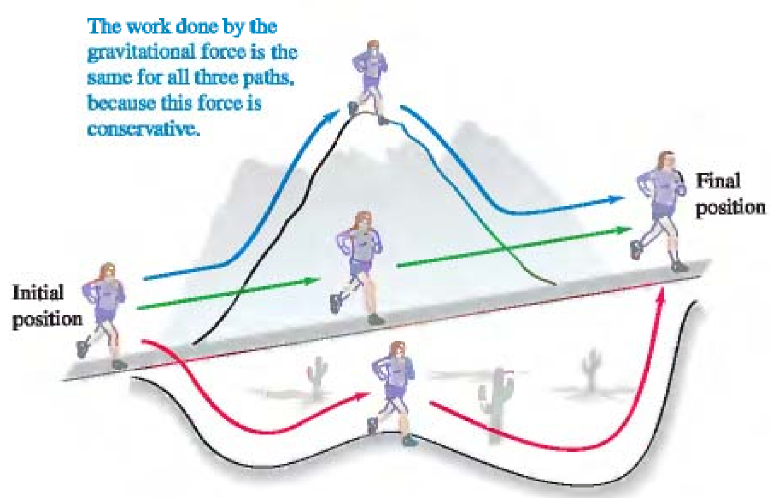
In our discussions of potential energy we have talked about "storing" kinetic energy by converting it to potential energy. We always have in mind that later we may retrieve it again as kinetic energy. For example, when you throw a ball up in the air, it slows down as kinetic energy is converted into potential energy. But on the way down, the conversion is reversed, and the ball speeds up as potential energy is converted back to kinetic energy. If there is no air resistance, the ball is moving just as fast when you catch it as when you threw it.

Another example is a glider moving on a frictionless horizontal air track that runs into a spring bumper at the end of the track. The glider stops as it compresses the spring and then bounces back. If there is no friction, the glider ends up with the same speed and kinetic energy it had before the collision. Again, there is a two-way conversion from kinetic to potential energy and back. In both cases we can define a potential-energy function so that the total mechanical energy, kinetic plus potential, is constant or *conserved* during the motion.

Conservative Forces

A force that offers this opportunity of two-way conversion between kinetic and potential energies is called a **conservative force.** We have seen two examples of conservative forces: the gravitational force and the spring force. (Later in this book we will study another conservative force, the electric force between charged objects.) An essential feature of conservative forces is that their work is always *reversible.* Anything that we deposit in the energy "bank" can later be withdrawn without loss. Another important aspect of conservative forces is that a body may move from point 1 to point 2 by various paths, but the work done by a conservative force is the same for all of these paths (Fig. 7.18). Thus, if a body

7.18 The work done by a conservative force such as gravity depends only on the end points of a path, not on the specific path taken between those points.



stays close to the surface of the earth, the gravitational force $m\vec{g}$ is independent of height, and the work done by this force depends only on the change in height. If the body moves around a closed path, ending at the same point where it started, the *total* work done by the gravitational force is always zero.

The work done by a conservative force *always* has four properties:

1. It can be expressed as the difference between the initial and final values of a *potential-energy* function.
2. It is reversible.
3. It is independent of the path of the body and depends only on the starting and ending points.
4. When the starting and ending points are the same, the total work is zero.

When the *only* forces that do work are conservative forces, the total mechanical energy $E = K + U$ is constant.

Nonconservative Forces

Not all forces are conservative. Consider the friction force acting on the crate sliding on a ramp in Example 7.6 (Section 7.1). When the body slides up and then back down to the starting point, the total work done on it by the friction force is *not* zero. When the direction of motion reverses, so does the friction force, and friction does *negative* work in *both* directions. When a car with its brakes locked skids across the pavement with decreasing speed (and decreasing kinetic energy), the lost kinetic energy cannot be recovered by reversing the motion or in any other way, and mechanical energy is *not* conserved. There is *no* potential-energy function for the friction force.

In the same way, the force of fluid resistance (see Section 5.3) is not conservative. If you throw a ball up in the air, air resistance does negative work on the ball while it's rising *and* while it's descending. The ball returns to your hand with less speed and less kinetic energy than when it left, and there is no way to get back the lost mechanical energy.

A force that is not conservative is called a **nonconservative force.** The work done by a nonconservative force *cannot* be represented by a potential-energy function. Some nonconservative forces, like kinetic friction or fluid resistance, cause mechanical energy to be lost or dissipated; a force of this kind is called a **dissipative force.** There are also nonconservative forces that *increase* mechanical energy. The fragments of an exploding firecracker fly off with very large kinetic energy, thanks to a chemical reaction of gunpowder with oxygen. The forces unleashed by this reaction are nonconservative because the process is not reversible. (The fragments never spontaneously reassemble themselves into a complete firecracker!)

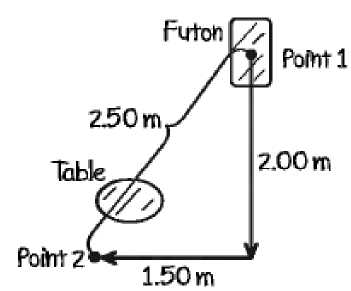
Example 7.10 Frictional work depends on the path

You are rearranging your furniture and wish to move a 40.0-kg futon 2.50 m across the room. However, the straight-line path is blocked by a heavy coffee table that you don't want to move. Instead, you slide the futon in a dogleg path over the floor; the doglegs are 2.00 m and 1.50 m long. Compared to the straight-line path, how much more work must you do to push the futon in the dogleg path? The coefficient of kinetic friction is 0.200.

SOLUTION

IDENTIFY: Here work is done both by you and by the force of friction, so we must use the energy relationship that includes forces other than elastic or gravitational forces. We'll use this relationship to find a connection between the work that *you* do and the work done by *friction.*

7.19 Our sketch for this problem.



SET UP: Figure 7.19 shows our sketch. The futon is at rest at both point 1 and point 2, so $K_1 = K_2 = 0$. There is no elastic potential

Continued

energy (there are no springs), and the gravitational potential energy does not change because the futon moves only horizontally, so $U_1 = U_2$. From Eq. (7.14) it follows that $W_{other} = 0$. The other work done on the futon is the sum of the positive work you do, W_{you}, and the negative work W_{fric} done by the kinetic friction force. Since the sum of these is zero, we have

$$W_{you} = -W_{fric}$$

Thus to determine W_{you}, we'll calculate the work done by friction.

EXECUTE: Because the floor is horizontal, the normal force on the futon equals its weight mg, and the magnitude of the friction force is $f_k = \mu_k n = \mu_k mg$. The work you must do over each path is then

$$\begin{aligned} W_{you} &= -W_{fric} = -(-f_k s) = +\mu_k mgs \\ &= (0.200)(40.0\ \text{kg})(9.80\ \text{m/s}^2)(2.50\ \text{m}) \\ &= 196\ \text{J} \qquad \text{(straight-line path)} \end{aligned}$$

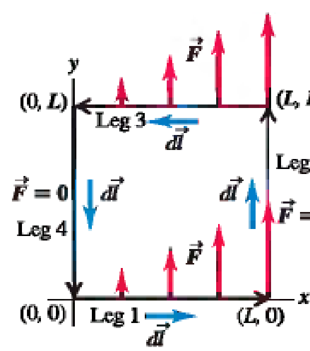
$$\begin{aligned} W_{you} &= -W_{fric} \\ &= (0.200)(40.0\ \text{kg})(9.80\ \text{m/s}^2)(2.00\ \text{m} + 1.50\ \text{m}) \\ &= 274\ \text{J} \qquad \text{(dogleg path)} \end{aligned}$$

The extra work you must do is $274\ \text{J} - 196\ \text{J} = 78\ \text{J}$.

EVALUATE: The work done by friction is $W_{fric} = -W_{you} = -196\ \text{J}$ on the straight-line path and $-274\ \text{J}$ on the dogleg path. The work done by friction depends on the path taken, which illustrates that friction is a *nonconservative* force.

Example 7.11 Conservative or nonconservative?

In a certain region of space the force on an electron is $\vec{F} = Cx\hat{j}$, where C is a positive constant. The electron moves in a counterclockwise direction around a square loop in the xy-plane (Fig. 7.20). The corners of the square are at $(x, y) = (0, 0)$, $(L, 0)$, (L, L), and $(0, L)$. Calculate the work done on the electron by the force $\vec{F}$ during one complete trip around the square. Is this force conservative or nonconservative?



7.20 An electron moving around a square loop while being acted on by the force $\vec{F} = Cx\hat{j}$.

SOLUTION

IDENTIFY: In Example 7.10 the force of friction was constant in magnitude and always opposite to the displacement, so it was easy to calculate the work done. Here, however, the force $\vec{F}$ is not constant and in general is not in the same direction as the displacement.

SET UP: To calculate the work done by the force $\vec{F}$, we'll use the more general expression for work, Eq. (6.14):

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{l}$$

where $d\vec{l}$ is an infinitesimal displacement. Let's calculate the work done on each leg of the square and then add the results to find the work done on the round trip.

EXECUTE: On the first leg, from $(0, 0)$ to $(L, 0)$, the force varies but is everywhere perpendicular to the displacement. So $\vec{F} \cdot d\vec{l} = 0$, and the work done on the first leg is $W_1 = 0$. The force has the same value $\vec{F} = CL\hat{j}$ everywhere on the second leg from $(L, 0)$ to (L, L). The displacement on this leg is in the $+y$-direction, so $d\vec{l} = dy\hat{j}$ and

$$\vec{F} \cdot d\vec{l} = CL\hat{j} \cdot dy\hat{j} = CL\,dy$$

The work done on the second leg is then

$$W_2 = \int_{(L,0)}^{(L,L)} \vec{F} \cdot d\vec{l} = \int_{y=0}^{y=L} CL\,dy = CL\int_0^L dy = CL^2$$

On the third leg, from (L, L) to $(0, L)$, $\vec{F}$ is again perpendicular to the displacement so $W_3 = 0$. The force is zero on the final leg, from $(0, L)$ to $(0, 0)$, so no work is done and $W_4 = 0$. The work done by the force $\vec{F}$ on the round trip is

$$W = W_1 + W_2 + W_3 + W_4 = 0 + CL^2 + 0 + 0 = CL^2$$

The starting and ending points are the same, but the total work done by $\vec{F}$ is not zero. This is a nonconservative force; it *cannot* be represented by a potential-energy function.

EVALUATE: Because W is positive, the mechanical energy *increases* as the electron goes around the loop. This is not a mathematical curiosity; it's a description of what happens in an electrical generating plant. A loop of wire is moved through a magnetic field, which gives rise to a nonconservative force similar to the one in this example. Electrons in the wire gain energy as they move around the loop, and this energy is carried via transmission lines to the consumer. (We'll discuss how this works in detail in Chapter 29.)

If the electron went around the loop clockwise instead of counterclockwise, the force $\vec{F}$ would be unaffected but the direction of each infinitesimal displacement $d\vec{l}$ would reverse. Thus the sign of work would also reverse, and the work for a clockwise round trip would be $W = -CL^2$. This is a different behavior than the nonconservative friction force. When a body slides over a stationary surface with friction, the work done by friction is always negative, no matter what the direction of motion (see Example 7.6 in Section 7.1).

The Law of Conservation of Energy

Nonconservative forces cannot be represented in terms of potential energy. But we can describe the effects of these forces in terms of kinds of energy other than kinetic and potential energy. When a car with locked brakes skids to a stop, the tires and the road surface both become hotter. The energy associated with this change in the state of the materials is called **internal energy.** Raising the temperature of a body increases its internal energy; lowering the body's temperature decreases its internal energy.

To see the significance of internal energy, let's consider a block sliding on a rough surface. Friction does *negative* work on the block as it slides, and the change in internal energy of the block and surface (both of which get hotter) is *positive.* Careful experiments show that the increase in the internal energy is *exactly* equal to the absolute value of the work done by friction. In other words,

$$\Delta U_{\mathrm{int}} = -W_{\mathrm{other}}$$

where ΔU_{int} is the change in internal energy. If we substitute this into Eq. (7.7) or (7.14), we find

$$K_1 + U_1 - \Delta U_{\mathrm{int}} = K_2 + U_2$$

Writing $\Delta K = K_2 - K_1$ and $\Delta U = U_2 - U_1$, we can finally express this as

$$\Delta K + \Delta U + \Delta U_{\mathrm{int}} = 0 \qquad \text{(law of conservation of energy)} \qquad (7.15)$$

This remarkable statement is the general form of the **law of conservation of energy.** In a given process, the kinetic energy, potential energy, and internal energy of a system may all change. But the *sum* of those changes is always zero. If there is a decrease in one form of energy, it is made up for by an increase in the other forms (Fig. 7.21). When we expand our definition of energy to include internal energy, Eq. (7.15) says: *Energy is never created or destroyed; it only changes form.* No exception to this rule has ever been found.

? The concept of work has been banished from Eq. (7.15); instead, it suggests that we think purely in terms of the conversion of energy from one form to another. For example, when you throw a baseball straight up, you convert a portion of the internal energy of your molecules into kinetic energy of the baseball. This is converted into gravitational potential energy as the ball climbs and back to kinetic energy as the ball falls. If there is air resistance, part of the energy is used to heat up the air and the ball and increase their internal energy. Energy is converted back into the kinetic form as the ball falls. If you catch the ball in your hand, whatever energy was not lost to the air once again becomes internal energy; the ball and your hand are now warmer than they were at the beginning.

In Chapters 19 and 20, we will study the relationship of internal energy to temperature changes, heat, and work. This is the heart of the area of physics called *thermodynamics.*

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5.7 Modified Atwood Machine

7.21 When 1 liter of gasoline is burned in an automotive engine, it releases 3.3×10^7 J of internal energy. Hence $\Delta U_{\mathrm{int}} = -3.3 \times 10^7$ J, where the minus sign means that the amount of energy stored in the gasoline has decreased. This energy can be converted into kinetic energy (making the car go faster) or into potential energy (enabling the car to climb uphill).

Example 7.12 Work done by friction

Let's look again at Example 7.5 (Section 7.1), in which your cousin Throcky skateboards down a curved ramp. He starts with zero kinetic energy and 735 J of potential energy, and at the bottom he has 450 J of kinetic energy and zero potential energy. So $\Delta K = +450$ J and $\Delta U = -735$ J. The work $W_{\mathrm{other}} = W_{\mathrm{fric}}$ done by the nonconservative friction forces is -285 J, so the change in internal energy is $\Delta U_{\mathrm{int}} = -W_{\mathrm{other}} = +285$ J. The wheels, the bearings, and the ramp all get a little warmer as Throcky rolls down. In accordance with Eq. (7.15), the sum of the energy changes equals zero:

$$\Delta K + \Delta U + \Delta U_{\mathrm{int}} = +450\,\mathrm{J} + (-735\,\mathrm{J}) + 285\,\mathrm{J} = 0$$

The total energy of the system (including nonmechanical forms of energy) is conserved.

Test Your Understanding of Section 7.3 In a hydroelectric generating station, falling water is used to drive turbines ("water wheels"), which in turn run electric generators. Compared to the amount of gravitational potential energy released by the falling water, how much electrical energy is produced? (i) the same; (ii) more; (iii) less.

7.4 Force and Potential Energy

For the two kinds of conservative forces (gravitational and elastic) we have studied, we started with a description of the behavior of the *force* and derived from that an expression for the *potential energy*. For example, for a body with mass m in a uniform gravitational field, the gravitational force is $F_y = -mg$. We found that the corresponding potential energy is $U(y) = mgy$. To stretch an ideal spring by a distance x, we exert a force equal to $+kx$. By Newton's third law the force that an ideal spring exerts on a body is opposite this, or $F_x = -kx$. The corresponding potential energy function is $U(x) = \frac{1}{2}kx^2$.

In studying physics, however, you'll encounter situations in which you are given an expression for the *potential energy* as a function of position and have to find the corresponding *force*. We'll see several examples of this kind when we study electric forces later in this book: it's often far easier to calculate the electric potential energy first and then determine the corresponding electric force afterward.

Here's how we find the force that corresponds to a given potential-energy expression. First let's consider motion along a straight line, with coordinate x. We denote the x-component of force, a function of x, by $F_x(x)$, and the potential energy as $U(x)$. This notation reminds us that both F_x and U are *functions* of x. Now we recall that in any displacement, the work W done by a conservative force equals the negative of the change ΔU in potential energy:

$$W = -\Delta U$$

Let's apply this to a small displacement Δx. The work done by the force $F_x(x)$ during this displacement is approximately equal to $F_x(x)\,\Delta x$. We have to say "approximately" because $F_x(x)$ may vary a little over the interval Δx. But it is at least approximately true that

$$F_x(x)\,\Delta x = -\Delta U \qquad \text{and} \qquad F_x(x) = -\frac{\Delta U}{\Delta x}$$

You can probably see what's coming. We take the limit as $\Delta x \rightarrow 0$; in this limit, the variation of F_x becomes negligible, and we have the exact relationship

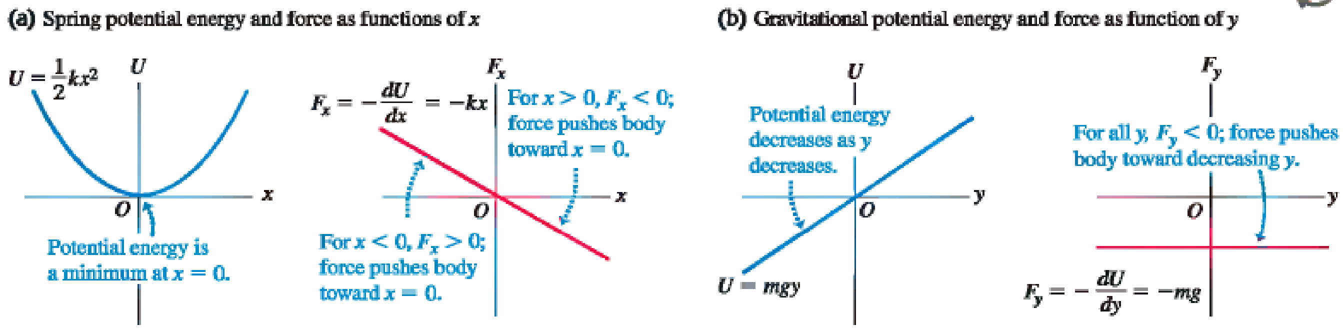
$$F_x(x) = -\frac{dU(x)}{dx} \qquad \text{(force from potential energy, one dimension)} \qquad (7.16)$$

This result makes sense; in regions where $U(x)$ changes most rapidly with x (that is, where $dU(x)/dx$ is large), the greatest amount of work is done during a given displacement, and this corresponds to a large force magnitude. Also, when $F_x(x)$ is in the positive x-direction, $U(x)$ *decreases* with increasing x. So $F_x(x)$ and $dU(x)/dx$ should indeed have opposite signs. The physical meaning of Eq. (7.16) is that *a conservative force always acts to push the system toward lower potential energy.*

As a check, let's consider the function for elastic potential energy, $U(x) = \frac{1}{2}kx^2$. Substituting this into Eq. (7.16) yields

$$F_x(x) = -\frac{d}{dx}\left(\frac{1}{2}kx^2\right) = -kx$$

7.22 A conservative force is the negative derivative of the corresponding potential energy.



which is the correct expression for the force exerted by an ideal spring (Fig. 7.22a). Similarly, for gravitational potential energy we have $U(y) = mgy$; taking care to change x to y for the choice of axis, we get $F_y = -dU/dy = -d(mgy)/dy = -mg$, which is the correct expression for gravitational force (Fig. 7.22b).

Example 7.13 An electric force and its potential energy

An electrically charged particle is held at rest at the point $x = 0$, while a second particle with equal charge is free to move along the positive x-axis. The potential energy of the system is

$$U(x) = \frac{C}{x}$$

where C is a positive constant that depends on the magnitude of the charges. Derive an expression for the x-component of force acting on the movable charged particle, as a function of its position.

SOLUTION

IDENTIFY: We are given the potential-energy function $U(x)$, and we want to find the force function $F_x(x)$.

SET UP: We'll use Eq. (7.16), $F_x(x) = -dU(x)/dx$.

EXECUTE: The derivative with respect to x of the function $1/x$ is $-1/x^2$. So the force on the movable charged particle for $x > 0$ is

$$F_x(x) = -\frac{dU(x)}{dx} = -C\left(-\frac{1}{x^2}\right) = \frac{C}{x^2}$$

EVALUATE: The x-component of force is positive, corresponding to a repulsion between like electric charges. The potential energy is very large when the particles are close together (small x) and approaches zero as the particles move farther apart (large x); the force pushes the movable particle toward large positive values of x, for which the potential energy is less. The force $F_x(x) = C/x^2$ gets weaker as the particles move farther apart (x increases). We'll study electric forces in greater detail in Chapter 21.

Force and Potential Energy in Three Dimensions

We can extend this analysis to three dimensions, where the particle may move in the x-, y-, or z-direction, or all at once, under the action of a conservative force that has components F_x, F_y, and F_z. Each component of force may be a function of the coordinates x, y, and z. The potential-energy function U is also a function of all three space coordinates. We can now use Eq. (7.16) to find each component of force. The potential-energy change ΔU when the particle moves a small distance Δx in the x-direction is again given by $-F_x\,\Delta x$; it doesn't depend on F_y and F_z, which represent force components that are perpendicular to the displacement and do no work. So we again have the approximate relationship

$$F_x = -\frac{\Delta U}{\Delta x}$$

The y- and z-components of force are determined in exactly the same way:

$$F_y = -\frac{\Delta U}{\Delta y} \qquad F_z = -\frac{\Delta U}{\Delta z}$$

To make these relationships exact, we take the limits $\Delta x \to 0$, $\Delta y \to 0$, and $\Delta z \to 0$ so that these ratios become derivatives. Because U may be a function of all three coordinates, we need to remember that when we calculate each of these derivatives, only one coordinate changes at a time. We compute the derivative of U with respect to x by assuming that y and z are constant and only x varies, and so on. Such a derivative is called a *partial derivative*. The usual notation for a partial derivative is $\partial U/\partial x$ and so on; the symbol ∂ is a modified d. So we write

$$F_x = -\frac{\partial U}{\partial x} \quad F_y = -\frac{\partial U}{\partial y} \quad F_z = -\frac{\partial U}{\partial z} \quad \text{(force from potential energy)} \tag{7.17}$$

We can use unit vectors to write a single compact vector expression for the force $\vec{F}$:

$$\vec{F} = -\left(\frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\hat{k}\right) \quad \text{(force from potential energy)} \tag{7.18}$$

The expression inside the parentheses represents a particular operation on the function U, in which we take the partial derivative of U with respect to each coordinate, multiply by the corresponding unit vector, and then take the vector sum. This operation is called the **gradient** of U and is often abbreviated as $\vec{\nabla}U$. Thus the force is the negative of the gradient of the potential-energy function:

$$\vec{F} = -\vec{\nabla}U \tag{7.19}$$

As a check, let's substitute into Eq. (7.19) the function $U = mgy$ for gravitational potential energy:

$$\vec{F} = -\vec{\nabla}(mgy) = -\left(\frac{\partial(mgy)}{\partial x}\hat{\imath} + \frac{\partial(mgy)}{\partial y}\hat{\jmath} + \frac{\partial(mgy)}{\partial z}\hat{k}\right) = (-mg)\hat{\jmath}$$

This is just the familiar expression for the gravitational force.

Example 7.14 Force and potential energy in two dimensions

A puck slides on a level, frictionless air-hockey table. The coordinates of the puck are x and y. It is acted on by a conservative force described by the potential-energy function

$$U(x, y) = \frac{1}{2}k(x^2 + y^2)$$

Derive an expression for the force acting on the puck, and find an expression for the magnitude of the force as a function of position.

SOLUTION

IDENTIFY: Starting with the function $U(x, y)$, we need to find the vector components and magnitude of the corresponding conservative force $\vec{F}$.

SET UP: We'll find the components of the force from $U(x, y)$ using Eq. (7.18). This function doesn't depend on z, so the partial derivative of U with respect to z is $\partial U/\partial z = 0$ and the force has no z-component. We'll then determine the magnitude of the force using the formula for the magnitude of a vector: $F = \sqrt{F_x^2 + F_y^2}$.

EXECUTE: The x- and y-components of the force are

$$F_x = -\frac{\partial U}{\partial x} = -kx \qquad F_y = -\frac{\partial U}{\partial y} = -ky$$

From Eq. (7.18) this corresponds to the vector expression

$$\vec{F} = -k(x\hat{\imath} + y\hat{\jmath})$$

Now $x\hat{\imath} + y\hat{\jmath}$ is just the position vector $\vec{r}$ of the particle, so we can rewrite this expression as $\vec{F} = -k\vec{r}$. This represents a force that at each point is opposite in direction to the position vector of the point—that is, a force that at each point is directed toward the origin. The potential energy is minimum at the origin, so again the force pushes in the direction of decreasing potential energy.

The *magnitude* of the force at any point is

$$F = \sqrt{(-kx)^2 + (-ky)^2} = k\sqrt{x^2 + y^2} = kr$$

where r is the particle's distance from the origin. This is the force that would be exerted on the puck if it were attached to one end of a spring that obeys Hooke's law and has a negligibly small length (compared to the other distances in the problem) when it is not stretched. (The other end is attached to the air-hockey table at the origin.)

EVALUATE: To check our result, note that the potential-energy function can also be expressed as $U = \frac{1}{2}kr^2$. Written this way, U is a function of a single coordinate r, so we can find the force using Eq. (7.16) with x replaced by r:

$$F_r = -\frac{dU}{dr} = -\frac{d}{dr}\left(\frac{1}{2}kr^2\right) = -kr$$

Just as we calculated above, the force has magnitude kr; the minus sign indicates that the force is radially inward (toward the origin).

Test Your Understanding of Section 7.4 A particle moving along the x-axis is acted on by a conservative force F_x. At a certain point, the force is zero. (a) Which of the following statements about the value of the potential-energy function $U(x)$ at that point is correct? (i) $U(x) = 0$; (ii) $U(x) > 0$; (iii) $U(x) < 0$; (iv) not enough information is given to decide. (b) Which of the following statements about the value of the derivative of $U(x)$ at that point is correct? (i) $dU(x)/dx = 0$; (ii) $dU(x)/dx > 0$; (iii) $dU(x)/dx < 0$; (iv) not enough information is given to decide.

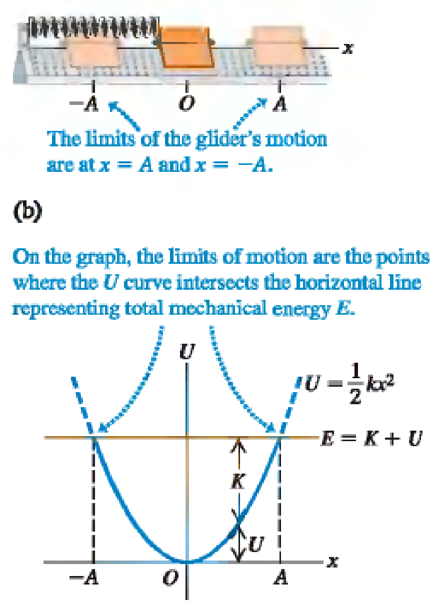
7.5 Energy Diagrams

When a particle moves along a straight line under the action of a conservative force, we can get a lot of insight into its possible motions by looking at the graph of the potential-energy function $U(x)$. Figure 7.23a shows a glider with mass m that moves along the x-axis on an air track. The spring exerts on the glider a force with x-component $F_x = -kx$. Figure 7.23b is a graph of the corresponding potential-energy function $U(x) = \frac{1}{2}kx^2$. If the elastic force of the spring is the *only* horizontal force acting on the glider, the total mechanical energy $E = K + U$ is constant, independent of x. A graph of E as a function of x is thus a straight horizontal line. We use the term **energy diagram** for a graph like this, which shows both the potential-energy function $U(x)$ and the energy of the particle subjected to the force that corresponds to $U(x)$.

7.23 (a) A glider on an air track. The spring exerts a force $F_x = -kx$. (b) The potential-energy function.

(a)

(b)



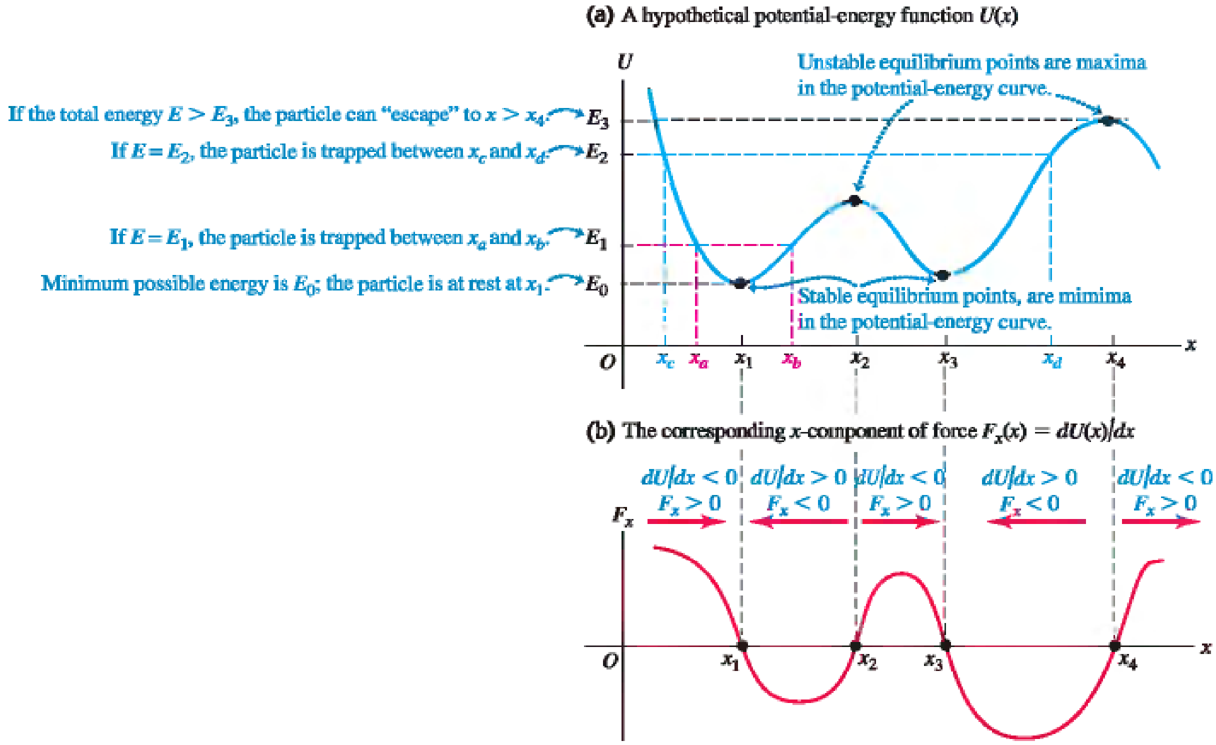
The vertical distance between the U and E graphs at each point represents the difference $E - U$, equal to the kinetic energy K at that point. We see that K is greatest at $x = 0$. It is zero at the values of x where the two graphs cross, labeled A and $-A$ in the diagram. Thus the speed v is greatest at $x = 0$, and it is zero at $x = \pm A$, the points of *maximum* possible displacement from $x = 0$ for a given value of the total energy E. The potential energy U can never be greater than the total energy E; if it were, K would be negative, and that's impossible. The motion is a back-and-forth oscillation between the points $x = A$ and $x = -A$.

At each point, the force F_x on the glider is equal to the negative of the slope of the $U(x)$ curve: $F_x = -dU/dx$ (see Fig. 7.22a). When the particle is at $x = 0$, the slope and the force are zero, so this is an *equilibrium* position. When x is positive, the slope of the $U(x)$ curve is positive and the force F_x is negative, directed toward the origin. When x is negative, the slope is negative and F_x is positive, again toward the origin. Such a force is called a *restoring force;* when the glider is displaced to either side of $x = 0$, the force tends to "restore" it back to $x = 0$. An analogous situation is a marble rolling around in a round-bottomed bowl. We say that $x = 0$ is a point of **stable equilibrium.** More generally, *any minimum in a potential-energy curve is a stable equilibrium position.*

Figure 7.24a shows a hypothetical but more general potential-energy function $U(x)$. Figure 7.24b shows the corresponding force $F_x = -dU/dx$. Points x_1 and x_3 are stable equilibrium points. At each of these points, F_x is zero because the slope of the $U(x)$ curve is zero. When the particle is displaced to either side, the force pushes back toward the equilibrium point. The slope of the $U(x)$ curve is also zero at points x_2 and x_4, and these are also equilibrium points. But when the particle is displaced a little to the right of either point, the slope of the $U(x)$ curve becomes negative, corresponding to a positive F_x that tends to push the particle still farther from the point. When the particle is displaced a little to the left, F_x is negative, again pushing away from equilibrium. This is analogous to a marble rolling on the top of a bowling ball. Points x_2 and x_4 are called **unstable equilibrium** points; *any maximum in a potential-energy curve is an unstable equilibrium position.*

CAUTION **Potential energy and the direction of a conservative force** The direction of the force on a body is *not* determined by the sign of the potential energy U. Rather, it's the sign of $F_x = -dU/dx$ that matters. As we discussed in Section 7.1, the physically significant quantity is the *difference* in the value of U between two points, which is just

7.24 The maxima and minima of a potential-energy function $U(x)$ correspond to points where $F_x = 0$.



what the derivative $F_x = -dU/dx$ measures. This means that you can always add a constant to the potential-energy function without changing the physics of the situation.

If the total energy is E_1 and the particle is initially near x_1, it can move only in the region between x_a and x_b determined by the intersection of the E_1 and U graphs (Fig. 7.24a). Again, U cannot be greater than E_1 because K can't be negative. We speak of the particle as moving in a *potential well*, and x_a and x_b are the *turning points* of the particle's motion (since at these points, the particle stops and reverses direction). If we increase the total energy to the level E_2, the particle can move over a wider range, from x_c to x_d. If the total energy is greater than E_3, the particle can "escape" and move to indefinitely large values of x. At the other extreme, E_0 represents the least possible total energy the system can have.

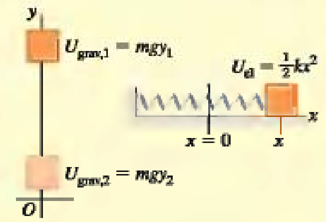
Test Your Understanding of Section 7.5 The curve in Fig. 7.24b has a maximum at a point between x_2 and x_3. Which statement correctly describes what happens to the particle when it is at this point? (i) The particle's acceleration is zero. (ii) The particle accelerates in the positive x-direction; the magnitude of the acceleration is less than at any other point between x_2 and x_3. (iii) The particle accelerates in the positive x-direction; the magnitude of the acceleration is greater than at any other point between x_2 and x_3. (iv) The particle accelerates in the negative x-direction; the magnitude of the acceleration is less than at any other point between x_2 and x_3. (v) The particle accelerates in the negative x-direction; the magnitude of the acceleration is greater than at any other point between x_2 and x_3.

CHAPTER 7 SUMMARY

Gravitational potential energy and elastic potential energy: The work done on a particle by a constant gravitational force can be represented as a change in the gravitational potential energy $U_{grav} = mgy$. This energy is a shared property of the particle and the earth. A potential energy is also associated with the elastic force $F_x = -kx$ exerted by an ideal spring, where x is the amount of stretch or compression. The work done by this force can be represented as a change in the elastic potential energy of the spring, $U_{el} = \frac{1}{2}kx^2$.

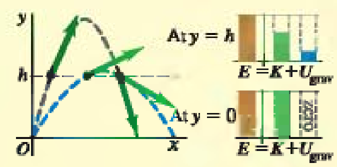
$$\begin{aligned} W_{grav} &= mgy_1 - mgy_2 \\ &= U_{grav,1} - U_{grav,2} \\ &= -\Delta U_{grav} \end{aligned} \qquad (7.1), (7.3)$$

$$\begin{aligned} W_{el} &= \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 \\ &= U_{el,1} - U_{el,2} = -\Delta U_{el} \end{aligned} \qquad (7.10)$$



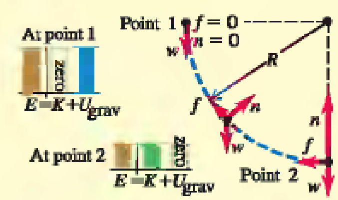
When total mechanical energy is conserved: The total potential energy U is the sum of the gravitational and elastic potential energy: $U = U_{grav} + U_{el}$. If no forces other than the gravitational and elastic forces do work on a particle, the sum of kinetic and potential energy is conserved. This sum $E = K + U$ is called the total mechanical energy. (See Examples 7.1, 7.3, 7.4, and 7.7.)

$$K_1 + U_1 = K_2 + U_2 \qquad (7.4), (7.11)$$



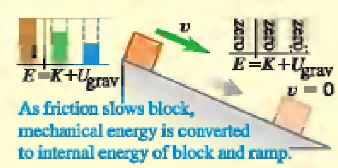
When total mechanical energy is not conserved: When forces other than the gravitational and elastic forces do work on a particle, the work W_{other} done by these other forces equals the change in total mechanical energy (kinetic energy plus total potential energy). (See Examples 7.2, 7.5, 7.6, 7.8, and 7.9.)

$$K_1 + U_1 + W_{other} = K_2 + U_2 \qquad (7.14)$$



Conservative forces, nonconservative forces, and the law of conservation of energy: All forces are either conservative or nonconservative. A conservative force is one for which the work–kinetic energy relationship is completely reversible. The work of a conservative force can always be represented by a potential-energy function, but the work of a nonconservative force cannot. The work done by nonconservative forces manifests itself as changes in the internal energy of bodies. The sum of kinetic, potential, and internal energy is always conserved. (See Examples 7.10–7.12.)

$$\Delta K + \Delta U + \Delta U_{int} = 0 \qquad (7.15)$$

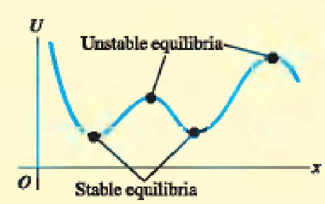


Determining force from potential energy: For motion along a straight line, a conservative force $F_x(x)$ is the negative derivative of its associated potential-energy function U. In three dimensions, the components of a conservative force are negative partial derivatives of U. (See Examples 7.13 and 7.14.)

$$F_x(x) = -\frac{dU(x)}{dx} \qquad (7.16)$$

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y}$$
$$F_z = -\frac{\partial U}{\partial y} \qquad (7.17)$$

$$\vec{F} = -\left(\frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\hat{k}\right) \qquad (7.18)$$



Key Terms

potential energy, *214*
gravitational potential energy, *214*
total mechanical energy, *215*
conservation of mechanical energy, *215*
elastic potential energy, *223*
conservative force, *228*
nonconservative force, *229*
dissipative force, *229*
internal energy, *231*
law of conservation of energy, *231*
gradient, *234*
energy diagram, *235*
stable equilibrium, *235*
unstable equilibrium, *236*

Answer to Chapter Opening Question ?

Gravity is doing positive work on the diver, since this force is in the same downward direction as his displacement. This corresponds to a decrease in gravitational potential energy. The water is doing negative work on the diver; it exerts an upward force of fluid resistance as he moves downward. This corresponds to an increase in internal energy of the diver and the water (see Section 7.3).

Answers to Test Your Understanding Questions

7.1 Answer: (iii) The initial kinetic energy $K_1 = 0$, the initial potential energy $U_1 = mgy_1$, and the final potential energy $U_2 = mgy_2$ are the same for both blocks. Mechanical energy is conserved in both cases, so the final kinetic energy $K = \frac{1}{2}mv_2^2$ is also the same for both blocks. Hence the speed at the right-hand end is the *same* in both cases!

7.2 Answer: (iii) The elevator is still moving downward, so the kinetic energy K is positive (remember that K can never be negative); the elevator is below point 1, so $y < 0$ and $U_{grav} < 0$; and the spring is compressed, so $U_{el} > 0$.

7.3 Answer: (iii) Because of friction in the turbines and between the water and turbines, some of the potential energy goes into raising the temperatures of the water and the mechanism.

7.4 Answers: (a) (iv), (b) (i) If $F_x = 0$ at a point, then the derivative of $U(x)$ must be zero at that point because $F_x = -dU(x)/dx$. However, this tells us absolutely nothing about the *value* of $U(x)$ at that point.

7.5 Answers: (iii) Figure 7.24b shows the x-component of force, F_x. Where this is maximum (most positive), the x-component of force and the x-acceleration have more positive values than at adjacent values of x.

PROBLEMS

For instructor-assigned homework, go to **www.masteringphysics.com**

Discussion Questions

Q7.1. A baseball is thrown straight up with initial speed v_0. If air resistance cannot be ignored, when the ball returns to its initial height its speed is less than v_0. Explain why, using energy concepts.

Q7.2. A projectile has the same initial kinetic energy no matter what the angle of projection. Why doesn't it rise to the same maximum height in each case?

Q7.3. Does an object's speed at the bottom of a frictionless ramp depend on the shape of the ramp or just on its height? Explain. What if the ramp is *not* frictionless?

Q7.4. An egg is released from rest from the roof of a building and falls to the ground. Its fall is observed by a student on the roof of the building, who uses coordinates with origin at the roof, and by a student on the ground, who uses coordinates with origin at the ground. Do the two students assign the same or different values to the initial gravitational potential energy, the final gravitational potential energy, the change in gravitational potential energy, and the kinetic energy of the egg just before it strikes the ground? Explain.

Q7.5. A physics teacher had a bowling ball suspended from a very long rope attached to the high ceiling of a large lecture hall. To illustrate his faith in conservation of energy, he would back up to one side of the stage, pull the ball far to one side until the taut rope brought it just to the end of his nose, and then release it. The massive ball would swing in a mighty arc across the stage and then return to stop momentarily just in front of the nose of the stationary, unflinching teacher. However, one day after the demonstration he looked up just in time to see a student at the other side of the stage *push* the ball away from his nose as he tried to duplicate the demonstration. Tell the rest of the story and explain the reason for the potentially tragic outcome.

Q7.6. Lost Energy? The principle of the conservation of energy tells us that energy is never lost, but only changes from one form to another. Yet in many ordinary situations, energy may appear to be lost. In each case, explain what happens to the "lost" energy. (a) A box sliding on the floor comes to a halt due to friction. How did friction take away its kinetic energy, and what happened to that energy? (b) A car stops when you apply the brakes. What happened to its kinetic energy? (c) Air resistance uses up some of the original gravitational potential energy of a falling object. What type of energy did the "lost" potential energy become? (d) When a returning space shuttle touches down on the runway, it has lost almost all its kinetic energy and gravitational potential energy. Where did all that energy go?

Q7.7. Is it possible for a frictional force to *increase* the mechanical energy of a system? If so, give examples.

Q7.6. A woman bounces on a trampoline, going a little higher with each bounce. Explain how she increases the total mechanical energy.

Q7.9. Fractured Physics. People often call their electric bill a *power* bill, yet the quantity on which the bill is based is expressed in *kilowatt-hours.* What are people really being billed for?

Q7.10. A rock of mass m and a rock of mass $2m$ are both released from rest at the same height and feel no air resistance as they fall. Which statements about these rocks are true? (There may be more than one correct choice.) (a) Both have the same initial gravitational potential energy. (b) Both have the same kinetic energy when they reach the ground. (c) Both reach the ground with the same speed. (d) When it reaches the ground, the heavier rock has twice the kinetic energy of the lighter one. (e) When it reaches the ground, the heavier rock has four times the kinetic energy of the lighter one.

Q7.11. On a friction-free ice pond, a hockey puck is pressed against (but not attached to) a fixed ideal spring, compressing the spring by a distance x_0. The maximum energy stored in the spring is U_0, the maximum speed the puck gains after being released is v_0, and its maximum kinetic energy is K_0. Now the puck is pressed so it compresses the spring twice as far as before. In this case, (a) what is the maximum potential energy stored in the spring (in terms of U_0), and (b) what are the puck's maximum kinetic energy and speed (in terms of K_0 and x_0)?

Q7.12. When people are cold, they often rub their hands together to warm them up. How does doing this produce heat? Where did the heat come from?

Q7.13. You often hear it said that most of our energy ultimately comes from the sun. Trace each of the following energies back to the sun. (a) the kinetic energy of a jet plane; (b) the potential energy gained by a mountain climber; (c) the electrical energy used to run a computer; (d) the electrical energy from a hydroelectric plant.

Q7.14. A box slides down a ramp and work is done on the box by the forces of gravity and friction. Can the work of each of these forces be expressed in terms of the change in a potential-energy function? For each force explain why or why not.

Q7.15. In physical terms, explain why friction is a nonconservative force. Does it store energy for future use?

Q7.16. A compressed spring is clamped in its compressed position and then is dissolved in acid. What becomes of its potential energy?

Q7.17. Since only changes in potential energy are important in any problem, a student decides to let the elastic potential energy of a spring be zero when the spring is stretched a distance x_1. The student decides, therefore, to let $U = \frac{1}{2}k(x - x_1)^2$. Is this correct? Explain.

Q7.18. Figure 7.22a shows the potential-energy function for the force $F_x = -kx$. Sketch the potential-energy function for the force $F_x = +kx$. For this force, is $x = 0$ a point of equilibrium? Is this equilibrium stable or unstable? Explain.

Q7.19. Figure 7.22b shows the potential-energy function associated with the gravitational force between an object and the earth. Use this graph to explain why objects always fall toward the earth when they are released.

Q7.20. For a system of two particles we often let the potential energy for the force between the particles approach zero as the separation of the particles approaches infinity. If this choice is made, explain why the potential energy at noninfinite separation is positive if the particles repel one another and negative if they attract.

Q7.21. Explain why the points $x = A$ and $x = -A$ in Fig. 7.23b are called *turning points*. How are the values of E and U related at a turning point?

Q7.22. A particle is in *neutral equilibrium* if the net force on it is zero and remains zero if the particle is displaced slightly in any direction. Sketch the potential-energy function near a point of neutral equilibrium, for the case of one-dimensional motion. Give an example of an object in neutral equilibrium.

Q7.23. The net force on a particle of mass m has the potential-energy function graphed in Fig. 7.24a. If the total energy is E_1, graph the speed v of the particle versus its position x. At what value of x is the speed greatest? Sketch v versus x if the total energy is E_2.

Q7.24. The potential-energy function for a force $\vec{F}$ is $U = \alpha x^3$, where α is a positive constant. What is the direction of $\vec{F}$?

Exercises

Section 7.1 Gravitational Potential Energy

7.1. In one day, a 75-kg mountain climber ascends from the 1500-m level on a vertical cliff to the top at 2400 m. The next day, she descends from the top to the base of the cliff, which is at an elevation of 1350 m. What is her change in gravitational potential energy (a) on the first day and (b) on the second day?

7.2. A 5.00-kg sack of flour is lifted vertically at a constant speed of 3.50 m/s through a height of 15.0 m. (a) How great a force is required? (b) How much work is done on the sack by the lifting force? What becomes of this work?

7.3. A 120-kg mail bag hangs by a vertical rope 3.5 m long. A postal worker then displaces the bag to a position 2.0 m sideways from its original position, always keeping the rope taut. (a) What horizontal force is necessary to hold the bag in the new position? (b) As the bag is moved to this position, how much work is done (i) by the rope and (ii) by the worker?

7.4. A 72.0-kg swimmer jumps into the old swimming hole from a diving board 3.25 m above the water. Use energy conservation to find his speed just he hits the water (a) if he just holds his nose and drops in, (b) if he bravely jumps straight up (but just beyond the board!) at 2.50 m/s, and (c) if he manages to jump downward at 2.50 m/s.

7.5. A baseball is thrown from the roof of a 22.0-m-tall building with an initial velocity of magnitude 12.0 m/s and directed at an angle of 53.1° above the horizontal. (a) What is the speed of the ball just before it strikes the ground? Use energy methods and ignore air resistance. (b) What is the answer for part (a) if the initial velocity is at an angle of 53.1° *below* the horizontal? (c) If the effects of air resistance are included, will part (a) or (b) give the higher speed?

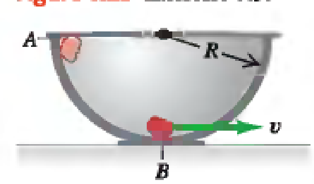
7.6. A crate of mass M starts from rest at the top of a frictionless ramp inclined at an angle α above the horizontal. Find its speed at the bottom of the ramp, a distance d from where it started. Do this in two ways: (a) Take the level at which the potential energy is zero to be at the bottom of the ramp with y positive upward. (b) Take the zero level for potential energy to be at the top of the ramp with y positive upward. (c) Why did the normal force not enter into your solution?

7.7. Answer part (b) of Example 7.6 (Section 7.1) by applying Eq. (7.7) to points 2 and 3, rather than to points 1 and 3 as was done in the example.

7.8. An empty crate is given an initial push down a ramp, starting it with a speed v_0, and reaches the bottom with speed v and kinetic energy K. Some books are now placed in the crate, so that the total mass is quadrupled. The coefficient of kinetic friction is constant and air resistance is negligible. Starting again with v_0 at the top of the ramp, what are the speed and kinetic energy at the bottom? Explain the reasoning behind your answers.

7.9. A small rock with mass 0.20 kg is released from rest at point A, which is at the top edge of a large, hemispherical bowl with radius $R = 0.50$ m (Fig. 7.25). Assume that the size of the rock is small compared to R, so that the rock can be treated as a particle, and assume that the rock slides rather than rolls. The work done by friction on the rock when it moves from point A to point B at the bottom of the bowl has magnitude 0.22 J. (a) Between points A and B, how much work is done on the rock by (i) the normal force and (ii) gravity? (b) What is the speed of the rock as it reaches point B? (c) Of the three forces acting on the rock as it slides down the bowl, which (if any) are constant and which are not? Explain. (d) Just as the rock reaches point B, what is the normal force on it due to the bottom of the bowl?

Figure 7.25 Exercise 7.9.



7.10. A stone of mass m is thrown upward at an angle θ above the horizontal and feels no appreciable air resistance. Use conservation of energy to show that at its highest point, it is a distance $v_0^2(\sin^2\theta)/2g$ above the point where it was launched. (*Hint:* $v_0^2 = v_{0x}^2 + v_{0y}^2$.)

7.11. You are testing a new amusement park roller coaster with an empty car with mass 120 kg. One part of the track is a vertical loop with radius 12.0 m. At the bottom of the loop (point A) the car has speed 25.0 m/s, and at the top of the loop (point B) it has speed 8.0 m/s. As the car rolls from point A to point B, how much work is done by friction?

7.12. Tarzan and Jane. Tarzan, in one tree, sights Jane in another tree. He grabs the end of a vine with length 20 m that makes an angle of 45° with the vertical, steps off his tree limb, and swings down and then up to Jane's open arms. When he arrives, his vine makes an angle of 30° with the vertical. Determine whether he gives her a tender embrace or knocks her off her limb by calculating Tarzan's speed just before he reaches Jane. You can ignore air resistance and the mass of the vine.

7.13. A 10.0-kg microwave oven is pushed 8.00 m up the sloping surface of a loading ramp inclined at an angle of 36.9° above the horizontal, by a constant force $\vec{F}$ with a magnitude 110 N and acting parallel to the ramp. The coefficient of kinetic friction between the oven and the ramp is 0.250. (a) What is the work done on the oven by the force $\vec{F}$? (b) What is the work done on the oven by the friction force? (c) Compute the increase in potential energy for the oven. (d) Use your answers to parts (a), (b), and (c) to calculate the increase in the oven's kinetic energy. (e) Use $\sum\vec{F} = m\vec{a}$ to calculate the acceleration of the oven. Assuming that the oven is initially at rest, use the acceleration to calculate the oven's speed after traveling 8.00 m. From this, compute the increase in the oven's kinetic energy, and compare it to the answer you got in part (d).

7.14. Pendulum. A small rock with mass 0.12 kg is fastened to a massless string with length 0.80 m to form a pendulum. The pendulum is swinging so as to make a maximum angle of 45° with the vertical. Air resistance is negligible. (a) What is the speed of the rock when the string passes through the vertical position? (b) What is the tension in the string when it makes an angle of 45° with the vertical? (c) What is the tension in the string as it passes through the vertical?

Section 7.2 Elastic Potential Energy

7.15. A force of 800 N stretches a certain spring a distance of 0.200 m. (a) What is the potential energy of the spring when it is stretched 0.200 m? (b) What is its potential energy when it is compressed 5.00 cm?

7.16. An ideal spring of negligible mass is 12.00 cm long when nothing is attached to it. When you hang a 3.15-kg weight from it, you measure its length to be 13.40 cm. If you wanted to store 10.0 J of potential energy in this spring, what would be its *total* length? Assume that it continues to obey Hooke's law.

7.17. A spring stores potential energy U_0 when it is compressed a distance x_0 from its uncompressed length. (a) In terms of U_0, how much energy does it store when it is compressed (i) twice as much and (ii) half as much? (b) In terms of x_0, how much must it be compressed from its uncompressed length to store (i) twice as much energy and (ii) half as much energy?

7.18. A slingshot will shoot a 10-g pebble 22.0 m straight up. (a) How much potential energy is stored in the slingshot's rubber band? (b) With the same potential energy stored in the rubber band, how high can the slingshot shoot a 25-g pebble? (c) What physical effects did you ignore in solving this problem?

7.19. A spring of negligible mass has force constant $k =$ 1600 N/m. (a) How far must the spring be compressed for 3.20 J of potential energy to be stored in it? (b) You place the spring vertically with one end on the floor. You then drop a 1.20-kg book onto it from a height of 0.80 m above the top of the spring. Find the maximum distance the spring will be compressed.

7.20. A 1.20-kg piece of cheese is placed on a vertical spring of negligible mass and force constant $k = 1800$ N/m that is compressed 15.0 cm. When the spring is released, how high does the cheese rise from this initial position? (The cheese and the spring are *not* attached.)

7.21. Consider the glider of Example 7.7 (Section 7.2) and Fig. 7.16. As in the example, the glider is released from rest with the spring stretched 0.100 m. What is the displacement x of the glider from its equilibrium position when its speed is 0.20 m/s? (You should get more than one answer. Explain why.)

7.22. Consider the glider of Example 7.7 (Section 7.2) and Fig. 7.16. (a) As in the example, the glider is released from rest with the spring stretched 0.100 m. What is the speed of the glider when it returns to $x = 0$? (b) What must the initial displacement of the glider be if its maximum speed in the subsequent motion is to be 2.50 m/s?

7.23. A 2.50-kg mass is pushed against a horizontal spring of force constant 25.0 N/cm on a frictionless air table. The spring is attached to the tabletop, and the mass is not attached to the spring in any way. When the spring has been compressed enough to store 11.5 J of potential energy in it, the mass is suddenly released from rest. (a) Find the greatest speed the mass reaches. When does this occur? (b) What is the greatest acceleration of the mass, and when does it occur?

7.24. (a) For the elevator of Example 7.9 (Section 7.2), what is the speed of the elevator after it has moved downward 1.00 m from point 1 in Fig. 7.17? (b) When the elevator is 1.00 m below point 1 in Fig. 7.17, what is its acceleration?

7.25. You are asked to design a spring that will give a 1160-kg satellite a speed of 2.50 m/s relative to an orbiting space shuttle. Your spring is to give the satellite a maximum acceleration of $5.00g$. The spring's mass, the recoil kinetic energy of the shuttle, and changes in gravitational potential energy will all be negligible. (a) What must the force constant of the spring be? (b) What distance must the spring be compressed?

Section 7.3 Conservative and Nonconservative Forces

7.26. A 75-kg roofer climbs a vertical 7.0-m ladder to the flat roof of a house. He then walks 12 m on the roof, climbs down another vertical 7.0-m ladder, and finally walks on the ground back to his starting point. How much work is done on him by gravity (a) as he climbs up; (b) as he climbs down; (c) as he walks on the roof and on the ground? (d) What is the total work done on him by gravity during this round trip? (e) On the basis of your answer to part (d), would you say that gravity is a conservative or nonconservative force? Explain.

7.27. A 10.0-kg box is pulled by a horizontal wire in a circle on a rough horizontal surface for which the coefficient of kinetic friction is 0.250. Calculate the work done by friction during one complete circular trip if the radius is (a) 2.00 m and (b) 4.00 m. (c) On the basis of the results you just obtained, would you say that friction is a conservative or nonconservative force? Explain.

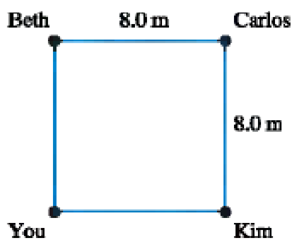
7.28. In an experiment, one of the forces exerted on a proton is $\vec{F} = -\alpha x^2\hat{\imath}$, where $\alpha = 12$ N/m^2. (a) How much work does $\vec{F}$ do when the proton moves along the straight-line path from the point

$(0.10 \text{ m}, 0)$ to the point $(0.10 \text{ m}, 0.40 \text{ m})$? (b) Along the straight-line path from the point $(0.10 \text{ m}, 0)$ to the point $(0.30 \text{ m}, 0)$? (c) Along the straight-line path from the point $(0.30 \text{ m}, 0)$ to the point $(0.10 \text{ m}, 0)$? (d) Is the force $\vec{F}$ conservative? Explain. If $\vec{F}$ is conservative, what is the potential-energy function for it? Let $U = 0$ when $x = 0$.

7.29. A 0.60-kg book slides on a horizontal table. The kinetic friction force on the book has magnitude 1.2 N. (a) How much work is done on the book by friction during a displacement of 3.0 m to the left? (b) The book now slides 3.0 m to the right, returning to its starting point. During this second 3.0-m displacement, how much work is done on the book by friction? (c) What is the total work done on the book by friction during the complete round trip? (d) On the basis of your answer to part (c), would you say that the friction force is conservative or nonconservative? Explain.

7.30. You and three friends stand at the corners of a square whose sides are 8.0 m long in the middle of the gym floor, as shown in Fig. 7.26. You take your physics book and push it from one person to the other. The book has a mass of 1.5 kg, and the coefficient of kinetic friction between the book and the floor is $\mu_k = 0.25$. (a) The book slides from you to Beth and then from Beth to Carlos, along the lines connecting these people. What is the work done by friction during this displacement? (b) You slide the book from you to Carlos along the diagonal of the square. What is the work done by friction during this displacement? (c) You slide the book to Kim who then slides it back to you. What is the total work done by friction during this motion of the book? (d) Is the friction force on the book conservative or nonconservative? Explain.

Figure 7.26 Exercise 7.30.



7.31. A block with mass m is attached to an ideal spring that has force constant k. (a) The block moves from x_1 to x_2, where $x_2 > x_1$. How much work does the spring force do during this displacement? (b) The block moves from x_1 to x_2 and then from x_2 to x_1. How much work does the spring force do during the displacement from x_2 to x_1? What is the total work done by the spring during the entire $x_1 \rightarrow x_2 \rightarrow x_1$ displacement? Explain why you got the answer you did. (c) The block moves from x_1 to x_3, where $x_3 > x_2$. How much work does the spring force do during this displacement? The block then moves from x_3 to x_2. How much work does the spring force do during this displacement? What is the total work done by the spring force during the $x_1 \rightarrow x_3 \rightarrow x_2$ displacement? Compare your answer to the answer in part (a), where the starting and ending points are the same but the path is different.

Section 7.4 Force and Potential Energy

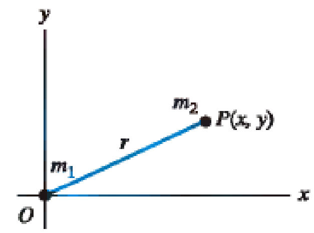
7.32. The potential energy of a pair of hydrogen atoms separated by a large distance x is given by $U(x) = -C_6/x^6$, where C_6 is a positive constant. What is the force that one atom exerts on the other? Is this force attractive or repulsive?

7.33. A force parallel to the x-axis acts on a particle moving along the x-axis. This force produces potential energy $U(x)$ given by $U(x) = \alpha x^4$, where $\alpha = 1.20 \text{ J/m}^4$. What is the force (magnitude and direction) when the particle is at $x = -0.800$ m?

7.34. Gravity in One Dimension. Two point masses, m_1 and m_2, lie on the x-axis, with m_1 held in place at the origin and m_2 at position x and free to move. The gravitational potential energy of these masses is found to be $U(x) = -Gm_1m_2/x$, where G is a constant (called the *gravitational constant*). You'll learn more about gravitation in Chapter 12. Find the x-component of the force acting on m_2 due to m_1. Is this force attractive or repulsive? How do you know?

7.35. Gravity in Two Dimensions. Two point masses, m_1 and m_2, lie in the xy-plane, with m_1 held in place at the origin and m_2 free to move a distance r away at a point P having coordinates x and y (Fig. 7.27). The gravitational potential energy of these masses is found to be $U(r) = -Gm_1m_2/r$, where G is the gravitational constant. (a) Show that the components of the force on m_2 due to m_1 are

Figure 7.27 Exercise 7.35.



$$F_x = -\frac{Gm_1m_2x}{(x^2 + y^2)^{3/2}} \quad \text{and} \quad F_y = -\frac{Gm_1m_2y}{(x^2 + y^2)^{3/2}}$$

(*Hint:* First write r in terms of x and y.) (b) Show that the magnitude of the force on m_2 is $F = Gm_1m_2/r^2$. (c) Does m_1 attract or repel m_2? How do you know?

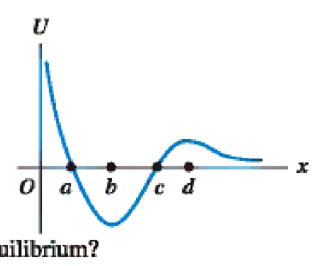
7.36. An object moving in the xy-plane is acted on by a conservative force described by the potential-energy function $U(x, y) = \alpha(1/x^2 + 1/y^2)$, where α is a positive constant. Derive an expression for the force expressed in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

Section 7.5 Energy Diagrams

7.37. The potential energy of two atoms in a diatomic molecule is approximated by $U(r) = a/r^{12} - b/r^6$, where r is the spacing between atoms and a and b are positive constants. (a) Find the force $F(r)$ on one atom as a function of r. Make two graphs, one of $U(r)$ versus r and one of $F(r)$ versus r. (b) Find the equilibrium distance between the two atoms. Is this equilibrium stable? (c) Suppose the distance between the two atoms is equal to the equilibrium distance found in part (b). What minimum energy must be added to the molecule to *dissociate* it—that is, to separate the two atoms to an infinite distance apart? This is called the *dissociation energy* of the molecule. (d) For the molecule CO, the equilibrium distance between the carbon and oxygen atoms is 1.13×10^{-10} m and the dissociation energy is 1.54×10^{-18} J per molecule. Find the values of the constants a and b.

7.38. A marble moves along the x-axis. The potential-energy function is shown in Fig. 7.28. (a) At which of the labeled x-coordinates is the force on the marble zero? (b) Which of the labeled x-coordinates is a position of stable equilibrium? (c) Which of the labeled x-coordinates is a position of unstable equilibrium?

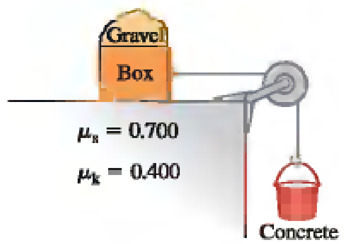
Figure 7.28 Exercise 7.38.



Problems

7.38. At a construction site, a 65.0-kg bucket of concrete hangs from a light (but strong) cable that passes over a light friction-free pulley and is connected to an 80.0-kg box on a horizontal roof (Fig. 7.29). The cable pulls horizontally on the box, and a 50.0-kg bag of gravel rests on top of the box. The coefficients of friction between the box and roof are shown. (a) Find the friction force on the bag of gravel and on the box. (b) Suddenly a worker picks up the bag of gravel. Use energy conservation to find the speed of the bucket after it has descended 2.00 m, from rest. (You can check your answer by solving this problem using Newton's laws.)

Figure 7.29 Problem 7.39.

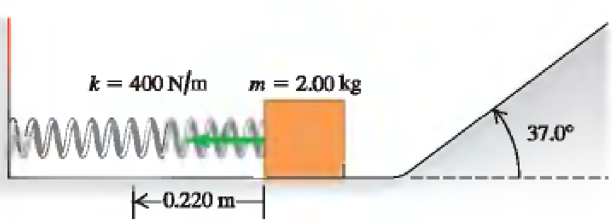


7.40. Two blocks with different mass are attached to either end of a light rope that passes over a light, frictionless pulley that is suspended from the ceiling. The masses are released from rest, and the more massive one starts to descend. After this block has descended 1.20 m, its speed is 3.00 m/s. If the total mass of the two blocks is 15.0 kg, what is the mass of each block?

7.41. Legal Physics. In an auto accident, a car hit a pedestrian and the driver then slammed on the brakes to stop the car. During the subsequent trial, the driver's lawyer claimed that he was obeying the posted 35 mi/h speed limit, but that the legal speed was too high to allow him to see and react to the pedestrian in time. You have been called in as the state's expert witness. Your investigation of the accident found that the skid marks made while the brakes were applied were 280 ft long, and the tread on the tires produced a coefficient of kinetic friction of 0.30 with the road. (a) In your testimony in court, will you say that the driver was obeying the posted speed? You must be able to back up your conclusion with clear reasoning because one of the lawyers will surely cross-examine you. (b) If the driver's speeding ticket were \$10 for each mile per hour he was driving above the posted speed limit, would he have to pay a fine? If so, how much would it be?

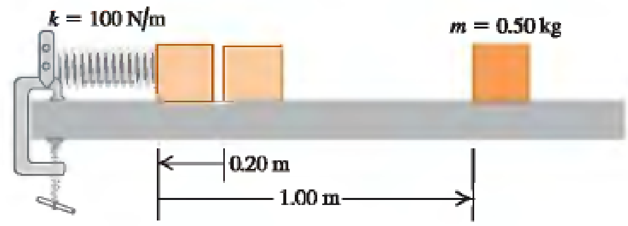
7.42. A 2.00-kg block is pushed against a spring with negligible mass and force constant $k = 400$ N/m, compressing it 0.220 m. When the block is released, it moves along a frictionless, horizontal surface and then up a frictionless incline with slope 37.0° (Fig. 7.30). (a) What is the speed of the block as it slides along the horizontal surface after having left the spring? (b) How far does the block travel up the incline before starting to slide back down?

Figure 7.30 Problem 7.42.



7.43. A block with mass 0.50 kg is forced against a horizontal spring of negligible mass, compressing the spring a distance of 0.20 m (Fig. 7.31). When released, the block moves on a horizontal tabletop for 1.00 m before coming to rest. The spring constant k is 100 N/m. What is the coefficient of kinetic friction μ_k between the block and the tabletop?

Figure 7.31 Problem 7.43.

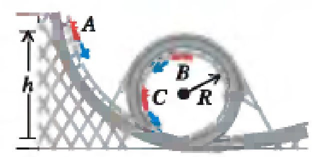


7.44. On a horizontal surface, a crate with mass 50.0 kg is placed against a spring that stores 360 J of energy. The spring is released, and the crate slides 5.60 m before coming to rest. What is the speed of the crate when it is 2.00 m from its initial position?

7.45. Bouncing Ball. A 650-gram rubber ball is dropped from an initial height of 2.50 m, and on each bounce it returns to 75% of its previous height. (a) What is the initial mechanical energy of the ball, just after it is released from its initial height? (b) How much mechanical energy does the ball lose during its first bounce? What happens to this energy? (c) How much mechanical energy is lost during the second bounce?

7.46. Riding a Loop-the-Loop. A car in an amusement park ride rolls without friction around the track shown in Fig. 7.32. It starts from rest at point A at a height h above the bottom of the loop. Treat the car as a particle. (a) What is the minimum value of h (in terms of R) such that the car moves around the loop without falling off at the top (point B)? (b) If $h = 3.50R$ and $R = 20.0$ m, compute the speed, radial acceleration, and tangential acceleration of the passengers when the car is at point C, which is at the end of a horizontal diameter. Show these acceleration components in a diagram, approximately to scale.

Figure 7.32 Problem 7.46.



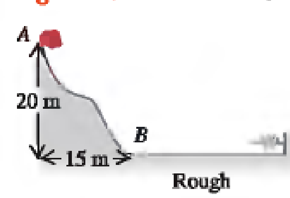
7.47. A 2.0-kg piece of wood slides on the surface shown in Fig. 7.33. The curved sides are perfectly smooth, but the rough horizontal bottom is 30 m long and has a kinetic friction coefficient of 0.20 with the wood. The piece of wood starts from rest 4.0 m above the rough bottom. (a) Where will this wood eventually come to rest? (b) For the motion from the initial release until the piece of wood comes to rest, what is the total amount of work done by friction?

Figure 7.33 Problem 7.47.

7.48. Up and Down the Hill. A 28-kg rock approaches the foot of a hill with a speed of 15 m/s. This hill slopes upward at a constant angle of 40.0° above the horizontal. The coefficients of static and kinetic friction between the hill and the rock are 0.75 and 0.20, respectively. (a) Use energy conservation to find the maximum height above the foot of the hill reached by the rock. (b) Will the rock remain at rest at its highest point, or will it slide back down the hill? (c) If the rock does slide back down, find its speed when it returns to the bottom of the hill.

7.49. A 15.0-kg stone slides down a snow-covered hill (Fig. 7.34), leaving point A with a speed of 10.0 m/s. There is no friction on the hill between points A and B, but there is friction on the level ground at the bottom of the hill, between B and the wall. After

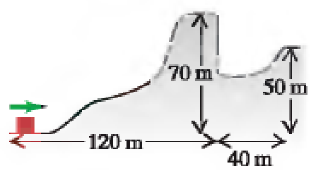
Figure 7.34 Problem 7.49.



entering the rough horizontal region, the stone travels 100 m and then runs into a very long, light spring with force constant 2.00 N/m. The coefficients of kinetic and static friction between the stone and the horizontal ground are 0.20 and 0.80, respectively. (a) What is the speed of the stone when it reaches point B? (b) How far will the stone compress the spring? (c) Will the stone move again after it has been stopped by the spring?

7.50. A 2.8-kg block slides over the smooth, icy hill shown in Fig. 7.35. The top of the hill is horizontal and 70 m higher than its base. What minimum speed must the block have at the base of the hill so that it will not fall into the pit on the far side of the hill?

Figure 7.35 Problem 7.50.



7.51. Bungee Jump. A bungee cord is 30.0 m long and, when stretched a distance x, it exerts a restoring force of magnitude kx. Your father-in-law (mass 95.0 kg) stands on a platform 45.0 m above the ground, and one end of the cord is tied securely to his ankle and the other end to the platform. You have promised him that when he steps off the platform he will fall a maximum distance of only 41.0 m before the cord stops him. You had several bungee cords to select from, and you tested them by stretching them out, tying one end to a tree, and pulling on the other end with a force of 380.0 N. When you do this, what distance will the bungee cord that you should select have stretched?

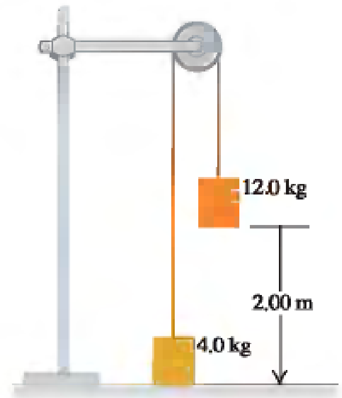
7.52. Ski Jump Ramp. You are designing a ski jump ramp for the next Winter Olympics. You need to calculate the vertical height h from the starting gate to the bottom of the ramp. The skiers push off hard with their ski poles at the start, just above the starting gate, so they typically have a speed of 2.0 m/s as they reach the gate. For safety, the skiers should have a speed of no more than 30.0 m/s when they reach the bottom of the ramp. You determine that for a 85.0-kg skier with good form, friction and air resistance will do total work of magnitude 4000 J on him during his run down the slope. What is the maximum height h for which the maximum safe speed will not be exceeded?

7.53. The Great Sandini is a 60-kg circus performer who is shot from a cannon (actually a spring gun). You don't find many men of his caliber, so you help him design a new gun. This new gun has a very large spring with a very small mass and a force constant of 1100 N/m that he will compress with a force of 4400 N. The inside of the gun barrel is coated with Teflon, so the average friction force will be only 40 N during the 4.0 m he moves in the barrel. At what speed will he emerge from the end of the barrel, 2.5 m above his initial rest position?

7.54. You are designing a delivery ramp for crates containing exercise equipment. The 1470-N crates will move at 1.8 m/s at the top of a ramp that slopes downward at 22.0°. The ramp exerts a 550-N kinetic friction force on each crate, and the maximum static friction force also has this value. Each crate will compress a spring at the bottom of the ramp and will come to rest after traveling a total distance of 8.0 m along the ramp. Once stopped, a crate must not rebound back up the ramp. Calculate the force constant of the spring that will be needed in order to meet the design criteria.

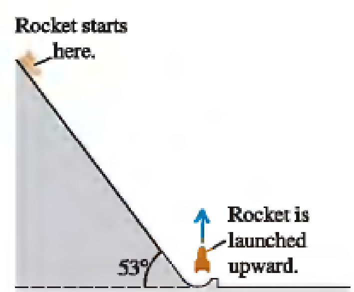
7.55. A system of two paint buckets connected by a lightweight rope is released from rest with the 12.0-kg bucket 2.00 m above the floor (Fig. 7.36). Use the principle of conservation of energy to find the speed with which this bucket strikes the floor. You can ignore friction and the mass of the pulley.

Figure 7.36 Problem 7.55.



7.56. A 1500-kg rocket is to be launched with an initial upward speed of 50.0 m/s. In order to assist its engines, the engineers will start it from rest on a ramp that rises 53° above the horizontal (Fig. 7.37). At the bottom, the ramp turns upward and launches the rocket vertically. The engines provide a constant forward thrust of 2000 N, and friction with the ramp surface is a constant 500 N. How far from the base of the ramp should the rocket start, as measured along the surface of the ramp?

Figure 7.37 Problem 7.56.



7.57. A machine part of mass m is attached to a horizontal ideal spring of force constant k that is attached to the edge of a friction-free horizontal surface. The part is pushed against the spring, compressing it a distance x_0, and then released from rest. Find the maximum (a) speed and (b) acceleration of the machine part. (c) Where in the motion do the maxima in parts (a) and (b) occur? (d) What will be the maximum extension of the spring? (e) Describe the subsequent motion of this machine part. Will it ever stop permanently?

7.58. A wooden rod of negligible mass and length 80.0 cm is pivoted about a horizontal axis through its center. A white rat with mass 0.500 kg clings to one end of the stick, and a mouse with mass 0.200 kg clings to the other end. The system is released from rest with the rod horizontal. If the animals can manage to hold on, what are their speeds as the rod swings through a vertical position?

7.59. A 0.100-kg potato is tied to a string with length 2.50 m, and the other end of the string is tied to a rigid support. The potato is held straight out horizontally from the point of support, with the string pulled taut, and is then released. (a) What is the speed of the potato at the lowest point of its motion? (b) What is the tension in the string at this point?

7.60. These data are from a computer simulation for a batted baseball with mass 0.145 kg, including air resistance:

t	x	y	v_x	v_y
0	0	0	30.0 m/s	40.0 m/s
3.05 s	70.2 m	53.6 m	18.6 m/s	0
6.59 s	124.4 m	0	11.9 m/s	−28.7 m/s

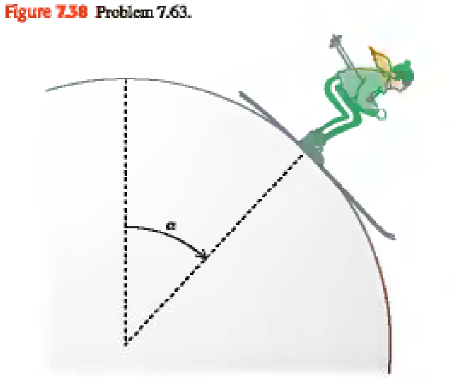
(a) How much work was done by the air on the baseball as it moved from its initial position to its maximum height? (b) How much work was done by the air on the baseball as it moved from its maximum height back to the starting elevation? (c) Explain why the magnitude of the answer in part (b) is smaller than the magnitude of the answer in part (a).

7.61. Down the Pole. A fireman of mass m slides a distance d down a pole. He starts from rest. He moves as fast at the bottom as if he had stepped off a platform a distance $h \le d$ above the ground and descended with negligible air resistance. (a) What average friction force did the fireman exert on the pole? Does your answer make sense in the special cases of $h = d$ and $h = 0$? (b) Find a numerical value for the average friction force a 75-kg fireman exerts, for $d = 2.5$ m and $h = 1.0$ m. (c) In terms of g, h, and d, what is the speed of the fireman when he is a distance y above the bottom of the pole?

7.62. A 60.0-kg skier starts from rest at the top of a ski slope 65.0 m high. (a) If frictional forces do −10.5 kJ of work on her as she descends, how fast is she going at the bottom of the slope? (b) Now moving horizontally, the skier crosses a patch of soft snow, where $\mu_k = 0.20$. If the patch is 82.0 m wide and the average force of air resistance on the skier is 160 N, how fast is she going after crossing the patch? (c) The skier hits a snowdrift and penetrates 2.5 m into it before coming to a stop. What is the average force exerted on her by the snowdrift as it stops her?

7.63. A skier starts at the top of a very large, frictionless snowball, with a very small initial speed, and skis straight down the side (Fig. 7.38). At what point does she lose contact with the snowball and fly off at a tangent? That is, at the instant she loses contact with the snowball, what angle α does a radial line from the center of the snowball to the skier make with the vertical?

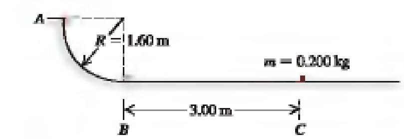
Figure 7.38 Problem 7.63.



7.64. A rock is tied to a cord and the other end of the cord is held fixed. The rock is given an initial tangential velocity that causes it to rotate in a vertical circle. Prove that the tension in the cord at the lowest point exceeds the tension at the highest point by six times the weight of the rock.

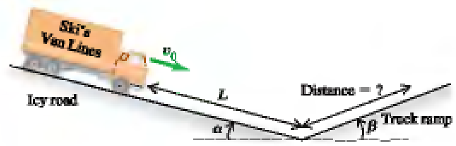
7.65. In a truck-loading station at a post office, a small 0.200-kg package is released from rest at point A on a track that is one-quarter of a circle with radius 1.60 m (Fig. 7.39). The size of the package is much less than 1.60 m, so the package can be treated as a particle. It slides down the track and reaches point B with a speed of 4.80 m/s. From point B, it slides on a level surface a distance of 3.00 m to point C, where it comes to rest. (a) What is the coefficient of kinetic friction on the horizontal surface? (b) How much work is done on the package by friction as it slides down the circular arc from A to B?

Figure 7.39 Problem 7.65.



7.66. A truck with mass m has a brake failure while going down an icy mountain road of constant downward slope angle α (Fig. 7.40). Initially the truck is moving downhill at speed v_0. After careening downhill a distance L with negligible friction, the truck driver steers the runaway vehicle onto a runaway truck ramp of constant upward slope angle β. The truck ramp has a soft sand surface for which the coefficient of rolling friction is μ_r. What is the distance that the truck moves up the ramp before coming to a halt? Solve using energy methods.

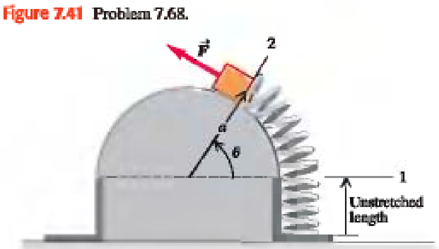
Figure 7.40 Problem 7.66.



7.67. A certain spring is found *not* to obey Hooke's law; it exerts a restoring force $F_x(x) = -\alpha x - \beta x^2$ if it is stretched or compressed, where $\alpha = 60.0$ N/m and $\beta = 18.0$ N/m^2. The mass of the spring is negligible. (a) Calculate the potential-energy function $U(x)$ for this spring. Let $U = 0$ when $x = 0$. (b) An object with mass 0.900 kg on a frictionless, horizontal surface is attached to this spring, pulled a distance 1.00 m to the right (the $+x$-direction) to stretch the spring, and released. What is the speed of the object when it is 0.50 m to the right of the $x = 0$ equilibrium position?

7.68. A variable force $\vec{F}$ is maintained tangent to a frictionless, semicircular surface (Fig. 7.41). By slow variations in the force, a

Figure 7.41 Problem 7.68.



block with weight w is moved, and the spring to which it is attached is stretched from position 1 to position 2. The spring has negligible mass and force constant k. The end of the spring moves in an arc of radius a. Calculate the work done by the force $\vec{F}$.

7.69. A 0.150-kg block of ice is placed against a horizontal, compressed spring mounted on a horizontal tabletop that is 1.20 m above the floor. The spring has force constant 1900 N/m and is initially compressed 0.045 m. The mass of the spring is negligible. The spring is released, and the block slides along the table, goes off the edge, and travels to the floor. If there is negligible friction between the block of ice and the tabletop, what is the speed of the block of ice when it reaches the floor?

7.70. A 3.00-kg block is connected to two ideal horizontal springs having force constants $k_1 = 25.0$ N/cm and $k_2 = 20.0$ N/cm (Fig. 7.42). The system is initially in equilibrium on a horizontal, frictionless surface. The block is now pushed 15.0 cm to the right and released from rest. (a) What is the maximum speed of the block? Where in the motion does the maximum speed occur? (b) What is the maximum compression of spring 1?

Figure 7.42 Problem 7.70.

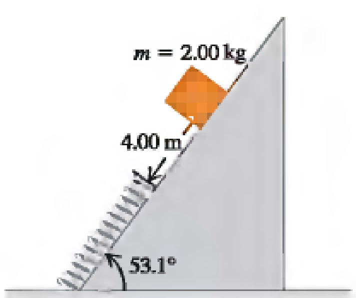
7.71. An experimental apparatus with mass m is placed on a vertical spring of negligible mass and pushed down until the spring is compressed a distance x. The apparatus is then released and reaches its maximum height at a distance h above the point where it is released. The apparatus is not attached to the spring, and at its maximum height it is no longer in contact with the spring. The maximum magnitude of acceleration the apparatus can have without being damaged is a, where $a > g$. (a) What should the force constant of the spring be? (b) What distance x must the spring be compressed initially?

7.72. If a fish is attached to a vertical spring and slowly lowered to its equilibrium position, it is found to stretch the spring by an amount d. If the same fish is attached to the end of the unstretched spring and then allowed to fall from rest, through what maximum distance does it stretch the spring? (*Hint:* Calculate the force constant of the spring in terms of the distance d and the mass m of the fish.)

7.73. A wooden block with mass 1.50 kg is placed against a compressed spring at the bottom of an incline of slope 30.0° (point A). When the spring is released, it projects the block up the incline. At point B, a distance of 6.00 m up the incline from A, the block is moving up the incline at 7.00 m/s and is no longer in contact with the spring. The coefficient of kinetic friction between the block and the incline is $\mu_k = 0.50$. The mass of the spring is negligible. Calculate the amount of potential energy that was initially stored in the spring.

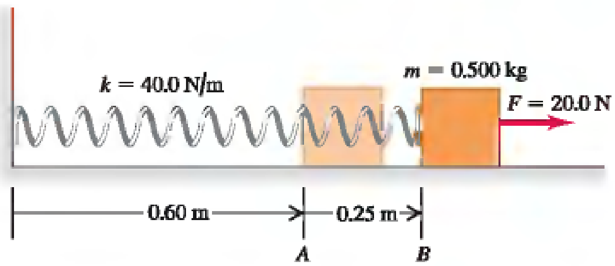
7.74. A 2.00-kg package is released on a 53.1° incline, 4.00 m from a long spring with force constant 120 N/m that is attached at the bottom of the incline (Fig. 7.43). The coefficients of friction between the package and the incline are $\mu_s = 0.40$ and $\mu_k = 0.20$. The mass of the spring is negligible. (a) What is the speed of the package just before it reaches the spring? (b) What is the maximum compression of the spring? (c) The package rebounds back up the incline. How close does it get to its initial position?

Figure 7.43 Problem 7.74.



7.75. A 0.500-kg block, attached to a spring with length 0.60 m and force constant 40.0 N/m, is at rest with the back of the block at point A on a frictionless, horizontal air table (Fig. 7.44). The mass of the spring is negligible. You move the block to the right along the surface by pulling with a constant 20.0-N horizontal force. (a) What is the block's speed when the back of the block reaches point B, which is 0.25 m to the right of point A? (b) When the back of the block reaches point B, you let go of the block. In the subsequent motion, how close does the block get to the wall where the left end of the spring is attached?

Figure 7.44 Problem 7.75.



7.76. Fraternity Physics. The brothers of Iota Eta Pi fraternity build a platform, supported at all four corners by vertical springs, in the basement of their frat house. A brave fraternity brother wearing a football helmet stands in the middle of the platform; his weight compresses the springs by 0.18 m. Then four of his fraternity brothers, pushing down at the corners of the platform, compress the springs another 0.53 m until the top of the brave brother's helmet is 0.90 m below the basement ceiling. They then simultaneously release the platform. You can ignore the masses of the springs and platform. (a) When the dust clears, the fraternity asks you to calculate their fraternity brother's speed just before his helmet hit the flimsy ceiling. (b) Without the ceiling, how high would he have gone? (c) In discussing their probation, the dean of students suggests that the next time they try this, they do it outdoors on another planet. Would the answer to part (b) be the same if this stunt were performed on a planet with a different value of g? Assume that the fraternity brothers push the platform down 0.53 m as before. Explain your reasoning.

7.77. A particle with mass m is acted on by a conservative force and moves along a path given by $x = x_0 \cos\omega_0 t$ and $y = y_0 \sin\omega_0 t$, where x_0, y_0, and ω_0 are constants. (a) Find the components of the force that acts on the particle. (b) Find the potential energy of the particle as a function of x and y. Take $U = 0$ when $x = 0$ and $y = 0$. (c) Find the total energy of the particle when (i) $x = x_0$, $y = 0$ and (ii) $x = 0$, $y = y_0$.

7.76. When it is burned, 1 gallon of gasoline produces 1.3×10^8 J of energy. A 1500-kg car accelerates from rest to 37 m/s in 10 s. The engine of this car is only 15% efficient (which is typical), meaning that only 15% of the energy from the combustion of the gasoline is used to accelerate the car. The rest goes into things like the internal kinetic energy of the engine parts as well as heating of the exhaust air and engine. (a) How many gallons of gasoline does this car use during the acceleration? (b) How many such accelerations will it take to burn up 1 gallon of gas?

7.79. A hydroelectric dam holds back a lake of surface area 3.0×10^6 m^2 that has vertical sides below the water level. The water level in the lake is 150 m above the base of the dam. When the water passes through turbines at the base of the dam, its mechanical energy is converted into electrical energy with 90% efficiency. (a) If gravitational potential energy is taken to be zero at the base of the dam, how much energy is stored in the top meter of the water in the lake? The density of water is 1000 kg/m^3. (b) What volume of water must pass through the dam to produce 1000 kilowatt-hours of electrical energy? What distance does the level of water in the lake fall when this much water passes through the dam?

7.80. How much total energy is stored in the lake in Problem 7.79? As in that problem, take the gravitational potential energy to be zero at the base of the dam. Express your answer in joules and in kilowatt-hours. (*Hint:* Break the lake up into infinitesimal horizontal layers of thickness dy, and integrate to find the total potential energy.)

7.81. Gravity in Three Dimensions. A point mass m_1 is held in place at the origin, and another point mass m_2 is free to move a distance r away at a point P having coordinates x, y, and z. The gravitational potential energy of these masses is found to be $U(r) = -Gm_1m_2/r$, where G is the gravitational constant (see Exercises 7.34 and 7.35). (a) Show that the components of the force on m_2 due to m_1 are

$$F_x = -\frac{Gm_1m_2x}{(x^2+y^2+z^2)^{3/2}} \qquad F_y = -\frac{Gm_1m_2y}{(x^2+y^2+z^2)^{3/2}}$$

$$F_z = -\frac{Gm_1m_2z}{(x^2+y^2+z^2)^{3/2}}$$

(*Hint:* First write r in terms of x, y, and z.) (b) Show that the magnitude of the force on m_2 is $F = Gm_1m_2/r^2$. (c) Does m_1 attract or repel m_2? How do you know?

7.82. (a) Is the force $\vec{F} = Cy^2\hat{\jmath}$, where C is a negative constant with units of N/m^2, conservative or nonconservative? Justify your answer. (b) Is the force $\vec{F} = Cy^2\hat{\imath}$, where C is a negative constant with units of N/m^2, conservative or nonconservative? Justify your answer.

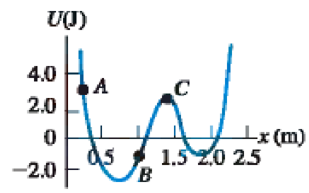
7.83. A cutting tool under microprocessor control has several forces acting on it. One force is $\vec{F} = -\alpha xy^2\hat{\jmath}$, a force in the negative y-direction whose magnitude depends on the position of the tool. The constant is $\alpha = 2.50$ N/m^3. Consider the displacement of the tool from the origin to the point $x = 3.00$ m, $y = 3.00$ m. (a) Calculate the work done on the tool by $\vec{F}$ if this displacement is along the straight line $y = x$ that connects these two points. (b) Calculate the work done on the tool by $\vec{F}$ if the tool is first moved out along the x-axis to the point $x = 3.00$ m, $y = 0$ and then moved parallel to the y-axis to the point $x = 3.00$ m, $y = 3.00$ m. (c) Compare the work done by $\vec{F}$ along these two paths. Is $\vec{F}$ conservative or nonconservative? Explain.

7.84. An object has several forces acting on it. One force is $\vec{F} = \alpha xy\hat{\imath}$, a force in the x-direction whose magnitude depends on the position of the object. (See Problem 6.96.) The constant is $\alpha = 2.00$ N/m^2. The object moves along the following path: (1) It starts at the origin and moves along the y-axis to the point $x = 0$, $y = 1.50$ m; (2) it moves parallel to the x-axis to the point $x = 1.50$ m, $y = 1.50$ m; (3) it moves parallel to the y-axis to the point $x = 1.50$ m, $y = 0$; (4) it moves parallel to the x-axis back to the origin. (a) Sketch this path in the xy-plane. (b) Calculate the work done on the object by $\vec{F}$ for each leg of the path and for the complete round trip. (c) Is $\vec{F}$ conservative or nonconservative? Explain.

7.85. A Hooke's law force $-kx$ and a constant conservative force F in the $+x$-direction act on an atomic ion. (a) Show that a possible potential-energy function for this combination of forces is $U(x) = \frac{1}{2}kx^2 - Fx - F^2/2k$. Is this the *only* possible function? Explain. (b) Find the stable equilibrium position. (c) Graph $U(x)$ (in units of F^2/k) versus x (in units of F/k) for values of x between $-5F/k$ and $5F/k$. (d) Are there any unstable equilibrium positions? (e) If the total energy is $E = F^2/k$, what are the maximum and minimum valnes of x that the ion reaches in its motion? If the ion has mass m, find its maximum speed if the total energy is $E = F^2/k$. For what value of x is the speed maximum?

7.86. A particle moves along the x-axis while acted on by a single conservative force parallel to the x-axis. The force corresponds to the potential-energy function graphed in Fig. 7.45. The particle is released from rest at point A. (a) What is the direction of the force on the particle when it is at point A? (b) At point B? (c) At what value of x is the kinetic energy of the particle a maximum? (d) What is the force on the particle when it is at point C? (e) What is the largest value of x reached by the particle during its motion? (f) What value or values of x correspond to points of stable equilibrium? (g) Of unstable equilibrium?

Figure 7.45 Problem 7.86.



Challenge Problem

7.87. A proton with mass m moves in one dimension. The potential-energy function is $U(x) = \alpha/x^2 - \beta/x$, where α and β are positive constants. The proton is released from rest at $x_0 = \alpha/\beta$. (a) Show that $U(x)$ can be written as

$$U(x) = \frac{\alpha}{x_0^2}\left[\left(\frac{x_0}{x}\right)^2 - \frac{x_0}{x}\right]$$

Graph $U(x)$. Calculate $U(x_0)$ and thereby locate the point x_0 on the graph. (b) Calculate $v(x)$, the speed of the proton as a function of position. Graph $v(x)$ and give a qualitative description of the motion. (c) For what value of x is the speed of the proton a maximum? What is the value of that maximum speed? (d) What is the force on the proton at the point in part (c)? (e) Let the proton be released instead at $x_1 = 3\alpha/\beta$. Locate the point x_1 on the graph of $U(x)$. Calculate $v(x)$ and give a qualitative description of the motion. (f) For each release point ($x = x_0$ and $x = x_1$), what are the maximum and minimum values of x reached during the motion?

MOMENTUM, IMPULSE, AND COLLISIONS

8

? Which could potentially cause you the greater injury: being tackled by a lightweight, fast-moving football player, or being tackled by a player with double the mass but moving at half the speed?

Learning Goals

By studying this chapter, you will learn:

- the meaning of the momentum of a particle, and how the impulse of the net force acting on a particle causes its momentum to change.
- the conditions under which the total momentum of a system of particles is constant (conserved).
- how to solve problems in which two bodies collide with each other.
- the important distinction among elastic, inelastic, and completely inelastic collisions.
- the definition of the center of mass of a system, and what determines how the center of mass moves.
- how to analyze situations such as rocket propulsion in which the mass of a body changes as it moves.

There are many questions involving forces that cannot be answered by directly applying Newton's second law, $\sum\vec{F} = m\vec{a}$. For example, when an 18-wheeler collides head-on with a compact car, what determines which way the wreckage moves after the collision? In playing pool, how do you decide how to aim the cue ball in order to knock the eight ball into the pocket? And when a meteorite collides with the earth, how much of the meteorite's kinetic energy is released in the impact?

A common theme of all these questions is that they involve forces about which we know very little: the forces between the car and the 18-wheeler, between the two pool balls, or between the meteorite and the earth. Remarkably, we will find in this chapter that we don't have to know *anything* about these forces to answer questions of this kind!

Our approach uses two new concepts, ***momentum*** and ***impulse***, and a new conservation law, ***conservation of momentum***. This conservation law is every bit as important as that of conservation of energy. The law of conservation of momentum is valid even in situations in which Newton's laws are inadequate, such as bodies moving at very high speeds (near the speed of light) or objects on a very small scale (such as the constituents of atoms). Within the domain of Newtonian mechanics, conservation of momentum enables us to analyze many situations that would be very difficult if we tried to use Newton's laws directly. Among these are *collision* problems, in which two bodies collide and can exert very large forces on each other for a short time.

8.1 Momentum and Impulse

In Chapter 6 we re-expressed Newton's second law for a particle, $\sum\vec{F} = m\vec{a}$, in terms of the work–energy theorem. This theorem helped us tackle a great number of physics problems and led us to the law of conservation of energy. Let's now return to $\sum\vec{F} = m\vec{a}$ and see yet another useful way to restate this fundamental law.

Newton's Second Law in Terms of Momentum

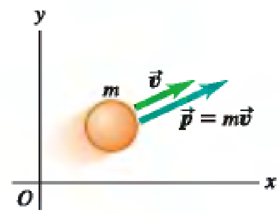
Consider a particle of constant mass m. (Later in this chapter we'll see how to deal with situations in which the mass of a body changes.) Because $\vec{a} = d\vec{v}/dt$, we can write Newton's second law for this particle as

$$\sum\vec{F} = m\frac{d\vec{v}}{dt} = \frac{d}{dt}(m\vec{v}) \tag{8.1}$$

We can take the mass m inside the derivative because it is constant. Thus Newton's second law says that the net force $\sum\vec{F}$ acting on a particle equals the time rate of change of the combination $m\vec{v}$, the product of the particle's mass and velocity. We'll call this combination the **momentum,** or **linear momentum,** of the particle. Using the symbol $\vec{p}$ for momentum, we have

$$\vec{p} = m\vec{v} \qquad \text{(definition of momentum)} \tag{8.2}$$

8.1 The velocity and momentum vectors of a particle.



The greater the mass m and speed v of a particle, the greater is its magnitude of momentum mv. Keep in mind, however, that momentum is a *vector* quantity with the same direction as the particle's velocity (Fig. 8.1). Hence a car driving north at 20 m/s and an identical car driving east at 20 m/s have the same *magnitude* of momentum (mv) but different momentum *vectors* $(m\vec{v})$ because their directions are different.

We often express the momentum of a particle in terms of its components. If the particle has velocity components v_x, v_y, and v_z, then its momentum components p_x, p_y, and p_z (which we also call the *x-momentum, y-momentum,* and *z-momentum*) are given by

$$p_x = mv_x \qquad p_y = mv_y \qquad p_z = mv_z \tag{8.3}$$

These three component equations are equivalent to Eq. (8.2).

The units of the magnitude of momentum are units of mass times speed; the SI units of momentum are kg · m/s. The plural of momentum is "momenta."

If we now substitute the definition of momentum, Eq. (8.2), into Eq. (8.1), we get

$$\sum\vec{F} = \frac{d\vec{p}}{dt} \qquad \text{(Newton's second law in terms of momentum)} \tag{8.4}$$

The net force (vector sum of all forces) acting on a particle equals the time rate of change of momentum of the particle. This, not $\sum\vec{F} = m\vec{a}$, is the form in which Newton originally stated his second law (although he called momentum the "quantity of motion"). This law is valid only in inertial frames of reference.

According to Eq. (8.4), a rapid change in momentum requires a large net force, while a gradual change in momentum requires less net force. This principle is used in the design of automobile safety devices such as air bags (Fig. 8.2).

8.2 If a fast-moving automobile stops suddenly in a collision, the driver's momentum (mass times velocity) changes from a large value to zero in a short time. An air bag causes the driver to lose momentum more gradually than would an abrupt collision with the steering wheel, reducing the force exerted on the driver as well as the possibility of injury.

The Impulse–Momentum Theorem

A particle's momentum $\vec{p} = m\vec{v}$ and its kinetic energy $K = \frac{1}{2}mv^2$ both depend on the mass and velocity of the particle. What is the fundamental difference between these two quantities? A purely mathematical answer is that momentum is a vector whose magnitude is proportional to speed, while kinetic energy is a scalar proportional to the speed squared. But to see the *physical* difference between momentum and kinetic energy, we must first define a quantity closely related to momentum called *impulse.*

Let's first consider a particle acted on by a *constant* net force $\sum\vec{F}$ during a time interval Δt from t_1 to t_2. (We'll look at the case of varying forces shortly.)

The **impulse** of the net force, denoted by $\vec{J}$, is defined to be the product of the net force and the time interval:

$$\vec{J} = \sum\vec{F}(t_2 - t_1) = \sum\vec{F}\,\Delta t \qquad \text{(assuming constant net force)} \tag{8.5}$$

Impulse is a vector quantity; its direction is the same as the net force $\sum\vec{F}$. Its magnitude is the product of the magnitude of the net force and the length of time that the net force acts. The SI unit of impulse is the newton-second $(\text{N}\cdot\text{s})$. Because $1\text{ N} = 1\text{ kg}\cdot\text{m/s}^2$, an alternative set of units for impulse is $\text{kg}\cdot\text{m/s}$, the same as the units of momentum.

To see what impulse is good for, let's go back to Newton's second law as restated in terms of momentum, Eq. (8.4). If the net force $\sum\vec{F}$ is constant, then $d\vec{p}/dt$ is also constant. In that case, $d\vec{p}/dt$ is equal to the *total* change in momentum $\vec{p}_2 - \vec{p}_1$ during the time interval $t_2 - t_1$, divided by the interval:

$$\sum\vec{F} = \frac{\vec{p}_2 - \vec{p}_1}{t_2 - t_1}$$

Multiplying this equation by $(t_2 - t_1)$, we have

$$\sum\vec{F}(t_2 - t_1) = \vec{p}_2 - \vec{p}_1$$

Comparing with Eq. (8.5), we end up with a result called the **impulse–momentum theorem:**

$$\vec{J} = \vec{p}_2 - \vec{p}_1 \qquad \text{(impulse–momentum theorem)} \tag{8.6}$$

The change in momentum of a particle during a time interval equals the impulse of the net force that acts on the particle during that interval.

The impulse–momentum theorem also holds when forces are not constant. To see this, we integrate both sides of Newton's second law $\sum\vec{F} = d\vec{p}/dt$ over time between the limits t_1 and t_2:

$$\int_{t_1}^{t_2} \sum\vec{F}\,dt = \int_{t_1}^{t_2} \frac{d\vec{p}}{dt}\,dt = \int_{\vec{p}_1}^{\vec{p}_2} d\vec{p} = \vec{p}_2 - \vec{p}_1$$

The integral on the left is defined to be the impulse $\vec{J}$ of the net force $\sum\vec{F}$ during this interval:

$$\vec{J} = \int_{t_1}^{t_2} \sum\vec{F}\,dt \qquad \text{(general definition of impulse)} \tag{8.7}$$

With this definition, the impulse–momentum theorem $\vec{J} = \vec{p}_2 - \vec{p}_1$, Eq. (8.6), is valid even when the net force $\sum\vec{F}$ varies with time.

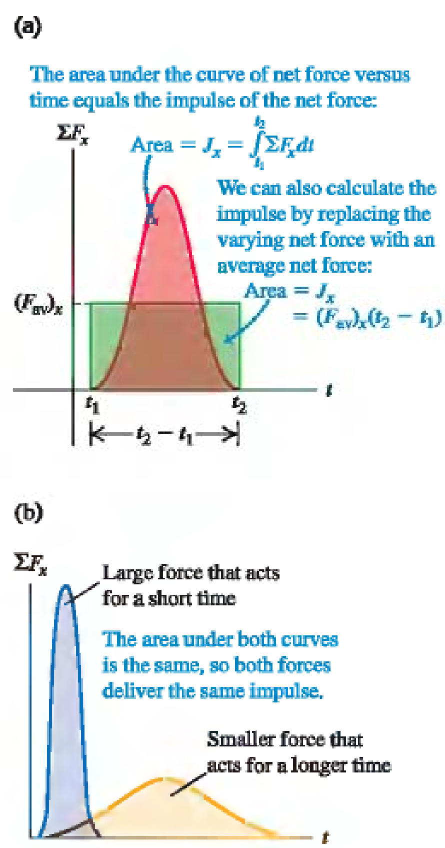
We can define an *average* net force $\vec{F}_{\text{av}}$ such that even when $\sum\vec{F}$ is not constant, the impulse $\vec{J}$ is given by

$$\vec{J} = \vec{F}_{\text{av}}(t_2 - t_1) \tag{8.8}$$

When $\sum\vec{F}$ is constant, $\sum\vec{F} = \vec{F}_{\text{av}}$ and Eq. (8.8) reduces to Eq. (8.5).

Figure 8.3a shows the x-component of net force $\sum F_x$ as a function of time during a collision. This might represent the force on a soccer ball that is in contact with a player's foot from time t_1 to t_2. The x-component of impulse during this interval is represented by the red area under the curve between t_1 and t_2. This area is equal to the green rectangular area bounded by t_1, t_2, and $(F_{\text{av}})_x$, so $(F_{\text{av}})_x(t_2 - t_1)$ is

8.3 The meaning of the area under a graph of $\sum F_x$ versus t.



equal to the impulse of the actual time-varying force during the same interval. Note that a large force acting for a short time can have the same impulse as a smaller force acting for a longer time if the areas under the force–time curves are the same (Fig. 8.3b). In this language, an antomobile airbag (Fig. 8.2) provides the same impulse to the driver as would the steering wheel or the dashboard by applying a weaker and less injurious force for a longer time.

Impulse and momentum are both vector quantities, and Eqs. (8.5)–(8.8) are all vector equations. In specific problems, it is often easiest to use them in component form:

$$\begin{aligned} J_x &= \int_{t_1}^{t_2} \Sigma F_x\,dt = (F_{av})_x(t_2 - t_1) = p_{2x} - p_{1x} = mv_{2x} - mv_{1x} \\ J_y &= \int_{t_1}^{t_2} \Sigma F_y\,dt = (F_{av})_y(t_2 - t_1) = p_{2y} - p_{1y} = mv_{2y} - mv_{1y} \end{aligned} \qquad (8.9)$$

and similarly for the z-component.

Momentum and Kinetic Energy Compared

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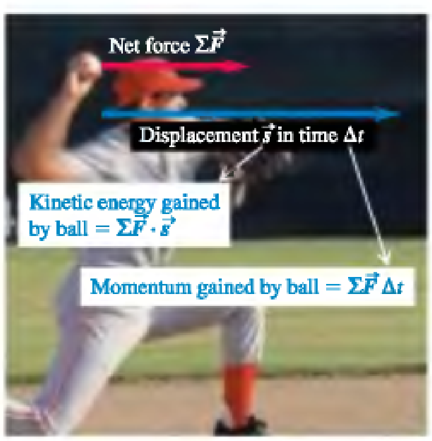
6.1 Momentum and Energy Change

We can now see the fundamental difference between momentum and kinetic energy. The impulse–momentum theorem $\vec{J} = \vec{p}_2 - \vec{p}_1$ says that changes in a particle's momentum are due to impulse, which depends on the *time* over which the net force acts. By contrast, the work–energy theorem $W_{tot} = K_2 - K_1$ tells us that kinetic energy changes when work is done on a particle; the total work depends on the *distance* over which the net force acts. Consider a particle that starts from rest at t_1 so that $\vec{v}_1 = \mathbf{0}$. Its initial momentum is $\vec{p}_1 = m\vec{v}_1 = \mathbf{0}$, and its initial kinetic energy is $K_1 = \frac{1}{2}mv_1^2 = 0$. Now let a constant net force equal to $\vec{F}$ act on that particle from time t_1 until time t_2. During this interval, the particle moves a distance s in the direction of the force. From Eq. (8.6), the particle's momentum at time t_2 is

$$\vec{p}_2 = \vec{p}_1 + \vec{J} = \vec{J}$$

where $\vec{J} = \vec{F}(t_2 - t_1)$ is the impulse that acts on the particle. So *the momentum of a particle equals the impulse that accelerated it from rest to its present speed;* impulse is the product of the net force that accelerated the particle and the *time* required for the acceleration. By comparison, the kinetic energy of the particle at t_2 is $K_2 = W_{tot} = Fs$, the total *work* done on the particle to accelerate it from rest. The total work is the product of the net force and the *distance* required to accelerate the particle (Fig. 8.4).

8.4 The *kinetic energy* of a pitched baseball is equal to the work the pitcher does on it (force multiplied by the distance the ball moves during the throw). The *momentum* of the ball is equal to the impulse the pitcher imparts to it (force multiplied by the time it took to bring the ball up to speed).



Here's an application of the distinction between momentum and kinetic energy. Suppose you have a choice between catching a 0.50-kg ball moving at 4.0 m/s or a 0.10-kg ball moving at 20 m/s. Which will be easier to catch? Both balls have the same magnitude of momentum, $p = mv = (0.50\text{ kg})(4.0\text{ m/s}) = (0.10\text{ kg})(20\text{ m/s}) = 2.0\text{ kg}\cdot\text{m/s}$. However, the two balls have different values of kinetic energy $K = \frac{1}{2}mv^2$; the large, slow-moving ball has $K = 4.0$ J, while the small, fast-moving ball has $K = 20$ J. Since the momentum is the same for both balls, both require the same *impulse* to be brought to rest. But stopping the 0.10-kg ball with your hand requires five times more *work* than stopping the 0.50-kg ball because the smaller ball has five times more kinetic energy. For a given force that you exert with your hand, it takes the same amount of time (the duration of the catch) to stop either ball, but your hand and arm will be pushed back five times farther if you choose to catch the small, fast-moving ball. To minimize arm strain, you should choose to catch the 0.50-kg ball with its lower kinetic energy.

Both the impulse–momentum and work–energy theorems are relationships between force and motion, and both rest on the foundation of Newton's laws. They are *integral* principles, relating the motion at two different times separated by a finite interval. By contrast, Newton's second law itself (in either of the forms $\Sigma\vec{F} = m\vec{a}$ or $\Sigma\vec{F} = d\vec{p}/dt$) is a *differential* principle, relating the forces to the rate of change of velocity or momentum at each instant.

Conceptual Example 8.1 Momentum versus kinetic energy

Consider again the race described in Conceptual Example 6.5 (Section 6.2) between two iceboats on a frictionless frozen lake. The iceboats have masses m and $2m$, and the wind exerts the same constant horizontal force $\vec{F}$ on each iceboat (see Fig. 6.14). The two iceboats start from rest and cross the finish line a distance s away. Which iceboat crosses the finish line with greater momentum?

SOLUTION

In Conceptual Example 6.5 we asked how the *kinetic energies* of the iceboats compare when they cross the finish line. The way to answer this was not to use the formula $K = \frac{1}{2}mv^2$, but to remember that a body's kinetic energy equals the total work done to accelerate it from rest. Both iceboats started from rest, and the total work done between the starting and finish lines was the same for both iceboats (because the net force and displacement were the same for both). Hence both iceboats cross the finish line with the same kinetic energy.

Similarly, the best way to compare the *momenta* of the iceboats is *not* to use the formula $\vec{p} = m\vec{v}$. By itself this formula isn't enough to determine which iceboat has greater momentum at the finish line. The iceboat of mass $2m$ has greater mass, which suggests greater momentum, but this iceboat crosses the finish line going slower than the other one, which suggests less momentum.

Instead, we use the idea that the momentum of each iceboat equals the impulse that accelerated it from rest. For each iceboat the downward force of gravity and the upward normal force add to zero, so the net force equals the constant horizontal wind force $\vec{F}$. Let Δt be the time an iceboat takes to reach the finish line, so that the impulse on the iceboat during that time is $\vec{J} = \vec{F}\,\Delta t$. Since the iceboat starts from rest, this equals the iceboat's momentum $\vec{p}$ at the finish line:

$$\vec{p} = \vec{F}\,\Delta t$$

Both iceboats are subjected to the same force $\vec{F}$, but they take different amounts of time Δt to reach the finish line. The iceboat of mass $2m$ accelerates more slowly and takes a longer time to travel the distance s; thus there is a greater impulse on this iceboat between the starting and finish lines. So the iceboat of mass $2m$ crosses the finish line with a greater magnitude of momentum than the iceboat of mass m (but with the same kinetic energy). Can you show that the iceboat of mass $2m$ has $\sqrt{2}$ times as much momentum at the finish line as the iceboat of mass m?

Example 8.2 A ball hits a wall

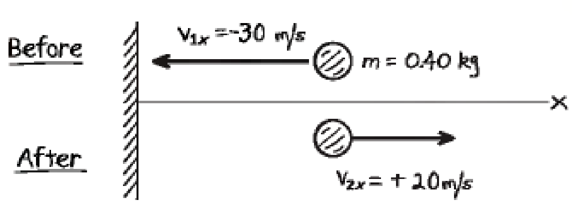
Suppose you throw a ball with a mass of 0.40 kg against a brick wall. It hits the wall moving horizontally to the left at 30 m/s and rebounds horizontally to the right at 20 m/s. (a) Find the impulse of the net force on the ball during its collision with the wall. (b) If the ball is in contact with the wall for 0.010 s, find the average horizontal force that the wall exerts on the ball during the impact.

SOLUTION

IDENTIFY: We're given enough information to determine the initial and final values of the ball's momentum, so we can use the impulse–momentum theorem to find the impulse. We'll then use the definition of impulse to determine the average force.

SET UP: Figure 8.5 shows our sketch. The motion is purely horizontal, so we need only a single axis. We'll take the x-axis to be horizontal and the positive direction to be to the right. Our target variable in part (a) is the x-component of impulse, J_x, which we'll find from the x-components of momentum before and after the impact, using Eqs. (8.9). In part (b), our target variable is the average x-component of force $(F_{av})_x$; once we know J_x, we can also find this force by using Eqs. (8.9).

8.5 Our sketch for this problem.



EXECUTE: (a) With our choice of x-axis, the initial and final x-components of momentum of the ball are

$$p_{1x} = mv_{1x} = (0.40\,\text{kg})(-30\,\text{m/s}) = -12\,\text{kg}\cdot\text{m/s}$$
$$p_{2x} = mv_{2x} = (0.40\,\text{kg})(+20\,\text{m/s}) = +8.0\,\text{kg}\cdot\text{m/s}$$

From the x-equation in Eqs. (8.9), the x-component of impulse equals the *change* in the x-momentum:

$$\begin{aligned} J_x &= p_{2x} - p_{1x} \\ &= 8.0\,\text{kg}\cdot\text{m/s} - (-12\,\text{kg}\cdot\text{m/s}) = 20\,\text{kg}\cdot\text{m/s} = 20\,\text{N}\cdot\text{s} \end{aligned}$$

(b) The collision time is $t_2 - t_1 = \Delta t = 0.010\,\text{s}$. From the x-equation in Eqs. (8.9), $J_x = (F_{av})_x(t_2 - t_1) = (F_{av})_x\,\Delta t$, so

$$(F_{av})_x = \frac{J_x}{\Delta t} = \frac{20\,\text{N}\cdot\text{s}}{0.010\,\text{s}} = 2000\,\text{N}$$

EVALUATE: The x-component of impulse is positive—that is, to the right in Fig. 8.5. This is as it should be: The impulse represents the "kick" that the wall imparts to the ball, and this "kick" is certainly to the right.

CAUTION **Momentum is a vector** Because momentum is a vector, we had to include the negative sign in p_{1x}. Had we carelessly omitted it, we would have calculated the impulse to be $8.0\,\text{kg}\cdot\text{m/s} - (12\,\text{kg}\cdot\text{m/s}) = -4\,\text{kg}\cdot\text{m/s}$. This incorrect answer would say that the wall had somehow given the ball a kick to the *left!* Make sure that you account for the *direction* of momentum in your calculations.

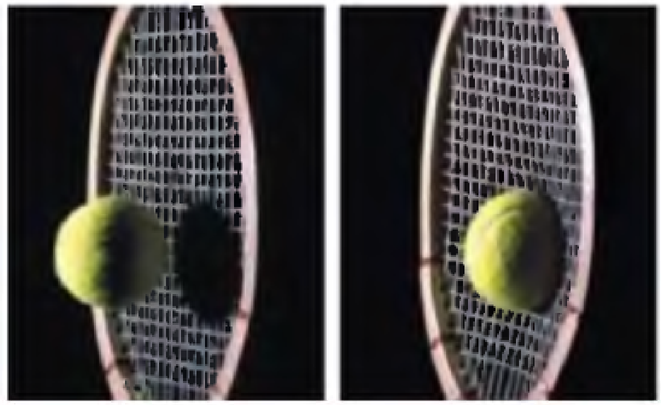
The force that the wall exerts on the ball has to have a large magnitude of 2000 N (equal to 450 lb, or the weight of a 200-kg

Continued

object) to change the ball's momentum in such a short time interval. Other forces that act on the ball during the collision are very weak by comparison; for instance, the gravitational force is only 3.9 N. Thus, during the brief time that the collision lasts, we can ignore all other forces on the ball to a very good approximation. Figure 8.6 is a photograph showing the impact of a tennis ball and racket.

Note that the 2000-N value we calculated is just the *average* horizontal force that the wall exerts on the ball during the impact. It corresponds to the horizontal line $(F_{av})_x$ in Fig. 8.3a. The horizontal force is zero before impact, rises to a maximum, and then decreases to zero when the ball loses contact with the wall. If the ball is relatively rigid, like a baseball or golf ball, the collision lasts a short time and the maximum force is large, as in the blue curve in Fig. 8.3b. If the ball is softer, like a tennis ball, the collision time is longer and the maximum force is less, as in the orange curve in Fig. 8.3b.

8.6 Typically, a tennis ball is in contact with the racket for approximately 0.01 s. The ball flattens noticeably due to the tremendous force exerted by the racket.



Example 8.3 Kicking a soccer ball

A soccer ball has a mass of 0.40 kg. Initially, it is moving to the left at 20 m/s, but then it is kicked and given a velocity at 45° upward and to the right, with a magnitude of 30 m/s (Fig. 8.7a). Find the impulse of the net force and the average net force, assuming a collision time $\Delta t = 0.010$ s.

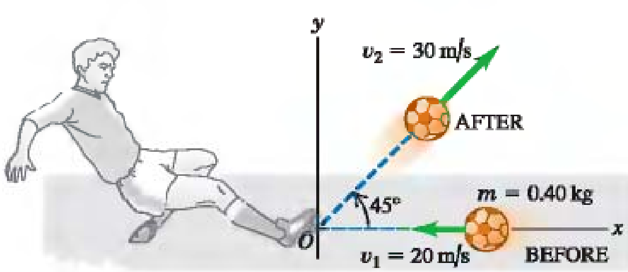
SOLUTION

IDENTIFY: This example uses the same principles as Example 8.2. The key difference is that the initial and final velocities are not along the same line, so we have to be careful to treat momentum and impulse as vector quantities, using their x- and y-components.

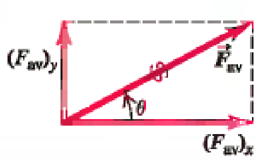
SET UP: We take the x-axis to be horizontally to the right and the y-axis to be vertically upward. Our target variables are the components of the net impulse on the ball, J_x and J_y, and the components of the average net force on the ball, $(F_{av})_x$ and $(F_{av})_y$. We'll find them using the x- and y-components of Eqs. (8.9).

8.7 (a) Kicking a soccer ball. (b) Finding the average force on the ball from its components.

(a) Before-and-after diagram



(b) Average force on the ball



EXECUTE: With our choice of axes, we find the ball's velocity components before (subscript 1) and after (subscript 2) it is kicked:

$$v_{1x} = -20\text{ m/s} \qquad v_{1y} = 0$$
$$v_{2x} = v_{2y} = (30\text{ m/s})(0.707) = 21.2\text{ m/s}$$
$$(\text{since } \cos 45° = \sin 45° = 0.707)$$

The x-component of impulse is equal to the x-component of momentum change, and the same is true for the y-components:

$$J_x = p_{2x} - p_{1x} = m(v_{2x} - v_{1x})$$
$$= (0.40\text{ kg})[21.2\text{ m/s} - (-20\text{ m/s})] = 16.5\text{ kg}\cdot\text{m/s}$$
$$J_y = p_{2y} - p_{1y} = m(v_{2y} - v_{1y})$$
$$= (0.40\text{ kg})(21.2\text{ m/s} - 0) = 8.5\text{ kg}\cdot\text{m/s}$$

The components of the average net force on the ball are

$$(F_{av})_x = \frac{J_x}{\Delta t} = 1650\text{ N} \qquad (F_{av})_y = \frac{J_y}{\Delta t} = 850\text{ N}$$

The magnitude and direction of the average force are

$$F_{av} = \sqrt{(1650\text{ N})^2 + (850\text{ N})^2} = 1.9 \times 10^3\text{ N}$$
$$\theta = \arctan\frac{850\text{ N}}{1650\text{ N}} = 27°$$

where θ is measured upward from the $+x$-axis (Fig. 8.7b). Note that because the ball was not initially at rest, the ball's final velocity does *not* have the same direction as the average force that acted on it.

EVALUATE: The average net force $\vec{F}_{av}$ includes the effects of the force of gravity, although these are small; the weight of the ball is only 3.9 N. As in Example 8.2, the average force acting during the collision is exerted almost entirely by the object that the ball hit (in this case, the soccer player's foot).

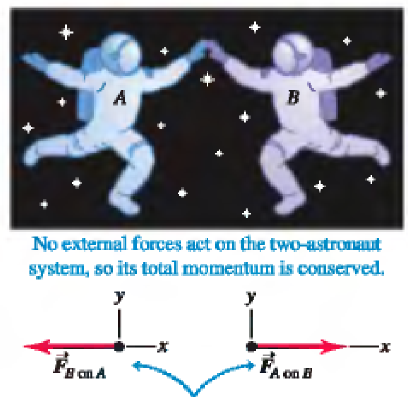
Test Your Understanding of Section 8.1 Rank the following situations according to the magnitude of the impulse of the net force, from largest value to smallest value. In each situation a 1000-kg automobile is moving along a straight east–west road. (i) The automobile is initially moving east at 25 m/s and comes to a stop in 10 s. (ii) The automobile is initially moving east at 25 m/s and comes to a stop in 5 s. (iii) The automobile is initially at rest, and a 2000-N net force toward the east is applied to it for 10 s. (iv) The automobile is initially moving east at 25 m/s, and a 2000-N net force toward the west is applied to it for 10 s. (v) The automobile is initially moving east at 25 m/s. Over a 30-s period, the automobile reverses direction and ends up moving west at 25 m/s.

8.2 Conservation of Momentum

The concept of momentum is particularly important in situations in which we have two or more *interacting* bodies. To see why, let's consider first an idealized system consisting of two bodies that interact with each other but not with anything else—for example, two astronauts who touch each other as they float freely in the zero-gravity environment of outer space (Fig. 8.8). Think of the astronauts as particles. Each particle exerts a force on the other; according to Newton's third law, the two forces are always equal in magnitude and opposite in direction. Hence, the *impulses* that act on the two particles are equal and opposite, and the changes in momentum of the two particles are equal and opposite.

Let's go over that again with some new terminology. For any system, the forces that the particles of the system exert on each other are called **internal forces.** Forces exerted on any part of the system by some object outside it are called **external forces.** For the system shown in Fig. 8.8, the internal forces are $\vec{F}_{B \text{ on } A}$, exerted by particle B on particle A, and $\vec{F}_{A \text{ on } B}$, exerted by particle A on particle B. There are *no* external forces; when this is the case, we have an **isolated system.**

8.8 Two astronauts push each other as they float freely in the zero-gravity environment of space.



The net force on particle A is $\vec{F}_{B \text{ on } A}$ and the net force on particle B is $\vec{F}_{A \text{ on } B}$, so from Eq. (8.4) the rates of change of the momenta of the two particles are

$$\vec{F}_{B \text{ on } A} = \frac{d\vec{p}_A}{dt} \qquad \vec{F}_{A \text{ on } B} = \frac{d\vec{p}_B}{dt} \tag{8.10}$$

The momentum of each particle changes, but these changes are related to each other by Newton's third law: The two forces $\vec{F}_{B \text{ on } A}$ and $\vec{F}_{A \text{ on } B}$ are always equal in magnitude and opposite in direction. That is, $\vec{F}_{B \text{ on } A} = -\vec{F}_{A \text{ on } B}$, so $\vec{F}_{B \text{ on } A} + \vec{F}_{A \text{ on } B} = \mathbf{0}$. Adding together the two equations in Eq. (8.10), we have

$$\vec{F}_{B \text{ on } A} + \vec{F}_{A \text{ on } B} = \frac{d\vec{p}_A}{dt} + \frac{d\vec{p}_B}{dt} = \frac{d(\vec{p}_A + \vec{p}_B)}{dt} = \mathbf{0} \tag{8.11}$$

The rates of change of the two momenta are equal and opposite, so the rate of change of the vector sum $\vec{p}_A + \vec{p}_B$ is zero. We now define the **total momentum** $\vec{P}$ of the system of two particles as the vector sum of the momenta of the individual particles; that is,

$$\vec{P} = \vec{p}_A + \vec{p}_B \tag{8.12}$$

Then Eq. (8.11) becomes, finally,

$$\vec{F}_{B \text{ on } A} + \vec{F}_{A \text{ on } B} = \frac{d\vec{P}}{dt} = \mathbf{0} \tag{8.13}$$

The time rate of change of the *total* momentum $\vec{P}$ is zero. Hence the total momentum of the system is constant, even though the individual momenta of the particles that make up the system can change.

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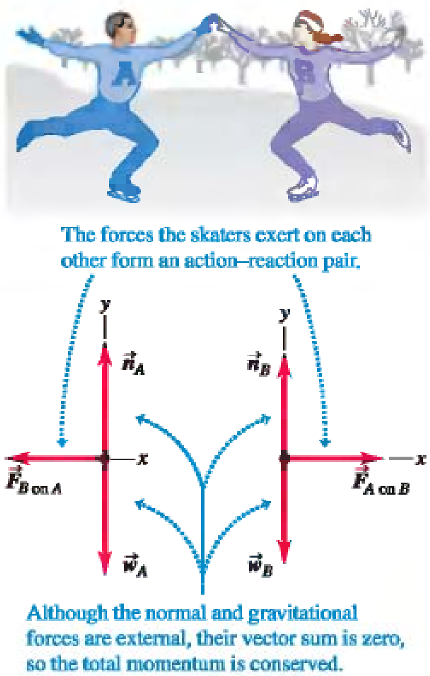
6.3 Momentum Conservation and Collisions
6.7 Explosion Problems
6.10 Pendulum Person-Projectile Bowling

8.9 Two ice skaters push each other as they skate on a frictionless, horizontal surface. (Compare to Fig. 8.8.)



If external forces are also present, they must be included on the left side of Eq. (8.13) along with the internal forces. Then the total momentum is, in general, not constant. But if the vector sum of the external forces is zero, as in Fig. 8.9, these forces don't contribute to the sum, and $d\vec{P}/dt$ is again zero. Thus we have the following general result:

If the vector sum of the external forces on a system is zero, the total momentum of the system is constant.

This is the simplest form of the **principle of conservation of momentum.** This principle is a direct consequence of Newton's third law. What makes this principle useful is that it doesn't depend on the detailed nature of the internal forces that act between members of the system. This means that we can apply conservation of momentum even if (as is often the case) we know very little about the internal forces. We have used Newton's second law to derive this principle, so we have to be careful to use it only in inertial frames of reference.

We can generalize this principle for a system that contains any number of particles $A, B, C, \ldots$ interacting only with each other. The total momentum of such a system is

$$\vec{P} = \vec{p}_A + \vec{p}_B + \cdots = m_A\vec{v}_A + m_B\vec{v}_B + \cdots \quad \text{(total momentum of a system of particles)} \tag{8.14}$$

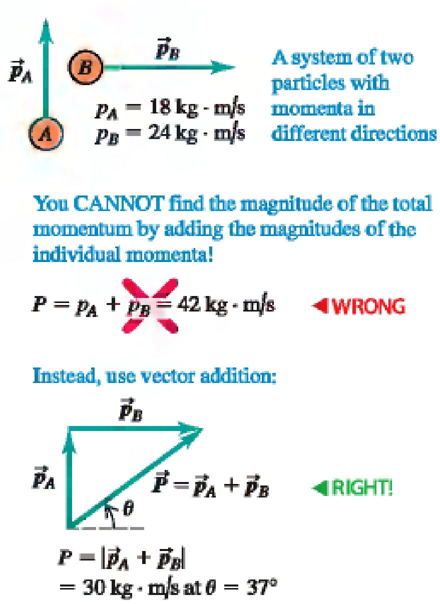
We make the same argument as before: The total rate of change of momentum of the system due to each action–reaction pair of internal forces is zero. Thus the total rate of change of momentum of the entire system is zero whenever the vector sum of the external forces acting on it is zero. The internal forces can change the momenta of individual particles in the system but not the *total* momentum of the system.

CAUTION Conservation of momentum means conservation of its components When you apply the conservation of momentum to a system, remember that momentum is a *vector* quantity. Hence you must use vector addition to compute the total momentum of a system (Fig. 8.10). Using components is usually the simplest method. If p_{Ax}, p_{Ay}, and p_{Az} are the components of momentum of particle A, and similarly for the other particles, then Eq. (8.14) is equivalent to the component equations

$$\begin{aligned} P_x &= p_{Ax} + p_{Bx} + \cdots \\ P_y &= p_{Ay} + p_{By} + \cdots \\ P_z &= p_{Az} + p_{Bz} + \cdots \end{aligned} \tag{8.15}$$

If the vector sum of the external forces on the system is zero, then P_x, P_y, and P_z are all constant.

8.10 When applying conservation of momentum, remember that momentum is a vector quantity!



In some ways the principle of conservation of momentum is more general than the principle of conservation of mechanical energy. For example, mechanical energy is conserved only when the internal forces are *conservative*—that is, when the forces allow two-way conversion between kinetic and potential energy—but conservation of momentum is valid even when the internal forces are *not* conservative. In this chapter we will analyze situations in which both momentum and mechanical energy are conserved, and others in which only momentum is conserved. These two principles play a fundamental role in all areas of physics, and we will encounter them throughout our study of physics.

Problem-Solving Strategy 8.1 Conservation of Momentum

IDENTIFY *the relevant concepts:* Before applying conservation of momentum to a problem, you must decide whether momentum *is* conserved! This will be true *only* if the vector sum of the external forces acting on the system of particles is zero. If this is not the case, you can't use conservation of momentum.

SET UP *the problem using the following steps:*

1. Define a coordinate system and show it in a sketch, including the positive direction for each axis. Often it is easiest to choose the x-axis in the direction of one of the initial velocities. Make sure you are using an inertial frame of reference. Most of the problems in this chapter deal with two-dimensional situations, in which the vectors have only x- and y-components, but this strategy can be generalized to include z-components when necessary.
2. Treat each body as a particle. Draw "before" and "after" sketches, and include vectors on each to represent all known velocities. Label the vectors with magnitudes, angles, components, or whatever information is given, and give each unknown magnitude, angle, or component an algebraic symbol. It's helpful to use the subscripts 1 and 2 for velocities before and after the interaction, respectively, and use letters (not numbers) to label each particle.
3. As always, identify the target variable(s) from among the unknowns.

EXECUTE *the solution as follows:*

1. Write an equation in symbols equating the total *initial* x-component of momentum (that is, before the interaction) to the total *final* x-component of momentum (that is, after the interaction), using $p_x = mv_x$ for each particle. Write another equation for the y-components, using $p_y = mv_y$ for each particle. (*Never* add the x- and y-components of velocity or momentum together in the same equation!) Even when all motions are along a line (such as the x-axis), the components of velocity along this line can be positive or negative; be careful with signs!
2. Solve these equations to determine whatever results are required. In some problems you will have to convert from the x- and y-components of a velocity to its magnitude and direction, or the reverse.
3. In some problems, energy considerations give additional relationships among the various velocities, as we will see later in this chapter.

EVALUATE *your answer:* Does your answer make physical sense? If your target variable is a certain body's momentum, check that the direction of the momentum is reasonable.

Example 8.4 Recoil of a rifle

A marksman holds a rifle of mass $m_R = 3.00\text{ kg}$ loosely in his hands, so as to let it recoil freely when fired. He fires a bullet of mass $m_B = 5.00\text{ g}$ horizontally with a velocity relative to the ground of $v_{Bx} = 300\text{ m/s}$. What is the recoil velocity v_{Rx} of the rifle? What are the final momentum and kinetic energy of the bullet? Of the rifle?

SOLUTION

IDENTIFY: We consider an idealized model in which the horizontal forces the marksman exerts on the rifle are negligible. Then there is no net horizontal force on the system (the bullet and rifle) during the firing of the rifle, and so the total horizontal momentum of the system is the same before and after the rifle is fired (i.e., it is conserved).

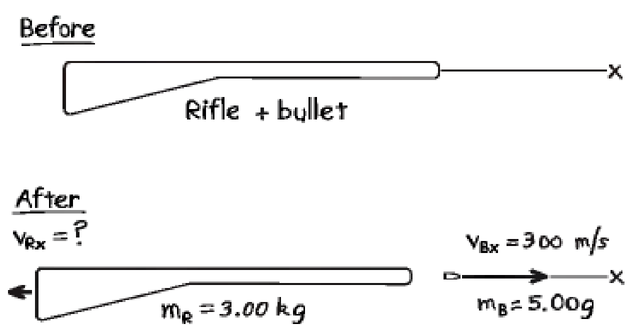
SET UP: Figure 8.11 shows our sketch. We take the positive x-axis to be the direction the rifle is aimed. Initially, both the rifle and the bullet are at rest, so the initial x-component of total momentum is zero. After the shot is fired, the bullet's x-momentum is $p_{Bx} = m_B v_{Bx}$ and the rifle's x-momentum is $p_{Rx} = m_R v_{Rx}$. Our target variables are v_{Rx}, p_{Bx}, p_{Rx}, and $K_B = \frac{1}{2}m_B v_{Bx}{}^2$ and $K_R = \frac{1}{2}m_R v_{Rx}{}^2$ (the final kinetic energies of the bullet and rifle, respectively).

EXECUTE: Conservation of the x-component of total momentum gives

$$P_x = 0 = m_B v_{Bx} + m_R v_{Rx}$$

$$v_{Rx} = -\frac{m_B}{m_R}v_{Bx} = -\left(\frac{0.00500\text{ kg}}{3.00\text{ kg}}\right)(300\text{ m/s}) = -0.500\text{ m/s}$$

8.11 Our sketch for this problem.



The negative sign means that the recoil is in the direction opposite to that of the bullet. If the butt of a rifle hit your shoulder at this speed, you'd feel it. It's more comfortable to hold the rifle tightly against your shoulder when you fire it; then m_R is replaced by the sum of your mass and the rifle's mass, and the recoil speed is much less.

The final momentum and kinetic energy of the bullet are

$$p_{Bx} = m_B v_{Bx} = (0.00500\text{ kg})(300\text{ m/s}) = 1.50\text{ kg}\cdot\text{m/s}$$
$$K_B = \tfrac{1}{2}m_B v_{Bx}{}^2 = \tfrac{1}{2}(0.00500\text{ kg})(300\text{ m/s})^2 = 225\text{ J}$$

For the rifle, the final momentum and kinetic energy are

$$p_{Rx} = m_R v_{Rx} = (3.00\text{ kg})(-0.500\text{ m/s}) = -1.50\text{ kg}\cdot\text{m/s}$$
$$K_R = \tfrac{1}{2}m_R v_{Rx}{}^2 = \tfrac{1}{2}(3.00\text{ kg})(-0.500\text{ m/s})^2 = 0.375\text{ J}$$

Continued

EVALUATE: The bullet and the rifle have equal and opposite *momenta* after the interaction. That's because they were subjected to equal and opposite interaction forces for the same amount of *time* (i.e., equal and opposite impulses). But the bullet acquires much greater *kinetic energy* than the rifle because the bullet travels a much greater *distance* than the rifle during the interaction. Thus the force on the bullet does more work than the force on the rifle. The ratio of the two kinetic energies, 600 : 1, is equal to the inverse ratio of the masses; in fact, it can be shown that this always happens in recoil situations. We leave the proof as a problem (see Exercise 8.22).

Our calculation doesn't depend on the details of how the rifle works. In a real rifle, the bullet is propelled forward by an explosive charge; if instead the rifle used a very stiff spring, the answers would have been exactly the same.

Example 8.5 Collision along a straight line

Two gliders move toward each other on a frictionless linear air track (Fig. 8.12a). After they collide (Fig. 8.12b), glider B moves away with a final velocity of $+2.0\ \text{m/s}$ (Fig. 8.12c). What is the final velocity of glider A? How do the changes in momentum and in velocity compare for the two gliders?

SOLUTION

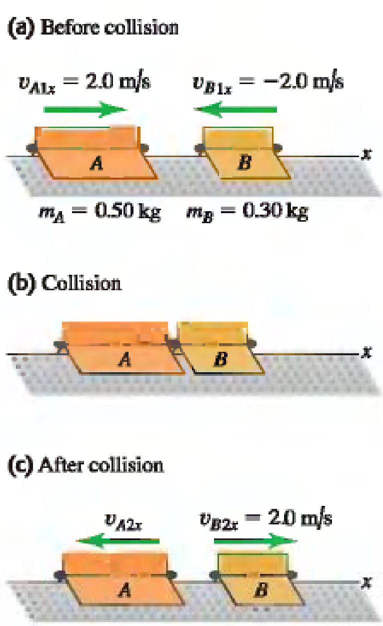
IDENTIFY: The total vertical force on each glider is zero; the net force on each glider is the horizontal force exerted on it by the other glider. The net *external* force on the two gliders together is zero, so the total momentum is conserved. (Compare Fig. 8.9.)

SET UP: We take the positive x-axis to be to the right, along the air track. We are given the masses and initial velocities of both gliders and the final velocity of glider B. Our target variables are v_{A2x}, the final x-component of velocity of glider A, and the changes in momentum and in velocity of the two gliders (the value after the collision minus the value before the collision).

EXECUTE: The x-component of total momentum before the collision is

$$\begin{aligned}P_x &= m_A v_{A1x} + m_B v_{B1x}\\ &= (0.50\ \text{kg})(2.0\ \text{m/s}) + (0.30\ \text{kg})(-2.0\ \text{m/s})\\ &= 0.40\ \text{kg}\cdot\text{m/s}\end{aligned}$$

8.12 Two gliders colliding on an air track.



This is positive (to the right in Fig. 8.12) because glider A has a greater magnitude of momentum before the collision than does glider B. The x-component of total momentum has the same value after the collision, so

$$P_x = m_A v_{A2x} + m_B v_{B2x}$$

Solving this equation for v_{A2x}, the final x-velocity of A, we find

$$\begin{aligned}v_{A2x} &= \frac{P_x - m_B v_{B2x}}{m_A} = \frac{0.40\ \text{kg}\cdot\text{m/s} - (0.30\ \text{kg})(2.0\ \text{m/s})}{0.50\ \text{kg}}\\ &= -0.40\ \text{m/s}\end{aligned}$$

The change in x-momentum of glider A is

$$\begin{aligned}m_A v_{A2x} - m_A v_{A1x} &= (0.50\ \text{kg})(-0.40\ \text{m/s})\\ &\quad - (0.50\ \text{kg})(2.0\ \text{m/s}) = -1.2\ \text{kg}\cdot\text{m/s}\end{aligned}$$

and the change in x-momentum of glider B is

$$\begin{aligned}m_B v_{B2x} - m_B v_{B1x} &= (0.30\ \text{kg})(2.0\ \text{m/s})\\ &\quad - (0.30\ \text{kg})(-2.0\ \text{m/s}) = +1.2\ \text{kg}\cdot\text{m/s}\end{aligned}$$

The two interacting gliders undergo changes in momentum that are equal in magnitude and opposite in direction. The same is *not* true of their changes in velocity, however. For A, $v_{A2x} - v_{A1x} = (-0.40\ \text{m/s}) - 2.0\ \text{m/s} = -2.4\ \text{m/s}$; for B, $v_{B2x} - v_{B1x} = 2.0\ \text{m/s} - (-2.0\ \text{m/s}) = +4.0\ \text{m/s}$.

EVALUATE: Why do the momentum changes have the same magnitude for the two gliders, but the velocity changes do not? By Newton's third law, both gliders were acted on for equal amounts of time by an interaction force of the same magnitude. Hence both gliders experienced impulses of the same magnitude, and therefore equal-magnitude changes in momentum. But by Newton's second law, the less massive glider (B) had a greater magnitude of acceleration and hence a greater velocity change.

Here's an application of these ideas. When a large truck collides with a car of normal size, both vehicles undergo equal changes in momentum. The occupants of the car, however, are subjected to greater acceleration (and greater chance of injury) than the occupants of the truck. An even more extreme example is what happens when a truck collides with an insect: The truck driver won't notice the resulting acceleration at all, but the insect surely will!

Example 8.6 Collision in a horizontal plane

Figure 8.13a shows two battling robots sliding on a frictionless surface. Robot A, with mass 20 kg, initially moves at 2.0 m/s parallel to the x-axis. It collides with robot B, which has mass 12 kg and is initially at rest. After the collision, robot A is moving at 1.0 m/s in a direction that makes an angle $\alpha = 30°$ with its initial direction (Fig. 8.13b). What is the final velocity of robot B?

SOLUTION

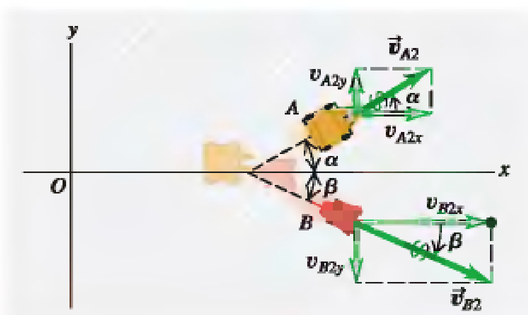
IDENTIFY: There are no horizontal (x or y) external forces, so the x-component and the y-component of the total momentum of the system are both conserved in the collision.

SET UP: Figure 8.13 shows the coordinate axes. The velocities are not all along a single line, so we have to treat momentum as a vector quantity. Momentum conservation requires that the sum of the x-components of momentum *before* the collision (subscript 1) must equal the sum *after* the collision (subscript 2), and similarly for the sums of the y-components. We write a separate momentum conservation equation for each component. Our target variable is $\vec{v}_{B2}$, the final velocity of robot B.

8.13 Views from above of the velocities (a) before and (b) after the collision.

(a) Before collision

(b) After collision



EXECUTE: Conservation of the x-component of total momentum says that

$$m_A v_{A1x} + m_B v_{B1x} = m_A v_{A2x} + m_B v_{B2x}$$

$$v_{B2x} = \frac{m_A v_{A1x} + m_B v_{B1x} - m_A v_{A2x}}{m_B}$$

$$= \frac{\begin{bmatrix}(20\text{ kg})(2.0\text{ m/s}) + (12\text{ kg})(0) \\ -(20\text{ kg})(1.0\text{ m/s})(\cos 30°)\end{bmatrix}}{12\text{ kg}}$$

$$= 1.89\text{ m/s}$$

Similarly, for the y-component of total momentum we have

$$m_A v_{A1y} + m_B v_{B1y} = m_A v_{A2y} + m_B v_{B2y}$$

$$v_{B2y} = \frac{m_A v_{A1y} + m_B v_{B1y} - m_A v_{A2y}}{m_B}$$

$$= \frac{\begin{bmatrix}(20\text{ kg})(0) + (12\text{ kg})(0) \\ -(20\text{ kg})(1.0\text{ m/s})(\sin 30°)\end{bmatrix}}{12\text{ kg}}$$

$$= -0.83\text{ m/s}$$

After the collision, robot B moves in the positive x-direction and the negative y-direction (Fig. 8.13b). The magnitude of $\vec{v}_{B2}$ is

$$v_{B2} = \sqrt{(1.89\text{ m/s})^2 + (-0.83\text{ m/s})^2} = 2.1\text{ m/s}$$

and the angle of its direction from the positive x-axis is

$$\beta = \arctan\frac{-0.83\text{ m/s}}{1.89\text{ m/s}} = -24°$$

EVALUATE: We can check our answer by looking at the values of momentum before and after the collision. Initially all of the momentum is in robot A, which has x-momentum $m_A v_{A1x} = (20\text{ kg})(2.0\text{ m/s}) = 40\text{ kg}\cdot\text{m/s}$ and zero y-momentum. After the collision, robot A has x-momentum $m_A v_{A2x} = (20\text{ kg})(1.0\text{ m/s})(\cos 30°) = 17\text{ kg}\cdot\text{m/s}$, while robot B has x-momentum $m_B v_{B2x} = (12\text{ kg})(1.89\text{ m/s}) = 23\text{ kg}\cdot\text{m/s}$; the total x-momentum is $40\text{ kg}\cdot\text{m/s}$, the same as before the collision (as it should be). In the y-direction, robot A acquires y-momentum $m_A v_{A2y} = (20\text{ kg})(1.0\text{ m/s})(\sin 30°) = 10\text{ kg}\cdot\text{m/s}$, while robot B acquires y-momentum of the same magnitude but opposite direction: $m_B v_{B2y} = (12\text{ kg})(-0.83\text{ m/s}) = -10\text{ kg}\cdot\text{m/s}$. Hence the *total* y-component of momentum after the collision has the same value (zero) as before the collision.

Test Your Understanding of Section 8.2 A spring-loaded toy sits at rest on a horizontal frictionless surface. When the spring releases, the toy breaks into three equal-mass pieces, A, B, and C, which slide along the surface. Piece A moves off in the negative x-direction, while piece B moves off in the negative y-direction. (a) What are the signs of the velocity components of piece C? (b) Which of the three pieces is moving the fastest?

8.3 Momentum Conservation and Collisions

To most people the term *collision* is likely to mean some sort of automotive disaster. We'll use it in that sense, but we'll also broaden the meaning to include any strong interaction between bodies that lasts a relatively short time. So we include

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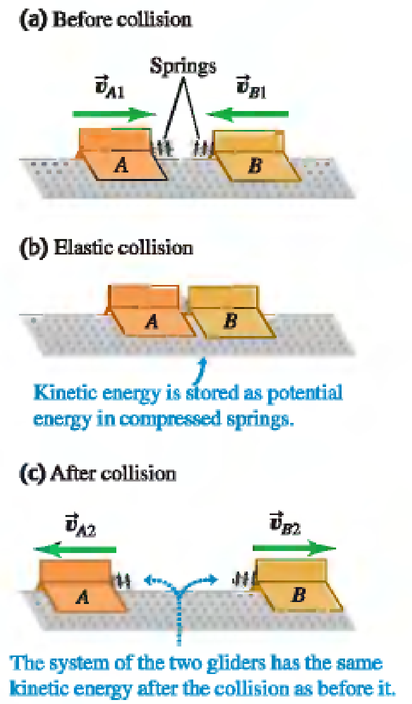
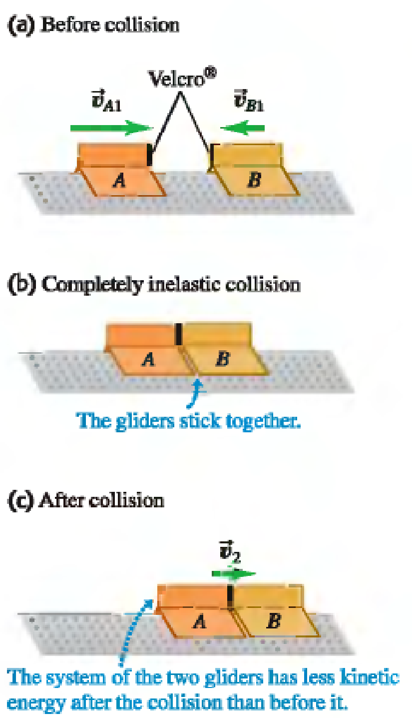
6.4 Collision Problems
6.8 Skier and Cart



8.14 Two gliders undergoing an elastic collision on a frictionless surface. Each glider has a steel spring bumper that exerts a conservative force on the other glider.



8.15 Two gliders undergoing a completely inelastic collision. The spring bumpers on the gliders are replaced by Velcro®, so the gliders stick together after collision.

not only car accidents but also balls colliding on a billiard table, neutrons hitting atomic nuclei in a nuclear reactor, the impact of a meteor on the Arizona desert, and a close encounter of a spacecraft with the planet Saturn.

If the forces between the bodies are much larger than any external forces, as is the case in most collisions, we can neglect the external forces entirely and treat the bodies as an *isolated* system. Then momentum is conserved and the total momentum of the system has the same value before and after the collision. Two cars colliding at an icy intersection provide a good example. Even two cars colliding on dry pavement can be treated as an isolated system during the collision if the forces between the cars are much larger than the friction forces of pavement against tires.

Elastic and Inelastic Collisions

If the forces between the bodies are also *conservative*, so that no mechanical energy is lost or gained in the collision, the total *kinetic* energy of the system is the same after the collision as before. Such a collision is called an **elastic collision.** A collision between two marbles or two billiard balls is almost completely elastic. Figure 8.14 shows a model for an elastic collision. When the gliders collide, their springs are momentarily compressed and some of the original kinetic energy is momentarily converted to elastic potential energy. Then the gliders bounce apart, the springs expand, and this potential energy is converted back to kinetic energy.

A collision in which the total kinetic energy after the collision is *less* than before the collision is called an **inelastic collision.** A meatball landing on a plate of spaghetti and a bullet embedding itself in a block of wood are examples of inelastic collisions. An inelastic collision in which the colliding bodies stick together and move as one body after the collision is often called a **completely inelastic collision.** Figure 8.15 shows an example; we have replaced the spring bumpers in Fig. 8.14 with Velcro®, which sticks the two bodies together.

CAUTION **An inelastic collision doesn't have to be *completely* inelastic** It's a common misconception that the *only* inelastic collisions are those in which the colliding bodies stick together. In fact, inelastic collisions include many situations in which the bodies do *not* stick. If two cars bounce off each other in a "fender bender," the work done to deform the fenders cannot be recovered as kinetic energy of the cars, so the collision is inelastic (Fig. 8.16).

Remember this rule: **In any collision in which external forces can be neglected, momentum is conserved and the total momentum before equals the total momentum after; in elastic collisions *only*, the total kinetic energy before equals the total kinetic energy after.**

Completely Inelastic Collisions

Let's look at what happens to momentum and kinetic energy in a *completely* inelastic collision of two bodies (A and B), as in Fig. 8.15. Because the two bodies stick together after the collision, they have the same final velocity $\vec{v}_2$:

$$\vec{v}_{A2} = \vec{v}_{B2} = \vec{v}_2$$

Conservation of momentum gives the relationship

$$m_A\vec{v}_{A1} + m_B\vec{v}_{B1} = (m_A + m_B)\vec{v}_2 \quad \text{(completely inelastic collision)} \quad (8.16)$$

If we know the masses and initial velocities, we can compute the common final velocity $\vec{v}_2$.

Suppose, for example, that a body with mass m_A and initial x-component of velocity v_{A1x} collides inelastically with a body with mass m_B that is initially at rest $(v_{B1x} = 0)$. From Eq. (8.16) the common x-component of velocity v_{2x} of both bodies after the collision is

$$v_{2x} = \frac{m_A}{m_A + m_B} v_{A1x} \quad \text{(completely inelastic collision, } B \text{ initially at rest)} \tag{8.17}$$

Let's verify that the total kinetic energy after this completely inelastic collision is less than before the collision. The motion is purely along the x-axis, so the kinetic energies K_1 and K_2 before and after the collision, respectively, are

$$K_1 = \tfrac{1}{2} m_A v_{A1x}{}^2$$

$$K_2 = \tfrac{1}{2}(m_A + m_B) v_{2x}{}^2 = \tfrac{1}{2}(m_A + m_B)\left(\frac{m_A}{m_A + m_B}\right)^2 v_{A1x}{}^2$$

The ratio of final to initial kinetic energy is

$$\frac{K_2}{K_1} = \frac{m_A}{m_A + m_B} \quad \text{(completely inelastic collision, } B \text{ initially at rest)} \tag{8.18}$$

The right side is always less than unity because the denominator is always greater than the numerator. Even when the initial velocity of m_B is not zero, it is not hard to verify that the kinetic energy after a completely inelastic collision is always less than before.

Please note: We don't recommend memorizing Eqs. (8.17) or (8.18). We derived them only to prove that kinetic energy is always lost in a completely inelastic collision.

8.16 Automobile collisions are intended to be inelastic, so that the structure of the car absorbs as much of the energy of the collision as possible. This absorbed energy cannot be recovered, since it goes into a permanent deformation of the car.

Example 8.7 A completely inelastic collision

Suppose we repeat the collision described in Example 8.5 (Section 8.2), but this time equip the gliders so that they stick together instead of bouncing apart after they collide. Their masses and initial velocities are the same as in Example 8.5. Find the common final x-velocity v_{2x}, and compare the initial and final kinetic energies.

SOLUTION

IDENTIFY: There are no external forces in the x-direction, so the x-component of momentum is conserved.

SET UP: Figure 8.17 shows our sketch. As in Example 8.5, we take the positive x-axis to point to the right. Our target variables are the final x-velocity v_{2x} and the initial and final kinetic energies of the system.

EXECUTE: From conservation of the x-component of momentum,

$$m_A v_{A1x} + m_B v_{B1x} = (m_A + m_B) v_{2x}$$

$$\begin{aligned} v_{2x} &= \frac{m_A v_{A1x} + m_B v_{B1x}}{m_A + m_B} \\ &= \frac{(0.50\,\text{kg})(2.0\,\text{m/s}) + (0.30\,\text{kg})(-2.0\,\text{m/s})}{0.50\,\text{kg} + 0.30\,\text{kg}} \\ &= 0.50\,\text{m/s} \end{aligned}$$

Because v_{2x} is positive, the gliders move together to the right (the $+x$-direction) after the collision. Before the collision, the kinetic energies of gliders A and B are

$$K_A = \tfrac{1}{2} m_A v_{A1x}{}^2 = \tfrac{1}{2}(0.50\,\text{kg})(2.0\,\text{m/s})^2 = 1.0\,\text{J}$$

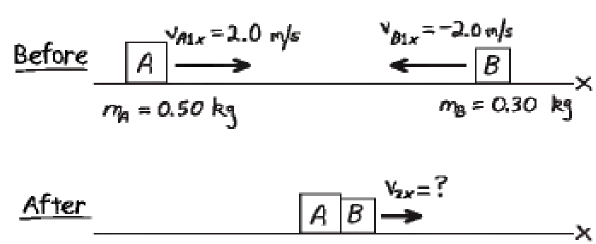
$$K_B = \tfrac{1}{2} m_B v_{B1x}{}^2 = \tfrac{1}{2}(0.30\,\text{kg})(-2.0\,\text{m/s})^2 = 0.60\,\text{J}$$

(Note that the kinetic energy of glider B is positive, even though the x-components of its velocity v_{B1x} and momentum $m_B v_{B1x}$ are both negative.) The *total* kinetic energy before the collision is 1.6 J. The kinetic energy after the collision is

$$\tfrac{1}{2}(m_A + m_B) v_{2x}{}^2 = \tfrac{1}{2}(0.50\,\text{kg} + 0.30\,\text{kg})(0.50\,\text{m/s})^2 = 0.10\,\text{J}$$

EVALUATE: The final kinetic energy is only $\frac{1}{16}$ of the original; $\frac{15}{16}$ is converted from mechanical energy to various other forms. If there is a ball of chewing gum between the gliders, it squashes and becomes warmer. If there is a spring between the gliders that is compressed as they lock together, then the energy is stored as potential energy of the spring. In both of these cases the *total* energy of the system is conserved, although *kinetic* energy is not. However, in an isolated system, momentum is *always* conserved, whether the collision is elastic or not.

8.17 Our sketch for this problem.



Example 8.8 The ballistic pendulum

Figure 8.18 shows a ballistic pendulum, a system for measuring the speed of a bullet. The bullet, with mass m_B, is fired into a block of wood with mass m_W, suspended like a pendulum, and makes a completely inelastic collision with it. After the impact of the bullet, the block swings up to a maximum height y. Given the values of y, m_B, and m_W, what is the initial speed v_1 of the bullet?

SOLUTION

IDENTIFY: We'll analyze this event in two stages: (1) the embedding of the bullet in the block and (2) the subsequent swinging of the block on its strings.

During the first stage, the bullet embeds itself in the block so quickly that the block has no time to move appreciably. The supporting strings remain nearly vertical, so negligible external horizontal force acts on the system of bullet plus block, and the *horizontal component of momentum* is conserved. Mechanical energy is *not* conserved in this stage because a nonconservative force does work (the force of friction between bullet and block).

In the second stage, after the collision, the block and bullet move as a unit. The only forces acting on this unit are gravity (a conservative force) and the string tensions (which do no work). Thus, as the block swings upward and to the right, *mechanical energy* is conserved. Momentum is *not* conserved during this stage because there is a net external force (the forces of gravity and string tension don't cancel when the strings are inclined).

SET UP: We take the positive x-axis to be to the right and the positive y-axis to be upward in Fig. 8.18. Our target variable is v_1. Another unknown quantity is the speed v_2 of the block and bullet as a unit just after the collision (that is, just at the end of the first stage). We'll use momentum conservation in the first stage to relate v_1 to v_2, and we'll use energy conservation in the second stage to relate v_2 to the (given) maximum height y.

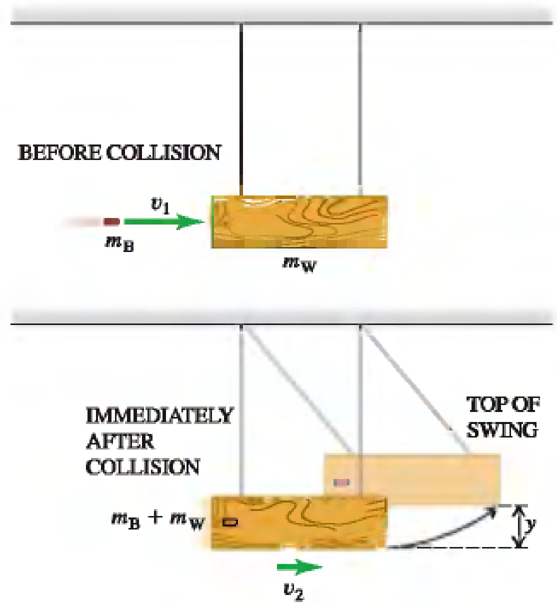
EXECUTE: In the first stage, the velocities are all in the positive x-direction. Momentum conservation gives

$$m_B v_1 = (m_B + m_W)v_2 \qquad v_1 = \frac{m_B + m_W}{m_B} v_2$$

At the beginning of the second stage, the block–bullet unit has kinetic energy $K = \frac{1}{2}(m_B + m_W)v_2^2$. [As in Eq. (8.18), this is less than the kinetic energy before the collision; the collision is inelastic!] The block–bullet unit swings up and comes to rest for an instant at a height y, where its kinetic energy is zero and the potential energy is $(m_B + m_W)gy$; it then swings back down. Energy conservation gives

$$\tfrac{1}{2}(m_B + m_W)v_2^2 = (m_B + m_W)gy \qquad v_2 = \sqrt{2gy}$$

8.18 A ballistic pendulum.



Now we substitute this expression into the momentum equation to find an expression for our target variable v_1:

$$v_1 = \frac{m_B + m_W}{m_B}\sqrt{2gy}$$

Hence measuring m_B, m_W, and y tells us the initial speed of the bullet.

EVALUATE: Let's check our answers by plugging in some realistic numbers. If $m_B = 5.00\text{ g} = 0.00500\text{ kg}$, $m_W = 2.00\text{ kg}$, and $y = 3.00\text{ cm} = 0.0300\text{ m}$, the initial speed of the bullet is

$$\begin{aligned} v_1 &= \frac{0.00500\text{ kg} + 2.00\text{ kg}}{0.00500\text{ kg}}\sqrt{2(9.80\text{ m/s}^2)(0.0300\text{ m})} \\ &= 307\text{ m/s} \end{aligned}$$

The speed v_2 of the block just after impact is

$$\begin{aligned} v_2 &= \sqrt{2gy} = \sqrt{2(9.80\text{ m/s}^2)(0.0300\text{ m})} \\ &= 0.767\text{ m/s} \end{aligned}$$

The kinetic energy of the bullet just before impact is $\frac{1}{2}(0.00500\text{ kg})(307\text{ m/s})^2 = 236\text{ J}$. Just after impact the kinetic energy of the bullet and block is $\frac{1}{2}(2.005\text{ kg})(0.767\text{ m/s})^2 = 0.589\text{ J}$. Nearly all the kinetic energy disappears as the wood splinters and the bullet and block become hotter.

Example 8.9 An automobile collision

A 1000-kg compact car is traveling north at 15 m/s when it collides with a 2000-kg truck traveling east at 10 m/s. All occupants are wearing seat belts and there are no injuries, but the two vehicles are thoroughly tangled and move away from the impact point as one mass. The insurance adjustor has asked you to find the velocity of the wreckage just after impact. What do you tell her?

SOLUTION

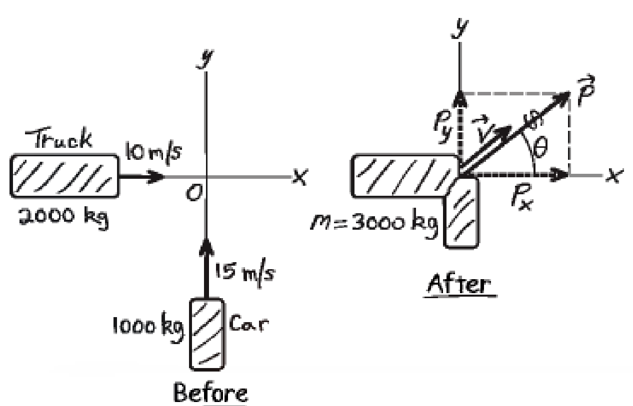
IDENTIFY: We'll assume that we can treat the cars as an isolated system during the collision. We can do so because the horizontal forces that the cars exert on each other during the collision have very large magnitudes, great enough to crumple the cars' metal skins. Compared with these forces, we can neglect any external forces such as friction. (We'll justify this assumption later.) Hence the momentum of the system of two cars has the same value just before and just after the collision.

SET UP: Figure 8.19 shows our sketch. We can find the total momentum before the collision, $\vec{P}$, using Eqs. (8.15) and the coordinate axes shown in Fig. 8.19. The momentum has the same value just after the collision; hence, once we've found $\vec{P}$, we'll be able to find the velocity $\vec{V}$ just after the collision (our second target variable) using the relationship $\vec{P} = M\vec{V}$, where M is the combined mass of the wreckage. We'll use the subscripts C and T for the car and truck, respectively.

EXECUTE: From Eqs. (8.15) the components of the total momentum $\vec{P}$ are

$$\begin{aligned} P_x &= p_{Cx} + p_{Tx} = m_C v_{Cx} + m_T v_{Tx} \\ &= (1000\ \text{kg})(0) + (2000\ \text{kg})(10\ \text{m/s}) \\ &= 2.0 \times 10^4\ \text{kg}\cdot\text{m/s} \\ P_y &= p_{Cy} + p_{Ty} = m_C v_{Cy} + m_T v_{Ty} \\ &= (1000\ \text{kg})(15\ \text{m/s}) + (2000\ \text{kg})(0) \\ &= 1.5 \times 10^4\ \text{kg}\cdot\text{m/s} \end{aligned}$$

8.19 Our sketch for this problem.



The magnitude of $\vec{P}$ is

$$\begin{aligned} P &= \sqrt{(2.0 \times 10^4\ \text{kg}\cdot\text{m/s})^2 + (1.5 \times 10^4\ \text{kg}\cdot\text{m/s})^2} \\ &= 2.5 \times 10^4\ \text{kg}\cdot\text{m/s} \end{aligned}$$

and its direction is given by the angle θ shown in Fig. 8.19, where

$$\tan\theta = \frac{P_y}{P_x} = \frac{1.5 \times 10^4\ \text{kg}\cdot\text{m/s}}{2.0 \times 10^4\ \text{kg}\cdot\text{m/s}} = 0.75 \qquad \theta = 37^\circ$$

The total momentum just after the collision is the same as just before. Assuming that no parts fall off, the total mass of wreckage is $M = m_C + m_T = 3000$ kg. From $\vec{P} = M\vec{V}$, the direction of the velocity $\vec{V}$ just after the collision is the same as that of the momentum, and its magnitude is

$$V = \frac{P}{M} = \frac{2.5 \times 10^4\ \text{kg}\cdot\text{m/s}}{3000\ \text{kg}} = 8.3\ \text{m/s}$$

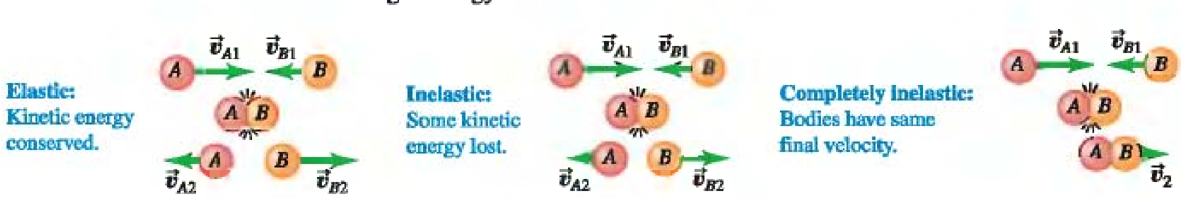
EVALUATE: This is an inelastic collision, so we expect the total kinetic energy to be less after the collision than before. Carry out the calculations yourself; you will find that the initial kinetic energy is 2.1×10^5 J and the final value is 1.0×10^5 J. More than half of the initial kinetic energy is converted to other forms.

We still need to justify our assumption that we can neglect the external forces on the vehicles during the collision. To do so, note that the mass of the truck is 2000 kg, its weight is about 20,000 N, and, if the coefficient of friction is about 0.5, the friction force when it slides across the pavement is about 10,000 N. The truck's kinetic energy just before the impact is $\frac{1}{2}(2000\ \text{kg})(10\ \text{m/s})^2 = 1.0 \times 10^5$ J. The car may crumple 0.2 m or so; to do -1.0×10^5 J of work on the car (required to stop it) in a distance of 0.2 m would require a force of 5.0×10^5 N, which is 50 times greater than the friction force. So it's reasonable to treat the external force of friction as negligible compared with the internal forces that the vehicles exert on each other.

Classifying Collisions

It's important to remember that we can classify collisions according to energy considerations (Fig. 8.20). A collision in which kinetic energy is conserved is called *elastic*. (We'll explore these in more depth in the next section.) A collision in which the total kinetic energy decreases is called *inelastic*. When the two bodies have a common final velocity, we say that the collision is *completely inelastic*. There are also cases in which the final kinetic energy is *greater* than the initial value. Rifle recoil, discussed in Example 8.4 (Section 8.2), is an example.

8.20 Collisions are classified according to energy considerations.



Finally, we emphasize again that we can sometimes use momentum conservation even when there are external forces acting on the system, if the net external force acting on the colliding bodies is small in comparison with the internal forces during the collision (as in Example 8.9)

Test Your Understanding of Section 8.3 For each situation, state whether the collision is elastic or inelastic. If it is inelastic, state whether it is completely inelastic. (a) You drop a ball from your hand. It collides with the floor and bounces back up so that it just reaches your hand. (b) You drop a different ball from your hand and let it collide with the ground. This ball bounces back up to half the height from which it was dropped. (c) You drop a ball of clay from your hand. When it collides with the ground, it stops.

8.4 Elastic Collisions

We saw in Section 8.3 that an *elastic collision* in an isolated system is one in which kinetic energy (as well as momentum) is conserved. Elastic collisions occur when the forces between the colliding bodies are *conservative.* When two billiard balls collide, they squash a little near the surface of contact, but then they spring back. Some of the kinetic energy is stored temporarily as elastic potential energy, but at the end it is reconverted to kinetic energy (Fig. 8.21).

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6.2 Collisions and Elasticity

6.7 Car Collisions: Two Dimensions

6.9 Pendulum Bashes Box

Let's look at an elastic collision between two bodies A and B. We start with a one-dimensional collision, in which all the velocities lie along the same line; we choose this line to be the x-axis. Each momentum and velocity then has only an x-component. We call the x-velocities before the collision v_{A1x} and v_{B1x}, and those after the collision v_{A2x} and v_{B2x}. From conservation of kinetic energy we have

$$\tfrac{1}{2}m_A v_{A1x}{}^2 + \tfrac{1}{2}m_B v_{B1x}{}^2 = \tfrac{1}{2}m_A v_{A2x}{}^2 + \tfrac{1}{2}m_B v_{B2x}{}^2$$

and conservation of momentum gives

$$m_A v_{A1x} + m_B v_{B1x} = m_A v_{A2x} + m_B v_{B2x}$$

If the masses m_A and m_B and the initial velocities v_{A1x} and v_{B1x} are known, we can solve these two equations to find the two final velocities v_{A2x} and v_{B2x}.

8.21 Billiard balls deform very little when they collide, and they quickly spring back from any deformation they do undergo. Hence the force of interaction between the balls is almost perfectly conservative, and the collision is almost perfectly elastic.

Elastic Collisions, One Body Initially at Rest

The general solution to the above equations is a little complicated, so we will concentrate on the particular case in which body B is at rest before the collision (so $v_{B1x} = 0$). Think of body B as a target for body A to hit. Then the kinetic energy and momentum conservation equations are, respectively,

$$\tfrac{1}{2}m_A v_{A1x}{}^2 = \tfrac{1}{2}m_A v_{A2x}{}^2 + \tfrac{1}{2}m_B v_{B2x}{}^2 \tag{8.19}$$

$$m_A v_{A1x} = m_A v_{A2x} + m_B v_{B2x} \tag{8.20}$$

We can solve for v_{A2x} and v_{B2x} in terms of the masses and the initial velocity v_{A1x}. This involves some fairly strenuous algebra, but it's worth it. No pain, no gain! The simplest approach is somewhat indirect, but along the way it uncovers an additional interesting feature of elastic collisions.

First we rearrange Eqs. (8.19) and (8.20) as follows:

$$m_B v_{B2x}{}^2 = m_A\left(v_{A1x}{}^2 - v_{A2x}{}^2\right) = m_A\left(v_{A1x} - v_{A2x}\right)\left(v_{A1x} + v_{A2x}\right) \tag{8.21}$$

$$m_B v_{B2x} = m_A\left(v_{A1x} - v_{A2x}\right) \tag{8.22}$$

Now we divide Eq. (8.21) by Eq. (8.22) to obtain

$$v_{B2x} = v_{A1x} + v_{A2x} \tag{8.23}$$

We substitute this expression back into Eq. (8.22) to eliminate v_{B2x} and then solve for v_{A2x}:

$$m_B(v_{A1x} + v_{A2x}) = m_A(v_{A1x} - v_{A2x})$$

$$v_{A2x} = \frac{m_A - m_B}{m_A + m_B} v_{A1x} \tag{8.24}$$

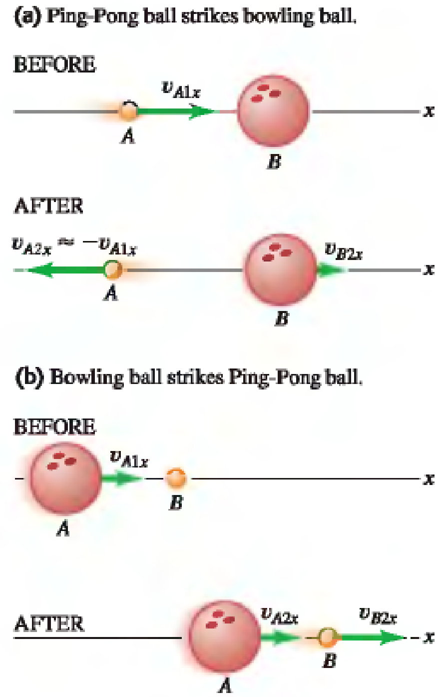
Finally, we substitute this result back into Eq. (8.23) to obtain

$$v_{B2x} = \frac{2m_A}{m_A + m_B} v_{A1x} \tag{8.25}$$

Now we can interpret the results. Suppose body A is a Ping-Pong ball and body B is a bowling ball. Then we expect A to bounce off after the collision with a velocity nearly equal to its original value but in the opposite direction (Fig. 8.22a), and we expect B's velocity to be much less. That's just what the equations predict. When m_A is much smaller than m_B, the fraction in Eq. (8.24) is approximately equal to (-1), so v_{A2x} is approximately equal to $-v_{A1x}$. The fraction in Eq. (8.25) is much smaller than unity, so v_{B2x} is much less than v_{A1x}. Figure 8.22b shows the opposite case, in which A is the bowling ball and B the Ping-Pong ball and m_A is much larger than m_B. What do you expect to happen then? Check your predictions against Eqs. (8.24) and (8.25).

Another interesting case occurs when the masses are equal (Fig. 8.23). If $m_A = m_B$, then Eqs. (8.24) and (8.25) give $v_{A2x} = 0$ and $v_{B2x} = v_{A1x}$. That is, the body that was moving stops dead; it gives all its momentum and kinetic energy to the body that was at rest. This behavior is familiar to all pool players.

8.22 Collisions between **(a)** a moving Ping-Pong ball and an initially stationary bowling ball, and **(b)** a moving bowling ball and an initially stationary Ping-Pong ball.



Elastic Collisions and Relative Velocity

Let's return to the more general case in which A and B have different masses. Equation (8.23) can be rewritten as

$$v_{A1x} = v_{B2x} - v_{A2x} \tag{8.26}$$

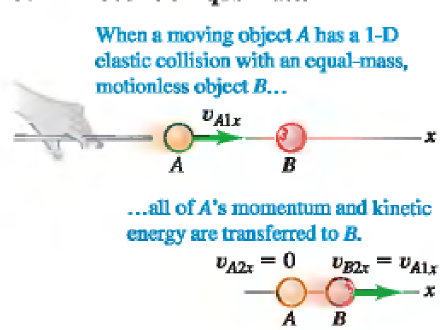
Here $v_{B2x} - v_{A2x}$ is the velocity of B relative to A *after* the collision; from Eq. (8.26), this equals v_{A1x}, which is the *negative* of the velocity of B relative to A *before* the collision. (We discussed relative velocity in Section 3.5.) The relative velocity has the same magnitude, but opposite sign, before and after the collision. The sign changes because A and B are approaching each other before the collision but moving apart after the collision. If we view this collision from a second coordinate system moving with constant velocity relative to the first, the velocities of the bodies are different but the *relative* velocities are the same. Hence our statement about relative velocities holds for *any* straight-line elastic collision, even when neither body is at rest initially. *In a straight-line elastic collision of two bodies, the relative velocities before and after the collision have the same magnitude but opposite sign.* This means that if B is moving before the collision, Eq. (8.26) becomes

$$v_{B2x} - v_{A2x} = -(v_{B1x} - v_{A1x}) \tag{8.27}$$

It turns out that a *vector* relationship similar to Eq. (8.27) is a general property of *all* elastic collisions, even when both bodies are moving initially and the velocities do not all lie along the same line. This result provides an alternative and equivalent definition of an elastic collision: *In an elastic collision, the relative velocity of the two bodies has the same magnitude before and after the collision.* Whenever this condition is satisfied, the total kinetic energy is also conserved.

When an elastic two-body collision isn't head-on, the velocities don't all lie along a single line. If they all lie in a plane, then each final velocity has two unknown components, and there are four unknowns in all. Conservation of energy and conservation of the x- and y-components of momentum give only three equations. To determine the final velocities uniquely, we need additional information, such as the direction or magnitude of one of the final velocities.

8.23 A one-dimensional elastic collision between bodies of equal mass.



Example 8.10 An elastic straight-line collision

We repeat the air-track experiment from Example 8.5 (Section 8.2), but now we add ideal spring bumpers to the gliders so that the collision is elastic. What are the velocities of A and B after the collision?

SOLUTION

IDENTIFY: As in Example 8.5, the net external force on the system of two gliders is zero, and the momentum of the system is conserved.

SET UP: Figure 8.24 shows our sketch. We again choose the positive x-axis to point to the right. We'll find our target variables, v_{A2x} and v_{B2x}, using Eq. (8.27) and the equation of momentum conservation.

EXECUTE: From conservation of momentum,

$$m_A v_{A1x} + m_B v_{B1x} = m_A v_{A2x} + m_B v_{B2x}$$

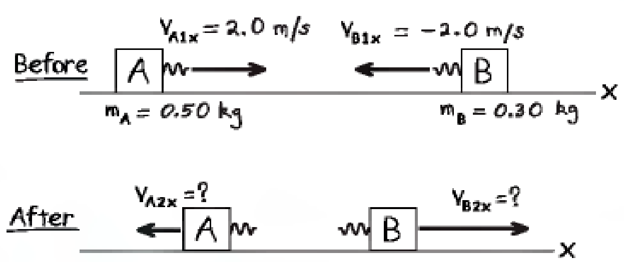
$$(0.50\ \text{kg})(2.0\ \text{m/s}) + (0.30\ \text{kg})(-2.0\ \text{m/s}) = (0.50\ \text{kg})v_{A2x} + (0.30\ \text{kg})v_{B2x}$$

$$0.50 v_{A2x} + 0.30 v_{B2x} = 0.40\ \text{m/s}$$

(In the last equation we divided through by the unit "kg.") From Eq. (8.27), the relative velocity relationship for an elastic collision, we have

$$v_{B2x} - v_{A2x} = -(v_{B1x} - v_{A1x})$$
$$= -(-2.0\ \text{m/s} - 2.0\ \text{m/s}) = 4.0\ \text{m/s}$$

8.24 Our sketch for this problem.



Before the collision, the velocity of B relative to A is to the left at 4.0 m/s; after the collision, the velocity of B relative to A is to the right at 4.0 m/s. Solving these equations simultaneously, we find

$$v_{A2x} = -1.0\ \text{m/s} \qquad v_{B2x} = 3.0\ \text{m/s}$$

EVALUATE: Both bodies reverse their directions of motion; A moves to the left at 1.0 m/s and B moves to the right at 3.0 m/s. This is different from the result of Example 8.5 because that collision was *not* elastic.

Note that unlike the situations shown in Fig. 8.22, the two gliders are *both* moving toward each other before the collision. Our results show that A (the more massive glider) moves slower after the collision than before the collision, and so loses kinetic energy. In contrast, B (the less massive glider) gains kinetic energy: It moves faster after the collision than before. The *total* kinetic energy after the elastic collision is

$$\tfrac{1}{2}(0.50\ \text{kg})(-1.0\ \text{m/s})^2 + \tfrac{1}{2}(0.30\ \text{kg})(3.0\ \text{m/s})^2 = 1.6\ \text{J}$$

As expected, this equals the total kinetic energy *before* the collision (which we calculated in Example 8.7 in Section 8.3). Thus kinetic energy is transferred from A to B in the collision, with none of it lost in the process. Much the same happens when a baseball player swings a bat and hits an oncoming baseball. The collision is nearly elastic, and the more massive bat transfers kinetic energy to the less massive baseball. The baseball leaves the bat with a much greater speed—perhaps enough to make a home run.

CAUTION Be careful with the elastic collision equations You might have been tempted to solve this problem using Eqs. (8.24) and (8.25). These equations apply *only* to situations in which body B is initially at rest, which isn't the case here. When in doubt, always solve the problem at hand using equations that are applicable to a broad variety of cases.

Example 8.11 Moderator in a nuclear reactor

The fission of uranium nuclei in a nuclear reactor produces high-speed neutrons. Before a neutron can trigger additional fissions, it has to be slowed down by collisions with nuclei in the *moderator* of the reactor. The first nuclear reactor (built in 1942 at the University of Chicago) and the reactor involved in the 1986 Chernobyl accident both used carbon (graphite) as the moderator material. Suppose a neutron (mass 1.0 u) traveling at 2.6×10^7 m/s undergoes a head-on elastic collision with a carbon nucleus (mass 12 u) initially at rest. The external forces during the collision are negligible. What are the velocities after the collision? (1 u is the *atomic mass unit,* equal to 1.66×10^{-27} kg.)

SOLUTION

IDENTIFY: We are given that the external forces can be neglected (so momentum is conserved in the collision) and that the collision is elastic (so kinetic energy is also conserved).

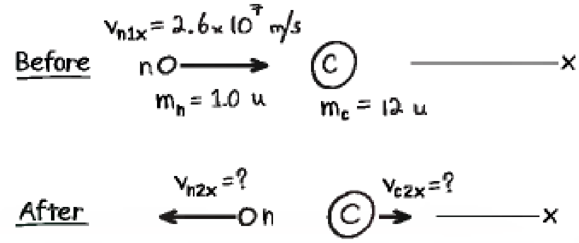
SET UP: Figure 8.25 shows our sketch. We take the x-axis to be in the direction in which the neutron is moving initially. Because the collision is head-on, both the neutron and the carbon nucleus move along this same axis after the collision. Furthermore, because one body is initially at rest, we can use Eqs. (8.24) and (8.25) with A replaced by n (for the neutron) and B replaced by C (for the carbon nucleus). We have $m_n = 1.0$ u, $m_C = 12$ u, and $v_{n1x} = 2.6 \times 10^7$ m/s, and we need to solve for the target variables v_{n2x} and v_{C2x} (the final velocities of the neutron and the carbon nucleus, respectively).

EXECUTE: We'll let you do the arithmetic; the results are

$$v_{n2x} = -2.2 \times 10^7\ \text{m/s} \qquad v_{C2x} = 0.4 \times 10^7\ \text{m/s}$$

EVALUATE: The neutron ends up with $\frac{11}{13}$ of its initial speed, and the speed of the recoiling carbon nucleus is $\frac{2}{13}$ of the neutron's initial speed. [These ratios are the factors $(m_n - m_C)/(m_n + m_C)$ and $2m_n/(m_n + m_C)$ that appear in Eqs. (8.24) and (8.25), with the subscripts revised for this problem.] Kinetic energy is proportional to speed squared, so the neutron's final kinetic energy is $(\frac{11}{13})^2$, or about 0.72 of its original value. If the neutron makes a second such collision, its kinetic energy is $(0.72)^2$, or about half its original value, and so on. After several collisions, the neutron will be moving quite slowly and will be able to trigger a fission reaction in a uranium nucleus.

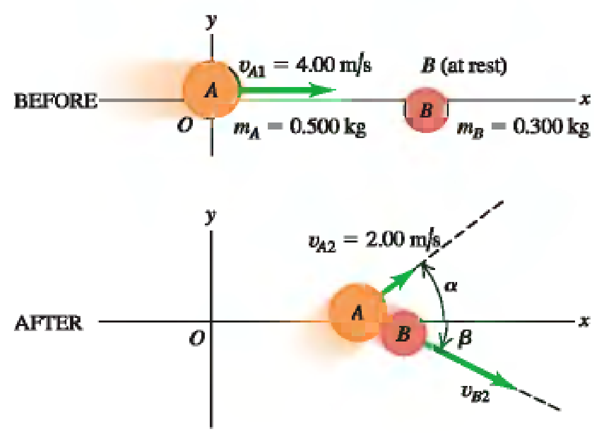
8.25 Our sketch for this problem.



Example 8.12 A two-dimensional elastic collision

Figure 8.26 shows an elastic collision of two pucks on a frictionless air-hockey table. Puck A has mass $m_A = 0.500$ kg and puck B has mass $m_B = 0.300$ kg. Puck A has an initial velocity of 4.00 m/s in the positive x-direction and a final velocity of 2.00 m/s in an unknown direction. Puck B is initially at rest. Find the final speed v_{B2} of puck B and the angles α and β in the figure.

8.26 An elastic collision that isn't head-on.



SOLUTION

IDENTIFY: Although the collision is elastic, it is *not* one-dimensional, so we can't use any of the one-dimensional formulas derived in this section. Instead, we'll use the equations for conservation of energy, conservation of x-momentum, and conservation of y-momentum.

SET UP: The target variables are given in the statement of the problem. We have three equations, which should be enough to solve for our three target variables.

EXECUTE: Because the collision is elastic, the initial and final kinetic energies are equal:

$$\tfrac{1}{2}m_A v_{A1}^2 = \tfrac{1}{2}m_A v_{A2}^2 + \tfrac{1}{2}m_B v_{B2}^2$$

$$v_{B2}^2 = \frac{m_A v_{A1}^2 - m_A v_{A2}^2}{m_B}$$

$$= \frac{(0.500\ \text{kg})(4.00\ \text{m/s})^2 - (0.500\ \text{kg})(2.00\ \text{m/s})^2}{0.300\ \text{kg}}$$

$$v_{B2} = 4.47\ \text{m/s}$$

Conservation of the x-component of total momentum gives

$$m_A v_{A1x} = m_A v_{A2x} + m_B v_{B2x}$$

$$(0.500\ \text{kg})(4.00\ \text{m/s}) = (0.500\ \text{kg})(2.00\ \text{m/s})(\cos\alpha) + (0.300\ \text{kg})(4.47\ \text{m/s})(\cos\beta)$$

and conservation of the y-component gives

$$0 = m_A v_{A2y} + m_B v_{B2y}$$

$$0 = (0.500\ \text{kg})(2.00\ \text{m/s})(\sin\alpha) - (0.300\ \text{kg})(4.47\ \text{m/s})(\sin\beta)$$

These are two simultaneous equations for α and β. The simplest solution is to eliminate β as follows: We solve the first equation for $\cos\beta$ and the second for $\sin\beta$; we then square each equation and add. Since $\sin^2\beta + \cos^2\beta = 1$, this eliminates β and leaves an equation that we can solve for $\cos\alpha$ and hence for α. We can then substitute this value back into either of the two equations and solve the result for β. We leave the details for you to work out in Exercise 8.44; the results are

$$\alpha = 36.9^\circ \qquad \beta = 26.6^\circ$$

EVALUATE: A quick way to check the answers is to make sure that the y-momentum, which was zero before the collision, is still zero after the collision. The y-momenta of the pucks are

$$p_{A2y} = (0.500\ \text{kg})(2.00\ \text{m/s})(\sin 36.9^\circ) = +0.600\ \text{kg}\cdot\text{m/s}$$

$$p_{B2y} = -(0.300\ \text{kg})(4.47\ \text{m/s})(\sin 26.6^\circ) = -0.600\ \text{kg}\cdot\text{m/s}$$

The sum of these values is zero, as it should be.

Test Your Understanding of Section 8.4 Most present-day nuclear reactors use water as a moderator (see Example 8.11). Are water molecules (mass $m_w = 18.0$ u) a better or worse moderator than carbon atoms? (One advantage of water is that it also acts as a coolant for the reactor's radioactive core.)

8.5 Center of Mass

We can restate the principle of conservation of momentum in a useful way by using the concept of **center of mass.** Suppose we have several particles with masses m_1, m_2, and so on. Let the coordinates of m_1 be (x_1, y_1), those of m_2 be (x_2, y_2), and so on. We define the center of mass of the system as the point that has coordinates (x_{cm}, y_{cm}) given by

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_ix_i}{\sum_i m_i}$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_ix_i}{\sum_i m_i} \qquad \text{(center of mass)} \quad (8.28)$$

The position vector $\vec{r}_{cm}$ of the center of mass can be expressed in terms of the position vectors $\vec{r}_1, \vec{r}_2, \ldots$ of the particles as

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i} \qquad \text{(center of mass)} \quad (8.29)$$

In statistical language, the center of mass is a *mass-weighted average* position of the particles.

Example 8.13 Center of mass of a water molecule

Figure 8.27 shows a simple model of the structure of a water molecule. The separation between atoms is $d = 9.57 \times 10^{-11}$ m. Each hydrogen atom has mass 1.0 u, and the oxygen atom has mass 16.0 u. Find the position of the center of mass.

SOLUTION

IDENTIFY: Nearly all the mass of each atom is concentrated in its nucleus, which is only about 10^{-5} times the overall radius of the atom. Hence we can safely represent each atom as a point particle.

SET UP: The coordinate system is shown in Fig. 8.27. We'll use Eqs. (8.28) to determine the coordinates x_{cm} and y_{cm}.

EXECUTE: The x-coordinate of each hydrogen atom is $d\cos(105°/2)$; the y-coordinates of the upper and lower hydrogen atoms are $+d\sin(105°/2)$ and $-d\sin(105°/2)$, respectively. The coordinates of the oxygen atom are $x = 0, y = 0$. From Eqs. (8.28) the x-coordinate of the center of mass is

$$x_{cm} = \frac{\begin{bmatrix}(1.0\,\text{u})(d\cos 52.5°) + (1.0\,\text{u}) \\ \times (d\cos 52.5°) + (16.0\,\text{u})(0)\end{bmatrix}}{1.0\,\text{u} + 1.0\,\text{u} + 16.0\,\text{u}} = 0.068d$$

and the y-coordinate is

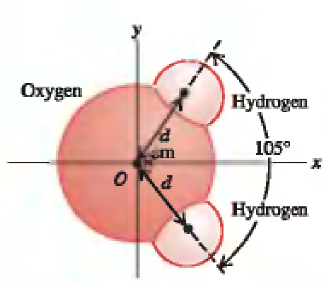
$$y_{cm} = \frac{\begin{bmatrix}(1.0\,\text{u})(d\sin 52.5°) + (1.0\,\text{u}) \\ \times (-d\sin 52.5°) + (16.0\,\text{u})(0)\end{bmatrix}}{1.0\,\text{u} + 1.0\,\text{u} + 16.0\,\text{u}} = 0$$

Substituting the value $d = 9.57 \times 10^{-11}$ m, we find

$$x_{cm} = (0.068)(9.57 \times 10^{-11}\,\text{m}) = 6.5 \times 10^{-12}\,\text{m}$$

EVALUATE: The center of mass is much closer to the oxygen atom than to either hydrogen atom because the oxygen atom is much more massive. Notice that the center of mass lies along the x-axis, which is the *axis of symmetry* of this molecule. If the molecule is rotated by 180° around this axis, it looks exactly the same as before. The position of the center of mass can't be affected by this rotation, so it must lie on the axis of symmetry.

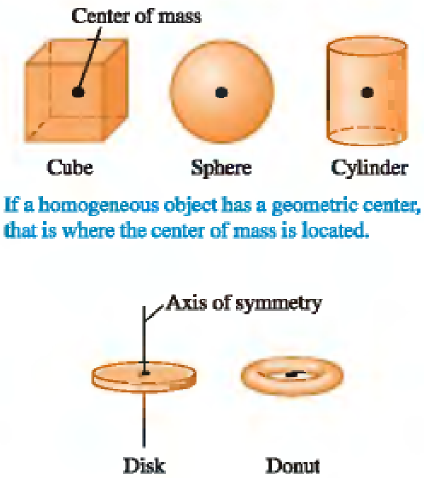
8.27 Where is the center of mass of a water molecule?



For solid bodies, in which we have (at least on a macroscopic level) a continuous distribution of matter, the sums in Eqs. (8.28) have to be replaced by integrals. The calculations can get quite involved, but we can say three general things about such problems (Fig. 8.28). First, whenever a homogeneous body has a geometric center, such as a billiard ball, a sugar cube, or a can of frozen orange juice, the center of mass is at the geometric center. Second, whenever a body has an axis of symmetry, such as a wheel or a pulley, the center of mass always lies on that axis. Third, there is no law that says the center of mass has to be within the body. For example, the center of mass of a donut is right in the middle of the hole.

We'll talk a little more about locating the center of mass in Chapter 11 in connection with the related concept of *center of gravity*.

8.28 Locating the center of mass of a symmetrical object.



Motion of the Center of Mass

To see the significance of the center of mass of a collection of particles, we must ask what happens to the center of mass when the particles move. The x- and y-components of velocity of the center of mass, $v_{\text{cm-}x}$ and $v_{\text{cm-}y}$, are the time derivatives of x_{cm} and y_{cm}. Also, dx_1/dt is the x-component of velocity of particle 1, and so on, so $dx_1/dt = v_{1x}$, and so on. Taking time derivatives of Eqs. (8.28), we get

$$v_{\text{cm-}x} = \frac{m_1 v_{1x} + m_2 v_{2x} + m_3 v_{3x} + \cdots}{m_1 + m_2 + m_3 + \cdots}$$
$$v_{\text{cm-}y} = \frac{m_1 v_{1y} + m_2 v_{2y} + m_3 v_{3y} + \cdots}{m_1 + m_2 + m_3 + \cdots} \tag{8.30}$$

These equations are equivalent to the single vector equation obtained by taking the time derivative of Eq. (8.29):

$$\vec{v}_{\text{cm}} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \tag{8.31}$$

We denote the *total* mass $m_1 + m_2 + \cdots$ by M. We can then rewrite Eq. (8.31) as

$$M\vec{v}_{\text{cm}} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots = \vec{P} \tag{8.32}$$

The right side is simply the total momentum $\vec{P}$ of the system. Thus we have proved that *the total momentum is equal to the total mass times the velocity of the center of mass.* When you catch a baseball, you are really catching a collection of a very large number of molecules of masses $m_1, m_2, m_3, \ldots$. The impulse you feel is due to the total momentum of this entire collection. But this impulse is the same as if you were catching a single particle of mass $M = m_1 + m_2 + m_3 + \cdots$ moving with velocity $\vec{v}_{\text{cm}}$, the velocity of the collection's center of mass. So Eq. (8.32) helps to justify representing an extended body as a particle.

For a system of particles on which the net external force is zero, so that the total momentum $\vec{P}$ is constant, the velocity of the center of mass $\vec{v}_{\text{cm}} = \vec{P}/M$ is also constant. Suppose we mark the center of mass of a wrench and then slide the wrench with a spinning motion across a smooth, horizontal tabletop (Fig. 8.29). The overall motion appears complicated, but the center of mass follows a straight line, as though all the mass were concentrated at that point.

8.29 The center of mass of this wrench is marked with a white dot. The net external force acting on the wrench is almost zero. As the wrench spins on a smooth horizontal surface, the center of mass moves in a straight line with nearly constant velocity.

Example 8.14 A tug-of-war on the ice

James and Ramon are standing 20.0 m apart on the slippery surface of a frozen pond. Ramon has mass 60.0 kg and James has mass 90.0 kg. Midway between the two men a mug of their favorite beverage sits on the ice. They pull on the ends of a light rope that is stretched between them. When James has moved 6.0 m toward the mug, how far and in what direction has Ramon moved?

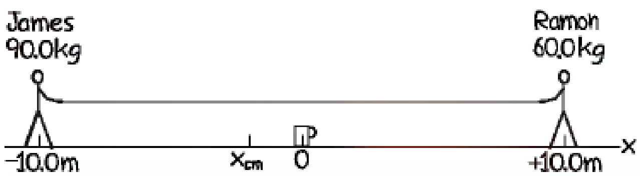
Continued

SOLUTION

IDENTIFY: The frozen surface is horizontal and essentially frictionless, so the net external force on the system of James, Ramon, and the rope is zero. Hence their total momentum is conserved. Initially there is no motion, so the total momentum is zero; thus the velocity of the center of mass is zero, and the center of mass remains at rest. We can use this to relate the positions of James and Ramon.

SET UP: Let's take the origin at the position of the mug, and let the $+x$-axis extend from the mug toward Ramon. Figure 8.30 shows our sketch. Since the rope is light, we can neglect its mass in calculating the position of the center of mass with Eq. (8.28).

8.30 Our sketch for this problem.



EXECUTE: The initial x-coordinates of James and Ramon are -10.0 m and $+10.0$ m, respectively, so the x-coordinate of the center of mass is

$$x_{cm} = \frac{(90.0\text{ kg})(-10.0\text{ m}) + (60.0\text{ kg})(10.0\text{ m})}{90.0\text{ kg} + 60.0\text{ kg}} = -2.0\text{ m}$$

When James moves 6.0 m toward the mug, his new x-coordinate is -4.0 m; we'll call Ramon's new x-coordinate x_2. The center of mass doesn't move, so

$$x_{cm} = \frac{(90.0\text{ kg})(-4.0\text{ m}) + (60.0\text{ kg})x_2}{90.0\text{ kg} + 60.0\text{ kg}} = -2.0\text{ m}$$

$$x_2 = 1.0\text{ m}$$

James has moved 6.0 m in the positive x-direction and is still 4.0 m from the mug, but Ramon has moved 9.0 m in the negative x-direction and is only 1.0 m from it.

EVALUATE: The ratio of how far each man moved, $(6.0\text{ m})/(9.0\text{ m}) = \frac{2}{3}$, equals the inverse ratio of their masses. Can you see why? If the two men keep moving (and if the surface is frictionless, they will!), Ramon will reach the mug first. This result is completely independent of how hard either person pulls; pulling harder just helps Ramon quench his thirst sooner.

External Forces and Center-of-Mass Motion

If the net external force on a system of particles is not zero, then total momentum is not conserved and the velocity of the center of mass changes. Let's look at the relationship between the motion of the center of mass and the forces acting on the system.

Equations (8.31) and (8.32) give the *velocity* of the center of mass in terms of the velocities of the individual particles. We take the time derivatives of these equations to show that the *accelerations* are related in the same way. Let $\vec{a}_{cm} = d\vec{v}_{cm}/dt$ be the acceleration of the center of mass; then we find

$$M\vec{a}_{cm} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots \tag{8.33}$$

Now $m_1\vec{a}_1$ is equal to the vector sum of forces on the first particle, and so on, so the right side of Eq. (8.33) is equal to the vector sum $\sum\vec{F}$ of *all* the forces on *all* the particles. Just as we did in Section 8.2, we can classify each force as *external* or *internal*. The sum of all forces on all the particles is then

$$\sum\vec{F} = \sum\vec{F}_{ext} + \sum\vec{F}_{int} = M\vec{a}_{cm}$$

Because of Newton's third law, the internal forces all cancel in pairs, and $\sum\vec{F}_{int} = \mathbf{0}$. What survives on the left side is the sum of only the *external* forces:

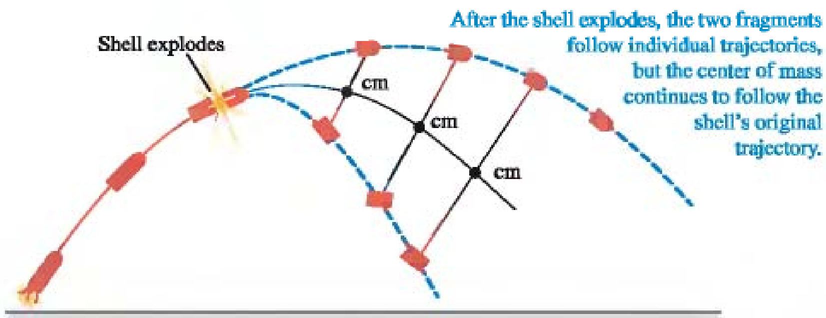
$$\sum\vec{F}_{ext} = M\vec{a}_{cm} \quad \text{(body or collection of particles)} \tag{8.34}$$

When a body or a collection of particles is acted on by external forces, the center of mass moves just as though all the mass were concentrated at that point and it were acted on by a net force equal to the sum of the external forces on the system.

This result may not sound very impressive, but in fact it is central to the whole subject of mechanics. In fact, we've been using this result all along; without it, we would not be able to represent an extended body as a point particle when we apply Newton's laws. It explains why only *external* forces can affect the motion of an extended body. If you pull upward on your belt, your belt exerts an equal downward force on your hands; these are *internal* forces that cancel and have no effect on the overall motion of your body.

8.31 (a) A shell explodes into two fragments in flight. If air resistance is ignored, the center of mass continues on the same trajectory as the shell's path before exploding. (b) The same effect occurs with exploding fireworks.

(a)



(b)

Suppose a cannon shell traveling in a parabolic trajectory (neglecting air resistance) explodes in flight, splitting into two fragments with equal mass (Fig. 8.31a). The fragments follow new parabolic paths, but the center of mass continues on the original parabolic trajectory, just as though all the mass were still concentrated at that point. A skyrocket exploding in air (Fig. 8.31b) is a spectacular example of this effect.

This property of the center of mass is important when we analyze the motion of rigid bodies. We describe the motion of an extended body as a combination of translational motion of the center of mass and rotational motion about an axis through the center of mass. We will return to this topic in Chapter 10. This property also plays an important role in the motion of astronomical objects. It's not correct to say that the moon orbits the earth; rather, the earth and moon both move in orbits around their center of mass.

There's one more useful way to describe the motion of a system of particles. Using $\vec{a}_{cm} = d\vec{v}_{cm}/dt$, we can rewrite Eq. (8.33) as

$$M\vec{a}_{cm} = M\frac{d\vec{v}_{cm}}{dt} = \frac{d(M\vec{v}_{cm})}{dt} = \frac{d\vec{P}}{dt} \tag{8.35}$$

The total system mass M is constant, so we're allowed to take it inside the derivative. Substituting Eq. (8.35) into Eq. (8.34), we find

$$\Sigma\vec{F}_{ext} = \frac{d\vec{P}}{dt} \quad \text{(extended body or system of particles)} \tag{8.36}$$

This equation looks like Eq. (8.4). The difference is that Eq. (8.36) describes a *system* of particles, such as an extended body, while Eq. (8.4) describes a single particle. The interactions between the particles that make up the system can change the individual momenta of the particles, but the *total* momentum $\vec{P}$ of the system can be changed only by external forces acting from outside the system.

Finally, we note that if the net external force is zero, Eq. (8.34) shows that the acceleration $\vec{a}_{cm}$ of the center of mass is zero. So the center-of-mass velocity $\vec{v}_{cm}$ is constant, as for the wrench in Fig. 8.29. From Eq. (8.36) the total momentum $\vec{P}$ is also constant. This reaffirms our statement in Section 8.3 of the principle of conservation of momentum.

Test Your Understanding of Section 8.5 Will the center of mass in Fig. 8.31a continue on the same parabolic trajectory even after one of the fragments hits the ground? Why or why not?

*8.6 Rocket Propulsion

ActivPhysics ONLINE 6.6 Saving an Astronaut

Momentum considerations are particularly useful for analyzing a system in which the masses of parts of the system change with time. In such cases we can't use Newton's second law $\sum \vec{F} = m\vec{a}$ directly because m changes. Rocket propulsion offers a typical and interesting example of this kind of analysis. A rocket is propelled forward by rearward ejection of burned fuel that initially was in the rocket (which is why rocket fuel is also called *propellant*). The forward force on the rocket is the reaction to the backward force on the ejected material. The total mass of the system is constant, but the mass of the rocket itself decreases as material is ejected.

As a simple example, consider a rocket fired in outer space, where there is no gravitational force and no air resistance. Let m denote the mass of the rocket, which will change as it expends fuel. We choose our x-axis to be along the rocket's direction of motion. Figure 8.32a shows the rocket at a time t, when its mass is m and its x-velocity relative to our coordinate system is v. (For simplicity, we will drop the subscript x in this discussion.) The x-component of total momentum at this instant is $P_1 = mv$. In a short time interval dt, the mass of the rocket changes by an amount dm. This is an inherently negative quantity because the rocket's mass m *decreases* with time. During dt, a *positive* mass $-dm$ of burned fuel is ejected from the rocket. Let v_{ex} be the exhaust *speed* of this material *relative to the rocket;* the burned fuel is ejected opposite the direction of motion, so its x-component of *velocity* relative to the rocket is $-v_{ex}$. The x-velocity v_{fuel} of the burned fuel relative to our coordinate system is then

$$v_{fuel} = v + (-v_{ex}) = v - v_{ex}$$

and the x-component of momentum of the ejected mass $(-dm)$ is

$$(-dm)v_{fuel} = (-dm)(v - v_{ex})$$

Figure 8.32b shows that at the end of the time interval dt, the x-velocity of the rocket and unburned fuel has increased to $v + dv$, and its mass has decreased to $m + dm$ (remember that dm is negative). The rocket's momentum at this time is

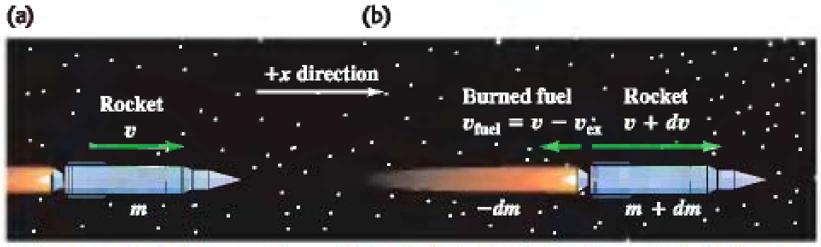
$$(m + dm)(v + dv)$$

Thus the *total* x-component of momentum P_2 of the rocket plus ejected fuel at time $t + dt$ is

$$P_2 = (m + dm)(v + dv) + (-dm)(v - v_{ex})$$

8.32 A rocket moving in gravity-free outer space at (a) time t and (b) time $t + dt$.

(a) (b)



At time t, the rocket has mass m and x-component of velocity v.

At time $t + dt$, the rocket has mass $m + dm$ (where dm is inherently *negative*) and x-component of velocity $v + dv$. The burned fuel has x-component of velocity $v_{fuel} = v - v_{ex}$ and mass $-dm$. (The minus sign is needed to make $-dm$ *positive* because dm is negative.)

According to our initial assumption, the rocket and fuel are an isolated system. Thus momentum is conserved, and the total x-component of momentum of the system must be the same at time t and at time $t + dt$: $P_1 = P_2$. Hence

$$mv = (m + dm)(v + dv) + (-dm)(v - v_{\text{ex}})$$

This can be simplified to

$$m\,dv = -dm\,v_{\text{ex}} - dm\,dv$$

We can neglect the term $(-dm\,dv)$ because it is a product of two small quantities and thus is much smaller than the other terms. Dropping this term, dividing by dt, and rearranging, we find

$$m\frac{dv}{dt} = -v_{\text{ex}}\frac{dm}{dt} \tag{8.37}$$

Now dv/dt is the acceleration of the rocket, so the left side of this equation (mass times acceleration) equals the net force F, or *thrust*, on the rocket,

$$F = -v_{\text{ex}}\frac{dm}{dt} \tag{8.38}$$

The thrust is proportional both to the relative speed v_{ex} of the ejected fuel and to the mass of fuel ejected per unit time, $-dm/dt$. (Remember that dm/dt is negative because it is the rate of change of the rocket's mass, so F is positive.)

The x-component of acceleration of the rocket is

$$a = \frac{dv}{dt} = -\frac{v_{\text{ex}}}{m}\frac{dm}{dt} \tag{8.39}$$

This is positive because v_{ex} is positive (remember, it's the exhaust *speed*) and dm/dt is negative. The rocket's mass m decreases continuously while the fuel is being consumed. If v_{ex} and dm/dt are constant, the acceleration increases until all the fuel is gone.

Equation (8.38) tells us that an effective rocket burns fuel at a rapid rate (large $-dm/dt$) and ejects the burned fuel at a high relative speed (large v_{ex}), as in Fig. 8.33. In the early days of rocket propulsion, people who didn't understand conservation of momentum thought that a rocket couldn't function in outer space because "it doesn't have anything to push against." On the contrary, rockets work *best* in outer space, where there is no air resistance! The launch vehicle in Fig. 8.33 is *not* "pushing against the ground" to get into the air.

If the exhaust speed v_{ex} is constant, we can integrate Eq. (8.39) to find a relationship between the velocity v at any time and the remaining mass m. At time $t = 0$, let the mass be m_0 and the velocity v_0. Then we rewrite Eq. (8.39) as

$$dv = -v_{\text{ex}}\frac{dm}{m}$$

We change the integration variables to v' and m', so we can use v and m as the upper limits (the final speed and mass). Then we integrate both sides, using limits v_0 to v and m_0 to m, and take the constant v_{ex} outside the integral:

$$\int_{v_0}^{v} dv' = -\int_{m_0}^{m} v_{\text{ex}}\frac{dm'}{m'} = -v_{\text{ex}}\int_{m_0}^{m}\frac{dm'}{m'}$$

$$v - v_0 = -v_{\text{ex}}\ln\frac{m}{m_0} = v_{\text{ex}}\ln\frac{m_0}{m} \tag{8.40}$$

The ratio m_0/m is the original mass divided by the mass after the fuel has been exhausted. In practical spacecraft this ratio is made as large as possible to maximize the speed gain, which means that the initial mass of the rocket is almost all fuel. The final velocity of the rocket will be greater in magnitude (and is often

8.33 To provide enough thrust to lift its payload into space, this Atlas V launch vehicle exhausts more than 1000 kg of burned fuel per second at speeds of nearly 4000 m/s.

much greater) than the relative speed v_{ex} if $\ln(m_0/m) > 1$—that is, if $m_0/m > e = 2.71828\ldots$

We've assumed throughout this analysis that the rocket is in gravity-free outer space. However, gravity must be taken into account when a rocket is launched from the surface of a planet, as in Fig. 8.33 (see Problem 8.110).

Example 8.15 Acceleration of a rocket

A rocket is in outer space, far from any planet, when the rocket engine is turned on. In the first second of firing, the rocket ejects $\frac{1}{120}$ of its mass with a relative speed of 2400 m/s. What is the rocket's initial acceleration?

SOLUTION

IDENTIFY: We are given the rocket's exhaust speed v_{ex}, but not its mass m or the rate of change of its mass dm/dt. However, we are told what fraction of the initial mass is lost during a given time interval, which should be enough.

SET UP: We'll use Eq. (8.39) to find the acceleration of the rocket.

EXECUTE: The initial rate of change of mass is

$$\frac{dm}{dt} = -\frac{m_0/120}{1\text{ s}} = -\frac{m_0}{120\text{ s}}$$

where m_0 is the initial $(t = 0)$ mass of the rocket. From Eq. (8.39) the initial acceleration is

$$a = -\frac{v_{ex}}{m_0}\frac{dm}{dt} = -\frac{2400\text{ m/s}}{m_0}\left(-\frac{m_0}{120\text{ s}}\right) = 20\text{ m/s}^2$$

EVALUATE: Note that the answer didn't depend on the value of m_0. If v_{ex} is the same, the initial acceleration is the same for a 120,000-kg spacecraft that ejects 1000 kg/s as for a 60-kg astronaut equipped with a small rocket that ejects 0.5 kg/s.

Example 8.16 Speed of a rocket

Suppose that $\frac{3}{4}$ of the initial mass m_0 of the rocket in Example 8.15 is fuel, so the final mass is $m = m_0/4$, and that the fuel is completely consumed at a constant rate in a total time $t = 90$ s. If the rocket starts from rest in our coordinate system, find its speed at the end of this time.

SOLUTION

IDENTIFY: We are given the initial velocity v_0 (equal to zero), the exhaust speed v_{ex}, and the final mass m in terms of the initial mass m_0.

SET UP: We'll use Eq. (8.40) directly to find the final speed v.

EXECUTE: We have $m_0/m = 4$, so from Eq. (8.40),

$$v = v_0 + v_{ex}\ln\frac{m_0}{m} = 0 + (2400\text{ m/s})(\ln 4) = 3327\text{ m/s}$$

EVALUATE: Let's examine what happens as the rocket gains speed. At the start of the flight, when the velocity of the rocket is zero, the ejected fuel is moving to the left, relative to our coordinate system, at 2400 m/s. At the end of the first second $(t = 1\text{ s})$, the rocket is moving at 20 m/s, and the fuel's speed relative to our system is 2380 m/s. During the next second the acceleration, given by Eq. (8.39), is a little greater. At $t = 2$ s, the rocket is moving a little faster than 40 m/s, and the fuel's speed is a little less than 2360 m/s. Detailed calculation shows that at about $t = 75.6$ s, the rocket's velocity v in our coordinate system equals 2400 m/s. The burned fuel ejected after this time moves *forward,* not backward, in our system. Since the final velocity of the rocket is 3327 m/s and the relative velocity is 2400 m/s, the last portion of the ejected fuel has a forward velocity (relative to our frame of reference) of $(3327 - 2400)\text{ m/s} = 927\text{ m/s}$. (To illustrate our point, we are using more figures than are significant.)

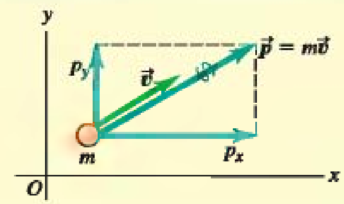
Test Your Understanding of Section 8.6 (a) If a rocket in gravity-free outer space has the same thrust at all times, is its acceleration constant, increasing, or decreasing? (b) If the rocket has the same acceleration at all times, is the thrust constant, increasing, or decreasing?

CHAPTER 8 SUMMARY

Momentum of a particle: The momentum $\vec{p}$ of a particle is a vector quantity equal to the product of the particle's mass m and velocity $\vec{v}$. Newton's second law says that the net force on a particle is equal to the rate of change of the particle's momentum.

$$\vec{p} = m\vec{v} \qquad (8.2)$$

$$\Sigma\vec{F} = \frac{d\vec{p}}{dt} \qquad (8.4)$$

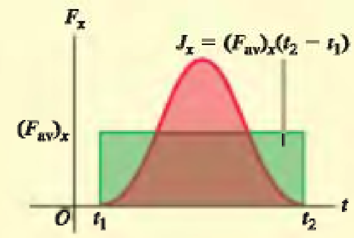


Impulse and momentum: If a constant net force $\Sigma\vec{F}$ acts on a particle for a time interval Δt from t_1 to t_2, the impulse $\vec{J}$ of the net force is the product of the net force and the time interval. If $\Sigma\vec{F}$ varies with time, $\vec{J}$ is the integral of the net force over the time interval. In any case, the change in a particle's momentum during a time interval equals the impulse of the net force that acted on the particle during that interval. The momentum of a particle equals the impulse that accelerated it from rest to its present speed. (See Examples 8.1–8.3.)

$$\vec{J} = \Sigma\vec{F}(t_2 - t_1) = \Sigma\vec{F}\,\Delta t \qquad (8.5)$$

$$\vec{J} = \int_{t_1}^{t_2} \Sigma\vec{F}\,dt \qquad (8.7)$$

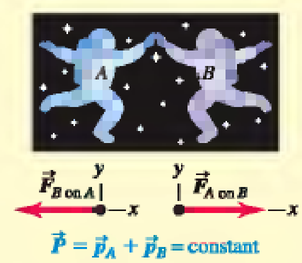
$$\vec{J} = \vec{p}_2 - \vec{p}_1 \qquad (8.6)$$



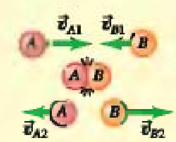
Conservation of momentum: An internal force is a force exerted by one part of a system on another. An external force is a force exerted on any part of a system by something outside the system. If the net external force on a system is zero, the total momentum of the system $\vec{P}$ (the vector sum of the momenta of the individual particles that make up the system) is constant, or conserved. Each component of total momentum is separately conserved. (See Examples 8.4–8.6)

$$\vec{P} = \vec{p}_A + \vec{p}_B + \cdots$$
$$= m_A\vec{v}_A + m_B\vec{v}_B + \cdots \qquad (8.14)$$

If $\Sigma\vec{F} = \mathbf{0}$, then $\vec{P}$ = constant.



Collisions: In collisions of all kinds, the initial and final total momenta are equal. In an elastic collision between two bodies, the initial and final total kinetic energies are also equal, and the initial and final relative velocities have the same magnitude. In an inelastic two-body collision, the total kinetic energy is less after the collision than before. If the two bodies have the same final velocity, the collision is completely inelastic. (See Examples 8.7–8.12.)

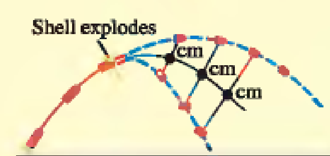


Center of mass: The position vector of the center of mass of a system of particles, $\vec{r}_{cm}$, is a weighted average of the positions $\vec{r}_1, \vec{r}_2, \ldots$ of the individual particles. The total momentum $\vec{P}$ of a system equals its total mass M multiplied by the velocity of its center of mass, $\vec{v}_{cm}$. The center of mass moves as though all the mass M were concentrated at that point. If the net external force on the system is zero, the center-of-mass velocity $\vec{v}_{cm}$ is constant. If the net external force is not zero, the center of mass accelerates as though it were a particle of mass M being acted on by the same net external force. (See Examples 8.13 and 8.14.)

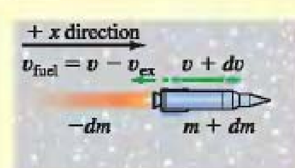
$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$
$$= \frac{\Sigma_i m_i\vec{r}_i}{\Sigma_i m_i} \qquad (8.29)$$

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots$$
$$= M\vec{v}_{cm} \qquad (8.32)$$

$$\Sigma\vec{F}_{ext} = M\vec{a}_{cm} \qquad (8.34)$$



Rocket propulsion: In rocket propulsion, the mass of a rocket changes as the fuel is used up and ejected from the rocket. Analysis of the motion of the rocket must include the momentum carried away by the spent fuel as well as the momentum of the rocket itself. (See Examples 8.15 and 8.16.)



Key Terms

momentum (linear momentum), *2*
impulse, *3*
impulse–momentum theorem, *3*
internal force, *7*
external force, *7*
isolated system, *7*
total momentum, *7*
principle of conservation of momentum, *8*
elastic collision, *12*
inelastic collision, *12*
completely inelastic collision, *12*
center of mass, *20*

Answer to Chapter Opening Question

The two players have the same magnitude of momentum $p = mv$ (the product of mass and speed), but the faster, lightweight player has twice as much kinetic energy $K = \frac{1}{2}mv^2$. Hence, the lightweight player can do twice as much work on you (and twice as much damage) in the process of coming to a halt (see Section 8.1).

Answers to Test Your Understanding Questions

8.1 Answer: (v), (i) and (ii) (tied for second place), (iii) and (iv) (tied for third place) We use two interpretations of the impulse of the net force: (1) the net force multiplied by the time that the net force acts, and (2) the change in momentum of the particle on which the net force acts. Which interpretation we use depends on what information we are given. We take the positive x-direction to be to the east. (i) The force is not given, so we use interpretation 2: $J_x = mv_{2x} - mv_{1x} = (1000\,\text{kg})(0) - (1000\,\text{kg})(25\,\text{m/s}) = -25{,}000\,\text{kg}\cdot\text{m/s}$, so the magnitude of the impulse is $25{,}000\,\text{kg}\cdot\text{m/s} = 25{,}000\,\text{N}\cdot\text{s}$. (ii) For the same reason as in (i), we use interpretation 2: $J_x = mv_{2x} - mv_{1x} = (1000\,\text{kg})(0) - (1000\,\text{kg})(25\,\text{m/s}) = -25{,}000\,\text{kg}\cdot\text{m/s}$, and the magnitude of the impulse is again $25{,}000\,\text{kg}\cdot\text{m/s} = 25{,}000\,\text{N}\cdot\text{s}$. (iii) The final velocity is not given, so we use interpretation 1: $J_x = (\sum F_x)_{\text{av}}(t_2 - t_1) = (2000\,\text{N})(10\,\text{s}) = 20{,}000\,\text{N}\cdot\text{s}$, so the magnitude of the impulse is $20{,}000\,\text{N}\cdot\text{s}$. (iv) For the same reason as in (iii), we use interpretation 1: $J_x = (\sum F_x)_{\text{av}}(t_2 - t_1) = (-2000\,\text{N})(10\,\text{s}) = -20{,}000\,\text{N}\cdot\text{s}$, so the magnitude of the impulse is $20{,}000\,\text{N}\cdot\text{s}$. (v) The force is not given, so we use interpretation 2: $J_x = mv_{2x} - mv_{1x} = (1000\,\text{kg})(-25\,\text{m/s}) - (1000\,\text{kg})(25\,\text{m/s}) = -50{,}000\,\text{kg}\cdot\text{m/s}$, so the magnitude of the impulse is $50{,}000\,\text{kg}\cdot\text{m/s} = 50{,}000\,\text{N}\cdot\text{s}$.

8.2 Answers: (a) $v_{C2x} > 0$, $v_{C2y} > 0$, (b) piece C There are no external horizontal forces, so the x- and y-components of the total momentum of the system are both conserved. Both components of the total momentum are zero before the spring releases, so they must be zero after the spring releases. Hence

$$P_x = 0 = m_A v_{A2x} + m_B v_{B2x} + m_C v_{C2x}$$
$$P_y = 0 = m_A v_{A2y} + m_B v_{B2y} + m_C v_{C2y}$$

We are given that $m_A = m_B = m_C$, $v_{A2x} < 0$, $v_{A2y} = 0$, $v_{B2x} = 0$, and $v_{B2y} < 0$. You can solve the above equations to show that $v_{C2x} = -v_{A2x} > 0$ and $v_{C2y} = -v_{B2y} > 0$, so the velocity components of piece C are both positive. Piece C has speed $\sqrt{v_{C2x}{}^2 + v_{C2y}{}^2} = \sqrt{v_{A2x}{}^2 + v_{B2y}{}^2}$, which is greater than the speed of either piece A or piece B.

8.3 Answers: (a) inelastic, (b) elastic, (c) completely inelastic In each case gravitational potential energy is converted to kinetic energy as the ball falls, and the collision is between the ball and the ground. In (a) all of the initial energy is converted back to gravitational potential energy, so no kinetic energy is lost in the bounce and the collision is elastic. In (b) there is less gravitational potential energy at the end than at the beginning, so some kinetic energy was lost in the bounce. Hence the collision is inelastic. In (c) the ball loses all the kinetic energy it has to give, the ball and the ground stick together, and the collision is completely inelastic.

8.4 Answer: worse After a collision with a water molecule initially at rest, the speed of the neutron is $|(m_n - m_w)/(m_n + m_w)| = |(1.0\,\text{u} - 18\,\text{u})/(1.0\,\text{u} + 18\,\text{u})| = \frac{17}{19}$ of its initial speed, and its kinetic energy is $(\frac{17}{19})^2 = 0.80$ of the initial value. Hence a water molecule is a worse moderator than a carbon atom, for which the corresponding numbers are $\frac{11}{13}$ and $(\frac{11}{13})^2 = 0.72$.

8.5 Answer: no If gravity is the only force acting on the system of two fragments, the center of mass will follow the parabolic trajectory of a freely falling object. Once a fragment lands, however, the ground exerts a normal force on that fragment. Hence the net force on the system has changed, and the trajectory of the center of mass changes in response.

8.6 Answers: (a) increasing, (b) decreasing From Eqs. (8.37) and (8.38), the thrust F is equal to $m(dv/dt)$, where m is the rocket's mass and dv/dt is its acceleration. Because m decreases with time, if the thrust F is constant, then the acceleration must increase with time (the same force acts on a smaller mass); if the acceleration dv/dt is constant, then the thrust must decrease with time (a smaller force is all that's needed to accelerate a smaller mass).

PROBLEMS

For instructor-assigned homework, go to www.masteringphysics.com

Discussion Questions

Q8.1. In splitting logs with a hammer and wedge, is a heavy hammer more effective than a lighter hammer? Why?

Q8.2. Suppose you catch a baseball and then someone invites you to catch a bowling ball with either the same momentum or the same kinetic energy as the baseball. Which would you choose? Explain.

Q8.3. When rain falls from the sky, what happens to its momentum as it hits the ground? Is your answer also valid for Newton's famous apple?

Q8.4. A car has the same kinetic energy when it is traveling south at 30 m/s as when it is traveling northwest at 30 m/s. Is the momentum of the car the same in both cases? Explain.

Q8.5. A truck is accelerating as it speeds down the highway. One inertial frame of reference is attached to the ground with its origin at a fence post. A second frame of reference is attached to a police car that is traveling down the highway at constant velocity. Is the momentum of the truck the same in these two reference frames? Explain. Is the rate of change of the truck's momentum the same in these two frames? Explain.

Q8.6. When a large, heavy truck collides with a passenger car, the occupants of the car are more likely to be hurt than the truck driver. Why?

Q8.7. A woman holding a large rock stands on a frictionless, horizontal sheet of ice. She throws the rock with speed v_0 at an angle α above the horizontal. Consider the system consisting of the woman plus the rock. Is the momentum of the system conserved? Why or why not? Is any component of the momentum of the system conserved? Again, why or why not?

Q8.8. In Example 8.7 (Section 8.3), where the two gliders in Fig. 8.15 a stick together after the collision, the collision is inelastic because $K_2 < K_1$. In Example 8.5 (Section 8.2), is the collision inelastic? Explain.

Q8.9. In a completely inelastic collision between two objects, where the objects stick together after the collision, is it possible for the final kinetic energy of the system to be zero? If so, give an example in which this would occur. If the final kinetic energy is zero, what must the initial momentum of the system be? Is the initial kinetic energy of the system zero? Explain.

Q8.10. Since for a particle the kinetic energy is given by $K = \frac{1}{2}mv^2$ and the momentum by $\vec{p} = m\vec{v}$, it is easy to show that $K = p^2/2m$. How, then, is it possible to have an event during which the total momentum of the system is constant but the total kinetic energy changes?

Q8.11. In each of Examples 8.10, 8.11, and 8.12 (Section 8.4), verify that the relative velocity vector of the two bodies has the same magnitude before and after the collision. In each case what happens to the *direction* of the relative velocity vector?

Q8.12. A glass dropped on the floor is more likely to break if the floor is concrete than if it is wood. Why? (Refer to Fig. 8.3b.)

Q8.13. In Fig. 8.22b, the kinetic energy of the Ping-Pong ball is larger after its interaction with the bowling ball than before. From where does the extra energy come? Describe the event in terms of conservation of energy.

Q8.14. A machine gun is fired at a steel plate. Is the average force on the plate from the bullet impact greater if the bullets bounce off or if they are squashed and stick to the plate? Explain.

Q8.15. A net force of 4 N acts on an object initially at rest for 0.25 s and gives it a final speed of 5 m/s. How could a net force of 2 N produce the same final speed?

Q8.16. A net force with x-component $\sum F_x$ acts on an object from time t_1 to time t_2. The x-component of the momentum of the object is the same at t_1 as it is at t_2, but $\sum F_x$ is not zero at all times between t_1 and t_2. What can you say about the graph of $\sum F_x$ versus t?

Q8.17. A tennis player hits a tennis ball with a racket. Consider the system made up of the ball and the racket. Is the total momentum of the system the same just before and just after the hit? Is the total momentum just after the hit the same as 2 s later, when the ball is in midair at the high point of its trajectory? Explain any differences between the two cases.

Q8.18. In Example 8.4 (Section 8.2), consider the system consisting of the rifle plus the bullet. What is the speed of the system's center of mass after the rifle is fired? Explain.

Q8.19. An egg is released from rest from the roof of a building and falls to the ground. As the egg falls, what happens to the momentum of the system of the egg plus the earth?

Q8.20. A woman stands in the middle of a perfectly smooth, frictionless, frozen lake. She can set herself in motion by throwing things, but suppose she has nothing to throw. Can she propel herself to shore *without* throwing anything?

Q8.21. In a zero-gravity environment, can a rocket-propelled spaceship ever attain a speed greater than the relative speed with which the burnt fuel is exhausted?

Q8.22. When an object breaks into two pieces (explosion, radioactive decay, recoil, etc.), the lighter fragment gets more kinetic energy than the heavier one. This is a consequence of momentum conservation, but can you also explain it using Newton's laws of motion?

Q8.23. An apple falls from a tree and feels no air resistance. As it is falling, which of these statements about it are true? (a) Only its momentum is conserved; (b) only its mechanical energy is conserved, (c) both its momentum and its mechanical energy are conserved, (d) its kinetic energy is conserved.

Q8.24. Two pieces of clay collide and stick together. During the collision, which of these statements are true? (a) Only the momentum of the clay is conserved, (b) only the mechanical energy of the clay is conserved, (c) both the momentum and the mechanical energy of the clay are conserved, (d) the kinetic energy of the clay is conserved.

Q8.25. Two marbles are pressed together with a light ideal spring between them, but they are not attached to the spring in any way. They are then released on a frictionless horizontal table and soon move free of the spring. As the marbles are moving away from each other, which of these statements about them are true? (a) Only the momentum of the marbles is conserved, (b) only the mechanical energy of the marbles is conserved, (c) both the momentum and the mechanical energy of the marbles are conserved, (d) the kinetic energy of the marbles is conserved.

Q8.26. A very heavy SUV collides head-on with a very light compact car. Which of these statements about the collision are correct? (a) The amount of kinetic energy lost by the SUV is equal to the amount of kinetic energy gained by the compact, (b) the amount of momentum lost by the SUV is equal to the amount of momentum gained by the compact, (c) The compact feels a considerably greater force during the collision than the SUV does, (d) both cars lose the same amount of kinetic energy.

Exercises

Section 8.1 Momentum and Impulse

8.1. (a) What is the magnitude of the momentum of a 10,000-kg truck whose speed is 12.0 m/s? (b) What speed would a 2,000-kg SUV have to attain in order to have (i) the same momentum? (ii) the same kinetic energy?

8.2. In Conceptual Example 8.1 (Section 8.1), show that the iceboat with mass $2m$ has $\sqrt{2}$ times as much momentum at the finish line as does the iceboat with mass m.

8.3. (a) Show that the kinetic energy K and the momentum magnitude p of a particle with mass m are related by $K = p^2/2m$. (b) A 0.040-kg cardinal (*Richmondena cardinalis*) and a 0.145-kg baseball have the same kinetic energy. Which has the greater magnitude of momentum? What is the ratio of the cardinal's magnitude of momentum to the baseball's? (c) A 700-N man and a 450-N woman have the same momentum. Who has the greater kinetic energy? What is the ratio of the man's kinetic energy to that of the woman?

8.4. In a certain men's track and field event, the shotput has a mass of 7.30 kg and is released with a speed of 15.0 m/s at 40.0° above the horizontal over a man's straight left leg. What are the initial horizontal and vertical components of the momentum of this shotput?

8.5. One 110-kg football lineman is running to the right at 2.75 m/s while another 125-kg lineman is running directly toward him at

2.60 m/s. What are (a) the magnitude and direction of the net momentum of these two athletes, and (b) their total kinetic energy?

8.6. Two vehicles are approaching an intersection. One is a 2500-kg pickup traveling at 14.0 m/s from east to west (the $-x$-direction), and the other is a 1500-kg sedan going from south to north (the $+y$-direction at 23.0 m/s). (a) Find the x- and y-components of the net momentum of this system. (b) What are the magnitude and direction of the net momentum?

8.7. Force of a Golf Swing. A 0.0450-kg golf ball initially at rest is given a speed of 25.0 m/s when a club strikes. If the club and ball are in contact for 2.00 ms, what average force acts on the ball? Is the effect of the ball's weight during the time of contact significant? Why or why not?

8.8. Force of a Baseball Swing. A baseball has mass 0.145 kg. (a) If the velocity of a pitched ball has a magnitude of 45.0 m/s and the batted ball's velocity is 55.0 m/s in the opposite direction, find the magnitude of the change in momentum of the ball and of the impulse applied to it by the bat. (b) If the ball remains in contact with the bat for 2.00 ms, find the magnitude of the average force applied by the bat.

8.9. A 0.160-kg hockey puck is moving on an icy, frictionless, horizontal surface. At $t = 0$, the puck is moving to the right at 3.00 m/s. (a) Calculate the velocity of the puck (magnitude and direction) after a force of 25.0 N directed to the right has been applied for 0.050 s. (b) If, instead, a force of 12.0 N directed to the left is applied from $t = 0$ to $t = 0.050$ s, what is the final velocity of the puck?

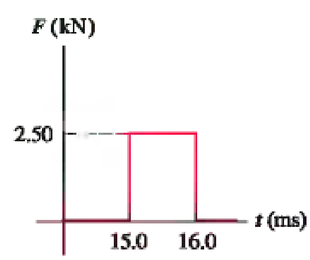
8.19. An engine of the orbital maneuvering system (OMS) on a space shuttle exerts a force of $(26{,}700\text{ N})\hat{j}$ for 3.90 s, exhausting a negligible mass of fuel relative to the 95,000-kg mass of the shuttle. (a) What is the impulse of the force for this 3.90 s? (b) What is the shuttle's change in momentum from this impulse? (c) What is the shuttle's change in velocity from this impulse? (d) Why can't we find the resulting change in the kinetic energy of the shuttle?

8.11. At time $t = 0$, a 2150-kg rocket in outer space fires an engine that exerts an increasing force on it in the $+x$-direction. This force obeys the equation $F_x = At^2$, where t is time, and has a magnitude of 781.25 N when $t = 1.25$ s. (a) Find the SI value of the constant A, including its units. (b) What impulse does the engine exert on the rocket during the 1.50-s interval starting 2.00 s after the engine is fired? (c) By how much does the rocket's velocity change during this interval?

8.12. A bat strikes a 0.145-kg baseball. Just before impact, the ball is traveling horizontally to the right at 50.0 m/s, and it leaves the bat traveling to the left at an angle of 30° above horizontal with a speed of 65.0 m/s. If the ball and bat are in contact for 1.75 ms, find the horizontal and vertical components of the average force on the ball.

8.13. A 2.00-kg stone is sliding to the right on a frictionless horizontal surface at 5.00 m/s when it is suddenly struck by an object that exerts a large horizontal force on it for a short period of time. The graph in Fig. 8.34 shows the magnitude of this force as a function of time. (a) What impulse does this force exert on the stone? (b) Just after the force stops acting, find the magnitude and direction of the stone's velocity if the force acts (i) to the right or (ii) to the left.

Figure 8.34 Exercise 8.13.



Section 8.2 Conservation of Momentum

8.14. A 68.5-kg astronaut is doing a repair in space on the orbiting space station. She throws a 2.25-kg tool away from her at 3.20 m/s relative to the space station. With what speed and in what direction will she begin to move?

8.15. Animal Propulsion. Squids and octopuses propel themselves by expelling water. They do this by keeping water in a cavity and then suddenly contracting the cavity to force out the water through an opening. A 6.50-kg squid (including the water in the cavity) at rest suddenly sees a dangerous predator. (a) If the squid has 1.75 kg of water in its cavity, at what speed must it expel this water to suddenly achieve a speed of 2.50 m/s to escape the predator? Neglect any drag effects of the surrounding water. (b) How much kinetic energy does the squid create by this maneuver?

8.16. You are standing on a sheet of ice that covers the football stadium parking lot in Buffalo; there is negligible friction between your feet and the ice. A friend throws you a 0.400-kg ball that is traveling horizontally at 10.0 m/s. Your mass is 70.0 kg. (a) If you catch the ball, with what speed do you and the ball move afterward? (b) If the ball hits you and bounces off your chest, so afterward it is moving horizontally at 8.0 m/s in the opposite direction, what is your speed after the collision?

8.17. On a frictionless, horizontal air table, puck A (with mass 0.250 kg) is moving toward puck B (with mass 0.350 kg), which is initially at rest. After the collision, puck A has a velocity of 0.120 m/s to the left, and puck B has a velocity of 0.650 m/s to the right. (a) What was the speed of puck A before the collision? (b) Calculate the change in the total kinetic energy of the system that occurs during the collision.

8.18. When cars are equipped with flexible bumpers, they will bounce off each other during low-speed collisions, thus causing less damage. In one such accident, a 1750-kg car traveling to the right at 1.50 m/s collides with a 1450-kg car going to the left at 1.10 m/s. Measurements show that the heavier car's speed just after the collision was 0.250 m/s in its original direction. You can ignore any road friction during the collision. (a) What was the speed of the lighter car just after the collision? (b) Calculate the change in the combined kinetic energy of the two-car system during this collision.

8.19. The expanding gases that leave the muzzle of a rifle also contribute to the recoil. A .30-caliber bullet has mass 0.00720 kg and a speed of 601 m/s relative to the muzzle when fired from a rifle that has mass 2.80 kg. The loosely held rifle recoils at a speed of 1.85 m/s relative to the earth. Find the momentum of the propellant gases in a coordinate system attached to the earth as they leave the muzzle of the rifle.

8.20. Block A in Fig. 8.35 has mass 1.00 kg, and block B has mass 3.00 kg. The blocks are forced together, compressing a spring S between them; then the system is released from rest on a level, frictionless surface. The spring, which has negligible mass, is not fastened to either block and drops to the surface after it has expanded. Block B acquires a speed of 1.20 m/s. (a) What is the final speed of block A? (b) How much potential energy was stored in the compressed spring?

Figure 8.35 Exercise 8.20.

8.21. A hunter on a frozen, essentially frictionless pond uses a rifle that shoots 4.20-g bullets at 965 m/s. The mass of the hunter (including his gun) is 72.5 kg, and the hunter holds tight to the gun after firing it. Find the recoil velocity of the hunter if he fires the rifle (a) horizontally and (b) at 56.0° above the horizontal.

8.22. An atomic nucleus suddenly bursts apart (fissions) into two pieces. Piece A, of mass m_A, travels off to the left with speed v_A. Piece B, of mass m_B, travels off to the right with speed v_B. (a) Use conservation of momentum to solve for v_B in terms of m_A, m_B, and v_A. (b) Use the results of part (a) to show that $K_A/K_B = m_B/m_A$, where K_A and K_B are the kinetic energies of the two pieces.

8.23. The nucleus of ^{214}Po decays radioactively by emitting an alpha particle (mass 6.65×10^{-27} kg) with kinetic energy 1.23×10^{-12} J, as measured in the laboratory reference frame. Assuming that the Po was initially at rest in this frame, find the recoil velocity of the nucleus that remains after the decay.

8.24. You are standing on a large sheet of frictionless ice and holding a large rock. In order to get off the ice, you throw the rock so it has velocity 12.0 m/s relative to the earth at an angle of 35.0° above the horizontal. If your mass is 70.0 kg and the rock's mass is 15.0 kg, what is your speed after you throw the rock (see Discussion Question Q8.7)?

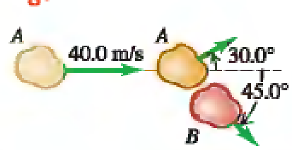
8.25. Two ice skaters, Daniel (mass 65.0 kg) and Rebecca (mass 45.0 kg), are practicing. Daniel stops to tie his shoelace and, while at rest, is struck by Rebecca, who is moving at 13.0 m/s before she collides with him. After the collision, Rebecca has a velocity of magnitude 8.00 m/s at an angle of 53.1° from her initial direction. Both skaters move on the frictionless, horizontal surface of the rink. (a) What are the magnitude and direction of Daniel's velocity after the collision? (b) What is the change in total kinetic energy of the two skaters as a result of the collision?

8.26. An astronaut in space cannot use a scale or balance to weigh objects because there is no gravity. But she does have devices to measure distance and time accurately. She knows her own mass is 78.4 kg, but she is unsure of the mass of a large gas canister in the airless rocket. When this canister is approaching her at 3.50 m/s, she pushes against it, which slows it down to 1.20 m/s (but does not reverse it) and gives her a speed of 2.40 m/s. What is the mass of this canister?

8.27. Changing Mass. An open-topped freight car with mass 24,000 kg is coasting without friction along a level track. It is raining very hard, and the rain is falling vertically downward. Originally, the car is empty and moving with a speed of 4.00 m/s. What is the speed of the car after it has collected 3000 kg of rainwater?

8.28. Asteroid Collision. Two asteroids of equal mass in the asteroid belt between Mars and Jupiter collide with a glancing blow. Asteroid A, which was initially traveling at 40.0 m/s, is deflected 30.0° from its original direction, while asteroid B travels at 45.0° to the original direction of A (Fig. 8.36). (a) Find the speed of each asteroid after the collision. (b) What fraction of the original kinetic energy of asteroid A dissipates during this collision?

Figure 8.36 Exercise 8.28.



Section 8.3 Momentum Conservation and Collisions

8.29. A 15.0-kg fish swimming at 1.10 m/s suddenly gobbles up a 4.50-kg fish that is initially stationary. Neglect any drag effects of the water. (a) Find the speed of the large fish just after it eats the small one. (b) How much mechanical energy was dissipated during this meal?

8.30. Two fun-loving otters are sliding toward each other on a muddy (and hence frictionless) horizontal surface. One of them, of mass 7.50 kg, is sliding to the left at 5.00 m/s, while the other, of mass 5.75 kg, is slipping to the right at 6.00 m/s. They hold fast to each other after they collide. (a) Find the magnitude and direction of the velocity of these free-spirited otters right after they collide. (b) How much mechanical energy dissipates during this play?

8.31. Deep Impact Mission. In July 2005, NASA's "Deep Impact" mission crashed a 372-kg probe directly onto the surface of the comet Tempel 1, hitting the surface at 37,000 km/h. The original speed of the comet at that time was about 40,000 km/h, and its mass was estimated to be in the range $(0.10–2.5) \times 10^{14}$ kg. Use the smallest value of the estimated mass. (a) What change in the comet's velocity did this collision produce? Would this change be noticeable? (b) Suppose this comet were to hit the earth and fuse with it. By how much would it change our planet's velocity? Would this change be noticeable? (The mass of the earth is 5.97×10^{24} kg.)

8.32. A 1050-kg sports car is moving westbound at 15.0 m/s on a level road when it collides with a 6320-kg truck driving east on the same road at 10.0 m/s. The two vehicles remain locked together after the collision. (a) What is the velocity (magnitude and direction) of the two vehicles just after the collision? (b) At what speed should the truck have been moving so that it and car are both stopped in the collision? (c) Find the change in kinetic energy of the system of two vehicles for the situations of part (a) and part (b). For which situation is the change in kinetic energy greater in magnitude?

8.33. On a very muddy football field, a 110-kg linebacker tackles an 85-kg halfback. Immediately before the collision, the linebacker is slipping with a velocity of 8.8 m/s north and the halfback is sliding with a velocity of 7.2 m/s east. What is the velocity (magnitude and direction) at which the two players move together immediately after the collision?

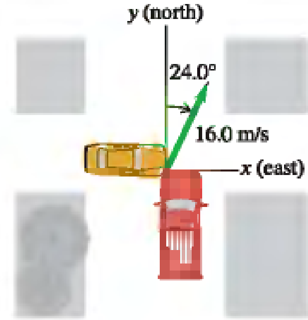
8.34. Two skaters collide and grab on to each other on frictionless ice. One of them, of mass 70.0 kg, is moving to the right at 2.00 m/s, while the other, of mass 65.0 kg, is moving to the left at 2.50 m/s. What are the magnitude and direction of the velocity of these skaters just after they collide?

8.35. Two cars, one a compact with mass 1200 kg and the other a large gas-guzzler with mass 3000 kg, collide head-on at typical freeway speeds. (a) Which car has a greater magnitude of momentum change? Which car has a greater velocity change? (b) If the larger car changes its velocity by Δv, calculate the change in the velocity of the small car in terms of Δv. (c) Which car's occupants would you expect to sustain greater injuries? Explain.

8.36. Bird Defense. To protect their young in the nest, peregrine falcons will fly into birds of prey (such as ravens) at high speed. In one such episode, a 600-g falcon flying at 20.0 m/s hit a 1.50-kg raven flying at 9.0 m/s. The falcon hit the raven at right angles to its original path and bounced back at 5.0 m/s. (These figures were estimated by the author as he watched this attack occur in northern New Mexico.) (a) By what angle did the falcon change the raven's direction of motion? (b) What was the raven's speed right after the collision?

8.37. At the intersection of Texas Avenue and University Drive, a yellow subcompact car with mass 950 kg traveling east on University collides with a red pickup truck with mass 1900 kg that is traveling north on Texas and ran a red light (Fig. 8.37). The two vehicles stick together as a result of the collision, and the wreckage slides at 16.0 m/s in the direction 24.0° east of north. Calculate the

Figure 8.37 Exercise 8.37.



speed of each vehicle before the collision. The collision occurs during a heavy rainstorm; you can ignore friction forces between the vehicles and the wet road.

8.38. A 5.00-g bullet is fired horizontally into a 1.20-kg wooden block resting on a horizontal surface. The coefficient of kinetic friction between block and surface is 0.20. The bullet remains embedded in the block, which is observed to slide 0.230 m along the surface before stopping. What was the initial speed of the bullet?

8.39. A Ballistic Pendulum. A 12.0-g rifle bullet is fired with a speed of 380 m/s into a ballistic pendulum with mass 6.00 kg, suspended from a cord 70.0 cm long (see Example 8.8 in Section 8.3). Compute (a) the vertical height through which the pendulum rises, (b) the initial kinetic energy of the bullet, and (c) the kinetic energy of the bullet and pendulum immediately after the bullet becomes embedded in the pendulum.

8.40. You and your friends are doing physics experiments on a frozen pond that serves as a frictionless, horizontal surface. Sam, with mass 80.0 kg, is given a push and slides eastward. Abigail, with mass 50.0 kg, is sent sliding northward. They collide, and after the collision Sam is moving at 37.0° north of east with a speed of 6.00 m/s and Abigail is moving at 23.0° south of east with a speed of 9.00 m/s. (a) What was the speed of each person before the collision? (b) By how much did the total kinetic energy of the two people decrease during the collision?

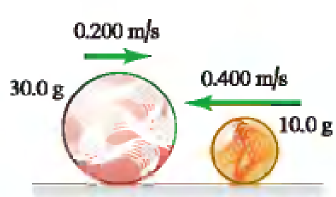
Section 8.4 Elastic Collisions

8.41. Blocks A (mass 2.00 kg) and B (mass 10.00 kg) move on a frictionless, horizontal surface. Initially, block B is at rest and block A is moving toward it at 2.00 m/s. The blocks are equipped with ideal spring bumpers, as in Example 8.10. The collision is head-on, so all motion before and after the collision is along a straight line. (a) Find the maximum energy stored in the spring bumpers and the velocity of each block at that time. (b) Find the velocity of each block after they have moved apart.

8.42. A 0.150-kg glider is moving to the right on a frictionless, horizontal air track with a speed of 0.80 m/s. It has a head-on collision with a 0.300-kg glider that is moving to the left with a speed of 2.20 m/s. Find the final velocity (magnitude and direction) of each glider if the collision is elastic.

8.43. A 10.0-g marble slides to the left with a velocity of magnitude 0.400 m/s on the frictionless, horizontal surface of an icy New York sidewalk and has a head-on, elastic collision with a larger 30.0-g marble sliding to the right with a velocity of magnitude 0.200 m/s (Fig. 8.38). (a) Find the velocity of each marble (magnitude and direction) after the collision. (Since the collision is head-on, all the motion is along a line.) (b) Calculate the *change in momentum* (that is, the momentum after the collision minus the momentum before the collision) for each marble. Compare the values you get for each marble. (c) Calculate the *change in kinetic energy* (that is, the kinetic energy after the collision minus the kinetic energy before the collision) for each marble. Compare the values you get for each marble.

Figure 8.38 Exercise 8.43.



8.44. Supply the details of the calculation of α and β in Example 8.12 (Section 8.4).

8.45. Moderators. Canadian nuclear reactors use *heavy water* moderators in which elastic collisions occur between the neutrons and deuterons of mass 2.0 u (see Example 8.11 in Section 8.4). (a) What is the speed of a neutron, expressed as a fraction of its original speed, after a head-on, elastic collision with a deuteron that is initially at rest? (b) What is its kinetic energy, expressed as a fraction of its original kinetic energy? (c) How many such successive collisions will reduce the speed of a neutron to 1/59,000 of its original value?

8.46. You are at the controls of a particle accelerator, sending a beam of 1.50×10^7 m/s protons (mass m) at a gas target of an unknown element. Your detector tells you that some protons bounce straight back after a collision with one of the nuclei of the unknown element. All such protons rebound with a speed of 1.20×10^7 m/s. Assume that the initial speed of the target nucleus is negligible and the collision is elastic. (a) Find the mass of one nucleus of the unknown element. Express your answer in terms of the proton mass m. (b) What is the speed of the unknown nucleus immediately after such a collision?

Section 8.5 Center of Mass

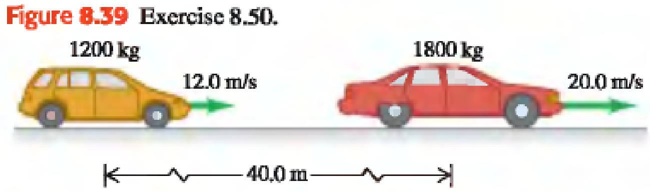
8.47. Three odd-shaped blocks of chocolate have the following masses and center-of-mass coordinates: (1) 0.300 kg, $(0.200\ \text{m}, 0.300\ \text{m})$; (2) 0.400 kg, $(0.100\ \text{m}, -0.400\ \text{m})$; (3) 0.200 kg, $(-0.300\ \text{m}, 0.600\ \text{m})$. Find the coordinates of the center of mass of the system of three chocolate blocks.

8.48. Find the position of the center of mass of the system of the sun and Jupiter. (Since Jupiter is more massive than the rest of the planets combined, this is essentially the position of the center of mass of the solar system.) Does the center of mass lie inside or outside the sun? Use the data in Appendix F.

8.49. Pluto and Charon. Pluto's diameter is approximately 2370 km, and the diameter of its satellite Charon is 1250 km. Although the distance varies, they are often about 19,700 km apart, center-to-center. Assuming that both Pluto and Charon have the same composition and hence the same average density, find the location of the center of mass of this system relative to the center of Pluto.

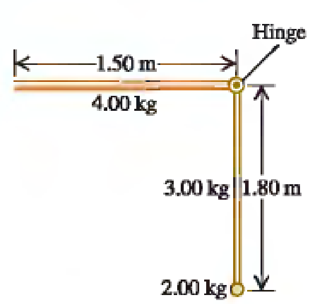
8.50. A 1200-kg station wagon is moving along a straight highway at 12.0 m/s. Another car, with mass 1800 kg and speed 20.0 m/s, has its center of mass 40.0 m ahead of the center of mass of the station wagon (Fig. 8.39). (a) Find the position of the center of mass of the system consisting of the two automobiles. (b) Find the magnitude of the total momentum of the system from the given data. (c) Find the speed of the center of mass of the system. (d) Find the total momentum of the system, using the speed of the center of mass. Compare your result with that of part (b).

Figure 8.39 Exercise 8.50.



8.51. A machine part consists of a thin, uniform 4.00-kg bar that is 1.50 m long, hinged perpendicular to a similar vertical bar of mass 3.00 kg and length 1.80 m. The longer bar has a small but dense 2.00-kg ball at one end (Fig. 8.40). By what distance will the center of mass of this part move horizontally and vertically if the vertical bar is pivoted counterclockwise through 90° to make the entire part horizontal?

Figure 8.40 Exercise 8.51.



8.52. At one instant, the center of mass of a system of two particles is located on the x-axis at $x = 2.0\text{ m}$ and has a velocity of $(5.0\text{ m/s})\hat{\imath}$. One of the particles is at the origin. The other particle has a mass of 0.10 kg and is at rest on the x-axis at $x = 8.0\text{ m}$. (a) What is the mass of the particle at the origin? (b) Calculate the total momentum of this system. (c) What is the velocity of the particle at the origin?

8.53. In Example 8.14 (Section 8.5), Ramon pulls on the rope to give himself a speed of 0.70 m/s. What is James's speed?

8.54. A system consists of two particles. At $t = 0$ one particle is at the origin; the other, which has a mass of 0.50 kg, is on the y-axis at $y = 6.0\text{ m}$. At $t = 0$ the center of mass of the system is on the y-axis at $y = 2.4\text{ m}$. The velocity of the center of mass is given by $(0.75\text{ m/s}^3)t^2\hat{\imath}$. (a) Find the total mass of the system. (b) Find the acceleration of the center of mass at any time t. (c) Find the net external force acting on the system at $t = 3.0\text{ s}$.

8.55. A radio-controlled model airplane has a momentum given by $[(-0.75\text{ kg}\cdot\text{m/s}^3)t^2 + (3.0\text{ kg}\cdot\text{m/s})]\hat{\imath} + (0.25\text{ kg}\cdot\text{m/s}^2)t\hat{\jmath}$. What are the x-, y-, and z-components of the net force on the airplane?

*Section 8.6 Rocket Propulsion

***8.56.** A small rocket burns 0.0500 kg of fuel per second, ejecting it as a gas with a velocity relative to the rocket of magnitude 1600 m/s. (a) What is the thrust of the rocket? (b) Would the rocket operate in outer space where there is no atmosphere? If so, how would you steer it? Could you brake it?

***8.57.** A 70-kg astronaut floating in space in a 110-kg MMU (manned maneuvering unit) experiences an acceleration of 0.029 m/s^2 when he fires one of the MMU's thrusters. (a) If the speed of the escaping N_2 gas relative to the astronaut is 490 m/s, how much gas is used by the thruster in 5.0 s? (b) What is the thrust of the thruster?

***8.58.** A rocket is fired in deep space, where gravity is negligible. If the rocket has an initial mass of 6000 kg and ejects gas at a relative velocity of magnitude 2000 m/s, how much gas must it eject in the first second to have an initial acceleration of 25.0 m/s^2?

***8.58.** A rocket is fired in deep space, where gravity is negligible. In the first second it ejects $\frac{1}{160}$ of its mass as exhaust gas and has an acceleration of 15.0 m/s^2. What is the speed of the exhaust gas relative to the rocket?

***8.60.** A C6-5 model rocket engine has an impnlse of $10.0\text{ N}\cdot\text{s}$ for 1.70 s, while burning 0.0125 kg of propellant. It has a maximum thrust of 13.3 N. The initial mass of the engine plus propellant is 0.0258 kg. (a) What fraction of the maximum thrust is the average thrust? (b) Calculate the relative speed of the exhaust gases, assuming it is constant. (c) Assuming that the relative speed of the exhaust gases is constant, find the final speed of the engine if it was attached to a very light frame and fired from rest in gravity-free outer space.

***8.61.** A single-stage rocket is fired from rest from a deep-space platform, where gravity is negligible. If the rocket burns its fuel in 50.0 s and the relative speed of the exhaust gas is $v_{ex} = 2100\text{ m/s}$, what must the mass ratio m_0/m be for a final speed v of 8.00 km/s (about equal to the orbital speed of an earth satellite)?

***8.62.** Obviously, we can make rockets to go very fast, but what is a reasonable top speed? Assume that a rocket is fired from rest at a space station in deep space, where gravity is negligible. (a) If the rocket ejects gas at a relative speed of 2000 m/s and you want the rocket's speed eventually to be $1.00 \times 10^{-3}c$, where c is the speed of light, what fraction of the initial mass of the rocket and fuel is *not* fuel? (b) What is this fraction if the final speed is to be 3000 m/s?

Problems

8.63. A steel ball with mass 40.0 g is dropped from a height of 2.00 m onto a horizontal steel slab. The ball rebounds to a height of 1.60 m. (a) Calculate the impulse delivered to the ball during impact. (b) If the ball is in contact with the slab for 2.00 ms, find the average force on the ball during impact.

8.64. In a volcanic eruption, a 2400-kg boulder is thrown vertically upward into the air. At its highest point, it suddenly explodes (due to trapped gases) into two fragments, one being three times the mass of the other. The lighter fragment starts out with only horizontal velocity and lands 274 m directly north of the point of the explosion. Where will the other fragment land? Neglect any air resistance.

8.65. Just before it is struck by a racket, a tennis ball weighing 0.560 N has a velocity of $(20.0\text{ m/s})\hat{\imath} - (4.0\text{ m/s})\hat{\jmath}$. During the 3.00 ms that the racket and ball are in contact, the net force on the ball is constant and equal to $-(380\text{ N})\hat{\imath} + (110\text{ N})\hat{\jmath}$. (a) What are the x- and y-components of the impulse of the net force applied to the ball? (b) What are the x- and y-components of the final velocity of the ball?

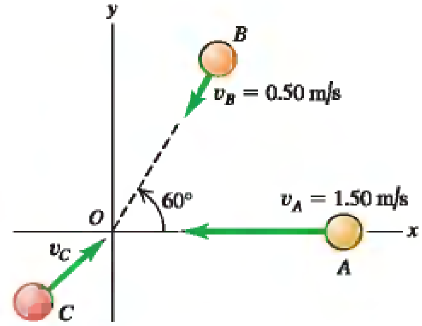
8.66. Three coupled railroad cars roll along and couple with a fourth car, which is initially at rest. These four cars roll along and couple with a fifth car initially at rest. This process continues until the speed of the final collection of railroad cars is one-fifth the speed of the initial three railroad cars. All the cars are identical. Ignoring friction, how many cars are in the final collection?

8.67. A 1500-kg blue convertible is traveling south, and a 2000-kg red SUV is traveling west. If the total momentum of the system consisting of the two cars is $8000\text{ kg}\cdot\text{m/s}$ directed at 60.0° west of south, what is the speed of each vehicle?

8.68. Three identical pucks on a horizontal air table have repelling magnets. They are held together and then released simultaneously. Each has the same speed at any instant. One puck moves due west. What is the direction of the velocity of each of the other two pucks?

8.69. Spheres A (mass 0.020 kg), B (mass 0.030 kg), and C (mass 0.050 kg) are approaching the origin as they slide on a frictionless air table (Fig. 8.41). The initial velocities of A and B are given in the figure. All three spheres arrive at the origin at the same time and stick together. (a) What must the x- and y-components of the initial velocity of C be if all three objects are to end up moving at 0.50 m/s in the $+x$-direction after the collision? (b) If C has the velocity found in part (a), what is the change in the kinetic energy of the system of three spheres as a result of the collision?

Figure 8.41 Problem 8.69.



8.70. A railroad handcar is moving along straight, frictionless tracks with negligible air resistance. In the following cases, the car initially has a total mass (car and contents) of 200 kg and is traveling east with a velocity of magnitude 5.00 m/s. Find the *final* *velocity* of the car in each case, assuming that the handcar does not leave the tracks. (a) A 25.0-kg mass is thrown sideways out of the car with a velocity of magnitude 2.00 m/s relative to the car's initial velocity. (b) A 25.0-kg mass is thrown backward out of the car with a velocity of 5.00 m/s relative to the initial motion of the car. (c) A 25.0-kg mass is thrown into the car with a velocity of 6.00 m/s relative to the ground and opposite in direction to the initial velocity of the car.

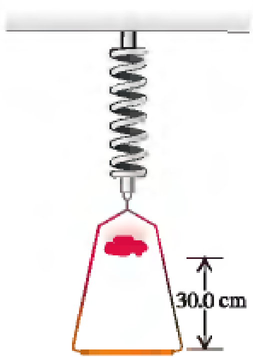
8.71. Changing Mass. A railroad hopper car filled with sand is rolling with an initial speed of 15.0 m/s on straight, horizontal tracks. You can ignore frictional forces on the railroad car. The total mass of the car plus sand is 85,000 kg. The hopper door is not fully closed so sand leaks out the bottom. After 20 min, 13,000 kg of sand has leaked out. Then what is the speed of the railroad car? (Compare your analysis with that used to solve Exercise 8.27.)

8.72. At a classic auto show, a 840-kg 1955 Nash Metropolitan motors by at 9.0 m/s, followed by a 1620-kg 1957 Packard Clipper purring past at 5.0 m/s. (a) Which car has the greater kinetic energy? What is the ratio of the kinetic energy of the Nash to that of the Packard? (b) Which car has the greater magnitude of momentum? What is the ratio of the magnitude of momentum of the Nash to that of the Packard? (c) Let F_N be the net force required to stop the Nash in time t, and let F_P be the net force required to stop the Packard in the same time. Which is larger: F_N or F_P? What is the ratio F_N/F_P of these two forces? (d) Now let F_N be the net force required to stop the Nash in a distance d, and let F_P be the net force required to stop the Packard in the same distance. Which is larger: F_N or F_P? What is the ratio F_N/F_P?

8.73. A soldier on a firing range fires an eight-shot burst from an assault weapon at a full automatic rate of 1000 rounds per minute. Each bullet has a mass of 7.45 g and a speed of 293 m/s relative to the ground as it leaves the barrel of the weapon. Calculate the average recoil force exerted on the weapon during that burst.

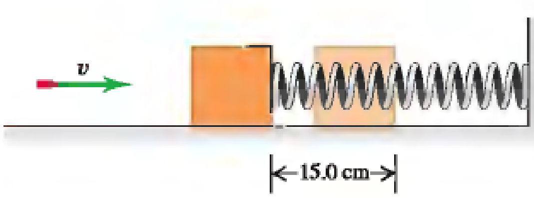
8.74. A 0.150-kg frame, when suspended from a coil spring, stretches the spring 0.050 m. A 0.200-kg lump of putty is dropped from rest onto the frame from a height of 30.0 cm (Fig. 8.42). Find the maximum distance the frame moves downward from its initial position.

Figure 8.42 Problem 8.74.



8.75. A rifle bullet with mass 8.00 g strikes and embeds itself in a block with mass 0.992 kg that rests on a frictionless, horizontal surface and is attached to a coil spring (Fig. 8.43). The impact compresses the spring 15.0 cm. Calibration of the spring shows that a force of 0.750 N is required to compress the spring 0.250 cm. (a) Find the magnitude of the block's velocity just after impact. (b) What was the initial speed of the bullet?

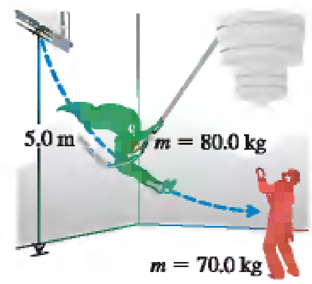
Figure 8.43 Problem 8.75.



8.76. A Ricocheting Bullet. 0.100-kg stone rests on a frictionless, horizontal surface. A bullet of mass 6.00 g, traveling horizontally at 350 m/s, strikes the stone and rebounds horizontally at right angles to its original direction with a speed of 250 m/s. (a) Compute the magnitude and direction of the velocity of the stone after it is struck. (b) Is the collision perfectly elastic?

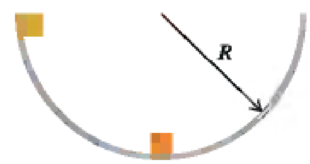
8.77. A movie stuntman (mass 80.0 kg) stands on a window ledge 5.0 m above the floor (Fig. 8.44). Grabbing a rope attached to a chandelier, he swings down to grapple with the movie's villain (mass 70.0 kg), who is standing directly under the chandelier. (Assume that the stuntman's center of mass moves downward 5.0 m. He releases the rope just as he reaches the villain.) (a) With what speed do the entwined foes start to slide across the floor? (b) If the coefficient of kinetic friction of their bodies with the floor is $\mu_k = 0.250$, how far do they slide?

Figure 8.44 Problem 8.77.



8.78. Two identical masses are released from rest in a smooth hemispherical bowl of radius R, from the positions shown in Fig. 8.45. You can ignore friction between the masses and the surface of the bowl. If they stick together when they collide, how high above the bottom of the bowl will the masses go after colliding?

Figure 8.45 Problem 8.78.



8.79. A ball with mass M, moving horizontally at 5.00 m/s, collides elastically with a block with mass $3M$ that is initially hanging at rest from the ceiling on the end of a 50.0-cm wire. Find the maximum angle through which the block swings after it is hit.

8.80. A 20.00-kg lead sphere is hanging from a hook by a thin wire 3.50 m long, and is free to swing in a complete circle. Suddenly it is struck horizontally by a 5.00-kg steel dart that embeds itself in the lead sphere. What must be the minimum initial speed of the dart so that the combination makes a complete circular loop after the collision?

8.81. An 8.00-kg ball, hanging from the ceiling by a light wire 135 cm long, is struck in an elastic collision by a 2.00-kg ball moving horizontally at 5.00 m/s just before the collision. Find the tension in the wire just after the collision.

8.82. A rubber ball of mass m is released from rest at height h above the floor. After its first bounce, it rises to 90% of its original height. What impulse (magnitude and direction) does the floor exert on this ball during its first bounce? Express your answer in terms of the variables m and h.

8.83. A 4.00-g bullet, traveling horizontally with a velocity of magnitude 400 m/s, is fired into a wooden block with mass 0.800 kg, initially at rest on a level surface. The bullet passes through the block and emerges with its speed reduced to 120 m/s. The block slides a distance of 45.0 cm along the surface from its initial position. (a) What is the coefficient of kinetic friction between block and surface? (b) What is the decrease in kinetic energy of the bullet? (c) What is the kinetic energy of the block at the instant after the bullet passes through it?

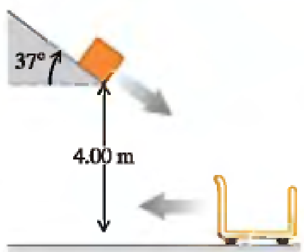
8.84. A 5.00-g bullet is shot *through* a 1.00-kg wood block suspended on a string 2.00 m long. The center of mass of the block rises a distance of 0.45 cm. Find the speed of the bullet as it emerges from the block if its initial speed is 450 m/s.

8.85. A neutron with mass m makes a head-on, elastic collision with a nucleus of mass M, which is initially at rest. (a) Show that if the neutron's initial kinetic energy is K_0, the kinetic energy that it loses during the collision is $4mMK_0/(M+m)^2$. (b) For what value of M does the incident neutron lose the most energy? (c) When M has the value calculated in part (b), what is the speed of the neutron after the collision?

8.86. Energy Sharing in Elastic Collisions. A stationary object with mass m_B is struck head-on by an object with mass m_A that is moving initially at speed v_0. (a) If the collision is elastic, what percentage of the original energy does each object have after the collision? (b) What does your answer in part (a) give for the special cases (i) $m_A = m_B$ and (ii) $m_A = 5m_B$? (c) For what values, if any, of the mass ratio m_A/m_B is the original kinetic energy shared equally by the two objects after the collision?

8.87. In a shipping company distribution center, an open cart of mass 50.0 kg is rolling to the left at a speed of 5.00 m/s (Fig. 8.46). You can ignore friction between the cart and the floor. A 15.0-kg package slides down a chute that is inclined at 37° from the horizontal and leaves the end of the chute with a speed of 3.00 m/s. The package lands in the cart and they roll off together. If the lower end of the chute is a vertical distance of 4.00 m above the bottom of the cart, what are (a) the speed of the package just before it lands in the cart and (b) the final speed of the cart?

Figure 8.46 Problem 8.87.



8.88. A blue puck with mass 0.0400 kg, sliding with a velocity of magnitude 0.200 m/s on a frictionless, horizontal air table, makes a perfectly elastic, head-on collision with a red puck with mass m, initially at rest. After the collision, the velocity of the blue puck is 0.050 m/s in the same direction as its initial velocity. Find (a) the velocity (magnitude and direction) of the red puck after the collision; and (b) the mass m of the red puck.

8.89. Two asteroids with masses m_A and m_B are moving with velocities $\vec{v}_A$ and $\vec{v}_B$ with respect to an astronomer in a space vehicle. (a) Show that the total kinetic energy as measured by the astronomer is

$$K = \tfrac{1}{2}Mv_{\text{cm}}{}^2 + \tfrac{1}{2}(m_A v_A'^2 + m_B v_B'^2)$$

with $\vec{v}_{\text{cm}}$ and M defined as in Section 8.5, $\vec{v}_A' = \vec{v}_A - \vec{v}_{\text{cm}}$, and $\vec{v}_B' = \vec{v}_B - \vec{v}_{\text{cm}}$. In this expression the total kinetic energy of the two asteroids is the energy associated with their center of mass plus the energy associated with the internal motion relative to the center of mass. (b) If the asteroids collide, what is the *minimum* possible kinetic energy they can have after the collision, as measured by the astronomer? Explain.

8.90. Suppose you hold a small ball in contact with, and directly over, the center of a large ball. If you then drop the small ball a short time after dropping the large ball, the small ball rebounds with surprising speed. To show the extreme case, ignore air resistance and suppose the large ball makes an elastic collision with the floor and then rebounds to make an elastic collision with the still-descending small ball. Just before the collision between the two balls, the large ball is moving upward with velocity $\vec{v}$ and the small ball has velocity $-\vec{v}$. (Do you see why?) Assume the large ball has a much greater mass than the small ball. (a) What is the velocity of the small ball immediately after its collision with the large ball? (b) From the answer to part (a), what is the ratio of the small ball's rebound distance to the distance it fell before the collision?

8.91. Jack and Jill are standing on a crate at rest on the frictionless, horizontal surface of a frozen pond. Jack has mass 75.0 kg, Jill has mass 45.0 kg, and the crate has mass 15.0 kg. They remember that they must fetch a pail of water, so each jumps horizontally from the top of the crate. Just after each jumps, that person is moving away from the crate with a speed of 4.00 m/s relative to the crate. (a) What is the final speed of the crate if both Jack and Jill jump simultaneously and in the same direction? (*Hint:* Use an inertial coordinate system attached to the ground.) (b) What is the final speed of the crate if Jack jumps first and then a few seconds later Jill jumps in the same direction? (c) What is the final speed of the crate if Jill jumps first and then Jack, again in the same direction?

8.92. Energy Sharing. An object with mass m, initially at rest, explodes into two fragments, one with mass m_A and the other with mass m_B, where $m_A + m_B = m$. (a) If energy Q is released in the explosion, how much kinetic energy does each fragment have immediately after the collision? (b) What percentage of the total energy released does each fragment get when one fragment has four times the mass of the other?

8.93. Neutron Decay. A neutron at rest decays (breaks up) to a proton and an electron. Energy is released in the decay and appears as kinetic energy of the proton and electron. The mass of a proton is 1836 times the mass of an electron. What fraction of the total energy released goes into the kinetic energy of the proton?

8.94. A ^{232}Th (thorium) nucleus at rest decays to a ^{228}Ra (radium) nucleus with the emission of an alpha particle. The total kinetic energy of the decay fragments is 6.54×10^{-13} J. An alpha particle has 1.76% of the mass of a ^{228}Ra nucleus. Calculate the kinetic energy of (a) the recoiling ^{228}Ra nucleus and (b) the alpha particle.

8.95. Antineutrino. In beta decay, a nucleus emits an electron. A ^{210}Bi (bismuth) nucleus at rest undergoes beta decay to ^{210}Po (polonium). Suppose the emitted electron moves to the right with a momentum of 5.60×10^{-22} kg · m/s. The ^{210}Po nucleus, with mass 3.50×10^{-25} kg, recoils to the left at a speed of 1.14×10^{-3} m/s. Momentum conservation requires that a second particle, called an antineutrino, must also be emitted. Calculate the magnitude and direction of the momentum of the antineutrino that is emitted in this decay.

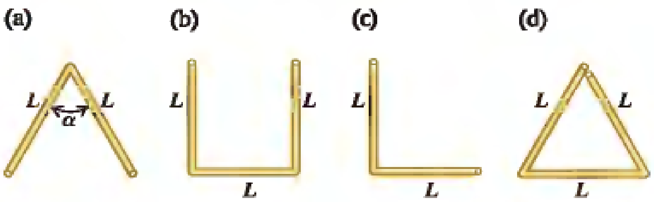
8.96. A proton moving with speed v_{A1} in the $+x$-direction makes an elastic, off-center collision with an identical proton originally at rest. After impact, the first proton moves with speed v_{A2} in the first quadrant at an angle α with the x-axis, and the second moves with speed v_{B2} in the fourth quadrant at an angle β with the x-axis (Fig. 8.13). (a) Write the equations expressing conservation of linear momentum in the x- and y-directions. (b) Square the equations from part (a) and add them. (c) Now introduce the fact that the collision is elastic. (d) Prove that $\alpha + \beta = \pi/2$. (You have shown that this equation is obeyed in any elastic, off-center collision between objects of equal mass when one object is initially at rest.)

8.97. Hockey puck B rests on a smooth ice surface and is struck by a second puck A, which has the same mass. Puck A is initially traveling at 15.0 m/s and is deflected 25.0° from its initial direction. Assume that the collision is perfectly elastic. Find the final speed of each puck and the direction of B's velocity after the collision. [*Hint:* Use the relationship derived in part (d) of Problem 8.96.]

8.90. Jonathan and Jane are sitting in a sleigh that is at rest on frictionless ice. Jonathan's weight is 800 N, Jane's weight is 600 N, and that of the sleigh is 1000 N. They see a poisonous spider on the floor of the sleigh and immediately jump off. Jonathan jumps to the left with a velocity of 5.00 m/s at 30.0° above the horizontal (relative to the ice), and Jane jumps to the right at 7.00 m/s at 36.9° above the horizontal (relative to the ice). Calculate the sleigh's horizontal velocity (magnitude and direction) after they jump out.

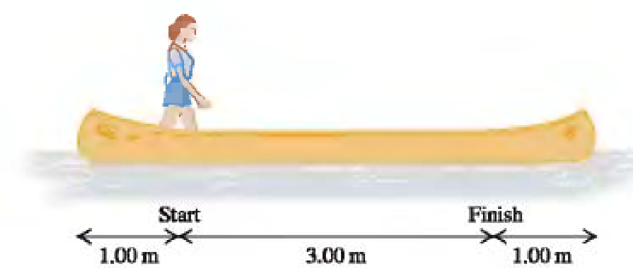
8.99. The objects in Fig. 8.47 are constructed of uniform wire bent into the shapes shown. Find the position of the center of mass of each.

Figure 8.47 Problem 8.99.



8.100. A 45.0-kg woman stands up in a 60.0-kg canoe 5.00 m long. She walks from a point 1.00 m from one end to a point 1.00 m from the other end (Fig. 8.48). If you ignore resistance to motion of the canoe in the water, how far does the canoe move during this process?

Figure 8.48 Problem 8.100.



8.101. You are standing on a concrete slab that in turn is resting on a frozen lake. Assume there is no friction between the slab and the ice. The slab has a weight five times your weight. If you begin walking forward at 2.00 m/s relative to the ice, with what speed, relative to the ice, does the slab move?

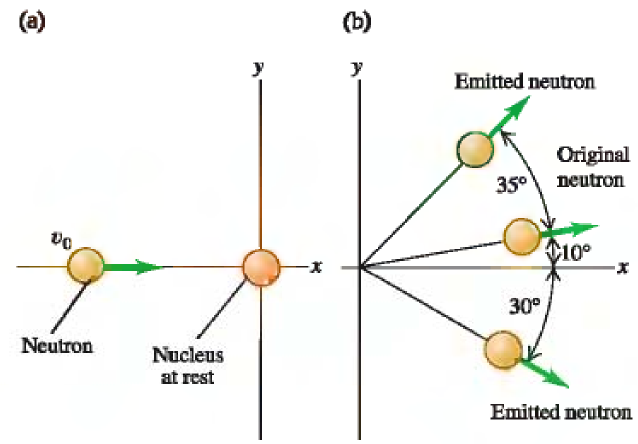
8.102. A 20.0-kg projectile is fired at an angle of 60.0° above the horizontal with a speed of 80.0 m/s. At the highest point of its trajectory, the projectile explodes into two fragments with equal mass, one of which falls vertically with zero initial speed. You can ignore air resistance. (a) How far from the point of firing does the other fragment strike if the terrain is level? (b) How much energy is released during the explosion?

8.103. A fireworks rocket is fired vertically upward. At its maximum height of 80.0 m, it explodes and breaks into two pieces, one with mass 1.40 kg and the other with mass 0.28 kg. In the explosion, 860 J of chemical energy is converted to kinetic energy of the two fragments. (a) What is the speed of each fragment just after the explosion? (b) It is observed that the two fragments hit the ground at the same time. What is the distance between the points on the ground where they land? Assume that the ground is level and air resistance can be ignored.

8.104. A 12.0-kg shell is launched at an angle of 55.0° above the horizontal with an initial speed of 150 m/s. When it is at its highest point, the shell exploded into two fragments, one three times heavier than the other. The two fragments reach the ground at the same time. Assume that air resistance can be ignored. If the heavier fragment lands back at the same point from which the shell was launched, where will the lighter fragment land and how much energy was released in the explosion?

8.105. A Nuclear Reaction. Fission, the process that supplies energy in nuclear power plants, occurs when a heavy nucleus is split into two medium-sized nuclei. One such reaction occurs when a neutron colliding with a ^{235}U (uranium) nucleus splits that nucleus into a ^{141}Ba (barium) nucleus and a ^{92}Kr (krypton) nucleus. In this reaction, two neutrons also are split off from the original ^{235}U. Before the collision, the arrangement is as shown in Fig. 8.49a. After the collision, the ^{141}Ba nucleus is moving in the $+z$-direction and the ^{92}Kr nucleus in the $-z$-direction. The three neutrons are moving in the xy-plane, as shown in Fig. 8.49b. If the incoming neutron has an initial velocity of magnitude 3.0×10^3 m/s and a final velocity of magnitude 2.0×10^3 m/s in the directions shown, what are the speeds of the other two neutrons, and what can you say about the speeds of the ^{141}Ba and ^{92}Kr nuclei? (The mass of the ^{141}Ba nucleus is approximately 2.3×10^{-25} kg, and the mass of ^{92}Kr is about 1.5×10^{-25} kg.)

Figure 8.49 Problem 8.105.



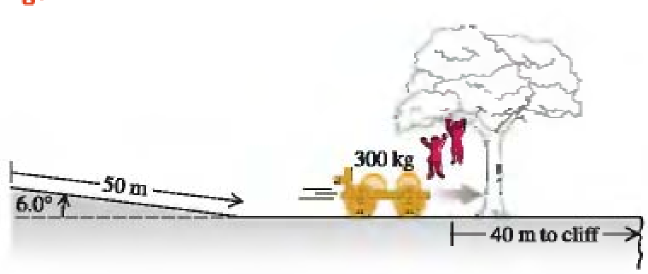
8.106. Center-of-Mass Coordinate System. Puck A (mass m_A) is moving on a frictionless, horizontal air table in the $+x$-direction with velocity $\vec{v}_{A1}$ and makes an elastic, head-on collision with puck B (mass m_B), which is initially at rest. After the collision, both pucks are moving along the x-axis. (a) Calculate the velocity of the center of mass of the two-puck system before the collision. (b) Consider a coordinate system whose origin is at the center of mass and moves with it. Is this an inertial reference frame? (c) What are the initial velocities $\vec{u}_{A1}$ and $\vec{u}_{B1}$ of the two pucks in this center-of-mass reference frame? What is the total momentum in this frame? (d) Use conservation of momentum and energy, applied in the center-of-mass reference frame, to relate the final momentum of each puck to its initial momentum and thus the final velocity of each puck to its initial velocity. Your results should show that a one-dimensional, elastic collision has a very simple description in center-of-mass coordinates. (e) Let $m_A = 0.400$ kg, $m_B = 0.200$ kg, and $v_{A1} = 6.00$ m/s. Find the center-of-mass velocities $\vec{u}_{A1}$ and $\vec{u}_{B1}$, apply the simple result found in part (d), and transform back to velocities in a stationary frame to find the final velocities of the pucks. Does your result agree with Eqs. (8.24) and (8.25)?

8.107. The *coefficient of restitution* ϵ for a head-on collision is defined as the ratio of the relative speed after the collision to the relative speed before. (a) What is ϵ for a completely inelastic collision? (b) What is ϵ for an elastic collision? (c) A ball is dropped from a height h onto a stationary surface and rebounds back to a height H_1. Show that $\epsilon = \sqrt{H_1/h}$. (d) A properly inflated basketball should have a coefficient of restitution of 0.85. When dropped from a height of 1.2 m above a solid wood floor, to what height should a properly inflated basketball bounce? (e) The height of the first bounce is H_1. If ϵ is constant, show that the height of the nth bounce is $H_n = \epsilon^{2n}h$. (f) If ϵ is constant, what is the height of the eighth bounce of a properly inflated basketball dropped from 1.2 m?

8.108. Binding Energy of the Hydrogen Molecule. When two hydrogen atoms of mass m combine to form a diatomic hydrogen molecule (H_2), the potential energy of the system after they combine is $-\Delta$, where Δ is a positive quantity called the *binding energy* of the molecule. (a) Show that in a collision that involves only two hydrogen atoms, it is *impossible* to form an H_2 molecule because momentum and energy cannot simultaneously be conserved. (*Hint:* If you can show this to be true in one frame of reference, then it is true in all frames of reference. Can you see why?) (b) An H_2 molecule can be formed in a collision that involves *three* hydrogen atoms. Suppose that before such a collision, each of the three atoms has speed 1.00×10^3 m/s, and they are approaching at 120° angles so that at any instant, the atoms lie at the corners of an equilateral triangle. Find the speeds of the H_2 molecule and of the single hydrogen atom that remains after the collision. The binding energy of H_2 is $\Delta = 7.23 \times 10^{-19}$ J, and the mass of the hydrogen atom is 1.67×10^{-27} kg.

8.109. A wagon with two boxes of gold, having total mass 300 kg, is cut loose from the horses by an outlaw when the wagon is at rest 50 m up a 6.0° slope (Fig. 8.50). The outlaw plans to have the wagon roll down the slope and across the level ground, and then fall into a canyon where his confederates wait. But in a tree 40 m from the canyon edge wait the Lone Ranger (mass 75.0 kg) and Tonto (mass 60.0 kg). They drop vertically into the wagon as it passes beneath them. (a) If they require 5.0 s to grab the gold and jump out, will they make it before the wagon goes over the edge? The wagon rolls with negligible friction. (b) When the two heroes drop into the wagon, is the kinetic energy of the system of the heroes plus the wagon conserved? If not, does it increase or decrease, and by how much?

Figure 8.50 Problem 8.109.



***8.110.** In Section 8.6, we considered a rocket fired in outer space where there is no air resistance and where gravity is negligible. Suppose instead that the rocket is accelerating vertically upward from rest on the earth's surface. Continue to ignore air resistance and consider only that part of the motion where the altitude of the rocket is small so that g may be assumed to be constant. (a) How is Eq. (8.37) modified by the presence of the gravity force? (b) Derive an expression for the acceleration a of the rocket, analogous to Eq. (8.39). (c) What is the acceleration of the rocket in Example 8.15 (Section 8.6) if it is near the earth's surface rather than in outer space? You can ignore air resistance. (d) Find the speed of the rocket in Example 8.16 (Section 8.6) after 90 s if the rocket is fired from the earth's surface rather than in outer space. You can ignore air resistance. How does your answer compare with the rocket speed calculated in Example 8.16?

***8.111. A Multistage Rocket.** Suppose the first stage of a two-stage rocket has total mass 12,000 kg, of which 9000 kg is fuel. The total mass of the second stage is 1000 kg, of which 700 kg is fuel. Assume that the relative speed v_{ex} of ejected material is constant, and ignore any effect of gravity. (The effect of gravity is small during the firing period if the rate of fuel consumption is large.) (a) Suppose the entire fuel supply carried by the two-stage rocket is utilized in a single-stage rocket with the same total mass of 13,000 kg. In terms of v_{ex}, what is the speed of the rocket, starting from rest, when its fuel is exhausted? (b) For the two-stage rocket, what is the speed when the fuel of the first stage is exhausted if the first stage carries the second stage with it to this point? This speed then becomes the initial speed of the second stage. At this point, the second stage separates from the first stage. (c) What is the final speed of the second stage? (d) What value of v_{ex} is required to give the second stage of the rocket a speed of 7.00 km/s?

***8.112.** For the rocket described in Examples 8.15 and 8.16 (Section 8.6), the mass of the rocket as a function of time is

$$m(t) = \begin{cases} m_0 & \text{for } t < 0 \\ m_0\left(1 - \dfrac{t}{120\text{ s}}\right) & \text{for } 0 \le t \le 90\text{ s} \\ m_0/4 & \text{for } t \ge 90\text{ s} \end{cases}$$

(a) Calculate and graph the velocity of the rocket as a function of time from $t = 0$ to $t = 100$ s. (b) Calculate and graph the acceleration of the rocket as a function of time from $t = 0$ to $t = 100$ s. (c) A 75-kg astronaut lies on a reclined chair during the firing of the rocket. What is the maximum net force exerted by the chair on the astronaut during the firing? How does your answer compare with her weight on earth?

Challenge Problems

8.113. In Section 8.5 we calculated the center of mass by considering objects composed of a *finite* number of point masses or objects that, by symmetry, could be represented by a finite number of point masses. For a solid object whose mass distribution does not allow for a simple determination of the center of mass by symmetry, the sums of Eqs. (8.28) must be generalized to integrals

$$x_{cm} = \frac{1}{M}\int x\,dm \qquad y_{cm} = \frac{1}{M}\int y\,dm$$

where x and y are the coordinates of the small piece of the object that has mass dm. The integration is over the whole of the object. Consider a thin rod of length L, mass M, and cross-sectional area A. Let the origin of the coordinates be at the left end of the rod and the positive x-axis lie along the rod. (a) If the density $\rho = M/V$ of the object is uniform, perform the integration described above to show that the x-coordinate of the center of mass of the rod is at its geometrical center. (b) If the density of the object varies linearly with x—that is, $\rho = \alpha x$, where α is a positive constant—calculate the x-coordinate of the rod's center of mass.

8.114. Use the methods of Challenge Problem 8.113 to calculate the x- and y-coordinates of the center of mass of a semicircular metal plate with uniform density ρ and thickness t. Let the radius of the plate be a. The mass of the plate is thus $M = \frac{1}{2}\rho\pi a^2 t$. Use the coordinate system indicated in Fig. 8.51.

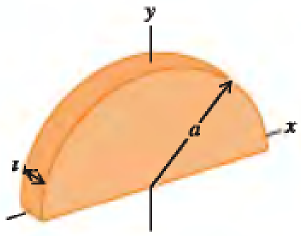


Figure 8.51 Challenge Problem 8.114.

8.115. One-fourth of a rope of length l is hanging down over the edge of a frictionless table. The rope has a uniform, linear density (mass per unit length) λ (Greek lambda), and the end already on the table is held by a person. How much work does the person do when she pulls on the rope to raise the rest of the rope slowly onto the table? Do the problem in two ways as follows. (a) Find the force that the person must exert to raise the rope and from this the work done. Note that this force is variable because at different times, different amounts of rope are hanging over the edge. (b) Suppose the segment of the rope initially hanging over the edge of the table has all of its mass concentrated at its center of mass. Find the work necessary to raise this to table height. You will probably find this approach simpler than that of part (a). How do the answers compare, and why is this so?

***8.116 A Variable-Mass Raindrop.** In a rocket-propulsion problem the mass is variable. Another such problem is a raindrop falling through a cloud of small water droplets. Some of these small droplets adhere to the raindrop, thereby *increasing* its mass as it falls. The force on the raindrop is

$$F_{\text{ext}} = \frac{dp}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}$$

Suppose the mass of the raindrop depends on the distance x that it has fallen. Then $m = kx$, where k is a constant, and $dm/dt = kv$. This gives, since $F_{\text{ext}} = mg$,

$$mg = m\frac{dv}{dt} + v(kv)$$

Or, dividing by k,

$$xg = x\frac{dv}{dt} + v^2$$

This is a differential equation that has a solution of the form $v = at$, where a is the acceleration and is constant. Take the initial velocity of the raindrop to be zero. (a) Using the proposed solution for v, find the acceleration a. (b) Find the distance the raindrop has fallen in $t = 3.00$ s. (c) Given that $k = 2.00$ g/m, find the mass of the raindrop at $t = 3.00$ s. For many more intriguing aspects of this problem, see K. S. Krane, *Amer. Jour. Phys.*, Vol. 49 (1981), pp. 113–117.

ROTATION OF RIGID BODIES

9

? All segments of a rotating helicopter blade have the same angular velocity and angular acceleration. Compared to a given blade segment, how many times greater is the linear speed of a second segment twice as far from the axis of rotation? How many times greater is the linear acceleration?

LEARNING GOALS

By studying this chapter, you will learn:

- How to describe the rotation of a rigid body in terms of angular coordinate, angular velocity, and angular acceleration.
- How to analyze rigid-body rotation when the angular acceleration is constant.
- How to relate the rotation of a rigid body to the linear velocity and linear acceleration of a point on the body.
- The meaning of a body's moment of inertia about a rotation axis, and how it relates to rotational kinetic energy.
- How to calculate the moment of inertia of various bodies.

What do the motions of a compact disc, a Ferris wheel, a circular saw blade, and a ceiling fan have in common? None of these can be represented adequately as a moving *point;* each involves a body that *rotates* about an axis that is stationary in some inertial frame of reference.

Rotation occurs at all scales, from the motion of electrons in atoms to the motions of entire galaxies. We need to develop some general methods for analyzing the motion of a rotating body. In this chapter and the next we consider bodies that have definite size and definite shape, and that in general can have rotational as well as translational motion.

Real-world bodies can be very complicated; the forces that act on them can deform them—stretching, twisting, and squeezing them. We'll neglect these deformations for now and assume that the body has a perfectly definite and unchanging shape and size. We call this idealized model a **rigid body.** This chapter and the next are mostly about rotational motion of a rigid body.

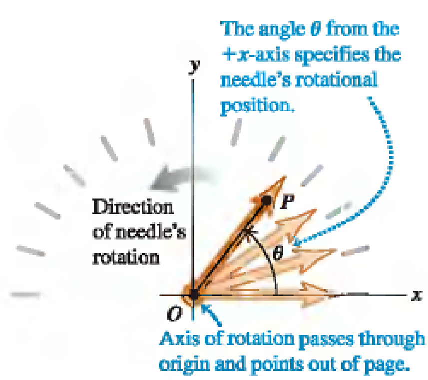
We begin with kinematic language for *describing* rotational motion. Next we look at the kinetic energy of rotation, the key to using energy methods for rotational motion. Then in Chapter 10 we'll develop dynamic principles that relate the forces on a body to its rotational motion.

9.1 Angular Velocity and Acceleration

In analyzing rotational motion, let's think first about a rigid body that rotates about a *fixed axis*—an axis that is at rest in some inertial frame of reference and does not change direction relative to that frame. The rotating rigid body might be a motor shaft, a chunk of beef on a barbecue skewer, or a merry-go-round.

Figure 9.1 shows a rigid body (in this case, the indicator needle of a speedometer) rotating about a fixed axis. The axis passes through point O and is

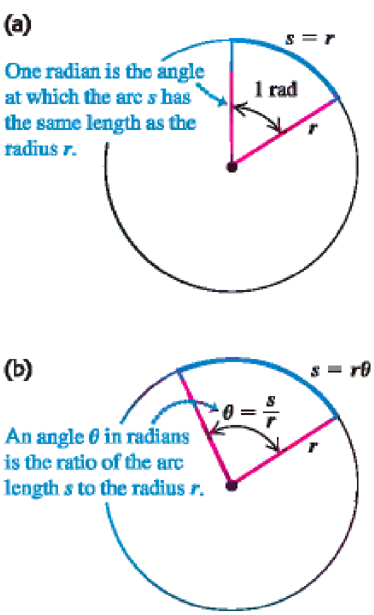
9.1 A speedometer needle (an example of a rigid body) rotating counterclockwise about a fixed axis.



perpendicular to the plane of the diagram, which we choose to call the xy-plane. One way to describe the rotation of this body would be to choose a particular point P on the body and to keep track of the x- and y-coordinates of this point. This isn't a terribly convenient method, since it takes two numbers (the two coordinates x and y) to specify the rotational position of the body. Instead, we notice that the line OP is fixed in the body and rotates with it. The angle θ that this line makes with the $+x$-axis describes the rotational position of the body; we will use this single quantity θ as a *coordinate* for rotation.

The angular coordinate θ of a rigid body rotating around a fixed axis can be positive or negative. If we choose positive angles to be measured counterclockwise from the positive x-axis, then the angle θ in Fig. 9.1 is positive. If we instead choose the positive rotation direction to be clockwise, then θ in Fig. 9.1 is negative. When we considered the motion of a particle along a straight line, it was essential to specify the direction of positive displacement along that line; when we discuss rotation around a fixed axis, it's just as essential to specify the direction of positive rotation.

9.2 Measuring angles in radians.



To describe rotational motion, the most natural way to measure the angle θ is not in degrees, but in **radians.** As shown in Fig. 9.2a, one radian (1 rad) is the angle subtended at the center of a circle by an arc with a length equal to the radius of the circle. In Fig. 9.2b an angle θ is subtended by an arc of length s on a circle of radius r. The value of θ (in radians) is equal to s divided by r:

$$\theta = \frac{s}{r} \quad \text{or} \quad s = r\theta \tag{9.1}$$

An angle in radians is the ratio of two lengths, so it is a pure number, without dimensions. If $s = 3.0$ m and $r = 2.0$ m, then $\theta = 1.5$, but we will often write this as 1.5 rad to distinguish it from an angle measured in degrees or revolutions.

The circumference of a circle (that is, the arc length all the way around the circle) is 2π times the radius, so there are 2π (about 6.283) radians in one complete revolution $(360°)$. Therefore

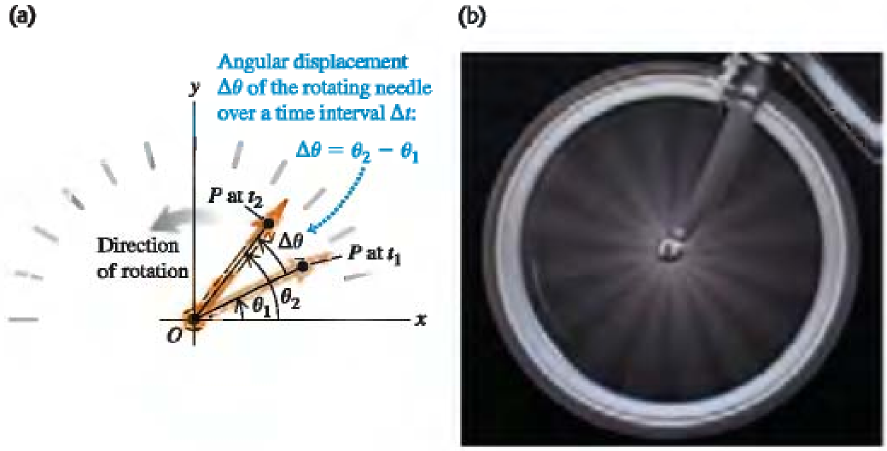
$$1\text{ rad} = \frac{360°}{2\pi} = 57.3°$$

Similarly, $180° = \pi$ rad, $90° = \pi/2$ rad, and so on. If we had insisted on measuring the angle θ in degrees, we would have needed to include an extra factor of $(2\pi/360)$ on the right-hand side of $s = r\theta$ in Eq. (9.1). By measuring angles in radians, we keep the relationship between angle and distance along an arc as simple as possible.

Angular Velocity

The coordinate θ shown in Fig. 9.1 specifies the rotational position of a rigid body at a given instant. We can describe the rotational *motion* of such a rigid body in terms of the rate of change of θ. We'll do this in an analogous way to our description of straight-line motion in Chapter 2. In Fig. 9.3a, a reference line OP in a rotating body makes an angle θ_1 with the $+x$-axis at time t_1. At a later time t_2 the angle has changed to θ_2. We define the **average angular velocity** $\omega_{\text{av-}z}$ (the Greek letter omega) of the body in the time interval $\Delta t = t_2 - t_1$ as the ratio of the **angular displacement** $\Delta\theta = \theta_2 - \theta_1$ to Δt:

$$\omega_{\text{av-}z} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t} \tag{9.2}$$



9.3 (a) Angular displacement $\Delta\theta$ of a rotating body. (b) Every part of a rotating rigid body has the same angular velocity $\Delta\theta/\Delta t$.

The subscript z indicates that the body in Fig. 9.3a is rotating about the z-axis, which is perpendicular to the plane of the diagram. The **instantaneous angular velocity** ω_z is the limit of $\omega_{\text{av-}z}$ as Δt approaches zero—that is, the derivative of θ with respect to t:

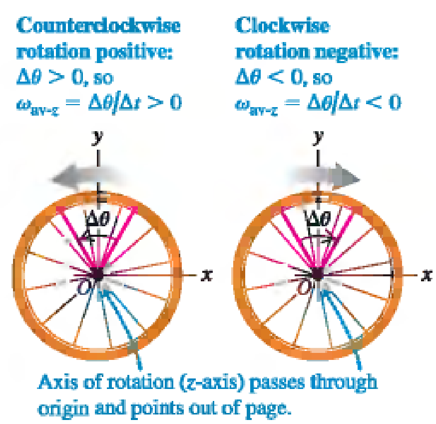
$$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad \text{(definition of angular velocity)} \qquad (9.3)$$

When we refer simply to "angular velocity," we mean the instantaneous angular velocity, not the average angular velocity.

The angular velocity ω_z can be positive or negative, depending on the direction in which the rigid body is rotating (Fig. 9.4). The angular *speed* ω, which we will use extensively in Sections 9.3 and 9.4, is the magnitude of angular velocity. Like ordinary (linear) speed v, the angular speed is never negative.

CAUTION **Angular velocity vs. linear velocity** Keep in mind the distinction between angular velocity ω_z and ordinary velocity, or *linear velocity*, v_x (see Section 2.2). If an object has a velocity v_x, the object as a whole is *moving* along the x-axis. By contrast, if an object has an angular velocity ω_z, then it is *rotating* around the z-axis. We do *not* mean that the object is moving along the z-axis.

9.4 A rigid body's average angular velocity (shown here) and instantaneous angular velocity can be positive or negative.



Different points on a rotating rigid body move different distances in a given time interval, depending on how far the point lies from the rotation axis. But because the body is rigid, *all* points rotate through the same angle in the same time (Fig. 9.3b). Hence *at any instant, every part of a rotating rigid body has the same angular velocity.* The angular velocity is positive if the body is rotating in the direction of increasing θ and negative if it is rotating in the direction of decreasing θ.

If the angle θ is in radians, the unit of angular velocity is the radian per second (rad/s). Other units, such as the revolution per minute (rev/min or rpm), are often used. Since 1 rev $= 2\pi$ rad, two useful conversions are

$$1 \text{ rev/s} = 2\pi \text{ rad/s} \qquad \text{and} \qquad 1 \text{ rev/min} = 1 \text{ rpm} = \frac{2\pi}{60} \text{ rad/s}$$

That is, 1 rad/s is about 10 rpm.

Example 9.1 Calculating angular velocity

The flywheel of a prototype car engine is under test. The angular position θ of the flywheel is given by

$$\theta = (2.0\ \text{rad/s}^3)t^3$$

The diameter of the flywheel is 0.36 m. (a) Find the angle θ, in radians and in degrees, at times $t_1 = 2.0$ s and $t_2 = 5.0$ s. (b) Find the distance that a particle on the rim moves during that time interval. (c) Find the average angular velocity, in rad/s and in rev/min (rpm), between $t_1 = 2.0$ s and $t_2 = 5.0$ s. (d) Find the instantaneous angular velocity at time $t = t_2 = 5.0$ s.

SOLUTION

IDENTIFY: We need to find the values θ_1 and θ_2 of the angular position at times t_1 and t_2, the angular displacement $\Delta\theta$ between t_1 and t_2, the distance traveled and the average angular velocity between t_1 and t_2, and the instantaneous angular velocity at t_2.

SET UP: We're given the angular position θ as a function of time, so we can easily find our first two target variables θ_1 and θ_2; the angular displacement $\Delta\theta$ is the difference between θ_1 and θ_2. Given $\Delta\theta$ we'll find the distance and the average angular velocity using Eqs. (9.1) and (9.2), respectively. To find the instantaneous angular velocity, we'll take the derivative of θ with respect to time, as in Eq. (9.3).

EXECUTE: (a) We substitute the values of t into the given equation:

$$\begin{aligned}\theta_1 &= (2.0\ \text{rad/s}^3)(2.0\ \text{s})^3 = 16\ \text{rad}\\ &= (16\ \text{rad})\frac{360^\circ}{2\pi\ \text{rad}} = 920^\circ\\ \theta_2 &= (2.0\ \text{rad/s}^3)(5.0\ \text{s})^3 = 250\ \text{rad}\\ &= (250\ \text{rad})\frac{360^\circ}{2\pi\ \text{rad}} = 14{,}000^\circ\end{aligned}$$

(b) The flywheel turns through an angular displacement of $\Delta\theta = \theta_2 - \theta_1 = 250\ \text{rad} - 16\ \text{rad} = 234\ \text{rad}$. The radius r is half the diameter, or 0.18 m. Equation (9.1) gives

$$s = r\theta = (0.18\ \text{m})(234\ \text{rad}) = 42\ \text{m}$$

To use Eq. (9.1), the angle *must* be expressed in radians. We drop "radians" from the unit for s because θ is really a dimensionless pure number; s is a distance and is measured in meters, the same unit as r.

(c) In Eq. (9.2) we have

$$\begin{aligned}\omega_{\text{av-}z} &= \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{250\ \text{rad} - 16\ \text{rad}}{5.0\ \text{s} - 2.0\ \text{s}} = 78\ \text{rad/s}\\ &= \left(78\frac{\text{rad}}{\text{s}}\right)\left(\frac{1\ \text{rev}}{2\pi\ \text{rad}}\right)\left(\frac{60\ \text{s}}{1\ \text{min}}\right) = 740\ \text{rev/min}\end{aligned}$$

(d) We use Eq. (9.3):

$$\begin{aligned}\omega_z &= \frac{d\theta}{dt} = \frac{d}{dt}[(2.0\ \text{rad/s}^3)t^3] = (2.0\ \text{rad/s}^3)(3t^2)\\ &= (6.0\ \text{rad/s}^3)t^2\end{aligned}$$

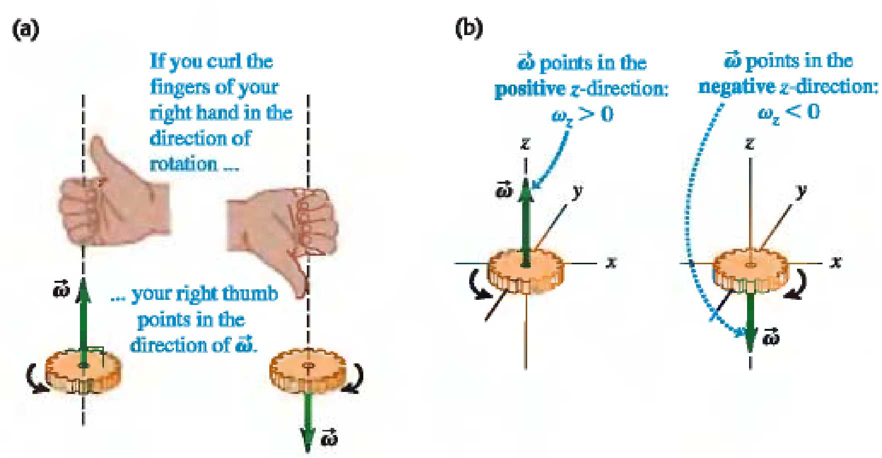
At time $t = 5.0$ s,

$$\omega_z = (6.0\ \text{rad/s}^3)(5.0\ \text{s})^2 = 150\ \text{rad/s}$$

EVALUATE: Our result in part (d) shows that ω_z is proportional to t^2 and hence increases with time. Our numerical results are consistent with this result: The 150-rad/s instantaneous angular velocity at $t = 5.0$ s is greater than the 78-rad/s average angular velocity for the 3.0-s interval leading up to that time (from $t_1 = 2.0$ s to $t_2 = 5.0$ s).

Angular Velocity As a Vector

As we have seen, our notation for the angular velocity ω_z about the z-axis is reminiscent of the notation v_x for the ordinary velocity along the x-axis (see Section 2.2). Just as v_x is the x-component of the velocity vector $\vec{v}$, ω_z is the z-component of an angular velocity *vector* $\vec{\omega}$ directed along the axis of rotation. As Fig. 9.5a



9.5 (a) The right-hand rule for the direction of the angular velocity vector $\vec{\omega}$. Reversing the direction of rotation reverses the direction of $\vec{\omega}$. (b) The sign of ω_z for rotation along the z-axis.

shows, the direction of $\vec{\omega}$ is given by the right-hand rule that we used to define the vector product in Section 1.10. If the rotation is about the z-axis, then $\vec{\omega}$ has only a z-component; this component is positive if $\vec{\omega}$ is along the positive z-axis and negative if $\vec{\omega}$ is along the negative z-axis (Fig. 9.5b).

The vector formulation is especially useful in situations in which the direction of the rotation axis *changes*. We'll examine such situations briefly at the end of Chapter 10. In this chapter, however, we'll consider only situations in which the rotation axis is fixed. Hence throughout this chapter we'll use "angular velocity" to refer to ω_z, the component of the angular velocity vector $\vec{\omega}$ along the axis.

Angular Acceleration

When the angular velocity of a rigid body changes, it has an *angular acceleration*. When you pedal your bicycle harder to make the wheels turn faster or apply the brakes to bring the wheels to a stop, you're giving the wheels an angular acceleration. You also impart an angular acceleration whenever you change the rotation speed of a piece of spinning machinery such as an automobile engine's crankshaft.

If ω_{1z} and ω_{2z} are the instantaneous angular velocities at times t_1 and t_2, we define the **average angular acceleration** $\alpha_{\text{av-}z}$ over the interval $\Delta t = t_2 - t_1$ as the change in angular velocity divided by Δt (Fig. 9.6):

$$\alpha_{\text{av-}z} = \frac{\omega_{2z} - \omega_{1z}}{t_2 - t_1} = \frac{\Delta\omega_z}{\Delta t} \tag{9.4}$$

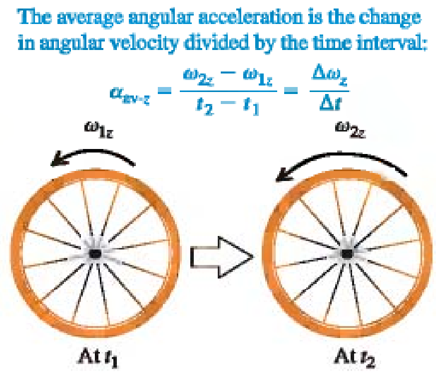
The **instantaneous angular acceleration** α_z is the limit of $\alpha_{\text{av-}z}$ as $\Delta t \to 0$:

$$\alpha_z = \lim_{\Delta t \to 0} \frac{\Delta\omega_z}{\Delta t} = \frac{d\omega_z}{dt} \quad \text{(definition of angular acceleration)} \tag{9.5}$$

The usual unit of angular acceleration is the radian per second per second, or rad/s^2. From now on we will use the term "angular acceleration" to mean the instantaneous angular acceleration rather than the average angular acceleration.

Because $\omega_z = d\theta/dt$, we can also express angular acceleration as the second derivative of the angular coordinate:

$$\alpha_z = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d^2\theta}{dt^2} \tag{9.6}$$



9.6 Calculating the average angular acceleration of a rotating rigid body.

You have probably noticed that we are using Greek letters for angular kinematic quantities: θ for angular position, ω_z for angular velocity, and α_z for angular acceleration. These are analogous to x for position, v_x for velocity, and a_x for acceleration, respectively, in straight-line motion. In each case, velocity is the rate of change of position with respect to time and acceleration is the rate of change of velocity with respect to time. We will sometimes use the terms "*linear* velocity" and "*linear* acceleration" for the familiar quantities we defined in Chapters 2 and 3 to distinguish clearly between these and the *angular* quantities introduced in this chapter.

In rotational motion, if the angular acceleration α_z is positive, then the angular velocity ω_z is increasing; if α_z is negative, then ω_z is decreasing. The rotation is speeding up if α_z and ω_z have the same sign and slowing down if α_z and ω_z have opposite signs. (These are exactly the same relationships as those between *linear* acceleration a_x and *linear* velocity v_x for straight-line motion; see Section 2.3.)

Example 9.2 Calculating angular acceleration

In Example 9.1 we found that the instantaneous angular velocity ω_z of the flywheel at any time t is given by

$$\omega_z = (6.0\ \text{rad/s}^3)t^2$$

(a) Find the average angular acceleration between $t_1 = 2.0$ s and $t_2 = 5.0$ s. (b) Find the instantaneous angular acceleration at time $t_2 = 5.0$ s.

SOLUTION

IDENTIFY: This example uses the definitions of average angular acceleration $\alpha_{\text{av-}z}$ and instantaneous angular acceleration α_z.

SET UP: We'll use Eqs. (9.4) and (9.5) to find the value of $\alpha_{\text{av-}z}$ between t_1 and t_2 and the value of α_z at $t = t_2$.

EXECUTE: (a) The values of ω_z at the two times are

$$\omega_{1z} = (6.0\ \text{rad/s}^3)(2.0\ \text{s})^2 = 24\ \text{rad/s}$$
$$\omega_{2z} = (6.0\ \text{rad/s}^3)(5.0\ \text{s})^2 = 150\ \text{rad/s}$$

From Eq. (9.4) the average angular acceleration is

$$\alpha_{\text{av-}z} = \frac{150\ \text{rad/s} - 24\ \text{rad/s}}{5.0\ \text{s} - 2.0\ \text{s}} = 42\ \text{rad/s}^2$$

(b) From Eq. (9.5) the instantaneous angular acceleration at any time t is

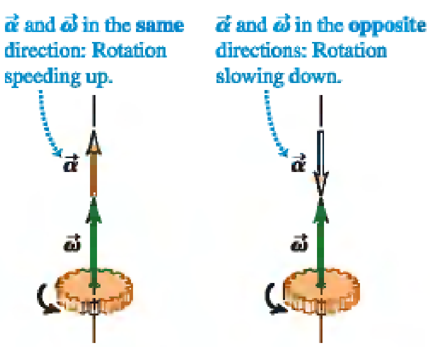
$$\alpha_z = \frac{d\omega_z}{dt} = \frac{d}{dt}[(6.0\ \text{rad/s}^3)(t^2)] = (6.0\ \text{rad/s}^3)(2t)$$
$$= (12\ \text{rad/s}^3)t$$

At time $t = 5.0$ s,

$$\alpha_z = (12\ \text{rad/s}^3)(5.0\ \text{s}) = 60\ \text{rad/s}^2$$

EVALUATE: Note that the angular acceleration is *not* constant in this situation. The angular velocity ω_z is always increasing because α_z is always positive. Furthermore, the rate at which the angular velocity increases is itself increasing, since α_z increases with time.

9.7 When the rotation axis is fixed, the angular acceleration and angular velocity vectors both lie along that axis.



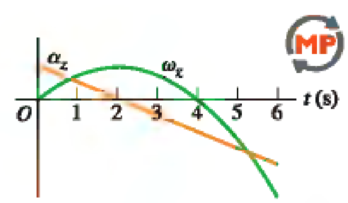
Angular Acceleration As a Vector

Just as we did for angular velocity, it's useful to define an angular acceleration *vector* $\vec{\alpha}$. Mathematically, $\vec{\alpha}$ is the time derivative of the angular velocity vector $\vec{\omega}$. If the object rotates around the fixed z-axis, then $\vec{\alpha}$ has only a z-component; the quantity α_z is just that component. In this case, $\vec{\alpha}$ is in the same direction as $\vec{\omega}$ if the rotation is speeding up and opposite to $\vec{\omega}$ if the rotation is slowing down (Fig. 9.7).

The angular acceleration vector will be particularly useful in Chapter 10 when we discuss what happens when the rotation axis can change direction. In this chapter, however, the rotation axis will always be fixed and we need use only the z-component α_z.

Test Your Understanding of Section 9.1

The figure shows a graph of ω_z and α_z versus time for a particular rotating body. (a) During which time intervals is the rotation speeding up? (i) $0 < t < 2$ s; (ii) $2\ \text{s} < t < 4$ s; (iii) $4\ \text{s} < t < 6$ s. (b) During which time intervals is the rotation slowing down? (i) $0 < t < 2$ s; (ii) $2\ \text{s} < t < 4$ s; (iii) $4\ \text{s} < 5 < 6$ s.



9.2 Rotation with Constant Angular Acceleration

In Chapter 2 we found that straight-line motion is particularly simple when the acceleration is constant. This is also true of rotational motion about a fixed axis. When the angular acceleration is constant, we can derive equations for angular velocity and angular position using exactly the same procedure that we used for straight-line motion in Section 2.4. In fact, the equations we are about to derive are identical to Eqs. (2.8), (2.12), (2.13), and (2.14) if we replace x with θ, v_x with ω_z, and a_x with α_z. We suggest that you review Section 2.4 before continuing.

Let ω_{0z} be the angular velocity of a rigid body at time $t = 0$, and let ω_z be its angular velocity at any later time t. The angular acceleration α_z is constant and equal to the average value for any interval. Using Eq. (9.4) with the interval from 0 to t, we find

$$\alpha_z = \frac{\omega_z - \omega_{0z}}{t - 0} \qquad \text{or}$$

$$\omega_z = \omega_{0z} + \alpha_z t \qquad \text{(constant angular acceleration only)} \tag{9.7}$$

The product $\alpha_z t$ is the total change in ω_z between $t = 0$ and the later time t; the angular velocity ω_z at time t is the sum of the initial value ω_{0z} and this total change.

With constant angular acceleration, the angular velocity changes at a uniform rate, so its average value between 0 and t is the average of the initial and final values:

$$\omega_{\text{av-}z} = \frac{\omega_{0z} + \omega_z}{2} \tag{9.8}$$

We also know that $\omega_{\text{av-}z}$ is the total angular displacement $(\theta - \theta_0)$ divided by the time interval $(t - 0)$:

$$\omega_{\text{av-}z} = \frac{\theta - \theta_0}{t - 0} \tag{9.9}$$

When we equate Eqs. (9.8) and (9.9) and multiply the result by t, we get

$$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t \qquad \text{(constant angular acceleration only)} \tag{9.10}$$

To obtain a relationship between θ and t that doesn't contain ω_z, we substitute Eq. (9.7) into Eq. (9.10):

$$\theta - \theta_0 = \frac{1}{2}[\omega_{0z} + (\omega_{0z} + \alpha_z t)]t \qquad \text{or}$$

$$\theta = \theta_0 + \omega_{0z}t + \frac{1}{2}\alpha_z t^2 \qquad \text{(constant angular acceleration only)} \tag{9.11}$$

That is, if at the initial time $t = 0$ the body is at angular position θ_0 and has angular velocity ω_{0z}, then its angular position θ at any later time t is the sum of three terms: its initial angular position θ_0, plus the rotation $\omega_{0z}t$ it would have if the angular velocity were constant, plus an additional rotation $\frac{1}{2}\alpha_z t^2$ caused by the changing angular velocity.

Following the same procedure as for straight-line motion in Section 2.4, we can combine Eqs. (9.7) and (9.11) to obtain a relationship between θ and ω_z that does not contain t. We invite you to work out the details, following the same procedure we used to get Eq. (2.13). (See Exercise 9.12.) In fact, because of the perfect analogy between straight-line and rotational quantities, we can simply take Eq. (2.13) and replace each straight-line quantity by its rotational analog. We get

$$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0) \qquad \text{(constant angular acceleration only)} \tag{9.12}$$

CAUTION **Constant angular acceleration** Keep in mind that all of these results are valid *only* when the angular acceleration α_z is *constant;* be careful not to try to apply them to problems in which α_z is *not* constant. Table 9.1 shows the analogy between Eqs. (9.7), (9.10), (9.11), and (9.12) for fixed-axis rotation with constant angular acceleration and the corresponding equations for straight-line motion with constant linear acceleration.

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7.7 Rotational Kinematics

Table 9.1 Comparison of Linear and Angular Motion with Constant Acceleration

Straight-Line Motion with Constant Linear Acceleration	Fixed-Axis Rotation with Constant Angular Acceleration
$a_x = \text{constant}$	$\alpha_z = \text{constant}$
$v_x = v_{0x} + a_x t$	$\omega_z = \omega_{0z} + \alpha_z t$
$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$	$\theta = \theta_0 + \omega_{0z}t + \frac{1}{2}\alpha_z t^2$
$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$	$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$
$x - x_0 = \frac{1}{2}(v_x + v_{0x})t$	$\theta - \theta_0 = \frac{1}{2}(\omega_z + \omega_{0z})t$

Example 9.3 Rotation with constant angular acceleration

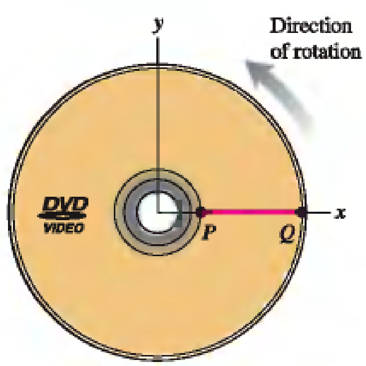
You have just finished watching a movie on DVD and the disc is slowing to a stop. The angular velocity of the disc at $t = 0$ is 27.5 rad/s and its angular acceleration is a constant $-10.0\ \text{rad/s}^2$. A line PQ on the surface of the disc lies along the $+x$-axis at $t = 0$ (Fig. 9.8). (a) What is the disc's angular velocity at $t = 0.300$ s? (b) What angle does the line PQ make with the $+x$-axis at this time?

SOLUTION

IDENTIFY: The angular acceleration of the disc is constant, so we can use any of the equations derived in this section. Our target variables are the angular velocity and the angular displacement at $t = 0.300$ s.

SET UP: We are given the initial angular velocity $\omega_{0z} = 27.5$ rad/s, the initial angle $\theta_0 = 0$ between the line PQ and the $+x$-axis, the angular acceleration $\alpha_z = -10.0\ \text{rad/s}^2$, and the time $t = 0.300$ s.

9.8 A line PQ on a rotating DVD at $t = 0$.



With this information it's easiest to use Eqs. (9.7) and (9.11) to find the target variables ω_z and θ, respectively.

EXECUTE: (a) From Eq. (9.7), at $t = 0.300$ s we have

$$\omega_z = \omega_{0z} + \alpha_z t = 27.5\ \text{rad/s} + (-10.0\ \text{rad/s}^2)(0.300\ \text{s}) = 24.5\ \text{rad/s}$$

(b) From Eq. (9.11),

$$\theta = \theta_0 + \omega_{0z}t + \frac{1}{2}\alpha_z t^2$$
$$= 0 + (27.5\ \text{rad/s})(0.300\ \text{s}) + \frac{1}{2}(-10.0\ \text{rad/s}^2)(0.300\ \text{s})^2$$
$$= 7.80\ \text{rad} = 7.80\ \text{rad}\left(\frac{1\ \text{rev}}{2\pi\ \text{rad}}\right) = 1.24\ \text{rev}$$

The DVD has turned through one complete revolution plus an additional 0.24 revolution—that is, through an additional angle of $(0.24\ \text{rev})(360°/\text{rev}) = 87°$. Hence the line PQ is at an angle of 87° with the $+x$-axis.

EVALUATE: Our answer to part (a) tells us that the angular velocity has decreased. This is as it should be, since α_z is negative. We can also use our answer for ω_z in part (a) to check our result for θ in part (b). To do so, we solve Eq. (9.12), $\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$, for the angle θ:

$$\theta = \theta_0 + \left(\frac{\omega_z^2 - \omega_{0z}^2}{2\alpha_z}\right)$$
$$= 0 + \frac{(24.5\ \text{rad/s})^2 - (27.5\ \text{rad/s})^2}{2(-10.0\ \text{rad/s}^2)} = 7.80\ \text{rad}$$

which agrees with the result we found earlier.

Test Your Understanding of Section 9.2 Suppose the DVD in Example 9.3 was initially spinning at twice the rate (55.0 rad/s rather than 27.5 rad/s) and slowed down at twice the rate $(-20.0\ \text{rad/s}^2$ rather than $-10.0\ \text{rad/s}^2)$. (a) Compared to the situation in Example 9.3, how long would it take the DVD to come to a stop? (i) the same amount of time; (ii) twice as much time; (iii) 4 times as much time; (iv) $\frac{1}{2}$ as much time; (v) $\frac{1}{4}$ as much time. (b) Compared to the situation in Example 9.3, through how many revolutions would the DVD rotate before coming to a stop? (i) the same number of revolutions; (ii) twice as many revolutions; (iii) 4 times as many revolutions; (iv) $\frac{1}{2}$ as many revolutions; (v) $\frac{1}{4}$ as many revolutions.

9.3 Relating Linear and Angular Kinematics

How do we find the linear speed and acceleration of a particular point in a rotating rigid body? We need to answer this question to proceed with our study of rotation. For example, to find the kinetic energy of a rotating body, we have to start from $K = \frac{1}{2}mv^2$ for a particle, and this requires knowing the speed v for each particle in the body. So it's worthwhile to develop general relationships between the *angular* speed and acceleration of a rigid body rotating about a fixed axis and the *linear* speed and acceleration of a specific point or particle in the body.

Linear Speed in Rigid-Body Rotation

When a rigid body rotates about a fixed axis, every particle in the body moves in a circular path. The circle lies in a plane perpendicular to the axis and is centered on the axis. The speed of a particle is directly proportional to the body's angular velocity; the faster the body rotates, the greater the speed of each particle. In Fig. 9.9, point P is a constant distance r from the axis of rotation, so it moves in a circle of radius r. At any time, the angle θ (in radians) and the arc length s are related by

$$s = r\theta$$

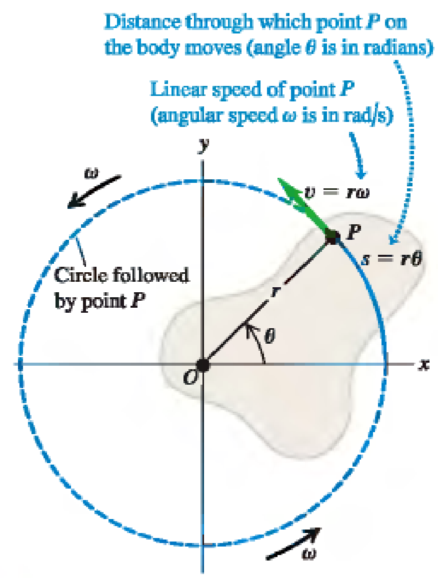
We take the time derivative of this, noting that r is constant for any specific particle, and take the absolute value of both sides:

$$\left|\frac{ds}{dt}\right| = r\left|\frac{d\theta}{dt}\right|$$

Now $|ds/dt|$ is the absolute value of the rate of change of arc length, which is equal to the instantaneous *linear* speed v of the particle. Analogously, $|d\theta/dt|$, the absolute value of the rate of change of the angle, is the instantaneous **angular speed** ω—that is, the magnitude of the instantaneous angular velocity in rad/s. Thus

$$v = r\omega \qquad \text{(relationship between linear and angular speeds)} \qquad (9.13)$$

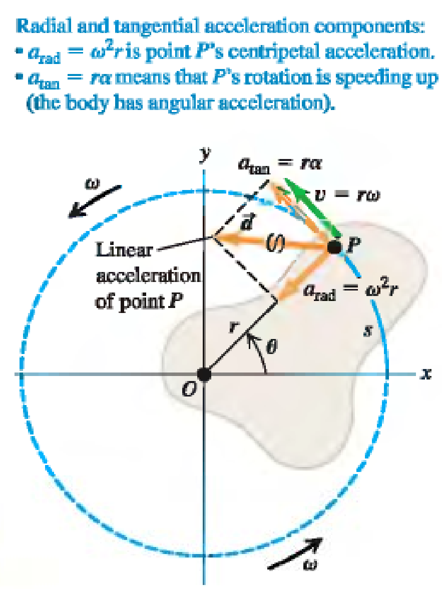
9.9 A rigid body rotating about a fixed axis through point O.



The farther a point is from the axis, the greater its linear speed. The *direction* of the linear velocity *vector* is tangent to its circular path at each point (Fig. 9.9).

CAUTION **Speed vs. velocity** Keep in mind the distinction between the linear and angular *speeds* v and ω, which appear in Eq. (9.13), and the linear and angular *velocities* v_x and ω_z. The quantities without subscripts, v and ω, are never negative; they are the magnitudes of the vectors $\vec{v}$ and $\vec{\omega}$, respectively, and their values tell you only how fast a particle is moving (v) or how fast a body is rotating (ω). The corresponding quantities with subscripts, v_x and ω_z, can be either positive or negative; their signs tell you the direction of the motion.

9.10 A rigid body whose rotation is speeding up. The acceleration of point P has a component a_{rad} toward the rotation axis (perpendicular to $\vec{v}$) and a component a_{tan} along the circle that point P follows (parallel to $\vec{v}$).



Linear Acceleration in Rigid-Body Rotation

We can represent the acceleration of a particle moving in a circle in terms of its centripetal and tangential components, a_{rad} and a_{tan} (Fig. 9.10), as we did in Section 3.4. It would be a good idea to review that section now. We found that the **tangential component of acceleration** a_{tan}, the component parallel to the instantaneous velocity, acts to change the *magnitude* of the particle's velocity (i.e., the speed) and is equal to the rate of change of speed. Taking the derivative of Eq. (9.13), we find

$$a_{tan} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha \qquad \text{(tangential acceleration of a point on a rotating body)} \qquad (9.14)$$

This component of a particle's acceleration is always tangent to the circular path of the particle.

The quantity $\alpha = d\omega/dt$ in Eq. (9.14) is the rate of change of the angular *speed*. It is not quite the same as $\alpha_z = d\omega_z/dt$, which is the rate of change of the angular *velocity*. For example, consider a body rotating so that its angular velocity vector points in the $-z$-direction (Fig. 9.5b). If the body is gaining angular speed at a rate of 10 rad/s per second, then $\alpha = 10 \text{ rad/s}^2$. But ω_z is negative and becoming more negative as the rotation gains speed, so $\alpha_z = -10 \text{ rad/s}^2$. The rule for rotation about a fixed axis is that α is equal to α_z if ω_z is positive but equal to $-\alpha_z$ if ω_z is negative.

The component of the particle's acceleration directed toward the rotation axis, the **centripetal component of acceleration** a_{rad}, is associated with the change of *direction* of the particle's velocity. In Section 3.4 we worked out the relationship $a_{rad} = v^2/r$. We can express this in terms of ω by using Eq. (9.13):

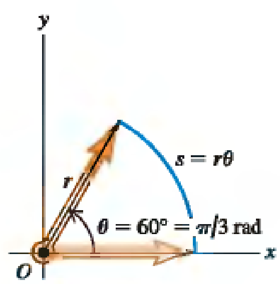
$$a_{rad} = \frac{v^2}{r} = \omega^2 r \quad \text{(centripetal acceleration of a point on a rotating body)} \tag{9.15}$$

This is true at each instant, *even when ω and v are not constant.* The centripetal component always points toward the axis of rotation.

The vector sum of the centripetal and tangential components of acceleration of a particle in a rotating body is the linear acceleration $\vec{a}$ (Fig. 9.10).

CAUTION **Use angles in radians in all equations** It's important to remember that Eq. (9.1), $s = r\theta$, is valid *only* when θ is measured in radians. The same is true of any equation derived from this, including Eqs. (9.13), (9.14), and (9.15). When you use these equations, you *must* express the angular quantities in radians, not revolutions or degrees (Fig. 9.11).

9.11 Always use radians when relating linear and angular quantities.



Equations (9.1), (9.13), and (9.14) also apply to any particle that has the same tangential velocity as a point in a rotating rigid body. For example, when a rope wound around a circular cylinder unwraps without stretching or slipping, its speed and acceleration at any instant are equal to the speed and tangential acceleration of the point at which it is tangent to the cylinder. The same principle holds for situations such as bicycle chains and sprockets, belts and pulleys that turn without slipping, and so on. We will have several opportunities to use these relationships later in this chapter and in Chapter 10. Note that Eq. (9.15) for the centripetal component a_{rad} is applicable to the rope or chain *only* at points that are in contact with the cylinder or sprocket. Other points do not have the same acceleration toward the center of the circle that points on the cylinder or sprocket have.

Example 9.4 Throwing a discus

A discus thrower moves the discus in a circle of radius 80.0 cm. At a certain instant, the thrower is spinning at an angular speed of 10.0 rad/s and the angular speed is increasing at 50.0 rad/s². At this instant, find the tangential and centripetal components of the acceleration of the discus and the magnitude of the acceleration.

SOLUTION

IDENTIFY: We model the discus as a particle traveling on a circular path (Fig. 9.12a), so we can use the ideas developed in this section.

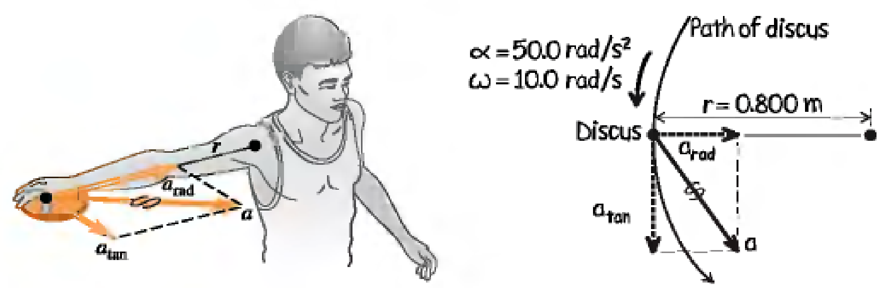
SET UP: We are given the radius $r = 0.800$ m, the angular speed $\omega = 10.0$ rad/s, and the rate of change of angular speed $\alpha = 50.0 \text{ rad/s}^2$. (Fig. 9.12b).The first two target variables are the acceleration components a_{tan} and a_{rad}, which we'll find with Eqs. (9.14) and (9.15), respectively. Given these components of the acceleration vector, we'll find its magnitude a (the third target variable) using the Pythagorean theorem.

EXECUTE: From Eqs. (9.14) and (9.15),

$$a_{tan} = r\alpha = (0.800 \text{ m})(50.0 \text{ rad/s}^2) = 40.0 \text{ m/s}^2$$
$$a_{rad} = \omega^2 r = (10.0 \text{ rad/s})^2(0.800 \text{ m}) = 80.0 \text{ m/s}^2$$

The magnitude of the acceleration vector is

$$a = \sqrt{a_{tan}^{\ 2} + a_{rad}^{\ 2}} = 89.4 \text{ m/s}^2$$



9.12 (a) Whirling a discus in a circle. (b) Our sketch showing the acceleration components for the discus.

EVALUATE: Note that we dropped the unit "radian" from our results for a_{tan}, a_{rad}, and a. We can do this because "radian" is a dimensionless quantity.

The magnitude a is about nine times g, the acceleration due to gravity. Can you show that if the angular speed doubles to 20.0 rad/s while α remains the same, the acceleration magnitude a increases to 322 m/s², or almost $33g$?

Example 9.5 Designing a propeller

You are asked to design an airplane propeller to turn at 2400 rpm. The forward airspeed of the plane is to be 75.0 m/s (270 km/h, or about 168 mi/h), and the speed of the tips of the propeller blades through the air must not exceed 270 m/s (Fig. 9.13a). (This is about 0.80 times the speed of sound in air. If the propeller tips were to move too close to the speed of sound, they would produce a tremendous amount of noise.) (a) What is the maximum radius the propeller can have? (b) With this radius, what is the acceleration of the propeller tip?

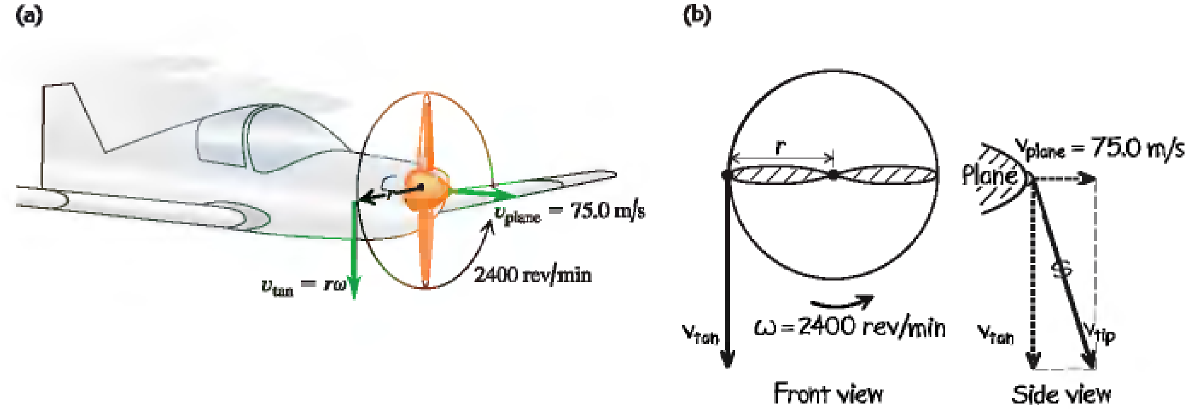
SOLUTION

IDENTIFY: The object of interest in this example is a particle at the tip of the propeller; our target variables are the particle's distance from the axis and its acceleration. Note that the speed of this particle through the air (which cannot exceed 270 m/s) is due to both the propeller's rotation *and* the forward motion of the airplane.

SET UP: As Fig. 9.13b shows, the velocity $\vec{v}_{tip}$ of a particle at the propeller tip is the vector sum of its tangential velocity due to the propeller's rotation (magnitude v_{tan}, given by Eq. (9.13)) and the forward velocity of the airplane (magnitude $v_{plane} = 75.0$ m/s). The rotation plane of the propeller is perpendicular to the direction of flight, so these two vectors are perpendicular and we can use the Pythagorean theorem to relate v_{tan} and v_{plane} to v_{tip}. We will then set $v_{tip} = 270$ m/s and solve for the radius r. Note that the angular speed of the propeller is constant, so the acceleration of the propeller tip has only a radial component; we'll find it using Eq. (9.15).

EXECUTE: We first convert ω to rad/s (see Fig. 9.11):

$$\omega = 2400 \text{ rpm} = \left(2400\,\frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right)$$
$$= 251 \text{ rad/s}$$



9.13 (a) A propeller-driven airplane in flight. (b) Our sketch showing the velocity components for the propeller tip.

Continued

(a) From Fig. 9.13b and Eq. (9.13), the velocity magnitude v_{total} is given by

$$v_{tip}^2 = v_{plane}^2 + v_{tan}^2 = v_{plane}^2 + r^2\omega^2 \quad \text{so}$$

$$r^2 = \frac{v_{tip}^2 - v_{plane}^2}{\omega^2} \quad \text{and} \quad r = \frac{\sqrt{v_{tip}^2 - v_{plane}^2}}{\omega}$$

If $v_{tip} = 270 \text{ m/s}$, the propeller radius is

$$r = \frac{\sqrt{(270 \text{ m/s})^2 - (75.0 \text{ m/s})^2}}{251 \text{ rad/s}} = 1.03 \text{ m}$$

(b) The centripetal acceleration is

$$a_{rad} = \omega^2 r$$
$$= (251 \text{ rad/s})^2(1.03 \text{ m}) = 6.5 \times 10^4 \text{ m/s}^2$$

The *tangential* acceleration is zero because the angular speed is constant.

EVALUATE: From $\sum\vec{F} = m\vec{a}$, the propeller must exert a force of 6.5×10^4 N on each kilogram of material at its tip! This is why propellers are made out of tough material, usually aluminum alloy.

Conceptual Example 9.6 Bicycle gears

How are the angular speeds of the two bicycle sprockets in Fig. 9.14 related to the number of teeth on each sprocket?

SOLUTION

The chain does not slip or stretch, so it moves at the same tangential speed v on both sprockets. From Eq. (9.13),

$$v = r_{front}\omega_{front} = r_{rear}\omega_{rear} \quad \text{so} \quad \frac{\omega_{rear}}{\omega_{front}} = \frac{r_{front}}{r_{rear}}$$

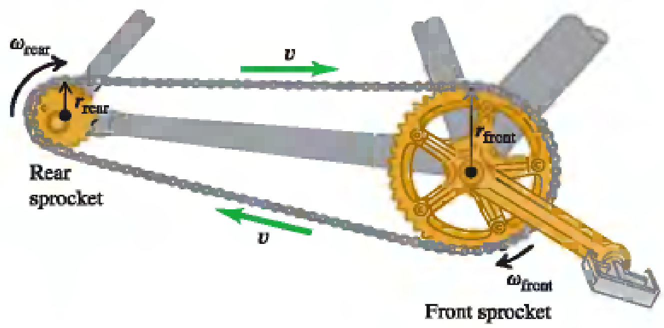
The angular speed is inversely proportional to the radius. This relationship also holds for pulleys connected by a belt, provided the belt doesn't slip. For chain sprockets the teeth must be equally spaced on the circumferences of both sprockets for the chain to mesh properly with both. Let N_{front} and N_{rear} be the numbers of teeth; the condition that the tooth spacing is the same on both sprockets is

$$\frac{2\pi r_{front}}{N_{front}} = \frac{2\pi r_{rear}}{N_{rear}} \quad \text{or} \quad \frac{r_{front}}{r_{rear}} = \frac{N_{front}}{N_{rear}}$$

Combining this with the other equation, we get

$$\frac{\omega_{rear}}{\omega_{front}} = \frac{N_{front}}{N_{rear}}$$

9.14 The sprockets and chain of a bicycle.



The angular speed of each sprocket is inversely proportional to the number of teeth. On a multispeed bike, you get the highest angular speed ω_{rear} of the rear wheel for a given pedaling rate ω_{front} when the ratio N_{front}/N_{rear} is maximum; this means using the largest-radius front sprocket (largest N_{front}) and the smallest-radius rear sprocket (smallest N_{rear}).

Test Your Understanding of Section 9.3 Information is stored on a CD or DVD (see Fig. 9.8) in a coded pattern of tiny pits. The pits are arranged in a track that spirals outward toward the rim of the disc. As the disc spins inside a player, the track is scanned at a constant *linear* speed. How must the rotation speed of the disc change as the player's scanning head moves over the track? (i) The rotation speed must increase. (ii) The rotation speed must decrease. (iii) The rotation speed must stay the same.

9.4 Energy in Rotational Motion

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7.7 Rotational Inertia

A rotating rigid body consists of mass in motion, so it has kinetic energy. As we will see, we can express this kinetic energy in terms of the body's angular speed and a new quantity, called *moment of inertia,* that depends on the body's mass and how the mass is distributed.

To begin, we think of a body as being made up of a large number of particles, with masses $m_1, m_2, \ldots$ at distances $r_1, r_2, \ldots$ from the axis of rotation. We label the particles with the index i: The mass of the ith particle is m_i and its distance from the axis of rotation is r_i. The particles don't necessarily all lie in the same plane, so we specify that r_i is the *perpendicular* distance from the axis to the ith particle.

When a rigid body rotates about a fixed axis, the speed v_i of the ith particle is given by Eq. (9.13), $v_i = r_i\omega$, where ω is the body's angular speed. Different particles have different values of r, but ω is the same for all (otherwise, the body wouldn't be rigid). The kinetic energy of the ith particle can be expressed as

$$\frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i r_i^2 \omega^2$$

The *total* kinetic energy of the body is the sum of the kinetic energies of all its particles:

$$K = \frac{1}{2}m_1 r_1^2 \omega^2 + \frac{1}{2}m_2 r_2^2 \omega^2 + \cdots = \sum_i \frac{1}{2}m_i r_i^2 \omega^2$$

Taking the common factor $\omega^2/2$ out of this expression, we get

$$K = \frac{1}{2}(m_1 r_1^2 + m_2 r_2^2 + \cdots)\omega^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

The quantity in parentheses, obtained by multiplying the mass of each particle by the square of its distance from the axis of rotation and adding these products, is denoted by I and is called the **moment of inertia** of the body for this rotation axis:

$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots = \sum_i m_i r_i^2 \qquad \text{(definition of moment of inertia)} \qquad (9.16)$$

The word "moment" means that I depends on how the body's mass is distributed in space; it has nothing to do with a "moment" of time. For a body with a given rotation axis and a given total mass, the greater the distance from the axis to the particles that make up the body, the greater the moment of inertia. In a rigid body, the distances r_i are all constant and I is independent of how the body rotates around the given axis. The SI unit of moment of inertia is the kilogram-meter^2 ($\text{kg} \cdot \text{m}^2$).

In terms of moment of inertia I, the **rotational kinetic energy** K of a rigid body is

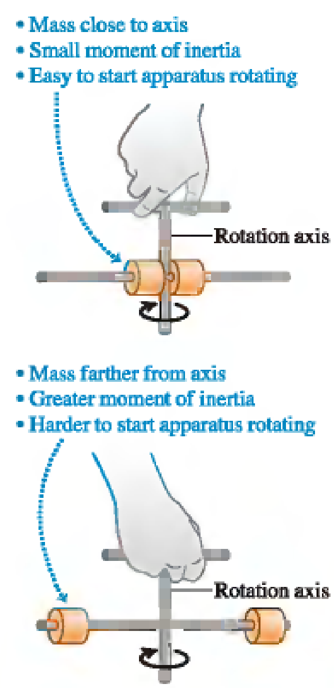
$$K = \frac{1}{2}I\omega^2 \qquad \text{(rotational kinetic energy of a rigid body)} \qquad (9.17)$$

The kinetic energy given by Eq. (9.17) is *not* a new form of energy; it's simply the sum of the kinetic energies of the individual particles that make up the rotating rigid body. To use Eq. (9.17), ω *must* be measured in radians per second, not revolutions or degrees per second, to give K in joules. That's because we used $v_i = r_i\omega$ in our derivation.

Equation (9.17) gives a simple physical interpretation of moment of inertia: *The greater the moment of inertia, the greater the kinetic energy of a rigid body rotating with a given angular speed* ω. We learned in Chapter 6 that the kinetic energy of a body equals the amount of work done to accelerate that body from rest. So the greater a body's moment of inertia, the harder it is to start the body rotating if it's at rest and the harder it is to stop its rotation if it's already rotating (Fig. 9.15). For this reason, I is also called the *rotational inertia*.

The next example shows how *changing* the rotation axis can affect the value of I.

9.15 An apparatus free to rotate around a vertical axis. To vary the moment of inertia, the two equal-mass cylinders can be locked into different positions on the horizontal shaft.



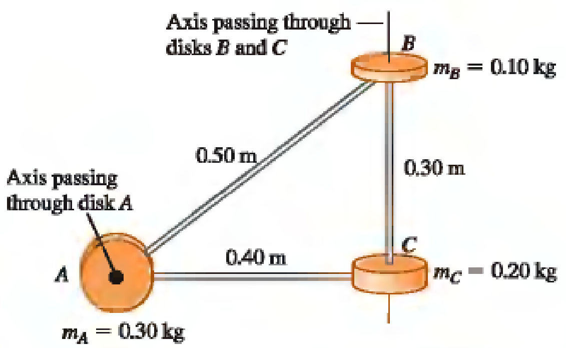
Example 9.7 Moments of inertia for different rotation axes

An engineer is designing a machine part consisting of three heavy disks linked by lightweight struts (Fig. 9.16). (a) What is the moment of inertia of this body about an axis through the center of disk A, perpendicular to the plane of the diagram? (b) What is the moment of inertia about an axis through the centers of disks B and C? (c) If the body rotates about an axis through A perpendicular to the plane of the diagram, with angular speed $\omega = 4.0$ rad/s, what is its kinetic energy?

SOLUTION

IDENTIFY: We'll consider the disks as massive particles and the lightweight struts as massless rods. Then we can use the ideas of this section to calculate the moment of inertia of this collection of three particles.

9.16 An oddly shaped machine part.



SET UP: In parts (a) and (b), we'll use Eq. (9.16) to find the moments of inertia for each of the two axes. Given the moment of inertia for axis A, we'll use Eq. (9.17) in part (c) to find the rotational kinetic energy.

EXECUTE: (a) The particle at point A lies *on* the axis. Its distance r from the axis is zero, so it contributes nothing to the moment of inertia. Equation (9.16) gives

$$I = \sum m_i r_i^2 = (0.10\text{ kg})(0.50\text{ m})^2 + (0.20\text{ kg})(0.40\text{ m})^2$$
$$= 0.057\text{ kg}\cdot\text{m}^2$$

(b) The particles at B and C both lie *on* the axis, so for them $r = 0$ and neither contributes to the moment of inertia. Only A contributes, and we have

$$I = \sum m_i r_i^2 = (0.30\text{ kg})(0.40\text{ m})^2 = 0.048\text{ kg}\cdot\text{m}^2$$

(c) From Eq. (9.17),

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}(0.057\text{ kg}\cdot\text{m}^2)(4.0\text{ rad/s})^2 = 0.46\text{ J}$$

EVALUATE: Our results show that the moment of inertia for the axis through A is greater than that for the axis through B and C. Hence, of the two axes, it's easier to make the machine part rotate about the axis through B and C.

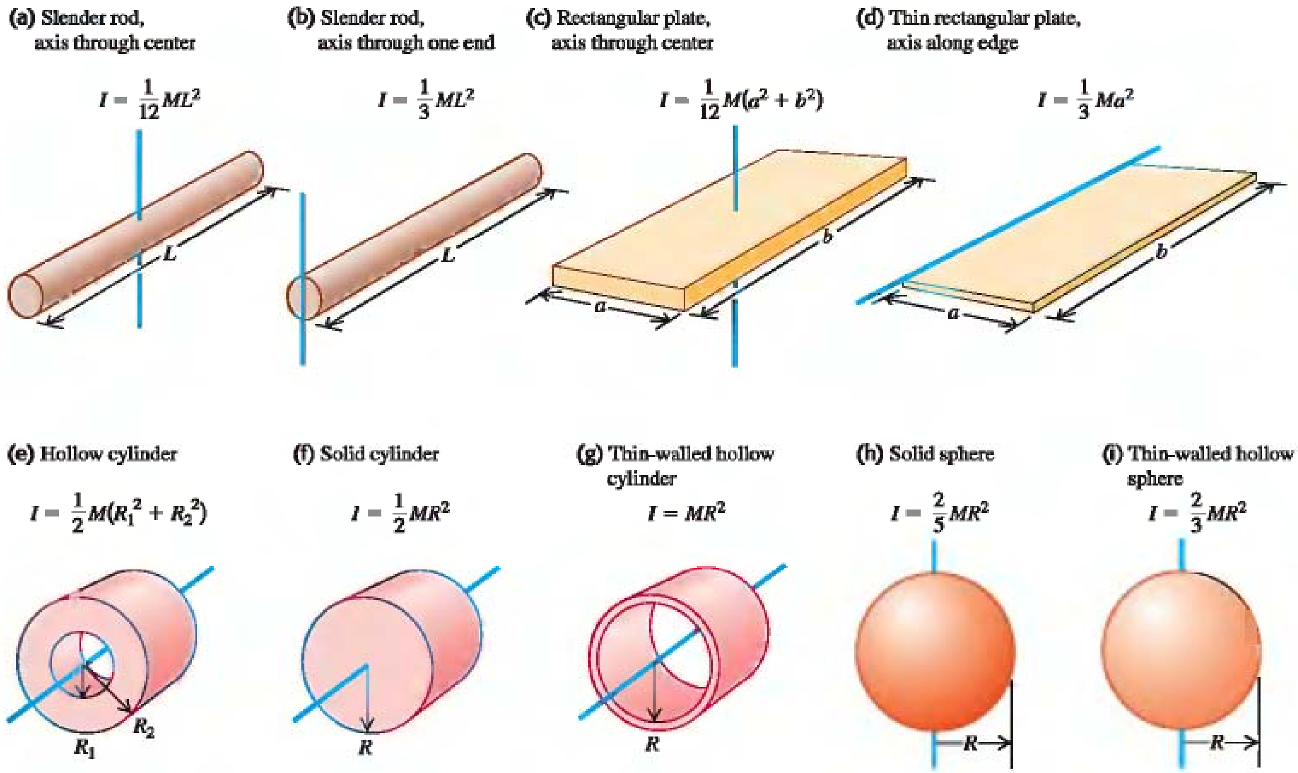
CAUTION **Moment of inertia depends on the choice of axis** The results of parts (a) and (b) of Example 9.7 show that the moment of inertia of a body depends on the location and orientation of the axis. It's not enough to just say, "The moment of inertia of this body is $0.048\text{ kg}\cdot\text{m}^2$." We have to be specific and say, "The moment of inertia of this body *about the axis through B and C* is $0.048\text{ kg}\cdot\text{m}^2$."

In Example 9.7 we represented the body as several point masses, and we evaluated the sum in Eq. (9.16) directly. When the body is a *continuous* distribution of matter, such as a solid cylinder or plate, the sum becomes an integral, and we need to use calculus to calculate the moment of inertia. We will give several examples of such calculations in Section 9.6; meanwhile, Table 9.2 gives moments of inertia for several familiar shapes in terms of their masses and dimensions. Each body shown in Table 9.2 is *uniform;* that is, the density has the same value at all points within the solid parts of the body.

CAUTION **Computing the moment of inertia** You may be tempted to try to compute the moment of inertia of a body by assuming that all the mass is concentrated at the center of mass and multiplying the total mass by the square of the distance from the center of mass to the axis. Resist that temptation; it doesn't work! For example, when a uniform thin rod of length L and mass M is pivoted about an axis through one end, perpendicular to the rod, the moment of inertia is $I = ML^2/3$ [case (b) in Table 9.2]. If we took the mass as concentrated at the center, a distance $L/2$ from the axis, we would obtain the *incorrect* result $I = M(L/2)^2 = ML^2/4$.

Now that we know how to calculate the kinetic energy of a rotating rigid body, we can apply the energy principles of Chapter 7 to rotational motion. Here are some points of strategy and some examples.

Table 9.2 Moments of Inertia of Various Bodies



Problem-Solving Strategy 9.1 Rotational Energy

IDENTIFY *the relevant concepts:* You can use work–energy relationships and conservation of energy to find relationships involving position and motion of a rigid body rotating around a fixed axis. As we saw in Chapter 7, the energy method is usually not helpful for problems that involve elapsed time. In Chapter 10 we'll see how to approach rotational problems of this kind.

SET UP *the problem* using the same steps as in Problem-Solving Strategy 7.1 (Section 7.1), with the following addition:

5. Many problems involve a rope or cable wrapped around a rotating rigid body, which functions as a pulley. In these situations, remember that the point on the pulley that contacts the rope has the same linear speed as the rope, provided the rope doesn't slip on the pulley. You can then take advantage of Eqs. (9.13) and (9.14), which relate the linear speed and tangential acceleration of a point on a rigid body to the angular velocity and angular acceleration of the body. Examples 9.8 and 9.9 illustrate this point.

EXECUTE *the solution:* As in Chapter 7, write expressions for the initial and final kinetic and potential energies (K_1, K_2, U_1, and U_2) and the nonconservative work W_{other} (if any). The new feature is rotational kinetic energy, which is expressed in terms of the body's moment of inertia I for the given axis and its angular speed ω $\left(K = \frac{1}{2}I\omega^2\right)$ instead of its mass m and speed v. Substitute these expressions into $K_1 + U_1 + W_{other} = K_2 + U_2$ (if nonconservative work is done) or $K_1 + U_1 = K_2 + U_2$ (if only conservative work is done) and solve for the target variable(s). As in Chapter 7, it's helpful to draw bar graphs showing the initial and final values of K, U, and $E = K + U$.

EVALUATE *your answer:* As always, check whether your answer makes physical sense.

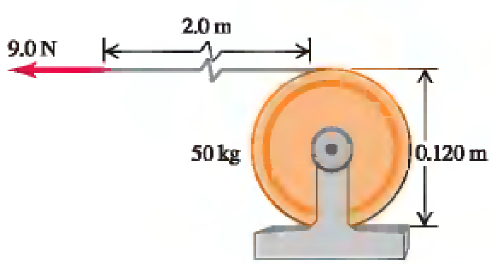
Example 9.8 An unwinding cable I

A light, flexible, nonstretching cable is wrapped several times around a winch drum, a solid cylinder of mass 50 kg and diameter 0.120 m, which rotates about a stationary horizontal axis held by frictionless bearings (Fig. 9.17). The free end of the cable is pulled with a constant 9.0-N force for a distance of 2.0 m. It unwinds without slipping and turns the cylinder. If the cylinder is initially at rest, find its final angular speed and the final speed of the cable.

SOLUTION

IDENTIFY: We will solve this problem using energy methods. Point 1 is when the cylinder first begins to move, and point 2 is when the cable has moved 2.0 m. We'll assume that the light cable is massless, so that only the cylinder has kinetic energy. The cylinder doesn't move vertically, so there are no changes in gravitational potential energy. There is friction between the cable and the cylinder, which is what makes the cylinder rotate when the cable is pulled. But because the cable doesn't slip, there is no sliding of the cable relative to the cylinder and no mechanical energy is lost in friction. Because the cable is massless, the force that the cable exerts on the cylinder rim is equal to the applied force F.

9.17 A cable unwinds from a cylinder (side view).



SET UP: The cylinder starts at rest, so the initial kinetic energy is $K_1 = 0$. Between points 1 and 2 the force F does work on the cylinder over a distance $s = 2.0$ m. As a result, the kinetic energy at point 2 is $K_2 = \frac{1}{2}I\omega^2$. One of our target variables is ω; the other is the speed of the cable at point 2, which is equal to the tangential speed v of the cylinder at that point. We'll find v from ω by using Eq. (9.13).

EXECUTE: The work done on the cylinder is $W_{\text{other}} = Fs = (9.0\text{ N})(2.0\text{ m}) = 18\text{ J}$. From Table 9.2 the moment of inertia is

$$I = \frac{1}{2}mR^2 = \frac{1}{2}(50\text{ kg})(0.060\text{ m})^2 = 0.090\text{ kg}\cdot\text{m}^2$$

(The radius R is half the diameter of the cylinder.) The relationship $K_1 + U_1 + W_{\text{other}} = K_2 + U_2$ then gives

$$0 + 0 + W_{\text{other}} = \frac{1}{2}I\omega^2 + 0$$

$$\omega = \sqrt{\frac{2W_{\text{other}}}{I}} = \sqrt{\frac{2(18\text{ J})}{0.090\text{ kg}\cdot\text{m}^2}} = 20\text{ rad/s}$$

The final tangential speed of the cylinder, and hence the final speed of the cable, is

$$v = R\omega = (0.060\text{ m})(20\text{ rad/s}) = 1.2\text{ m/s}$$

EVALUATE: If the mass of the cable can't be neglected, then some of the work done would go into the kinetic energy of the cable. Hence the cylinder would end up with less kinetic energy and a smaller angular speed than we calculated here.

Example 9.9 An unwinding cable II

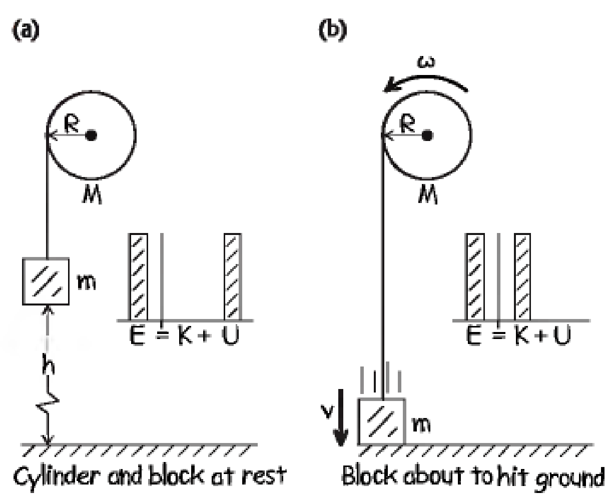
We wrap a light, flexible cable around a solid cylinder with mass M and radius R. The cylinder rotates with negligible friction about a stationary horizontal axis. We tie the free end of the cable to a block of mass m and release the object with no initial velocity at a distance h above the floor. As the block falls, the cable unwinds without stretching or slipping, turning the cylinder. Find the speed of the falling block and the angular speed of the cylinder just as the block strikes the floor.

SOLUTION

IDENTIFY: As in Example 9.8, the cable doesn't slip and friction does no work. The cable does no *net* work; at its upper end the force and displacement are in the same direction, and at its lower end they are in opposite directions. Thus the total work done by the two ends of the cable is zero. Hence only gravity does work, and so mechanical energy is conserved.

SET UP: Figure 9.18a shows the situation just before the block begins to fall. At this point the system has no kinetic energy, so

9.18 Our sketches for this problem.



$K_1 = 0$. We take the potential energy to be zero when the block is at floor level; then $U_1 = mgh$ and $U_2 = 0$. (We can ignore the gravitational potential energy for the rotating cylinder, since its height doesn't change.) Just before the block hits the floor (Fig. 9.18b), both the block and the cylinder have kinetic energy. The total kinetic energy K_2 at that instant is

$$K_2 = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

From Table 9.2 the moment of inertia of the cylinder is $I = \frac{1}{2}MR^2$. Also, v and ω are related by $v = R\omega$, since the speed of the falling block must be equal to the tangential speed at the outer surface of the cylinder. We'll use these relationships to solve for the target variables v and ω shown in Fig. 9.18b.

EXECUTE: We use our expressions for K_1, U_1, K_2, and U_2 and the relationship $\omega = v/R$ in the energy-conservation equation $K_1 + U_1 = K_2 + U_2$. We then solve for v:

$$0 + mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2 + 0 = \frac{1}{2}\left(m + \frac{1}{2}M\right)v^2$$

$$v = \sqrt{\frac{2gh}{1 + M/2m}}$$

The final angular speed of the cylinder is $\omega = v/R$.

EVALUATE: Let's check some particular cases. When M is much larger than m, v is very small, as we would expect. When M is much smaller than m, v is nearly equal to $\sqrt{2gh}$, which is the speed of a body that falls freely from an initial height h. Does it surprise you that v doesn't depend on the radius of the cylinder?

Gravitational Potential Energy for an Extended Body

In Example 9.9 the cable was of negligible mass, so we could ignore its kinetic energy as well as the gravitational potential energy associated with it. If the mass is *not* negligible, we need to know how to calculate the *gravitational potential energy* associated with such an extended body. If the acceleration of gravity g is the same at all points on the body, the gravitational potential energy is the same as though all the mass were concentrated at the center of mass of the body. Suppose we take the y-axis vertically upward. Then for a body with total mass M, the gravitational potential energy U is simply

$$U = Mgy_{cm} \quad \text{(gravitational potential energy for an extended body)} \quad (9.18)$$

where y_{cm} is the y-coordinate of the center of mass. This expression applies to any extended body, whether it is rigid or not (Fig. 9.19).

To prove Eq. (9.18), we again represent the body as a collection of mass elements m_i. The potential energy for element m_i is $m_i g y_i$, so the total potential energy is

$$U = m_1 g y_1 + m_2 g y_2 + \cdots = (m_1 y_1 + m_2 y_2 + \cdots)g$$

But from Eq. (8.28), which defines the coordinates of the center of mass,

$$m_1 y_1 + m_2 y_2 + \cdots = (m_1 + m_2 + \cdots)y_{cm} = My_{cm}$$

where $M = m_1 + m_2 + \cdots$ is the total mass. Combining this with the above expression for U, we find $U = Mgy_{cm}$ in agreement with Eq. (9.18).

We leave the application of Eq. (9.18) to the problems. We'll make use of this relationship in Chapter 10 in the analysis of rigid-body problems in which the axis of rotation moves.

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7.12 Woman and Flywheel Elevator—Energy Approach

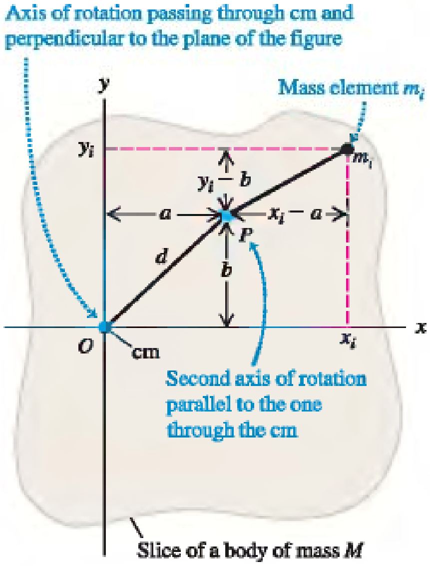
7.13 Rotoride—Energy Approach

9.19 In a technique called the "Fosbury flop" after its innovator, this athlete arches his body as he passes over the bar in the high jump. As a result, his center of mass actually passes *under* the bar. This technique requires a smaller increase in gravitational potential energy [Eq. (9.18)] than the older method of straddling the bar.

Test Your Understanding of Section 9.4 Suppose the cylinder and block in Example 9.9 have the same mass, so $m = M$. Just before the block strikes the floor, which statement is correct about the relationship between the kinetic energy of the falling block and the rotational kinetic energy of the cylinder? (i) The block has more kinetic energy than the cylinder. (ii) The block has less kinetic energy than the cylinder. (iii) The block and the cylinder have equal amounts of kinetic energy.

9.5 Parallel-Axis Theorem

We pointed out in Section 9.4 that a body doesn't have just one moment of inertia. In fact, it has infinitely many, because there are infinitely many axes about which it might rotate. But there is a simple relationship between the moment of inertia I_{cm} of a body of mass M about an axis through its center of



9.20 The mass element m_i has coordinates (x_i, y_i) with respect to an axis of rotation through the center of mass (cm) and coordinates $(x_i - a, y_i - b)$ with respect to the parallel axis through point P.

mass and the moment of inertia I_P about any other axis parallel to the original one but displaced from it by a distance d. This relationship, called the **parallel-axis theorem,** states that

$$I_P = I_{\text{cm}} + Md^2 \qquad \text{(parallel-axis theorem)} \qquad (9.19)$$

To prove this theorem, we consider two axes, both parallel to the z-axis, one through the center of mass and the other through a point P (Fig. 9.20). First we take a very thin slice of the body, parallel to the xy-plane and perpendicular to the z-axis. We take the origin of our coordinate system to be at the center of mass of the body; the coordinates of the center of mass are then $x_{\text{cm}} = y_{\text{cm}} = z_{\text{cm}} = 0$. The axis through the center of mass passes through this thin slice at point O, and the parallel axis passes through point P, whose x- and y-coordinates are (a, b). The distance of this axis from the axis through the center of mass is d, where $d^2 = a^2 + b^2$.

We can write an expression for the moment of inertia I_P about the axis through point P. Let m_i be a mass element in our slice, with coordinates (x_i, y_i, z_i). Then the moment of inertia I_{cm} of the slice about the axis through the center of mass (at O) is

$$I_{\text{cm}} = \sum_i m_i(x_i^2 + y_i^2)$$

The moment of inertia of the slice about the axis through P is

$$I_P = \sum_i m_i[(x_i - a)^2 + (y_i - b)^2]$$

These expressions don't involve the coordinates z_i measured perpendicular to the slices, so we can extend the sums to include *all* particles in *all* slices. Then I_P becomes the moment of inertia of the *entire* body for an axis through P. We then expand the squared terms and regroup, and obtain

$$I_P = \sum_i m_i(x_i^2 + y_i^2) - 2a\sum_i m_i x_i - 2b\sum_i m_i y_i + (a^2 + b^2)\sum_i m_i$$

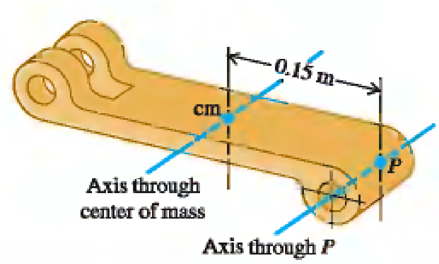
The first sum is I_{cm}. From Eq. (8.28), the definition of the center of mass, the second and third sums are proportional to x_{cm} and y_{cm}; these are zero because we have taken our origin to be the center of mass. The final term is d^2 multiplied by the total mass, or Md^2. This completes our proof that $I_P = I_{\text{cm}} + Md^2$.

As Eq. (9.19) shows, a rigid body has a lower moment of inertia about an axis through its center of mass than about any other parallel axis. Thus it's easier to start a body rotating if the rotation axis passes through the center of mass. This suggests that it's somehow most natural for a rotating body to rotate about an axis through its center of mass; we'll make this idea more quantitative in Chapter 10.

Example 9.10 Using the parallel-axis theorem

A part of a mechanical linkage (Fig. 9.21) has a mass of 3.6 kg. We measure its moment of inertia about an axis 0.15 m from its center of mass to be $I_P = 0.132\ \text{kg}\cdot\text{m}^2$. What is the moment of inertia I_{cm} about a parallel axis through the center of mass?

9.21 Calculating I_{cm} from a measurement of I_P.



SOLUTION

IDENTIFY: The parallel-axis theorem allows us to relate the moments of inertia I_{cm} and I_P through the two parallel axes.

SET UP: We'll use Eq. (9.19) to determine the target variable I_{cm}.

EXECUTE: Rearranging the equation and substituting the values,

$$\begin{aligned} I_{\text{cm}} &= I_P - Md^2 = 0.132\ \text{kg}\cdot\text{m}^2 - (3.6\ \text{kg})(0.15\ \text{m})^2 \\ &= 0.051\ \text{kg}\cdot\text{m}^2 \end{aligned}$$

EVALUATE: Our result shows that I_{cm} is less than I_P. This is as it should be: As we saw earlier, the moment of inertia for an axis through the center of mass is lower than for any other parallel axis.

Test Your Understanding of Section 9.5 A pool cue is a wooden rod with a uniform composition and tapered with a larger diameter at one end than at the other end. Use the parallel-axis theorem to decide whether a pool cue has a larger moment of inertia (i) for an axis through the thicker end of the rod and perpendicular to the length of the rod, or (ii) for an axis through the thinner end of the rod and perpendicular to the length of the rod.

*9.6 Moment-of-Inertia Calculations

NOTE: This optional section is for students who are familiar with integral calculus.

If a rigid body is a continuous distribution of mass—like a solid cylinder or a solid sphere—it cannot be represented by a few point masses. In this case the sum of masses and distances that defines the moment of inertia [Eq. (9.16)] becomes an integral. Imagine dividing the body into elements of mass dm that are very small, so that all points in a particular element are at essentially the same perpendicular distance from the axis of rotation. We call this distance r, as before. Then the moment of inertia is

$$I = \int r^2\,dm \tag{9.20}$$

To evaluate the integral, we have to represent r and dm in terms of the same integration variable. When the object is effectively one-dimensional, such as the slender rods (a) and (b) in Table 9.2, we can use a coordinate x along the length and relate dm to an increment dx. For a three-dimensional object it is usually easiest to express dm in terms of an element of volume dV and the *density* ρ of the body. Density is mass per unit volume, $\rho = dm/dV$, so we may also write Eq. (9.20) as

$$I = \int r^2 \rho\,dV$$

This expression tells us that a body's moment of inertia depends on how its density varies within its volume (Fig. 9.22). If the body is uniform in density, then we may take ρ outside the integral:

$$I = \rho \int r^2\,dV \tag{9.21}$$

To use this equation, we have to express the volume element dV in terms of the differentials of the integration variables, such as $dV = dx\,dy\,dz$. The element dV must always be chosen so that all points within it are at very nearly the same distance from the axis of rotation. The limits on the integral are determined by the shape and dimensions of the body. For regularly shaped bodies, this integration is often easy to do.

9.22 By measuring small variations in the orbits of satellites, geophysicists can measure the earth's moment of inertia. This tells us how our planet's mass is distributed within its interior. The data show that the earth is far denser at the core than in its outer layers.

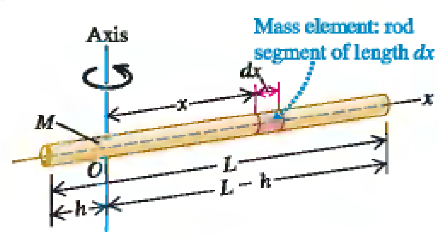
Example 9.11 Uniform thin rod, axis perpendicular to length

Figure 9.23 shows a slender uniform rod with mass M and length L. It might be a baton held by a twirler in a marching band (less the rubber end caps). Compute its moment of inertia about an axis through O, at an arbitrary distance h from one end.

SOLUTION

IDENTIFY: The rod is a continuous distribution of mass, so we must use integration to find the moment of inertia. We choose as an element of mass a short section of rod with length dx at a distance x from point O.

9.23 Finding the moment of inertia of a thin rod about an axis through O.



SET UP: The ratio of the mass dm of an element to the total mass M is equal to the ratio of its length dx to the total length L:

$$\frac{dm}{M} = \frac{dx}{L} \quad \text{so} \quad dm = \frac{M}{L}dx$$

We'll determine I from Eq. (9.20) with r replaced by x (see Fig. 9.23).

EXECUTE: Figure 9.23 shows that the integration limits on x are from $-h$ to $(L - h)$. Hence we obtain

$$I = \int x^2\,dm = \frac{M}{L}\int_{-h}^{L-h} x^2\,dx$$

$$= \left[\frac{M}{L}\left(\frac{x^3}{3}\right)\right]_{-h}^{L-h} = \frac{1}{3}M(L^2 - 3Lh + 3h^2)$$

EVALUATE: From this general expression we can find the moment of inertia about an axis through any point on the rod. For example, if the axis is at the left end, $h = 0$ and

$$I = \frac{1}{3}ML^2$$

If the axis is at the right end, we should get the same result. Putting $h = L$, we again get

$$I = \frac{1}{3}ML^2$$

If the axis passes through the center, the usual place for a twirled baton, then $h = L/2$ and

$$I = \frac{1}{12}ML^2$$

These results agree with the expressions in Table 9.2.

Example 9.12 Hollow or solid cylinder, rotating about axis of symmetry

Figure 9.24 shows a hollow, uniform cylinder with length L, inner radius R_1, and outer radius R_2. It might be a steel cylinder in a printing press or a sheet-steel rolling mill. Find the moment of inertia about the axis of symmetry of the cylinder.

SOLUTION

IDENTIFY: Again we must use integration to find the moment of inertia, but now we choose as a volume element a thin cylindrical shell of radius r, thickness dr, and length L. All parts of this element are at very nearly the same distance from the axis.

SET UP: The volume of the element is very nearly equal to that of a flat sheet with thickness dr, length L, and width $2\pi r$ (the circumference of the shell). Then

$$dm = \rho\,dV = \rho(2\pi rL\,dr)$$

We will use this expression in Eq. (9.20) and integrate from $r = R_1$ to $r = R_2$.

EXECUTE: The moment of inertia is given by

$$I = \int r^2\,dm = \int_{R_1}^{R_2} r^2\rho(2\pi rL\,dr)$$

$$= 2\pi\rho L\int_{R_1}^{R_2} r^3\,dr$$

$$= \frac{2\pi\rho L}{4}(R_2^4 - R_1^4)$$

$$= \frac{\pi\rho L}{2}(R_2^2 - R_1^2)(R_2^2 + R_1^2)$$

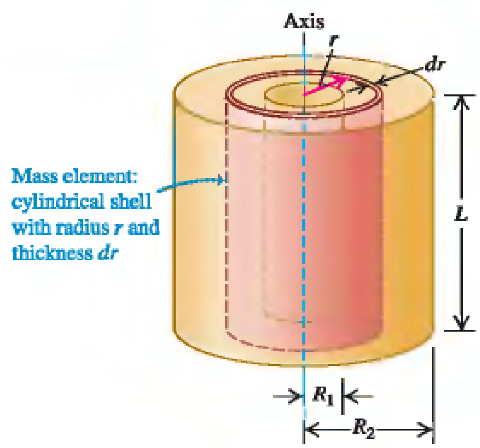
It is usually more convenient to express the moment of inertia in terms of the total mass M of the body, which is its density ρ multiplied by the total volume V. The volume is

$$V = \pi L(R_2^2 - R_1^2)$$

so the total mass M is

$$M = \rho V = \pi L\rho(R_2^2 - R_1^2)$$

9.24 Finding the moment of inertia of a hollow cylinder about its symmetry axis.



Hence the moment of inertia is

$$I = \frac{1}{2}M(R_1^2 + R_2^2)$$

EVALUATE: This agrees with Table 9.2, case (e). If the cylinder is solid, such as a lawn roller, $R_1 = 0$. Calling the outer radius R_2 simply R, we find that the moment of inertia of a solid cylinder of radius R is

$$I = \frac{1}{2}MR^2$$

If the cylinder has a very thin wall (like a pipe), R_1 and R_2 are very nearly equal; if R represents this common radius,

$$I = MR^2$$

We could have predicted this last result; in a thin-walled cylinder, all the mass is the same distance $r = R$ from the axis, so $I = \int r^2\,dm = R^2\int dm = MR^2$.

Example 9.13 Uniform sphere with radius R, axis through center

Find the moment of inertia of a solid, uniform sphere (like a billiard ball or ball bearing) about an axis through its center.

SOLUTION

IDENTIFY: To calculate the moment of inertia we divide the sphere into thin disks of thickness dx (Fig. 9.25), whose moment of inertia we know from Example 9.12. We'll integrate over these to find the total moment of inertia. The only tricky point is that the radius and mass of a disk depend on its distance x from the center of the sphere.

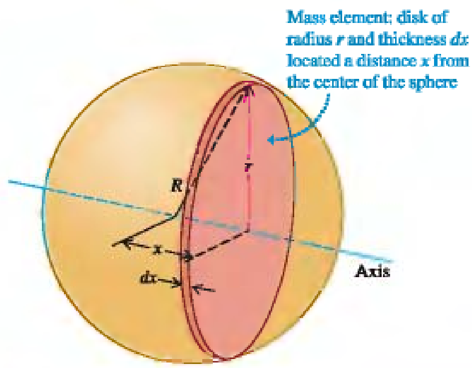
SET UP: The radius r of the disk shown in Fig. 9.25 is

$$r = \sqrt{R^2 - x^2}$$

Its volume is

$$dV = \pi r^2\, dx = \pi(R^2 - x^2)\, dx$$

9.25 Finding the moment of inertia of a sphere about an axis through its center.



and its mass is

$$dm = \rho\, dV = \pi\rho(R^2 - x^2)\, dx$$

EXECUTE: From Example 9.12, the moment of inertia of a disk of radius r and mass dm is

$$dI = \frac{1}{2}r^2\, dm = \frac{1}{2}\left(\sqrt{R^2 - x^2}\right)^2[\pi\rho(R^2 - x^2)\, dx]$$
$$= \frac{\pi\rho}{2}(R^2 - x^2)^2\, dx$$

Integrating this expression from $x = 0$ to $x = R$ gives the moment of inertia of the right hemisphere. The total I for the entire sphere, including both hemispheres, is just twice this:

$$I = (2)\frac{\pi\rho}{2}\int_0^R (R^2 - x^2)^2\, dx$$

Carrying out the integration, we obtain

$$I = \frac{8\pi\rho}{15}R^5$$

The mass M of the sphere of volume $V = 4\pi R^3/3$ is

$$M = \rho V = \frac{4\pi\rho R^3}{3}$$

By comparing the expressions for I and M, we find

$$I = \frac{2}{5}MR^2$$

EVALUATE: This result agrees with the expression in Table 9.2, case (h). Note that the moment of inertia of a solid sphere of mass M and radius R is less than the moment of inertia of a solid *cylinder* of the same mass and radius, $I = \frac{1}{2}MR^2$. The reason is that more of the sphere's mass is located close to the axis.

Test Your Understanding of Section 9.6 Two hollow cylinders have the same inner and outer radii and the same mass, but they have different lengths. One is made of low-density wood and the other of high-density lead. Which cylinder has the greater moment of inertia around its axis of symmetry? (i) the wood cylinder; (ii) the lead cylinder; (iii) the two moments of inertia are equal.

CHAPTER 9 SUMMARY

Rotational kinematics: When a rigid body rotates about a stationary axis (usually called the z-axis), its position is described by an angular coordinate θ. The angular velocity ω_z is the time derivative of θ, and the angular acceleration α_z is the time derivative of ω_z or the second derivative of θ. (See Examples 9.1 and 9.2.) If the angular acceleration is constant, then θ, ω_z, and α_z are related by simple kinematic equations analogous to those for straight-line motion with constant linear acceleration. (See Example 9.3.)

$$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad (9.3)$$

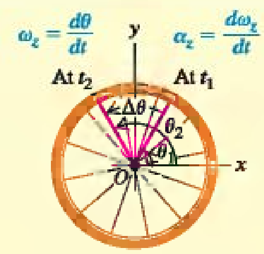
$$\alpha_z = \lim_{\Delta t \to 0} \frac{\Delta\omega_z}{\Delta t} = \frac{d\omega_z}{dt} = \frac{d^2\theta}{dt^2} \qquad (9.5)$$

$$\theta = \theta_0 + \omega_{0z}t + \frac{1}{2}\alpha_z t^2 \qquad (9.11)$$
(constant α_z only)

$$\theta - \theta_0 = \frac{1}{2}(\omega_z + \omega_{0z})t \qquad (9.10)$$
(constant α_z only)

$$\omega_z = \omega_{0z} + \alpha_z t \qquad (9.7)$$
(constant α_z only)

$$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0) \qquad (9.12)$$
(constant α_z only)

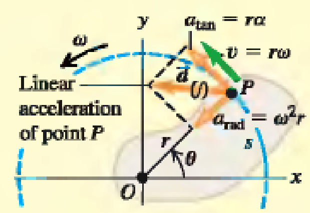


Relating linear and angular kinematics: The angular speed ω of a rigid body is the magnitude of its angular velocity. The rate of change of ω is $\alpha = d\omega/dt$. For a particle in the body a distance r from the rotation axis, the speed v and the components of the acceleration $\vec{a}$ are related to ω and α. (See Examples 9.4–9.6.)

$$v = r\omega \qquad (9.13)$$

$$a_{tan} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha \qquad (9.14)$$

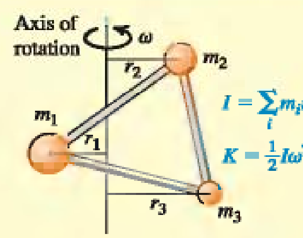
$$a_{rad} = \frac{v^2}{r} = \omega^2 r \qquad (9.15)$$



Moment of inertia and rotational kinetic energy: The moment of inertia I of a body about a given axis is a measure of its rotational inertia: The greater the value of I, the more difficult it is to change the state of the body's rotation. The moment of inertia can be expressed as a sum over the particles m_i that make up the body, each of which is at its own perpendicular distance r_i from the axis. The rotational kinetic energy of a rigid body rotating about a fixed axis depends on the angular speed ω and the moment of inertia I for that rotation axis. (See Examples 9.7–9.9.)

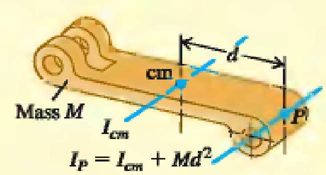
$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots = \sum_i m_i r_i^2 \qquad (9.16)$$

$$K = \frac{1}{2}I\omega^2 \qquad (9.17)$$



Calculating the moment of inertia: The parallel-axis theorem relates the moments of inertia of a rigid body of mass M about two parallel axes: an axis through the center of mass (moment of inertia I_{cm}) and a parallel axis a distance d from the first axis (moment of inertia I_P). (See Example 9.10.) If the body has a continuous mass distribution, the moment of inertia can be calculated by integration. (See Examples 9.11–9.13.)

$$I_P = I_{cm} + Md^2 \qquad (9.19)$$



Key Terms

rigid body, *285*
radian, *286*
average angular velocity, *286*
angular displacement, *286*
instantaneous angular velocity, *287*
average angular acceleration, *289*
instantaneous angular acceleration, *289*
angular speed, *293*
tangential component of acceleration, *293*
centripetal component of acceleration, *294*
moment of inertia, *297*
rotational kinetic energy, *297*
parallel-axis theorem, *302*

Answer to Chapter Opening Question ?

Both segments of the rigid blade have the same angular speed ω. From Eqs. (9.13) and (9.15), doubling the distance r for the same ω doubles the linear speed $v = r\omega$ and doubles the radial acceleration $a_{rad} = \omega^2 r$.

Answers to Test Your Understanding Questions

9.1 Answers: (a) (i) and (iii), (b) (ii) The rotation is speeding up when the angular velocity and angular acceleration have the same sign, and slowing down when they have opposite signs. Hence it is speeding up for $0 < t < 2$ s (ω_z and α_z are both positive) and for $4 \text{ s} < t < 6$ s (ω_z and α_z are both negative), but is slowing down for $2 \text{ s} < t < 4$ s (ω_z is positive and α_z is negative). Note that the body is rotating in one direction for $t < 4$ s (ω_z is positive) and in the opposite direction for $t > 4$ s (ω_z is negative).

9.2 Answers: (a) (i), (b) (ii) When the DVD comes to rest, $\omega_z = 0$. From Eq. (9.7), the *time* when this occurs is $t = (\omega_z - \omega_{0z})/\alpha_z = -\omega_{0z}/\alpha_z$ (this is a positive time because α_z is negative). If we double the initial angular velocity ω_{0z} and also double the angular acceleration α_z, their ratio is unchanged and the rotation stops in the same amount of time. The *angle* through which the DVD rotates is given by Eq. (9.10): $\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t = \frac{1}{2}\omega_{0z}t$ (since the final angular velocity is $\omega_z = 0$). The initial angular velocity ω_{0z} has been doubled but the time t is the same, so the angular displacement $\theta - \theta_0$ (and hence the number of revolutions) has doubled. You can also come to the same conclusion using Eq. (9.12).

9.3 Answer: (ii) From Eq. (9.13), $v = r\omega$. To maintain a constant linear speed v, the angular speed ω must decrease as the scanning head moves outward (greater r).

9.4 Answer: (i) The kinetic energy in the falling block is $\frac{1}{2}mv^2$, and the kinetic energy in the rotating cylinder is $\frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(\frac{v}{R}\right)^2 = \frac{1}{4}mv^2$. Hence the total kinetic energy of the system is $\frac{3}{4}mv^2$, of which two-thirds is in the block and one-third is in the cylinder.

9.5 Answer: (ii) More of the mass of the pool cue is concentrated at the thicker end, so the center of mass is closer to that end. The moment of inertia through a point P at either end is $I_P = I_{cm} + Md^2$; the thinner end is farther from the center of mass, so the distance d and the moment of inertia I_P are greater for the thinner end.

9.6 Answer: (iii) Our result from Example 9.12 does *not* depend on the cylinder length L. The moment of inertia depends only on the *radial* distribution of mass, not on its distribution along the axis.

PROBLEMS

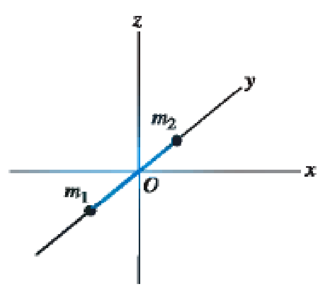
For instructor-assigned homework, go to www.masteringphysics.com

Discussion Questions

Q9.1. Which of the following formulas is valid if the angular acceleration of an object is *not* constant? Explain your reasoning in each case. (a) $v = r\omega$; (b) $a_{tan} = r\alpha$; (c) $\omega = \omega_0 + \alpha t$; (d) $a_{tan} = r\omega^2$; (e) $K = \frac{1}{2}I\omega^2$.

Q9.2. A diatomic molecule can be modeled as two point masses, m_1 and m_2, slightly separated (Fig. 9.26). If the molecule is oriented along the y-axis, it has kinetic energy K when it spins about the x-axis. What will its kinetic energy (in terms of K) be if it spins at the same angular speed about (a) the z-axis and (b) the y-axis?

Figure 9.26 Question Q9.2.



Q9.3. What is the difference between tangential and radial acceleration for a point on a rotating body?

Q9.4. In Fig. 9.14, all points on the chain have the same linear speed. Is the magnitude of the linear acceleration also the same for all points on the chain? How are the angular accelerations of the two sprockets related? Explain.

Q9.5. In Fig. 9.14, how are the radial accelerations of points at the teeth of the two sprockets related? Explain the reasoning behind your answer.

Q9.6. A flywheel rotates with constant angular velocity. Does a point on its rim have a tangential acceleration? A radial acceleration? Are these accelerations constant in magnitude? In direction? In each case give the reasoning behind your answer.

Q9.7. What is the purpose of the spin cycle of a washing machine? Explain in terms of acceleration components.

Q9.6. Although angular velocity and angular acceleration can be treated as vectors, the angular displacement θ, despite having a magnitude and a direction, cannot. This is because θ does not follow the commutative law of vector addition (Eq. 1.3). Prove this to yourself in the following way: Lay your physics textbook flat on the desk in front of you with the cover side up so you can read the writing on it. Rotate it through 90° about a horizontal axis so that the farthest edge comes toward you. Call this angular displacement θ_1. Then rotate it by 90° about a vertical axis so that the left edge comes toward you. Call this angular displacement θ_2. The spine of the book should now face you, with the writing on it oriented so that you can read it. Now start over again but carry out the two rotations in the reverse order. Do you get a different result? That is, does $\theta_1 + \theta_2$ equal $\theta_2 + \theta_1$? Now repeat this experiment but this time with an angle of 1° rather than 90°. Do you think that the infinitesimal displacement $d\vec{\theta}$ obeys the commutative law of addition and hence qualifies as a vector? If so, how is the direction of $d\vec{\theta}$ related to the direction of $\vec{\omega}$?

Q9.9. Can you think of a body that has the same moment of inertia for all possible axes? If so, give an example, and if not, explain why this is not possible. Can you think of a body that has the same moment of inertia for all axes passing through a certain point? If so, give an example and indicate where the point is located.

Q9.10. To maximize the moment of inertia of a flywheel while minimizing its weight, what shape and distribution of mass should it have? Explain.

Q9.11. How might you determine experimentally the moment of inertia of an irregularly shaped body about a given axis?

Q9.12. A cylindrical body has mass M and radius R. Can the mass be distributed within the body in such a way that its moment of inertia about its axis of symmetry is greater than MR^2? Explain.

Q9.13. Describe how you could use part (b) of Table 9.2 to derive the result in part (d).

Q9.14. A hollow spherical shell of radius R that is rotating about an axis through its center has rotational kinetic energy K. If you want to modify this sphere so that it has three times as much kinetic energy at the same angular speed while keeping the same mass, what should be its radius in terms of R?

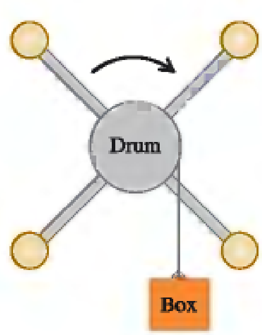
Q9.15. For the equations for I given in parts (a) and (b) of Table 9.2 to be valid, must the rod have a circular cross section? Is there any restriction on the size of the cross section for these equations to apply? Explain.

Q9.16. In part (d) of Table 9.2, the thickness of the plate must be much less than a for the expression given for I to apply. But in part (c), the expression given for I applies no matter how thick the plate is. Explain.

Q9.17. Two identical balls, A and B, are each attached to very light string, and each string is wrapped around the rim of a frictionless pulley of mass M. The only difference is that the pulley for ball A is a solid disk, while the one for ball B is a hollow disk, like part (e) in Table 9.2. If both balls are released from rest and fall the same distance, which one will have more kinetic energy, or will they have the same kinetic energy? Explain your reasoning.

Q9.18. An elaborate pulley consists of four identical balls at the ends of spokes extending out from a rotating drum (Fig. 9.27). A box is connected to a light thin rope wound around the rim of the drum. When it is released from rest, the box acquires a speed V after having fallen a distance d. Now the four balls are moved inward closer to the drum, and the box is again released from rest. After it has fallen a distance d, will its speed be equal to V, greater than V, or less than V? Show or explain why.

Figure 9.27 Question 9.18.



Q9.19. You can use any angular measure—radians, degrees, or revolutions—in some of the equations in Chapter 9, but you can use only radian measure in others. Identify those for which using radians is necessary and those for which it is not, and in each case give the reasoning behind your answer.

Q9.20. When calculating the moment of inertia of an object, can we treat all its mass as if it were concentrated at the center of mass of the object? Justify your answer.

Q9.21. A wheel is rotating about an axis perpendicular to the plane of the wheel and passing through the center of the wheel. The angular speed of the wheel is increasing at a constant rate. Point A is on the rim of the wheel and point B is midway between the rim and center of the wheel. For each of the following quantities, is its magnitude larger at point A, at point B, or is it the same at both points? (a) angular speed; (b) tangential speed; (c) angular acceleration; (d) tangential acceleration; (e) radial acceleration. Justify each of your answers.

Exercises

Section 9.1 Angular Velocity and Acceleration

9.1. (a) What angle in radians is subtended by an arc 1.50 m long on the circumference of a circle of radius 2.50 m? What is this angle in degrees? (b) An arc 14.0 cm long on the circumference of a circle subtends an angle of 128°. What is the radius of the circle? (c) The angle between two radii of a circle with radius 1.50 m is 0.700 rad. What length of arc is intercepted on the circumference of the circle by the two radii?

9.2. An airplane propeller is rotating at 1900 rpm (rev/min). (a) Compute the propeller's angular velocity in rad/s. (b) How many seconds does it take for the propeller to turn through 35°?

9.3. The angular velocity of a flywheel obeys the equation $\omega_z(t) = A + Bt^2$, where t is in seconds and A and B are constants having numerical values 2.75 (for A) and 1.50 (for B). (a) What are the units of A and B if ω is in rad/s? (b) What is the angular acceleration of the wheel at (i) $t = 0.00$ and (ii) $t = 5.00$ s? (c) Through what angle does the flywheel turn during the first 2.00 s? (*Hint:* See Section 2.6.)

9.4. A fan blade rotates with angular velocity given by $\omega_z(t) = \gamma - \beta t^2$, where $\gamma = 5.00$ rad/s and $\beta = 0.800$ rad/s^3. (a) Calculate the angular acceleration as a function of time. (b) Calculate the instantaneous angular acceleration α_z at $t = 3.00$ s and the average angular acceleration $\alpha_{\text{av-}z}$ for the time interval $t = 0$ to $t = 3.00$ s. How do these two quantities compare? If they are different, why are they different?

9.5. A child is pushing a merry-go-round. The angle through which the merry-go-round has turned varies with time according to $\theta(t) = \gamma t + \beta t^3$, where $\gamma = 0.400$ rad/s and $\beta = 0.0120$ rad/s^3. (a) Calculate the angular velocity of the merry-go-round as a function of time. (b) What is the initial value of the angular velocity? (c) Calculate the instantaneous value of the angular velocity ω_z at $t = 5.00$ s and the average angular velocity $\omega_{\text{av-}z}$ for the time interval $t = 0$ to $t = 5.00$ s. Show that $\omega_{\text{av-}z}$ is *not* equal to the average of the instantaneous angular velocities at $t = 0$ and $t = 5.00$ s, and explain why it is not.

9.6. At $t = 0$ the current to a dc electric motor is reversed, resulting in an angular displacement of the motor shaft given by $\theta(t) = (250\text{ rad/s})t - (20.0\text{ rad/s}^2)t^2 - (1.50\text{ rad/s}^3)t^3$. (a) At what time is the angular velocity of the motor shaft zero? (b) Calculate the angular acceleration at the instant that the motor shaft has zero angular velocity. (c) How many revolutions does the motor shaft turn through between the time when the current is reversed and the instant when the angular velocity is zero? (d) How fast was the motor shaft rotating at $t = 0$, when the current was reversed? (e) Calculate the average angular velocity for the time period from $t = 0$ to the time calculated in part (a).

9.7. The angle θ through which a disk drive turns is given by $\theta(t) = a + bt - ct^3$, where a, b, and c are constants, t is in seconds, and θ is in radians. When $t = 0$, $\theta = \pi/4$ rad and the angular velocity is 2.00 rad/s, and when $t = 1.50$ s, the angular acceleration is 1.25 rad/s^2. (a) Find a, b, and c, including their units. (b) What is the angular acceleration when $\theta = \pi/4$ rad? (c) What are θ and the angular velocity when the angular acceleration is 3.50 rad/s^2?

9.8. A wheel is rotating about an axis that is in the z-direction. The angular velocity ω_z is -6.00 rad/s at $t = 0$, increases linearly with time, and is $+8.00$ m/s at $t = 7.00$ s. We have taken counterclockwise rotation to be positive. (a) Is the angular acceleration during this time interval positive or negative? (b) During what time interval is the speed of the wheel increasing? Decreasing? (c) What is the angular displacement of the wheel at $t = 7.00$ s?

Section 9.2 Rotation with Constant Angular Acceleration

9.9. A bicycle wheel has an initial angular velocity of 1.50 rad/s. (a) If its angular acceleration is constant and equal to 0.300 rad/s^2, what is its angular velocity at $t = 2.50$ s? (b) Through what angle has the wheel turned between $t = 0$ and $t = 2.50$ s?

9.10. An electric fan is turned off, and its angular velocity decreases uniformly from 500 rev/min to 200 rev/min in 4.00 s. (a) Find the angular acceleration in rev/s^2 and the number of revolutions made by the motor in the 4.00-s interval. (b) How many more seconds are required for the fan to come to rest if the angular acceleration remains constant at the value calculated in part (a)?

9.11. The rotating blade of a blender turns with constant angular acceleration 1.50 rad/s^2. (a) How much time does it take to reach an angular velocity of 36.0 rad/s, starting from rest? (b) Through how many revolutions does the blade turn in this time interval?

9.12. (a) Derive Eq. (9.12) by combining Eqs. (9.7) and (9.11) to eliminate t. (b) The angular velocity of an airplane propeller increases from 12.0 rad/s to 16.0 rad/s while turning through 7.00 rad. What is the angular acceleration in rad/s^2?

9.13. A turntable rotates with a constant 2.25 rad/s^2 angular acceleration. After 4.00 s it has rotated through an angle of 60.0 rad. What was the angular velocity of the wheel at the beginning of the 4.00-s interval?

9.14. A circular saw blade 0.200 m in diameter starts from rest. In 6.00 s it accelerates with constant angular acceleration to an angular velocity of 140 rad/s. Find the angular acceleration and the angle through which the blade has turned.

9.15. A high-speed flywheel in a motor is spinning at 500 rpm when a power failure suddenly occurs. The flywheel has mass 40.0 kg and diameter 75.0 cm. The power is off for 30.0 s, and during this time the flywheel slows due to friction in its axle bearings. During the time the power is off, the flywheel makes 200 complete revolutions. (a) At what rate is the flywheel spinning when the power comes back on? (b) How long after the beginning of the power failure would it have taken the flywheel to stop if the power had not come back on, and how many revolutions would the wheel have made during this time?

9.16. A computer disk drive is turned on starting from rest and has constant angular acceleration. If it took 0.750 s for the drive to make its *second* complete revolution, (a) how long did it take to make the first complete revolution, and (b) what is its angular acceleration, in rad/s^2?

9.17. A safety device brings the blade of a power mower from an initial angular speed of ω_1 to rest in 1.00 revolution. At the same constant acceleration, how many revolutions would it take the blade to come to rest from an initial angular speed ω_3 that was three times as great, $\omega_3 = 3\omega_1$?

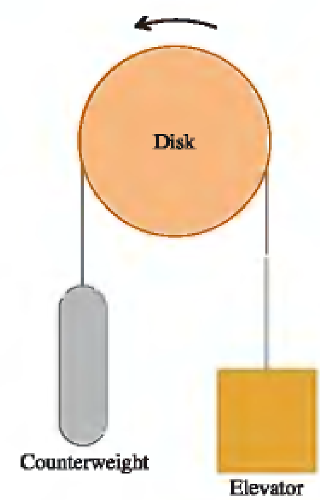
9.18. A straight piece of reflecting tape extends from the center of a wheel to its rim. You darken the room and use a camera and strobe unit that flashes once every 0.050 s to take pictures of the wheel as it rotates counterclockwise. You trigger the strobe so that the first flash $(t = 0)$ occurs when the tape is horizontal to the right at an angular displacement of zero. For the following situations draw a sketch of the photo you will get for the time exposure over five flashes (at $t = 0$, 0.050 s, 0.100 s, 0.150 s, and 0.200 s), and graph θ versus t and ω versus t for $t = 0$ to $t = 0.200$ s. (a) The angular velocity is constant at 10.0 rev/s. (b) The wheel starts from rest with a constant angular acceleration of 25.0 rev/s^2. (c) The wheel is rotating at 10.0 rev/s at $t = 0$ and changes angular velocity at a constant rate of -50.0 rev/s^2.

9.19. At $t = 0$ a grinding wheel has an angular velocity of 24.0 rad/s. It has a constant angular acceleration of 30.0 rad/s^2 until a circuit breaker trips at $t = 2.00$ s. From then on, it turns through 432 rad as it coasts to a stop at constant angular acceleration. (a) Through what total angle did the wheel turn between $t = 0$ and the time it stopped? (b) At what time did it stop? (c) What was its acceleration as it slowed down?

Section 9.3 Relating Linear and Angular Kinematics

9.20. In a charming 19th-century hotel, an old-style elevator is connected to a counterweight by a cable that passes over a rotating disk 2.50 m in diameter (Fig. 9.28). The elevator is raised and lowered by turning the disk, and the cable does not slip on the rim of the disk but turns with it. (a) At how many rpm must the disk turn to raise the elevator at 25.0 cm/s? (b) To start the elevator moving, it must be accelerated at $\frac{1}{8}g$. What must be the angular acceleration of the disk, in rad/s^2? (c) Through what angle (in radians and degrees) has the disk turned when it has raised the elevator 3.25 m between floors?

Figure 9.28 Exercise 9.20.



9.21. Using astronomical data from Appendix F, along with the fact that the earth spins on its axis once per day, calculate (a) the earth's orbital angular speed (in rad/s) due to its motion around the sun, (b) its angular speed (in rad/s) due to its axial spin, (c) the tangential speed of the earth around the sun (assuming a circular orbit), (d) the tangential speed of a point on the earth's equator due to the planet's axial spin, and (e) the radial and tangential acceleration components of the point in part (d).

9.22. Compact Disc. A compact disc (CD) stores music in a coded pattern of tiny pits 10^{-7} m deep. The pits are arranged in a track that spirals outward toward the rim of the disc; the inner and outer radii of this spiral are 25.0 mm and 58.0 mm, respectively. As the disc spins inside a CD player, the track is scanned at a constant *linear* speed of 1.25 m/s. (a) What is the angular speed of the CD when the innermost part of the track is scanned? The outermost part of the track? (b) The maximum playing time of a CD is 74.0 min. What would be the length of the track on such a maximum-duration CD if it were stretched out in a straight line? (c) What is the average angular acceleration of a maximum-duration CD during its 74.0-min playing time? Take the direction of rotation of the disc to be positive.

9.23. A wheel of diameter 40.0 cm starts from rest and rotates with a constant angular acceleration of 3.00 rad/s^2. At the instant the wheel has computed its second revolution, compute the radial acceleration of a point on the rim in two ways: (a) using the relationship $a_{rad} = \omega^2 r$ and (b) from the relationship $a_{rad} = v^2/r$.

9.24. Ultracentrifuge. Find the required angular speed (in rev/min) of an ultracentrifuge for the radial acceleration of a point 2.50 cm from the axis to equal 400,000g (that is, 400,000 times the acceleration due to gravity).

9.25. A flywheel with a radius of 0.300 m starts from rest and accelerates with a constant angular acceleration of 0.600 rad/s^2. Compute the magnitude of the tangential acceleration, the radial acceleration, and the resultant acceleration of a point on its rim (a) at the start; (b) after it has turned through 60.0°; (c) after it has turned through 120.0°.

9.26. An electric turntable 0.750 m in diameter is rotating about a fixed axis with an initial angular velocity of 0.250 rev/s and a constant angular acceleration of 0.900 rev/s^2. (a) Compute the angular velocity of the turntable after 0.200 s. (b) Through how many revolutions has the turntable spun in this time interval? (c) What is the tangential speed of a point on the rim of the turntable at $t = 0.200$ s? (d) What is the magnitude of the *resultant* acceleration of a point on the rim at $t = 0.200$ s?

9.27. Centrifuge. An advertisement claims that a centrifuge takes up only 0.127 m of bench space but can produce a radial acceleration of 3000g at 5000 rev/min. Calculate the required radius of the centrifuge. Is the claim realistic?

9.20. (a) Derive an equation for the radial acceleration that includes v and ω, but not r. (b) You are designing a merry-go-round for which a point on the rim will have a radial acceleration of 0.500 m/s^2 when the tangential velocity of that point has magnitude 2.00 m/s. What angular velocity is required to achieve these values?

9.29. Electric Drill. According to the shop manual, when drilling a 12.7-mm-diameter hole in wood, plastic, or aluminum, a drill should have a speed of 1250 rev/min. For a 12.7-mm-diameter drill bit turning at a constant 1250 rev/min, find (a) the maximum linear speed of any part of the hit and (b) the maximum radial acceleration of any part of the bit.

9.30. At $t = 3.00$ s a point on the rim of a 0.200-m-radius wheel has a tangential speed of 50.0 m/s as the wheel slows down with a tangential acceleration of constant magnitude 10.0 m/s^2. (a) Calculate the wheel's constant angular acceleration. (b) Calculate the angular velocities at $t = 3.00$ s and $t = 0$. (c) Through what angle did the wheel turn between $t = 0$ and $t = 3.00$ s? (d) At what time will the radial acceleration equal g?

9.31. The spin cycles of a washing machine have two angular speeds, 423 rev/min and 640 rev/min. The internal diameter of the drum is 0.470 m. (a) What is the ratio of the maximum radial force on the laundry for the higher angular speed to that for the lower speed? (b) What is the ratio of the maximum tangential speed of the laundry for the higher angular speed to that for the lower speed? (c) Find the laundry's maximum tangential speed and the maximum radial acceleration, in terms of g.

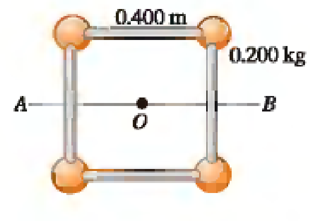
9.32. You are to design a rotating cylindrical axle to lift 800-N buckets of cement from the ground to a rooftop 78.0 m above the ground. The buckets will be attached to a hook on the free end of a cable that wraps around the rim of the axle; as the axle turns, the buckets will rise. (a) What should the diameter of the axle be in order to raise the buckets at a steady 2.00 cm/s when it is turning at 7.5 rpm? (b) If instead the axle must give the buckets an upward acceleration of 0.400 m/s^2, what should the angular acceleration of the axle be?

9.33. While riding a multispeed bicycle, the rider can select the radius of the rear sprocket that is fixed to the rear axle. The front sprocket of a bicycle has radius 12.0 cm. If the angular speed of the front sprocket is 0.600 rev/s, what is the radius of the rear sprocket for which the tangential speed of a point on the rim of the rear wheel will be 5.00 m/s? The rear wheel has radius 0.330 m.

Section 9.4 Energy in Rotational Motion

9.34. Four small spheres, each of which you can regard as a point of mass 0.200 kg, are arranged in a square 0.400 m on a side and connected by extremely light rods (Fig. 9.29). Find the moment of inertia of the system about an axis (a) through the center of the square, perpendicular to its plane (an axis through point O in the figure); (b) bisecting two opposite sides of the square (an axis along the line AB in the figure); (c) that passes through the centers of the upper left and lower right spheres and through point O.

Figure 9.29 Exercise 9.34.



9.35. Calculate the moment of inertia of each of the following uniform objects about the axes indicated. Consult Table 9.2 as needed. (a) A thin 2.50-kg rod of length 75.0 cm, about an axis perpendicular to it and passing through (i) one end and (ii) its center, and (iii) about an axis parallel to the rod and passing through it. (b) A 3.00-kg sphere 38.0 cm in diameter, about an axis through its center, if the sphere is (i) solid and (ii) a thin-walled hollow shell. (c) An 8.00-kg cylinder, of length 19.5 cm and diameter 12.0 cm, about the central axis of the cylinder, if the cylinder is (i) thin-walled and hollow, and (ii) solid.

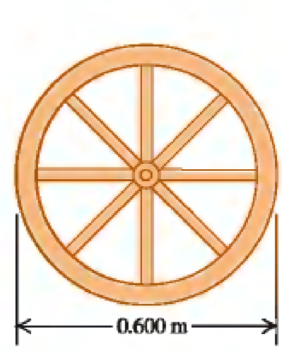
9.36. Small blocks, each with mass m, are clamped at the ends and at the center of a rod of length L and negligible mass. Compute the moment of inertia of the system about an axis perpendicular to the rod and passing through (a) the center of the rod and (b) a point one-fourth of the length from one end.

9.37. A uniform bar has two small balls glued to its ends. The bar is 2.00 m long and has mass 4.00 kg, while the balls each have mass 0.500 kg and can be treated as point masses. Find the moment of inertia of this combination about each of the following axes: (a) an axis perpendicular to the bar through its center; (b) an axis perpendicular to the bar through one of the balls; (c) an axis parallel to the bar through both balls; (d) an axis parallel to the bar and 0.500 m from it.

9.30. A twirler's baton is made of a slender metal cylinder of mass M and length L. Each end has a rubber cap of mass m, and you can accurately treat each cap as a particle in this problem. Find the total moment of inertia of the baton about the usual twirling axis (perpendicular to the baton through its center).

9.39. A wagon wheel is constructed as shown in Fig. 9.30. The radius of the wheel is 0.300 m, and the rim has mass 1.40 kg. Each of the eight spokes that lie along a diameter and

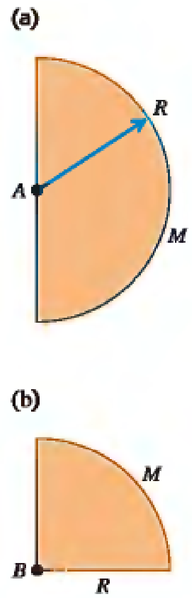
Figure 9.30 Exercise 9.39.



are 0.300 m long has mass 0.280 kg. What is the moment of inertia of the wheel about an axis through its center and perpendicular to the plane of the wheel? (Use the formulas given in Table 9.2.)

9.40. A uniform disk of radius R is cut in half so that the remaining half has mass M (Fig. 9.31a). (a) What is the moment of inertia of this half about an axis perpendicular to its plane through point A? (b) Why did your answer in part (a) come out the same as if this were a complete disk of mass M? (c) What would be the moment of inertia of a quarter disk of mass M and radius R about an axis perpendicular to its plane passing through point B (Fig. 9.31b)?

Figure 9.31 Exercise 9.40.



9.41. A compound disk of outside diameter 140.0 cm is made up of a uniform solid disk of radius 50.0 cm and area density 3.00 g/cm^2 surrounded by a concentric ring of inner radius 50.0 cm, outer radius 70.0 cm, and area density 2.00 g/cm^2. Find the moment of inertia of this object about an axis perpendicular to the plane of the object and passing through its center.

9.42. An airplane propeller is 2.08 m in length (from tip to tip) with mass 117 kg and is rotating at 2400 rpm (rev/min) about an axis through its center. You can model the propeller as a slender rod. (a) What is its rotational kinetic energy? (b) Suppose that, due to weight constraints, you had to reduce the propeller's mass to 75.0% of its original mass, but you still needed to keep the same size and kinetic energy. What would its angular speed have to be, in rpm?

9.43. Energy from the Moon? Suppose that some time in the future we decide to tap the moon's rotational energy for use on earth. In additional to the astronomical data in Appendix F, you may need to know that the moon spins on its axis once every 27.3 days. Assume that the moon is uniform throughout. (a) How much total energy could we get from the moon's rotation? (b) The world presently uses about 4.0×10^{20} J of energy per year. If in the future the world uses five times as much energy yearly, for how many years would the moon's rotation provide us energy? In light of your answer, does this seem like a cost-effective energy source in which to invest?

9.44. You need to design an industrial turntable that is 60.0 cm in diameter and has a kinetic energy of 0.250 J when turning at 45.0 rpm (rev/min). (a) What must be the moment of inertia of the turntable about the rotation axis? (b) If your workshop makes this turntable in the shape of a uniform solid disk, what must be its mass?

9.45. The flywheel of a gasoline engine is required to give up 500 J of kinetic energy while its angular velocity decreases from 650 rev/min to 520 rev/min. What moment of inertia is required?

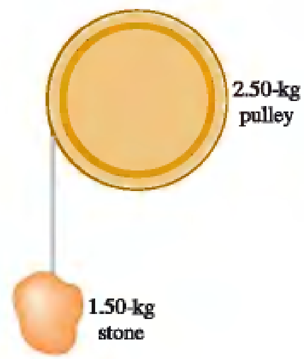
9.46. A light, flexible rope is wrapped several times around a *hollow* cylinder, with a weight of 40.0 N and a radius of 0.25 m, that rotates without friction about a fixed horizontal axis. The cylinder is attached to the axle by spokes of a negligible moment of inertia. The cylinder is initially at rest. The free end of the rope is pulled with a constant force P for a distance of 5.00 m, at which point the end of the rope is moving at 6.00 m/s. If the rope does not slip on the cylinder, what is the value of P?

9.47. Energy is to be stored in a 70.0-kg flywheel in the shape of a uniform solid disk with radius $R = 1.20$ m. To prevent structural failure of the flywheel, the maximum allowed radial acceleration of a point on its rim is 3500 m/s^2. What is the maximum kinetic energy that can be stored in the flywheel?

9.48. Suppose the solid cylinder in the apparatus described in Example 9.9 (Section 9.4) is replaced by a thin-walled, hollow cylinder with the same mass M and radius R. The cylinder is attached to the axle by spokes of a negligible moment of inertia. (a) Find the speed of the hanging mass m just as it strikes the floor. (b) Use energy concepts to explain why the answer to part (a) is different from the speed found in Example 9.9.

9.49. A frictionless pulley has the shape of a uniform solid disk of mass 2.50 kg and radius 20.0 cm. A 1.50-kg stone is attached to a very light wire that is wrapped around the rim of the pulley (Fig. 9.32), and the system is released from rest. (a) How far must the stone fall so that the pulley has 4.50 J of kinetic energy? (b) What percent of the total kinetic energy does the pulley have?

Figure 9.32 Exercise 9.49.



9.50. A bucket of mass m is tied to a massless cable that is wrapped around the outer rim of a frictionless uniform pulley of radius R, similar to the system shown in Fig. 9.32. In terms of the stated variables, what must be the moment of inertia of the pulley so that it always has half as much kinetic energy as the bucket?

9.51. How I Scales. If we multiply all the design dimensions of an object by a scaling factor f, its volume and mass will be multiplied by f^3. (a) By what factor will its moment of inertia be multiplied? (b) If a $\frac{1}{48}$-scale model has a rotational kinetic energy of 2.5 J, what will be the kinetic energy for the full-scale object of the same material rotating at the same angular velocity?

9.52. A uniform 2.00-m ladder of mass 9.00 kg is leaning against a vertical wall while making an angle of 53.0° with the floor. A worker pushes the ladder up against the wall until it is vertical. How much work did this person do against gravity?

9.53. A uniform 3.00-kg rope 24.0 m long lies on the ground at the top of a vertical cliff. A mountain climber at the top lets down half of it to help his partner climb up the cliff. What was the change in potential energy of the rope during this maneuver?

Section 9.5 Parallel-Axis Theorem

9.54. Find the moment of inertia of a hoop (a thin-walled, hollow ring) with mass M and radius R about an axis perpendicular to the hoop's plane at an edge.

9.55. About what axis will a uniform, balsa-wood sphere have the same moment of inertia as does a thin-walled, hollow, lead sphere of the same mass and radius, with the axis along a diameter?

9.56. Use the parallel-axis theorem to show that the moments of inertia given in parts (a) and (b) of Table 9.2 are consistent.

9.57. A thin, rectangular sheet of metal has mass M and sides of length a and b. Use the parallel-axis theorem to calculate the moment of inertia of the sheet for an axis that is perpendicular to the plane of the sheet and that passes through one corner of the sheet.

9.58. (a) For the thin rectangular plate shown in part (d) of Table 9.2, find the moment of inertia about an axis that lies in the plane of the plate, passes through the center of the plate, and is parallel to the axis shown in the figure. (b) Find the moment of inertia of the plate for an axis that lies in the plane of the plate, passes through the center of the plate, and is perpendicular to the axis in part (a).

9.59. A thin uniform rod of mass M and length L is bent at its center so that the two segments are now perpendicular to each other. Find its moment of inertia about an axis perpendicular to its plane and passing through (a) the point where the two segments meet and (b) the midpoint of the line connecting its two ends.

*Section 9.6 Moment-of-Inertia Calculations

***9.60.** Using the information in Table 9.2 and the parallel-axis theorem, find the moment of inertia of the slender rod with mass M and length L shown in Fig. 9.23 about an axis through O, at an arbitrary distance h from one end. Compare your result to that found by integration in Example 9.11 (Section 9.6).

***9.61.** Use Eq. (9.20) to calculate the moment of inertia of a uniform, solid disk with mass M and radius R for an axis perpendicular to the plane of the disk and passing through its center.

***9.62.** Use Eq. (9.20) to calculate the moment of inertia of a slender, uniform rod with mass M and length L abont an axis at one end, perpendicular to the rod.

***9.63.** A slender rod with length L has a mass per unit length that varies with distance from the left end, where $x = 0$, according to $dm/dx = \gamma x$, where γ has units of $\mathrm{kg/m^2}$. (a) Calculate the total mass of the rod in terms of γ and L. (b) Use Eq. (9.20) to calculate the moment of inertia of the rod for an axis at the left end, perpendicular to the rod. Use the expression you derived in part (a) to express I in terms of M and L. How does your result compare to that for a uniform rod? Explain this comparison. (c) Repeat part (b) for an axis at the right end of the rod. How do the results for parts (b) and (c) compare? Explain this result.

Problems

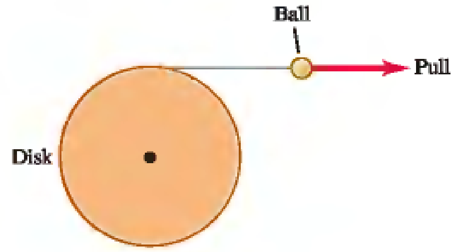
9.64. Sketch a wheel lying in the plane of your paper and rotating counterclockwise. Choose a point on the rim and draw a vector $\vec{r}$ from the center of the wheel to that point. (a) What is the direction of $\vec{\omega}$? (b) Show that the velocity of the point is $\vec{v} = \vec{\omega} \times \vec{r}$. (c) Show that the radial acceleration of the point is $\vec{a}_{\mathrm{rad}} = \vec{\omega} \times \vec{v} = \vec{\omega} \times (\vec{\omega} \times \vec{r})$ (see Exercise 9.28).

9.65. Trip to Mars. You are working on a project with NASA to launch a rocket to Mars, with the rocket blasting off from earth when earth and Mars are aligned along a straight line from the sun. If Mars is now 60° shead of earth in its orbit around the sun, when should you launch the rocket? (*Note:* All the planets orbit the sun in the same direction, 1 year on Mars is 1.9 earth-years, and assume circular orbits for both planets.)

9.66. A roller in a printing press turns through an angle $\theta(t)$ given by $\theta(t) = \gamma t^2 - \beta t^3$, where $\gamma = 3.20\ \mathrm{rad/s^2}$ and $\beta = 0.500\ \mathrm{rad/s^3}$. (a) Calculate the angular velocity of the roller as a function of time. (b) Calculate the angular acceleration of the roller as a function of time. (c) What is the maximum positive angular velocity, and at what value of t does it occur?

***9.67.** A disk of radius 25.0 cm is free to turn about an axle perpendicular to it through its center. It has very thin but strong string wrapped around its rim, and the string is attached to a ball that is pulled tangentially away from the rim of the disk (Fig. 9.33). The pull increases in magnitude and produces an acceleration of the ball that obeys the equation $a(t) = At$, where t is in seconds and A is a constant. The cylinder starts from rest, and at the end of the third second, the ball's acceleration is $1.80\ \mathrm{m/s^2}$. (a) Find A. (b) Express the angular acceleration of the disk as a function of time. (c) How much time after the disk has begun to turn does it reach an angular speed of 15.0 rad/s? (d) Through what angle has the disk turned just as it reaches 15.0 rad/s? (*Hint:* See Section 2.6.)

Figure 9.33 Problem 9.67.

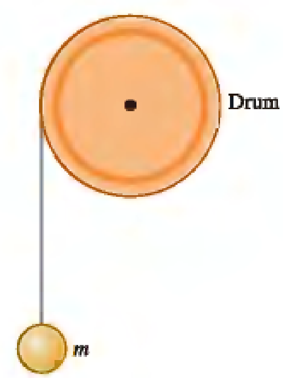


9.68. When a toy car is rapidly scooted across the floor, it stores energy in a flywheel. The car has mass 0.180 kg, and its flywheel has moment of inertia $4.00 \times 10^{-5}\ \mathrm{kg \cdot m^2}$. The car is 15.0 cm long. An advertisement claims that the car can travel at a scale speed of up to 700 km/h (440 mi/h). The scale speed is the speed of the toy car multiplied by the ratio of the length of an actual car to the length of the toy. Assume a length of 3.0 m for a real car. (a) For a scale speed of 700 km/h, what is the actual translational speed of the car? (b) If all the kinetic energy that is initially in the flywheel is converted to the translational kinetic energy of the toy, how much energy is originally stored in the flywheel? (c) What initial angular velocity of the flywheel was needed to store the amount of energy calculated in part (b)?

9.69. A classic 1957 Chevrolet Corvette of mass 1240 kg starts from rest and speeds up with a constant tangential acceleration of $3.00\ \mathrm{m/s^2}$ on a circular test track of radius 60.0 m. Treat the car as a particle. (a) What is its angular acceleration? (b) What is its angular speed 6.00 s after it starts? (c) What is its radial acceleration at this time? (d) Sketch a view from above showing the circular track, the car, the velocity vector, and the acceleration component vectors 6.00 s after the car starts. (e) What are the magnitudes of the total acceleration and net force for the car at this time? (f) What angle do the total acceleration and net force make with the car's velocity at this time?

9.70. Engineers are designing a system by which a falling mass m imparts kinetic energy to a rotating uniform drum to which it is attached by thin, very light wire wrapped around the rim of the drum (Fig. 9.34). There is no appreciable friction in the axle of the drum, and everything starts from rest. This system is being tested on earth, but it is to be used on Mars, where the acceleration due to gravity is $3.71\ \mathrm{m/s^2}$. In the earth tests, when m is set to 15.0 kg and allowed to fall through 5.00 m, it gives 250.0 J of kinetic energy to the drum. (a) If the system is operated on Mars, through what distance would the 15.0-0 mass have to fall to give the same amount of kinetic energy to the drum? (b) How fast would the 15.0-kg mass be moving on Mars just as the drum gained 250.0 J of kinetic energy?

Figure 9.34 Problem 9.70.



9.71. A vacuum cleaner belt is looped over a shaft of radius 0.45 cm and a wheel of radius 2.00 cm. The arrangement of the belt, shaft, and wheel is similar to that of the chain and sprockets in Fig. 9.14. The motor turns the shaft at 60.0 rev/s and the moving belt turns the wheel, which in turn is connected by another shaft to the roller that beats the dirt out of the rug being vacuumed. Assume that the belt doesn't slip on either the shaft or the wheel. (a) What is the speed of a point on the belt? (b) What is the angular velocity of the wheel, in rad/s?

9.72. The motor of a table saw is rotating at 3450 rev/min. A pulley attached to the motor shaft drives a second pulley of half the diameter by means of a V-belt. A circular saw blade of diameter 0.208 m is mounted on the same rotating shaft as the second pulley. (a) The operator is careless and the blade catches and throws back a small piece of wood. This piece of wood moves with linear speed equal to the tangential speed of the rim of the blade. What is this speed? (b) Calculate the radial acceleration of points on the outer edge of the blade to see why sawdust doesn't stick to its teeth.

9.73. A wheel changes its angular velocity with a constant angular acceleration while rotating about a fixed axis through its center. (a) Show that the change in the magnitude of the radial acceleration during any time interval of a point on the wheel is twice the product of the angular acceleration, the angular displacement, and the perpendicular distance of the point from the axis. (b) The radial acceleration of a point on the wheel that is 0.250 m from the axis changes from 25.0 m/s^2 to 85.0 m/s^2 as the wheel rotates through 15.0 rad. Calculate the tangential acceleration of this point. (c) Show that the change in the wheel's kinetic energy during any time interval is the product of the moment of inertia about the axis, the angular acceleration, and the angular displacement. (d) During the 15.0-rad angular displacement of part (b), the kinetic energy of the wheel increases from 20.0 J to 45.0 J. What is the moment of inertia of the wheel about the rotation axis?

9.74. A sphere consists of a solid wooden ball of uniform density 800 kg/m^3 and radius 0.20 m and is covered with a thin coating of lead foil with area density 20 kg/m^2. Calculate the moment of inertia of this sphere about an axis passing through its center.

9.75. Estimate your own moment of inertia about a vertical axis through the center of the top of your head when you are standing up straight with your arms outstretched. Make reasonable approximations and measure or estimate necessary quantities.

9.76. A thin uniform rod 50.0 cm long with mass 0.320 kg is bent at its center into a V shape, with a 70.0° angle at its vertex. Find the moment of inertia of this V-shaped object about an axis perpendicular to the plane of the V at its vertex.

9.77. It has been argued that power plants should make use of off-peak hours (such as late at night) to generate mechanical energy and store it until it is needed during peak load times, such as the middle of the day. One suggestion has been to store the energy in large flywheels spinning on nearly frictionless ball bearings. Consider a flywheel made of iron (density 7800 kg/m^3) in the shape of a 10.0-cm-thick uniform disk. (a) What would the diameter of such a disk need to be if it is to store 10.0 megajonles of kinetic energy when spinning at 90.0 rpm about an axis perpendicular to the disk at its center? (b) What would be the centripetal acceleration of a point on its rim when spinning at this rate?

9.76. While redesigning a rocket engine, you want to reduce its weight by replacing a solid spherical part with a hollow spherical shell of the same size. The parts rotate about an axis through their center. You need to make sure that the new part always has the same rotational kinetic energy as the original part had at any given rate of rotation. If the original part had mass M, what must be the mass of the new part?

9.79. The earth, which is not a uniform sphere, has a moment of inertia of $0.3308MR^2$ about an axis through its north and south poles. It takes the earth 86,164 s to spin once about this axis. Use Appendix F to calculate (a) the earth's kinetic energy due to its rotation about this axis and (b) the earth's kinetic energy due to its orbital motion around the sun. (c) Explain how the value of the earth's moment of inertia tells us that the mass of the earth is concentrated toward the planet's center.

9.68. A uniform, solid disk with mass m and radius R is pivoted about a horizontal axis through its center. A small object of the same mass m is glued to the rim of the disk. If the disk is released from rest with the small object at the end of a horizontal radius, find the angular speed when the small object is directly below the axis.

9.81. A metal sign for a car dealership is a thin, uniform right triangle with base length b and height h. The sign has mass M. (a) What is the moment of inertia of the sign for rotation about the side of length h? (b) If $M = 5.40$ kg, $b = 1.60$ m, and $h = 1.20$ m, what is the kinetic energy of the sign when it is rotating about an axis along the 1.20-m side at 2.00 rev/s?

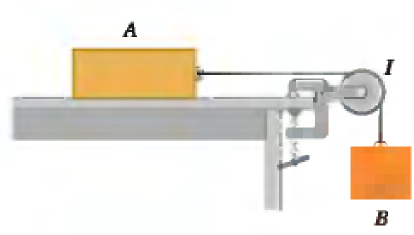
9.82. Measuring *I*. As an intern with an engineering firm, you are asked to measure the moment of inertia of a large wheel, for rotation about an axis through its center. Since you were a good physics student, you know what to do. You measure the diameter of the wheel to be 0.740 m and find that it weighs 280 N. You mount the wheel, using frictionless bearings, on a horizontal axis through the wheel's center. You wrap a light rope around the wheel and hang a 8.00-kg mass from the free end of the rope, as shown in Fig. 9.18. You release the mass from rest; the mass descends and the wheel turns as the rope unwinds. You find that the mass has speed 5.00 m/s after it has descended 2.00 m. (a) What is the moment of inertia of the wheel for an axis perpendicular to the wheel at its center? (b) Your boss tells you that a larger I is needed. He asks you to design a wheel of the same mass and radius that has $I = 19.0\ kg \cdot m^2$. How do you reply?

9.83. A meter stick with a mass of 0.160 kg is pivoted about one end so it can rotate without friction about a horizontal axis. The meter stick is held in a horizontal position and released. As it swings through the vertical, calculate (a) the change in gravitational potential energy that has occurred; (b) the angular speed of the stick; (c) the linear speed of the end of the stick opposite the axis. (d) Compare the answer in part (c) to the speed of a particle that has fallen 1.00 m, starting from rest.

9.84. Exactly one turn of a flexible rope with mass m is wrapped around a uniform cylinder with mass M and radius R. The cylinder rotates without friction about a horizontal axle along the cylinder axis. One end of the rope is attached to the cylinder. The cylinder starts with angular speed ω_0. After one revolution of the cylinder the rope has unwrapped and, at this instant, hangs vertically down, tangent to the cylinder. Find the angular speed of the cylinder and the linear speed of the lower end of the rope at this time. You can ignore the thickness of the rope. [*Hint:* Use Eq. (9.18).]

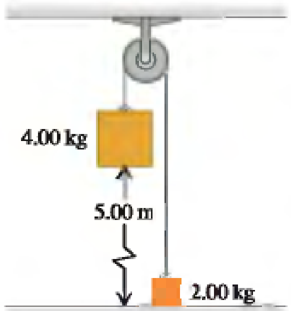
9.65. The pulley in Fig. 9.35 has radius R and a moment of inertia I. The rope does not slip over the pulley, and the pulley spins on a frictionless axle. The coefficient of kinetic friction between block A and the tabletop is μ_k. The system is released from rest, and block B descends. Block A has mass m_A and block B has mass m_B. Use energy methods to calculate the speed of block B as a function of the distance d that it has descended.

Figure 9.35 Problem 9.85.



9.68. The pulley in Fig. 9.36 has radius 0.160 m and moment of inertia $0.480\ \text{kg} \cdot \text{m}^2$. The rope does not slip on the pulley rim. Use energy methods to calculate the speed of the 4.00-kg block just before it strikes the floor.

Figure 9.36 Problem 9.86.

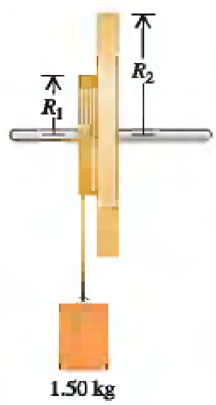


9.87. You hang a thin hoop with radius R over a nail at the rim of the hoop. You displace it to the side (within the plane of the hoop) through an angle β from its equilibrium position and let it go. What is its angular speed when it returns to its equilibrium position? [*Hint:* Use Eq. (9.18).]

9.68. A passenger bus in Zurich, Switzerland, derived its motive power from the energy stored in a large flywheel. The wheel was brought up to speed periodically, when the bus stopped at a station, by an electric motor, which could then be attached to the electric power lines. The flywheel was a solid cylinder with mass 1000 kg and diameter 1.80 m; its top angular speed was 3000 rev/min. (a) At this angular speed, what is the kinetic energy of the flywheel? (b) If the average power required to operate the bus is 1.86×10^4 W, how long could it operate between stops?

9.89. Two metal disks, one with radius $R_1 = 2.50$ cm and mass $M_1 = 0.80$ kg and the other with radius $R_2 = 5.00$ cm and mass $M_2 = 1.60$ kg, are welded together and mounted on a frictionless axis through their common center (Fig. 9.37). (a) What is the total moment of inertia of the two disks? (b) A light string is wrapped around the edge of the smaller disk, and a 1.50-kg block is suspended from the free end of the string. If the block is released from rest at a distance of 2.00 m above the floor, what is its speed just before it strikes the floor? (c) Repeat the calculation of part (b), this time with the string wrapped around the edge of the larger disk. In which case is the final speed of the block the greatest? Explain why this is so.

Figure 9.37 Problem 9.89.

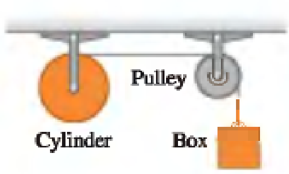


9.90. In the cylinder and mass combination described in Example 9.9 (Section 9.4), suppose the falling mass m is made of ideal rubber, so that no mechanical energy is lost when the mass hits the ground. (a) If the cylinder is originally not rotating and the mass m is released from rest at a height h above the ground, to what height will this mass rebound if it bounces straight back up from the floor? (b) Explain, in terms of energy, why the answer to part (a) is *less* than h.

9.91. In the system shown in Fig. 9.18, a 12.0-kg mass is released from rest and falls, causing the uniform 10.0-kg cylinder of diameter 30.0 cm to turn about a frictionless axle through its center. How far will the mass have to descend to give the cylinder 250 J of kinetic energy?

9.92. In Fig. 9.38, the cylinder and pulley turn without friction about stationary horizontal axles that pass through their centers. A light rope is wrapped around the cylinder, passes over the pulley, and has a 3.00-kg box suspended from its free end. There is no slipping between the rope and the pulley surface. The uniform cylinder has mass 5.00 kg and radius 40.0 cm. The pulley is a uniform disk with mass 2.00 kg and radius 20.0 cm. The box is released from rest and descends as the rope unwraps from the cylinder. Find the speed of the box when it has fallen 1.50 m.

Figure 9.38 Problem 9.92.



9.93. A thin, flat, uniform disk has mass M and radius R. A circular hole of radius $R/4$, centered at a point $R/2$ from the disk's center, is then punched in the disk. (a) Find the moment of inertia of the disk with the hole about an axis through the original center of the disk, perpendicular to the plane of the disk. (*Hint:* Find the moment of inertia of the piece punched from the disk.) (b) Find the moment of inertia of the disk with the hole about an axis through the center of the hole, perpendicular to the plane of the disk.

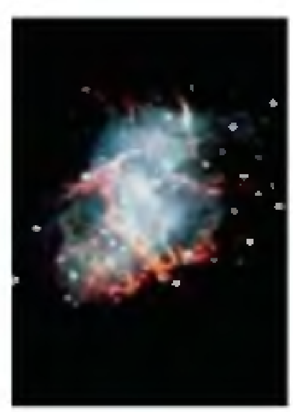
9.94. A pendulum is made of a uniform solid sphere with mass M and radius R suspended from the end of a light rod. The distance from the pivot at the upper end of the rod to the center of the sphere is L. The pendulum's moment of inertia I_P for rotation about the pivot is usually approximated as ML^2. (a) Use the parallel-axis theorem to show that if R is 5% of L and the mass of the rod is ignored, I_P is only 0.1% greater than ML^2. (b) If the mass of the rod is 1% of M and R is much less than L, what is the ratio of I_{rod} for an axis at the pivot to ML^2?

9.95. Perpendicular-Axis Theorem. Consider a rigid body that is a thin, plane sheet of arbitrary shape. Take the body to lie in the xy-plane and let the origin O of coordinates be located at any point within or outside the body. Let I_x and I_y be the moments of inertia about the x- and y-axes, and let I_O be the moment of inertia about an axis through O perpendicular to the plane. (a) By considering mass elements m_i with coordinates (x_i, y_i), show that $I_x + I_y = I_O$. This is called the perpendicular-axis theorem. Note that point O does not have to be the center of mass. (b) For a thin washer with mass M and with inner and outer radii R_1 and R_2, use the perpendicular-axis theorem to find the moment of inertia about an axis that is in the plane of the washer and that passes through its center. You may use the information in Table 9.2. (c) Use the perpendicular-axis theorem to show that for a thin, square sheet with mass M and side L, the moment of inertia about *any* axis in the plane of the sheet that passes through the center of the sheet is $\frac{1}{12}ML^2$. You may use the information in Table 9.2.

9.96. A thin, uniform rod is bent into a square of side length a. If the total mass is M, find the moment of inertia about an axis through the center and perpendicular to the plane of the square. (*Hint:* Use the parallel-axis theorem.)

***9.97.** A cylinder with radius R and mass M has density that increases linearly with distance r from the cylinder axis, $\rho = \alpha r$, where α is a positive constant. (a) Calculate the moment of inertia of the cylinder about a longitudinal axis through its center in terms of M and R. (b) Is your answer greater or smaller than the moment of inertia of a cylinder of the same mass and radius but of uniform density? Explain why this result makes qualitative sense.

Figure 9.39 Problem 9.98.



9.96. Neutron Stars and Supernova Remnants. The Crab Nebula is a cloud of glowing gas about 10 light-years across, located about 6500 light years from the earth (Fig. 9.39). It is the remnant of a star that underwent a *supernova explosion*, seen on earth in 1054 A.D. Energy is released by the

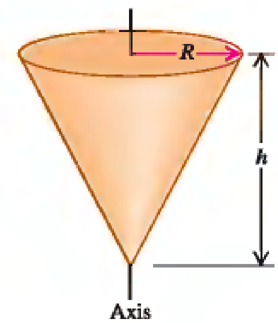
Crab Nebula at a rate of about 5×10^{31} W, about 10^5 times the rate at which the sun radiates energy. The Crab Nebula obtains its energy from the rotational kinetic energy of a rapidly spinning *neutron star* at its center. This object rotates once every 0.0331 s, and this period is increasing by 4.22×10^{-13} s for each second of time that elapses. (a) If the rate at which energy is lost by the neutron star is equal to the rate at which energy is released by the nebula, find the moment of inertia of the neutron star. (b) Theories of supernovae predict that the neutron star in the Crab Nebula has a mass about 1.4 times that of the sun. Modeling the neutron star as a solid uniform sphere, calculate its radius in kilometers. (c) What is the linear speed of a point on the equator of the neutron star? Compare to the speed of light. (d) Assume that the neutron star is uniform and calculate its density. Compare to the density of ordinary rock $(3000\ \text{kg/m}^3)$ and to the density of an atomic nucleus (about $10^{17}\ \text{kg/m}^3$). Justify the statement that a neutron star is essentially a large atomic nucleus.

Challenge Problems

9.99. The moment of inertia of a sphere with uniform density about an axis through its center is $\frac{2}{5}MR^2 = 0.400MR^2$. Satellite observations show that the earth's moment of inertia is $0.3308MR^2$. Geophysical data suggest the earth consists of five main regions: the inner core $(r = 0$ to $r = 1220\ \text{km})$ of average density $12{,}900\ \text{kg/m}^3$, the outer core $(r = 1220\ \text{km}$ to $r = 3480\ \text{km})$ of average density $10{,}900\ \text{kg/m}^3$, the lower mantle $(r = 3480\ \text{km}$ to $r = 5700\ \text{km})$ of average density $4900\ \text{kg/m}^3$, the upper mantle $(r = 5700\ \text{km}$ to $r = 6350\ \text{km})$ of average density $3600\ \text{kg/m}^3$, and the outer crust and oceans $(r = 6350\ \text{km}$ to $r = 6370\ \text{km})$ of average density $2400\ \text{kg/m}^3$. (a) Show that the moment of inertia about a diameter of a uniform spherical shell of inner radius R_1, outer radius R_2, and density ρ is $I = \rho(8\pi/15)(R_2^5 - R_1^5)$. (*Hint:* Form the shell by superposition of a sphere of density ρ and a smaller sphere of density $-\rho$.) (b) Check the given data by using them to calculate the mass of the earth. (c) Use the given data to calculate the earth's moment of inertia in terms of MR^2.

***9.100.** Calculate the moment of inertia of a uniform solid cone about an axis through its center (Fig. 9.40). The cone has mass M and altitude h. The radius of its circular base is R.

Figure 9.40 Challenge Problem 9.100.



9.101. On a compact disc (CD), music is coded in a pattern of tiny pits arranged in a track that spirals outward toward the rim of the disc. As the disc spins inside a CD player, the track is scanned at a constant *linear* speed of $v = 1.25\ \text{m/s}$. Because the radius of the track varies as it spirals outward, the *angular* speed of the disc must change as the CD is played. (See Exercise 9.22.) Let's see what angular acceleration is required to keep v constant. The equation of a spiral is $r(\theta) = r_0 + \beta\theta$, where r_0 is the radius of the spiral at $\theta = 0$ and β is a constant. On a CD, r_0 is the inner radius of the spiral track. If we take the rotation direction of the CD to be positive, β must be positive so that r increases as the disc turns and θ increases. (a) When the disc rotates through a small angle $d\theta$, the distance scanned along the track is $ds = r\,d\theta$. Using the above expression for $r(\theta)$, integrate ds to find the total distance s scanned along the track as a function of the total angle θ through which the disc has rotated. (b) Since the track is scanned at a constant linear speed v, the distance s found in part (a) is equal to vt. Use this to find θ as a function of time. There will be two solutions for θ; choose the positive one, and explain why this is the solution to choose. (c) Use your expression for $\theta(t)$ to find the angular velocity ω_z and the angular acceleration α_z as functions of time. Is α_z constant? (d) On a CD, the inner radius of the track is 25.0 mm, the track radius increases by 1.55 μm per revolution, and the playing time is 74.0 min. Find the values of r_0 and β, and find the total number of revolutions made during the playing time. (e) Using your results from parts (c) and (d), make graphs of ω_z (in rad/s) versus t and α_z (in rad/s^2) versus t between $t = 0$ and $t = 74.0$ min.

10 DYNAMICS OF ROTATIONAL MOTION

LEARNING GOALS

By studying this chapter, you will learn:

- What is meant by the torque produced by a force.
- How the net torque on a body affects the rotational motion of the body.
- How to analyze the motion of a body that both rotates and moves as a whole through space.
- How to solve problems that involve work and power for rotating bodies.
- What is meant by the angular momentum of a particle or of a rigid body.
- How the angular momentum of a system changes with time.
- Why a spinning gyroscope goes through the curious motion called precession.

? If this skydiver isn't touching the ground, how can he change his rotation speed? What physical principle is at work here?

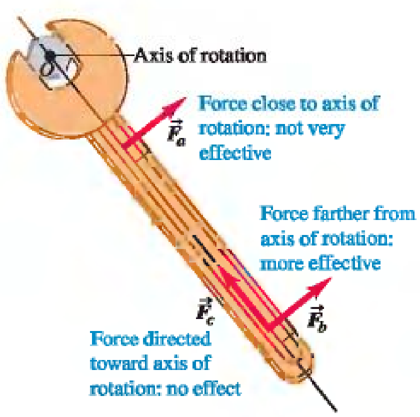
We learned in Chapters 4 and 5 that a net force applied to a body gives that body an acceleration. But what does it take to give a body an *angular* acceleration? That is, what does it take to start a stationary body rotating or to bring a spinning body to a halt? A force is required, but it must be applied in a way that gives a twisting or turning action.

In this chapter we will define a new physical quantity, *torque*, that describes the twisting or turning effort of a force. We'll find that the net torque acting on a rigid body determines its angular acceleration, in the same way that the net force on a body determines its linear acceleration. We'll also look at work and power in rotational motion so as to understand such problems as how energy is transmitted by the rotating drive shaft in a car. Finally, we will develop a new conservation principle, *conservation of angular momentum*, that is tremendously useful for understanding the rotational motion of both rigid and nonrigid bodies. We'll finish this chapter by studying *gyroscopes*, rotating devices that seemingly defy common sense and don't fall over when you might think they should—but that actually behave in perfect accordance with the dynamics of rotational motion.

10.1 Torque

We know that forces acting on a body can affect its **translational motion**—that is, the motion of the body as a whole through space. Now we want to learn which aspects of a force determine how effective it is in causing or changing *rotational* motion. The magnitude and direction of the force are important, but so is the point on the body where the force is applied. In Fig. 10.1 a wrench is being used to loosen a tight bolt. Force $\vec{F}_b$, applied near the end of the handle, is more effective than an equal force $\vec{F}_a$ applied near the bolt. Force $\vec{F}_c$ doesn't do any good at all; it's applied at the same point and has the same magnitude as $\vec{F}_b$, but it's directed along the length of the handle. The quantitative measure of the

10.1 Which of these three equal-magnitude forces is most likely to loosen the tight bolt?



tendency of a force to cause or change a body's rotational motion is called *torque;* we say that $\vec{F}_a$ applies a torque about point O to the wrench in Fig. 10.1, $\vec{F}_b$ applies a greater torque about O, and $\vec{F}_c$ applies zero torque about O.

Figure 10.2 shows three examples of how to calculate torque. The body in the figure can rotate about an axis that is perpendicular to the plane of the figure and passes through point O. Three forces, $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$, act on the body in the plane of the figure. The tendency of the first of these forces, $\vec{F}_1$, to cause a rotation about O depends on its magnitude F_1. It also depends on the *perpendicular* distance l_1 between point O and the **line of action** of the force (that is, the line along which the force vector lies). We call the distance l_1 the **lever arm** (or **moment arm**) of force $\vec{F}_1$ about O. The twisting effort is directly proportional to both F_1 and l_1, so we define the **torque** (or *moment*) of the force $\vec{F}_1$ with respect to O as the product $F_1 l_1$. We use the Greek letter τ (tau) for torque. In general, for a force of magnitude F whose line of action is a perpendicular distance l from O, the torque is

$$\tau = Fl \tag{10.1}$$

Physicists usually use the term "torque," while engineers usually use "moment" (unless they are talking about a rotating shaft). Both groups use the term "lever arm" or "moment arm" for the distance l.

The lever arm of $\vec{F}_1$ in Fig. 10.2 is the perpendicular distance l_1, and the lever arm of $\vec{F}_2$ is the perpendicular distance l_2. The line of action of $\vec{F}_3$ passes through point O, so the lever arm for $\vec{F}_3$ is zero and its torque with respect to O is zero. In the same way, force $\vec{F}_c$ in Fig. 10.1 has zero torque with respect to point O; $\vec{F}_b$ has a greater torque than $\vec{F}_a$ because its lever arm is greater.

CAUTION **Torque is always measured about a point** Note that torque is *always* defined with reference to a specific point. If we shift the position of this point, the torque of each force may also change. For example, the torque of force $\vec{F}_3$ in Fig. 10.2 is zero with respect to point O, but the torque of $\vec{F}_3$ is *not* zero about point A. It's not enough to refer to "the torque of $\vec{F}$"; you must say "the torque of $\vec{F}$ with respect to point X" or "the torque of $\vec{F}$ about point X."

Force $\vec{F}_1$ in Fig. 10.2 tends to cause *counterclockwise* rotation about O, while $\vec{F}_2$ tends to cause *clockwise* rotation. To distinguish between these two possibilities, we need to choose a positive sense of rotation. With the choice that *counterclockwise torques are positive and clockwise torques are negative,* the torques of $\vec{F}_1$ and $\vec{F}_2$ about O are

$$\tau_1 = +F_1 l_1 \qquad \tau_2 = -F_2 l_2$$

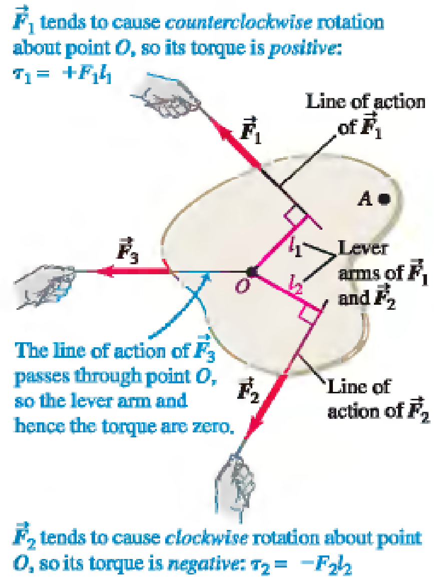
Figure 10.2 shows this choice for the sign of torque. We will often use the symbol $\circlearrowleft+$ to indicate our choice of the positive sense of rotation.

The SI unit of torque is the newton-meter. In our discussion of work and energy we called this combination the joule. But torque is *not* work or energy, and torque should be expressed in newton-meters, *not* joules.

Figure 10.3 shows a force $\vec{F}$ applied at a point P described by a position vector $\vec{r}$ with respect to the chosen point O. There are three ways to calculate the torque of this force:

1. Find the lever arm l and use $\tau = Fl$.
2. Determine the angle ϕ between the vectors $\vec{r}$ and $\vec{F}$; the lever arm is $r\sin\phi$, so $\tau = rF\sin\phi$.
3. Represent $\vec{F}$ in terms of a radial component F_{rad} along the direction of $\vec{r}$ and a tangential component F_{tan} at right angles, perpendicular to $\vec{r}$. (We call this a tangential component because if the body rotates, the point where the force acts moves in a circle, and this component is tangent to that circle.)

10.2 The torque of a force about a point is the product of the force magnitude and the lever arm of the force.



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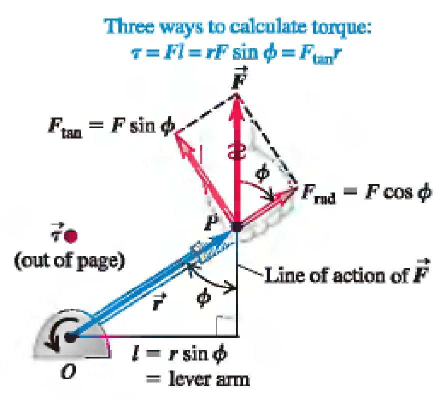
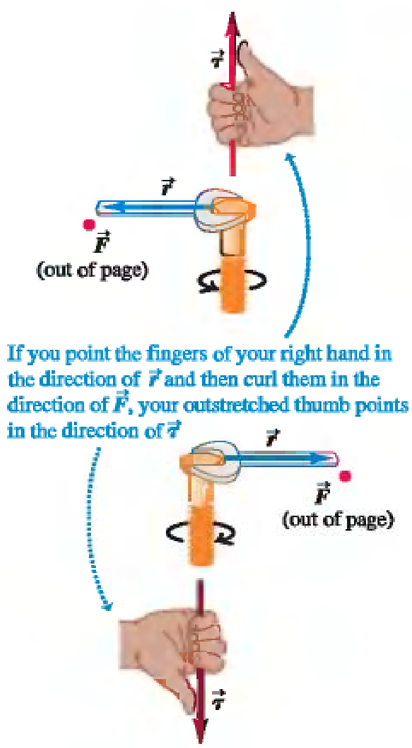
7.1 Calculating Torques

10.3 Three ways to calculate the torque of the force $\vec{F}$ about the point O. In this figure, $\vec{r}$ and $\vec{F}$ are in the plane of the page and the torque vector $\vec{\tau}$ points out of the page toward you.



10.4 The torque vector $\vec{\tau} = \vec{r} \times \vec{F}$ is directed along the axis of the bolt, perpendicular to both $\vec{r}$ and $\vec{F}$. The fingers of the right hand curl in the direction of the rotation that the torque tends to cause.



Then $F_{tan} = F\sin\phi$ and $\tau = r(F\sin\phi) = F_{tan}r$. The component F_{rad} produces *no* torque with respect to O because its lever arm with respect to that point is zero (compare to forces $\vec{F}_c$ in Fig. 10.1 and $\vec{F}_3$ in Fig. 10.2).

Summarizing these three expressions for torque, we have

$$\tau = Fl = rF\sin\phi = F_{tan}r \qquad \text{(magnitude of torque)} \qquad (10.2)$$

Torque as a Vector

We saw in Section 9.1 that angular velocity and angular acceleration can be represented as vectors; the same is true for torque. To see how to do this, note that the quantity $rF\sin\phi$ in Eq. (10.2) is the magnitude of the *vector product* $\vec{r} \times \vec{F}$ that we defined in Section 1.10. (You should go back and review that definition.) We now generalize the definition of torque as follows: When a force $\vec{F}$ acts at a point having a position vector $\vec{r}$ with respect to an origin O, as in Fig. 10.3, the torque $\vec{\tau}$ of the force with respect to O is the *vector* quantity

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \text{(definition of torque vector)} \qquad (10.3)$$

The torque as defined in Eq. (10.2) is just the magnitude of the torque vector $\vec{r} \times \vec{F}$. The direction of $\vec{\tau}$ is perpendicular to both $\vec{r}$ and $\vec{F}$. In particular, if both $\vec{r}$ and $\vec{F}$ lie in a plane perpendicular to the axis of rotation, as in Fig. 10.3, then the torque vector $\vec{\tau} = \vec{r} \times \vec{F}$ is directed along the axis of rotation, with a sense given by the right-hand rule (Fig. 1.29). Figure 10.4 shows the direction relationships.

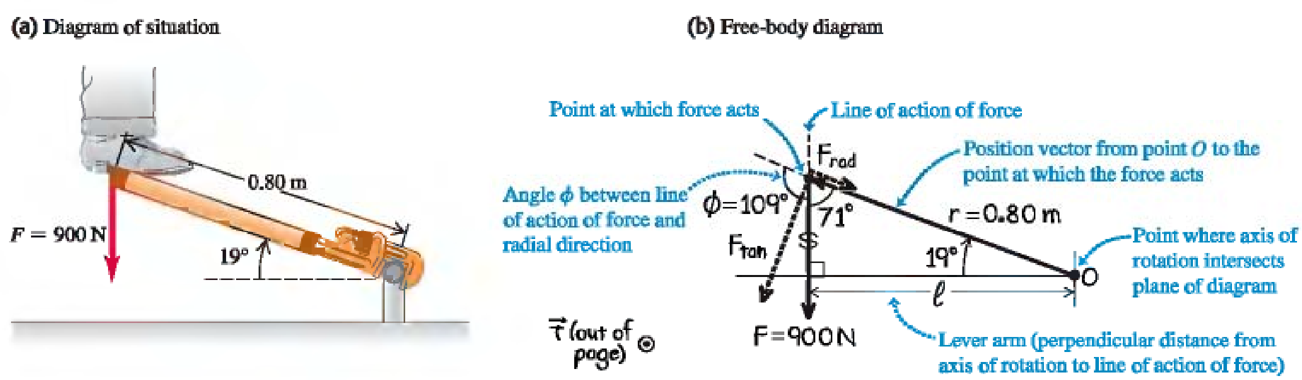
In diagrams that involve $\vec{r}$, $\vec{F}$, and $\vec{\tau}$, it's common to have one of the vectors oriented perpendicular to the page. (Indeed, by the very nature of the cross product, $\vec{\tau} = \vec{r} \times \vec{F}$ *must* be perpendicular to the plane of the vectors $\vec{r}$ and $\vec{F}$.) We use a dot (•) to represent a vector that points out of the page (see Fig. 10.3) and a cross (×) to represent a vector that points into the page.

In the following sections we will usually be concerned with rotation of a body about an axis oriented in a specified constant direction. In that case, only the component of torque along that axis is of interest, and we often call that component the torque with respect to the specified *axis*.

Example 10.1 Applying a torque

A weekend plumber, unable to loosen a pipe fitting, slips a piece of scrap pipe (a "cheater") over his wrench handle. He then applies his full weight of 900 N to the end of the cheater by standing on it. The distance from the center of the fitting to the point where the weight acts is 0.80 m, and the wrench handle and cheater make an angle of 19° with the horizontal (Fig. 10.5a). Find the magnitude and direction of the torque he applies about the center of the pipe fitting.

10.5 (a) A weekend plumber tries to loosen a pipe fitting by standing on a "cheater." (b) Our vector diagram to find the torque about O.



SOLUTION

IDENTIFY: Figure 10.5b shows the vectors $\vec{r}$ and $\vec{F}$ and the angle between them ($\phi = 109°$). We'll use our knowledge of these vectors to calculate the torque vector $\vec{\tau} = \vec{r} \times \vec{F}$.

SET UP: Equation (10.1) or (10.2) will tell us the magnitude of the torque, and the right-hand rule with Eq. (10.3) will tell us the torque direction.

EXECUTE: To use Eq. (10.1), we first calculate the lever arm l. As Fig. 10.5b shows,

$$l = (0.80\ \text{m})\sin 109° = (0.80\ \text{m})\sin 71° = 0.76\ \text{m}$$

Then Eq. (10.1) tells us that the magnitude of the torque is

$$\tau = Fl = (900\ \text{N})(0.76\ \text{m}) = 680\ \text{N}\cdot\text{m}$$

Or, from Eq. (10.2),

$$\tau = rF\sin\phi = (0.80\ \text{m})(900\ \text{N})(\sin 109°) = 680\ \text{N}\cdot\text{m}$$

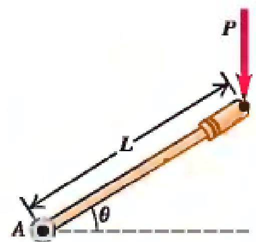
Alternatively, we can find $F_{\tan}$, the tangential component of $\vec{F}$. This is the component that acts perpendicular to $\vec{r}$ (that is, perpendicular to the "cheater"). The vector $\vec{r}$ is oriented 19° from the horizontal, so the perpendicular to $\vec{r}$ is oriented 19° from the vertical. Since $\vec{F}$ is vertical, this means $F_{\tan} = F(\cos 19°) = (900\ \text{N})(\cos 19°) = 851\ \text{N}$. Then the torque is

$$\tau = F_{\tan}r = (851\ \text{N})(0.80\ \text{m}) = 680\ \text{N}\cdot\text{m}$$

If you curl the fingers of your right hand from the direction of $\vec{r}$ (in the plane of Fig. 10.5b, to the left and up) into the direction of $\vec{F}$ (straight down), your right thumb points out of the plane of the figure. This is the direction of the torque $\vec{\tau}$.

EVALUATE: We've already checked our answer for the magnitude τ by calculating it in three different ways. To check our result for the direction of the torque, note that the force in Fig. 10.5 tends to produce a counterclockwise rotation about O. If you curl the fingers of your right hand in a counterclockwise direction, the thumb points out of the plane of Fig. 10.5, which is indeed the direction of the torque.

Test Your Understanding of Section 10.1 The figure shows a force P being applied to one end of a lever of length L. What is the magnitude of the torque of this force about point A? (i) $PL\sin\theta$; (ii) $PL\cos\theta$; (iii) $PL\tan\theta$.



10.2 Torque and Angular Acceleration for a Rigid Body

We are now ready to develop the fundamental relationship for the rotational dynamics of a rigid body. We will show that the angular acceleration of a rotating rigid body is directly proportional to the sum of the torque components along the axis of rotation. The proportionality factor is the moment of inertia.

To develop this relationship, we again imagine the body as being made up of a large number of particles. We choose the axis of rotation to be the z-axis; the first particle has mass m_1 and distance r_1 from this axis (Fig. 10.6). The *net force* $\vec{F}_1$ acting on this particle has a component $F_{1,\text{rad}}$ along the radial direction, a component $F_{1,\tan}$ that is tangent to the circle of radius r_1 in which the particle moves as the body rotates, and a component F_{1z} along the axis of rotation. Newton's second law for the tangential component is

$$F_{1,\tan} = m_1 a_{1,\tan} \tag{10.4}$$

We can express the tangential acceleration of the first particle in terms of the angular acceleration α_z of the body using Eq. (9.14): $a_{1,\tan} = r_1\alpha_z$. Using this relationship and multiplying both sides of Eq. (10.4) by r_1, we obtain

$$F_{1,\tan}r_1 = m_1 r_1^2 \alpha_z \tag{10.5}$$

From Eq. (10.2), $F_{1,\tan}r_1$ is just the *torque* of the net force with respect to the rotation axis, equal to the component τ_{1z} of the torque vector along the rotation axis. The subscript z is a reminder that the torque affects rotation around the z-axis, in the same way that the subscript on F_{1z} is a reminder that this force affects the motion of particle 1 along the z-axis.

Neither of the components $F_{1,\text{rad}}$ or F_{1z} contributes to the torque about the z-axis, since neither tends to change the particle's rotation about that axis. So

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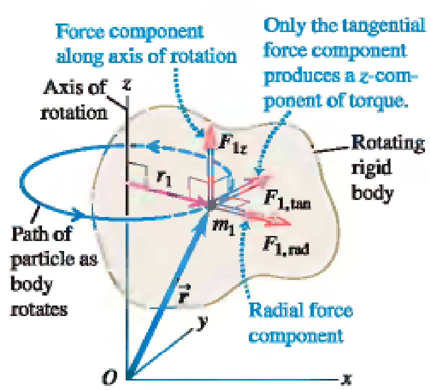
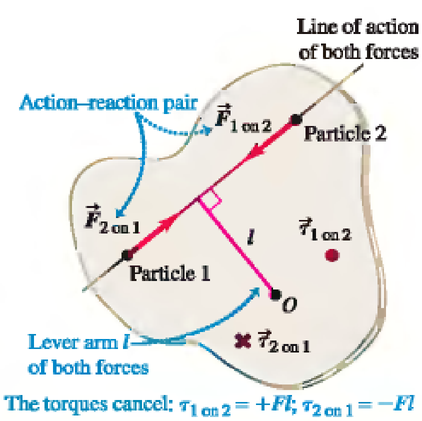
7.8 Rotoride—Dynamics Approach
7.9 Falling Ladder
7.10 Woman and Flywheel Elevator—Dynamics Approach

10.6 As a rigid body rotates around the z-axis, a net force $\vec{F}_1$ acts on one particle of the body. Only the force component $F_{1,\tan}$ can affect the rotation, because only $F_{1,\tan}$ exerts a torque about O with a z-component (along the rotation axis).



10.7 Loosening or tightening a screw requires giving it an angular acceleration and hence applying a torque. This is made easier by using a screwdriver with a large-radius handle, which provides a large lever arm for the force you apply with your hand.

10.8 Two particles in a rigid body exert equal and opposite forces on each other. If the forces act along the line joining the particles, the lever arms of the forces are the same and the torques due to the two forces are equal and opposite. Only *external* torques affect the body's rotation.



$\tau_{1z} = F_{1,\tan}r_1$ is the total torque acting on the particle with respect to the rotation axis. Also, $m_1r_1^2$ is I_1, the moment of inertia of the particle about the rotation axis. Hence we can rewrite Eq. (10.5) as

$$\tau_{1z} = I_1\alpha_z = m_1r_1^2\alpha_z$$

We write an equation like this for every particle in the body and then add all these equations:

$$\tau_{1z} + \tau_{2z} + \cdots = I_1\alpha_z + I_2\alpha_z + \cdots = m_1r_1^2\alpha_z + m_2r_2^2\alpha_z + \cdots$$

or

$$\sum \tau_{iz} = \left(\sum m_i r_i^2\right)\alpha_z \tag{10.6}$$

The left side of Eq. (10.6) is the sum of all the torques about the rotation axis that act on all the particles. The right side is $I = \sum m_i r_i^2$, the total moment of inertia about the rotation axis, multiplied by the angular acceleration α_z. Note that α_z is the same for every particle because this is a *rigid* body. Thus for the rigid body as a whole, Eq. (10.6) is the *rotational analog of Newton's second law:*

$$\sum \tau_z = I\alpha_z \tag{10.7}$$

(rotational analog of Newton's second law for a rigid body)

Just as Newton's second law says that the net force on a particle equals the particle's mass times its acceleration, Eq. (10.7) says that the net torque on a rigid body equals the body's moment of inertia about the rotation axis times its angular acceleration (Fig. 10.7).

Note that because our derivation assumed that the angular acceleration α_z is the same for all particles in the body, Eq. (10.7) is valid *only* for *rigid* bodies. Hence this equation doesn't apply to a rotating tank of water or a swirling tornado of air, different parts of which have different angular accelerations. Also note that since our derivation used Eq. (9.14), $a_{\tan} = r\alpha_z$, α_z must be measured in rad/s^2.

The torque on each particle is due to the net force on that particle, which is the vector sum of external and internal forces (see Section 8.2). According to Newton's third law, the *internal* forces that any pair of particles in the rigid body exert on each other are equal and opposite (Fig. 10.8). If these forces act along the line joining the two particles, their lever arms with respect to any axis are also equal. So the torques for each such pair are equal and opposite, and add to zero. Hence *all* the internal torques add to zero, so the sum $\sum \tau_z$ in Eq. (10.7) includes only the torques of the *external* forces.

Often, an important external force acting on a body is its *weight*. This force is not concentrated at a single point; it acts on every particle in the entire body. Nevertheless, it turns out that if $\vec{g}$ has the same value at all points, we always get the correct torque (about any specified axis) if we assume that all the weight is concentrated at the *center of mass* of the body. We will prove this statement in Chapter 11, but meanwhile we will use it for some of the problems in this chapter.

Problem-Solving Strategy 10.1 Rotational Dynamics for Rigid Bodies

MP

Our strategy for solving problems in rotational dynamics is very similar to Problem-Solving Strategy 5.2 (Section 5.2) for solving problems that involve Newton's second law.

IDENTIFY *the relevant concepts:* The equation $\sum \tau_z = I\alpha_z$ is useful whenever torques act on a rigid body—that is, whenever forces act on the body in such a way as to change its rotation.

In some cases you may be able to use an energy approach instead, as we did in Section 9.4. However, if the target variable is a force, a torque, an acceleration, an angular acceleration, or an elapsed time, using $\sum \tau_z = I\alpha_z$ is almost always the best approach.

SET UP *the problem* using the following steps:

1. Draw a sketch of the situation and select the body or bodies to be analyzed.
2. For each body, draw a free-body diagram and label unknown quantities with algebraic symbols. A new consideration is that

Continued

you must show the *shape* of the body accurately, including all dimensions and angles you will need for torque calculations.

3. Choose coordinate axes for each body and indicate a positive sense of rotation for each rotating body. If there is a linear acceleration, it's usually simplest to pick a positive axis in its direction. If you know the sense of α_z in advance, picking it as the positive sense of rotation simplifies the calculations.

EXECUTE *the solution* as follows:

1. For each body in the problem, decide whether it undergoes translational motion, rotational motion, or both. Then apply $\sum \vec{F} = m\vec{a}$ (as in Section 5.2), $\sum \tau_z = I\alpha_z$, or both to the body. Be careful to write separate equations of motion for each body.
2. There may be *geometrical* relationships between the motions of two or more bodies, as with a string that unwinds from a pulley while turning it or a wheel that rolls without slipping (to be discussed in Section 10.3). Express these relationships in algebraic form, usually as relationships between two linear accelerations or between a linear acceleration and an angular acceleration.
3. Check that the number of equations matches the number of unknown quantities. Then solve the equations to find the target variable(s).

EVALUATE *your answer:* Check that the algebraic signs of your results make sense. As an example, suppose the problem is about a spool of thread. If you are pulling thread off the spool, your answers should *not* tell you that the spool is turning in the direction that rolls the thread back on the spool! Whenever possible, check the results for special cases or extreme values of quantities. Ask yourself: "Does this result make sense?"

Example 10.2 An unwinding cable I

Figure 10.9a shows the same situation that we analyzed in Example 9.8 (Section 9.4) using energy methods. A cable is wrapped several times around a uniform solid cylinder that can rotate about its axis. The cylinder has diameter 0.120 m and mass 50 kg. The cable is pulled with a force of 9.0 N. Assuming that the cable unwinds without stretching or slipping, what is its acceleration?

SOLUTION

IDENTIFY: Our target variable is the acceleration of the cable, which we cannot find directly using the energy method of Section 9.4 (which does not involve acceleration). Instead, we'll apply rotational dynamics to the cylinder. To obtain the acceleration of the cable, we'll find a relationship between the motion of the cable and the motion of the rim of the cylinder.

SET UP: The cylinder rotates counterclockwise when the cable is pulled, so we take counterclockwise rotation to be positive. The net force on the cylinder must be zero because its center of mass remains at rest (Fig. 10.9b). The weight (magnitude Mg) and the normal force (magnitude n) exerted by the cylinder's bearings act along lines through the rotation axis. Hence these forces produce no torque with respect to that axis. The only torque about the rotation axis is due to the force F.

EXECUTE: The force F has a lever arm equal to the radius R of the cylinder: $l = R = 0.060$ m, so the torque due to F is $\tau_z = FR$. (This torque is positive as it tends to cause a counterclockwise rotation.) From Example 9.8, the moment of inertia of the cylinder about the rotation axis is $I = \frac{1}{2}MR^2$. Hence Eq. (10.7) gives us the *angular* acceleration of the cylinder:

$$\alpha_z = \frac{\tau_z}{I} = \frac{FR}{MR^2/2} = \frac{2F}{MR} = \frac{2(9.0\text{ N})}{(50\text{ kg})(0.060\text{ m})} = 6.0\text{ rad/s}^2$$

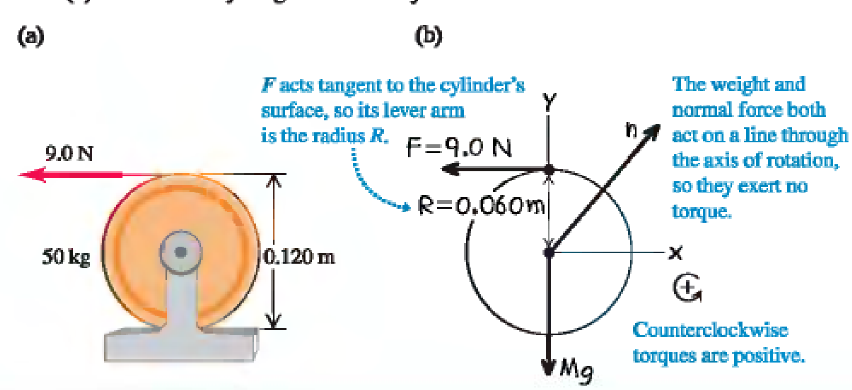
(Be certain to check that these units are correct. We can add the "rad" to our result because a radian is a dimensionless quantity.)

To get the *linear* acceleration of the cable, we need a kinematic relationship. We remarked in Section 9.3 that the acceleration of a cable unwinding from a cylinder is the same as the tangential component of acceleration of a point on the surface of the cylinder where the cable is tangent to it. This tangential acceleration is given by Eq. (9.14):

$$a_x = R\alpha = (0.060\text{ m})(6.0\text{ rad/s}^2) = 0.36\text{ m/s}^2$$

EVALUATE: Can you use this result, together with an equation from Chapter 2, to determine the speed of the cable after it has been pulled 2.0 m? Try it, and compare your result with Example 9.8, in which we found this speed using work and energy considerations.

10.9 (a) Cylinder and cable. (b) Our free-body diagram for the cylinder.



Example 10.3 An unwinding cable II

Let us revisit the situation that we analyzed in Example 9.9 (Section 9.4) using energy methods. This time, find the acceleration of the block of mass m.

SOLUTION

IDENTIFY: We'll apply translational dynamics to the hanging block and rotational dynamics to the cylinder. Because the cable doesn't slip on the cylinder, there is a relationship between the linear acceleration of the block (our target variable) and the angular acceleration of the cylinder.

SET UP: In Fig. 10.10, we sketch the situation and draw a free-body diagram for each body. We take the positive sense of rotation for the cylinder to be counterclockwise and the positive direction of the y-coordinate for the object to be downward.

EXECUTE: For the object, Newton's second law gives

$$\sum F_y = mg + (-T) = ma_y$$

For the cylinder, the weight Mg and the normal force n (exerted by the bearing) have no torques with respect to the rotation axis because they act along lines through that axis, just as in Example 10.2. The only torque is that due to the cable tension T. Applying Eq. (10.7) to the cylinder gives

$$\sum \tau_z = RT = I\alpha_z = \frac{1}{2}MR^2\alpha_z$$

As in Example 10.2, the acceleration of the cable is the same as the tangential acceleration of a point on the cylinder rim. According to Eq. (9.14), this acceleration is given by $a_y = a_{tan} = R\alpha_z$. We use this to replace $R\alpha_z$ with a_y in the cylinder equation above, and then divide by R; the result is

$$T = \frac{1}{2}Ma_y$$

Now we substitute this expression for T into Newton's second law for the object and solve for the acceleration a_y:

$$mg - \frac{1}{2}Ma_y = ma_y$$

$$a_y = \frac{g}{1 + M/2m}$$

EVALUATE: The acceleration is positive (in the downward direction) and less than g, as it should be, since the cable is holding the object back. To see how much force the cable exerts, substitute our expression for a_y back into Newton's second law for the object to find T:

$$T = mg - ma_y = mg - m\left(\frac{g}{1 + M/2m}\right) = \frac{mg}{1 + 2m/M}$$

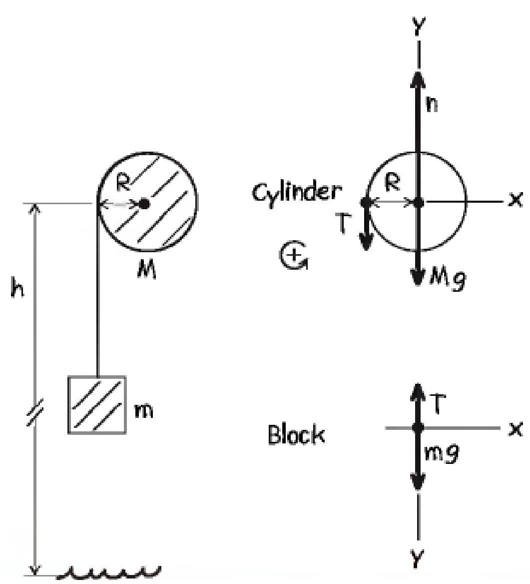
The tension in the cable is *not* equal to the weight mg of the object; if it were, the object could not accelerate.

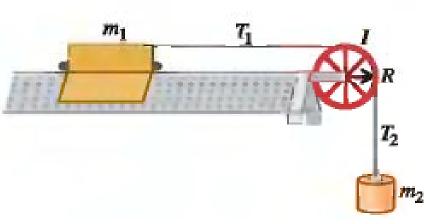
Let's check some particular cases. When M is much larger than m, the tension is nearly equal to mg, and the acceleration is correspondingly much less than g. When M is zero, $T = 0$ and $a_y = g$; the object then falls freely. If the object starts from rest ($v_{0y} = 0$) a height h above the floor, its y-velocity when it strikes the ground is given by $v_y^2 = v_{0y}^2 + 2a_y h = 2a_y h$, so $v_0 = 0$

$$v_y = \sqrt{2a_y h} = \sqrt{\frac{2gh}{1 + M/2m}}$$

This is the same result we obtained from energy considerations in Example 9.9.

10.10 (a) Our diagram of the situation. (b) Our free-body diagrams for the cylinder and the block. We assume the cable has negligible mass.



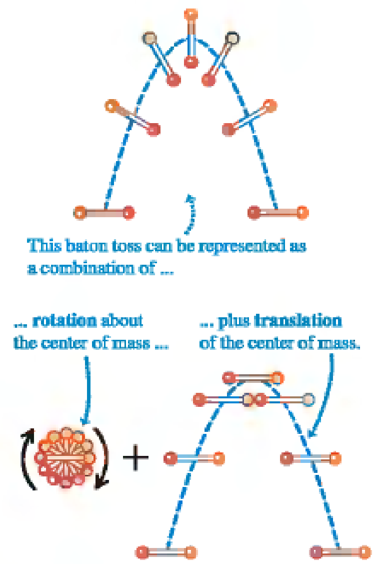


Test Your Understanding of Section 10.2 The figure shows a glider of mass m_1 that can slide without friction on a horizontal air track. It is attached to an object of mass m_2 by a massless string. The pulley has radius R and moment of inertia I about its axis of rotation. When released, the hanging object accelerates downward, the glider accelerates to the right, and the string turns the pulley without slipping or stretching. Rank the magnitudes of the following forces that act during the motion, in order from largest to smallest magnitude. (i) the tension force (magnitude T_1) in the horizontal part of the string; (ii) the tension force (magnitude T_2) in the vertical part of the string; (iii) the weight m_2g of the hanging object.

10.3 Rigid-Body Rotation About a Moving Axis

We can extend our analysis of the dynamics of rotational motion to some cases in which the axis of rotation moves. When that happens, the motion of the body is **combined translation and rotation.** The key to understanding such situations is this: Every possible motion of a rigid body can be represented as a combination of *translational motion of the center of mass* and *rotation about an axis through the center of mass.* This is true even when the center of mass accelerates, so that it is not at rest in any inertial frame. Figure 10.11 illustrates this for the motion of a tossed baton: The center of mass of the baton follows a parabolic curve, as though the baton were a particle located at the center of mass. Other examples of combined translational and rotational motions include a ball rolling down a hill and a yo-yo unwinding at the end of a string.

10.11 The motion of a rigid body is a combination of translational motion of the center of mass and rotation around the center of mass.



Combined Translation and Rotation: Energy Relationships

It's beyond the scope of this book to prove that the motion of a rigid body can always be divided into translation of the center of mass and rotation about the center of mass. But we can show that this is true for the *kinetic energy* of a rigid body that has both translational and rotational motions. In this case, the body's kinetic energy is the sum of a part $\frac{1}{2}Mv_{cm}^2$ associated with motion of the center of mass and a part $\frac{1}{2}I_{cm}\omega^2$ associated with rotation about an axis through the center of mass:

$$K = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 \qquad (10.8)$$

(rigid body with both translation and rotation)

To prove this relationship, we again imagine the rigid body to be made up of particles. Consider a typical particle with mass m_i as shown in Fig. 10.12. The velocity $\vec{v}_i$ of this particle relative to an inertial frame is the vector sum of the velocity $\vec{v}_{cm}$ of the center of mass and the velocity $\vec{v}_i'$ of the particle *relative to* the center of mass:

$$\vec{v}_i = \vec{v}_{cm} + \vec{v}_i' \qquad (10.9)$$

The kinetic energy K_i of this particle in the inertial frame is $\frac{1}{2}m_i v_i^2$, which we can also express as $\frac{1}{2}m_i(\vec{v}_i \cdot \vec{v}_i)$. Substituting Eq. (10.9) into this, we get

$$\begin{aligned} K_i &= \frac{1}{2}m_i(\vec{v}_{cm} + \vec{v}_i') \cdot (\vec{v}_{cm} + \vec{v}_i') \\ &= \frac{1}{2}m_i(\vec{v}_{cm} \cdot \vec{v}_{cm} + 2\vec{v}_{cm} \cdot \vec{v}_i' + \vec{v}_i' \cdot \vec{v}_i') \\ &= \frac{1}{2}m_i(v_{cm}^2 + 2\vec{v}_{cm} \cdot \vec{v}_i' + v_i'^2) \end{aligned}$$

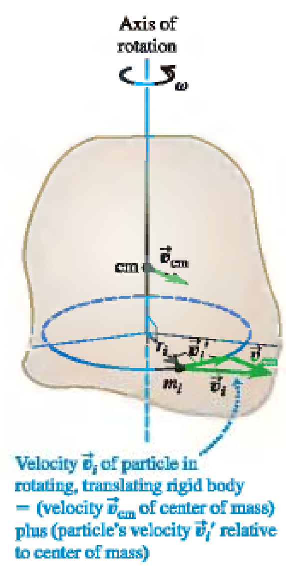
The total kinetic energy is the sum $\sum K_i$ for all the particles making up the body. Expressing the three terms in this equation as separate sums, we get

$$K = \sum K_i = \sum\left(\frac{1}{2}m_i v_{cm}^2\right) + \sum(m_i \vec{v}_{cm} \cdot \vec{v}_i') + \sum\left(\frac{1}{2}m_i v_i'^2\right)$$

The first and second terms have common factors that can be taken outside the sum:

$$K = \frac{1}{2}\left(\sum m_i\right)v_{cm}^2 + \vec{v}_{cm} \cdot \left(\sum m_i \vec{v}_i'\right) + \sum\left(\frac{1}{2}m_i v_i'^2\right) \qquad (10.10)$$

10.12 A rigid body with both translation and rotation.



Now comes the reward for our effort. In the first term, $\sum m_i$ is the total mass M. The second term is zero because $\sum m_i \vec{v}_i'$ is M times the velocity of the center of mass *relative to the center of mass*, and this is zero by definition. The last term is the sum of the kinetic energies of the particles computed by using their speeds with respect to the center of mass; this is just the kinetic energy of rotation around the center of mass. Using the same steps that led to Eq. (9.17) for the rotational kinetic energy of a rigid body, we can write this last term as $\frac{1}{2}I_{cm}\omega^2$, where I_{cm} is the moment of inertia with respect to the axis through the center of mass and ω is the angular speed. So Eq. (10.10) becomes Eq. (10.8):

$$K = \frac{1}{2}Mv_{cm}{}^2 + \frac{1}{2}I_{cm}\omega^2$$

Rolling Without Slipping

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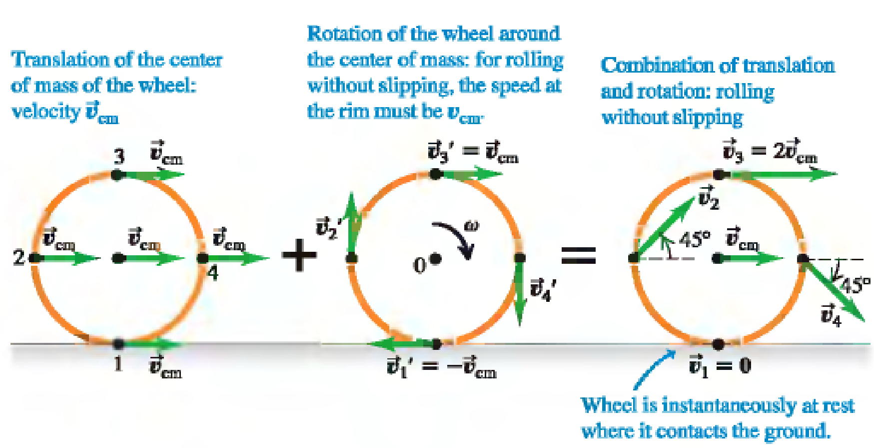
7.11 Race Between a Block and a Disk

An important case of combined translation and rotation is **rolling without slipping,** such as the motion of the wheel shown in Fig. 10.13. The wheel is symmetrical, so its center of mass is at its geometric center. We view the motion in an inertial frame of reference in which the surface on which the wheel rolls is at rest. In this frame, the point on the wheel that contacts the surface must be instantaneously *at rest* so that it does not slip. Hence the velocity $\vec{v}_1'$ of the point of contact relative to the center of mass must have the same magnitude but opposite direction as the center-of-mass velocity $\vec{v}_{cm}$. If the radius of the wheel is R and its angular speed about the center of mass is ω, then the magnitude of $\vec{v}_1'$ is $R\omega$; hence we must have

$$v_{cm} = R\omega \qquad \text{(condition for rolling without slipping)} \qquad (10.11)$$

As Fig. 10.13 shows, the velocity of a point on the wheel is the vector sum of the velocity of the center of mass and the velocity of the point relative to the center of mass. Thus while point 1, the point of contact, is instantaneously at rest, point 3 at the top of the wheel is moving forward *twice as fast* as the center of mass, and points 2 and 4 at the sides have velocities at 45° to the horizontal.

At any instant we can think of the wheel as rotating about an "instantaneous axis" of rotation that passes through the point of contact with the ground. The angular velocity ω is the same for this axis as for an axis through the center of mass; an observer at the center of mass sees the rim make the same number of revolutions per second as does an observer at the rim watching the center of mass spin around him. If we think of the motion of the rolling wheel in Fig. 10.13 in this way, the kinetic energy of the wheel is $K = \frac{1}{2}I_1\omega^2$, where I_1 is the moment of



10.13 The motion of a rolling wheel is the sum of the translational motion of the center of mass plus the rotational motion of the wheel around the center of mass.

inertia of the wheel about an axis through point 1. But by the parallel-axis theorem, Eq. (9.19), $I_1 = I_{cm} + MR^2$, where M is the total mass of the wheel and I_{cm} is the moment of inertia with respect to an axis through the center of mass. Using Eq. (10.11), the kinetic energy of the wheel is

$$K = \frac{1}{2}I_1\omega^2 = \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2}MR^2\omega^2 = \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2}Mv_{cm}^2$$

which is the same as Eq. (10.8).

CAUTION **Rolling without slipping** Note that the relationship $v_{cm} = R\omega$ holds *only* if there is rolling without slipping. When a drag racer first starts to move, the rear tires are spinning very fast even though the racer is hardly moving, so $R\omega$ is greater than v_{cm} (Fig. 10.14). If a driver applies the brakes too heavily so that the car skids, the tires will spin hardly at all and $R\omega$ is less than v_{cm}.

10.14 The smoke rising from this drag racer's rear tires shows that the tires are slipping on the road, so v_{cm} is *not* equal to $R\omega$.

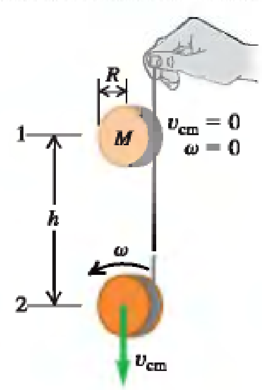
If a rigid body changes height as it moves, we must also consider gravitational potential energy. As we discussed in Section 9.4, the gravitational potential energy associated with any extended body of mass M, rigid or not, is the same as if we replace the body by a particle of mass M located at the body's center of mass. That is,

$$U = Mgy_{cm}$$

Example 10.4 Speed of a primitive yo-yo

A primitive yo-yo is made by wrapping a string several times around a solid cylinder with mass M and radius R (Fig. 10.15). You hold the end of the string stationary while releasing the cylinder with no initial motion. The string unwinds but does not slip or stretch as the cylinder drops and rotates. Use energy considerations to find the speed v_{cm} of the center of mass of the solid cylinder after it has dropped a distance h.

10.15 Calculating the speed of a primitive yo-yo.



SOLUTION

IDENTIFY: The upper end of the string is held fixed, not pulled upward, so the hand in Fig. 10.15 does no work on the system of string and cylinder. As in Example 9.8 (Section 9.4), there is friction between the string and the cylinder, but because the string never slips on the surface of the cylinder, no mechanical energy is lost. Thus we can use conservation of mechanical energy.

SET UP: The potential energies are $U_1 = Mgh$ and $U_2 = 0$. The string has no kinetic energy because it's massless. The initial kinetic energy of the cylinder is $K_1 = 0$, and its final kinetic energy K_2 is given by Eq. (10.8). The moment of inertia is $I = \frac{1}{2}MR^2$, and $\omega = v_{cm}/R$ because the cylinder does not slip on the string.

EXECUTE: From Eq. (10.8), the kinetic energy at point 2 is

$$K_2 = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v_{cm}}{R}\right)^2$$
$$= \frac{3}{4}Mv_{cm}^2$$

The kinetic energy is $1\frac{1}{2}$ times as great as it would be if the yo-yo were falling at speed v_{cm} without rotating. Two-thirds of the total kinetic energy ($\frac{1}{2}Mv_{cm}^2$) is translational and one-third ($\frac{1}{4}Mv_{cm}^2$) is rotational. Then, conservation of energy gives

$$K_1 + U_1 = K_2 + U_2$$
$$0 + Mgh = \frac{3}{4}Mv_{cm}^2 + 0$$

and

$$v_{cm} = \sqrt{\frac{4}{3}gh}$$

EVALUATE: This is less than the speed $\sqrt{2gh}$ that a dropped object would have, because one-third of the potential energy released as the cylinder falls appears as rotational kinetic energy.

Example 10.5 Race of the rolling bodies

In a physics lecture demonstration, an instructor "races" various round rigid bodies by releasing them from rest at the top of an inclined plane (Fig. 10.16). What shape should a body have to reach the bottom of the incline first?

SOLUTION

IDENTIFY: We can again use conservation of energy because there is no sliding of the rigid bodies over the inclined plane. Kinetic friction does no work if the bodies roll without slipping. We can also ignore the effects of *rolling friction,* introduced in Section 5.3, provided the bodies and the surface on which they roll are perfectly rigid. (Later in this section we'll explain why this is so.)

SET UP: Each body starts from rest at the top of an incline with height h, so $K_1 = 0$, $U_1 = Mgh$, and $U_2 = 0$. The kinetic energy at the bottom of the incline is given by Eq. (10.8). If the bodies roll without slipping, $\omega = v_{cm}/R$. We can express the moments of inertia of all the round bodies in Table 9.2 (about axes through their centers of mass) as $I_{cm} = cMR^2$, where c is a pure number less than or equal to 1 that depends on the shape of the body. Our goal is to find the value of c that gives the body the greatest speed at the bottom of the incline.

EXECUTE: From conservation of energy,

$$K_1 + U_1 = K_2 + U_2$$

$$0 + Mgh = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}cMR^2\left(\frac{v_{cm}}{R}\right)^2 + 0$$

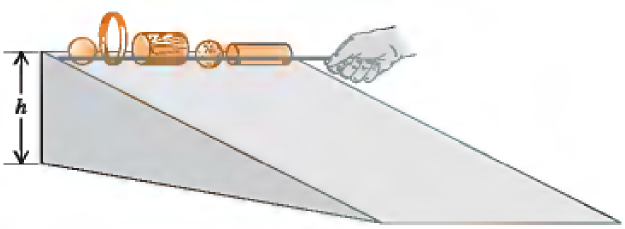
$$= \frac{1}{2}(1 + c)Mv_{cm}^2$$

Hence the speed at the bottom of the incline is

$$v_{cm} = \sqrt{\frac{2gh}{1 + c}}$$

EVALUATE: This is a fairly amazing result; the speed doesn't depend on either the mass M of the body or its radius R. All uniform solid cylinders have the same speed at the bottom, even if their masses and radii are different, because they have the same c. All solid spheres have the same speed, and so on. The smaller the value of c, the faster the body is moving at the bottom (and at any point on the way down). Small-c bodies always beat large-c bodies because they have less of their kinetic energy tied up in rotation and have more available for translation. Reading the values of c from Table 9.2, we see that the order of finish is as follows: any solid sphere, any solid cylinder, any thin-walled hollow sphere, and any thin-walled hollow cylinder.

10.16 Which body rolls down the incline fastest, and why?



Combined Translation and Rotation: Dynamics

We can also analyze the combined translational and rotational motions of a rigid body from the standpoint of dynamics. We showed in Section 8.5 that for a body with total mass M, the acceleration $\vec{a}_{cm}$ of the center of mass is the same as that of a point mass M acted on by all the external forces on the actual body:

$$\sum \vec{F}_{ext} = M\vec{a}_{cm} \qquad (10.12)$$

The rotational motion about the center of mass is described by the rotational analog of Newton's second law, Eq. (10.7):

$$\sum \tau_z = I_{cm}\alpha_z \qquad (10.13)$$

where I_{cm} is the moment of inertia with respect to an axis through the center of mass and the sum $\sum \tau_z$ includes all external torques with respect to this axis. It's not immediately obvious that Eq. (10.13) should apply to the motion of a translating rigid body; after all, our derivation of $\sum \tau_z = I\alpha_z$ in Section 10.2 assumed that the axis of rotation was stationary. But in fact, Eq. (10.13) is valid *even when the axis of rotation moves,* provided the following two conditions are met:

1. The axis through the center of mass must be an axis of symmetry.
2. The axis must not change direction.

10.17 The axle of a bicycle wheel passes through the wheel's center of mass and is an axis of symmetry. Hence the rotation of the wheel is described by Eq. (10.13), provided the bicycle doesn't turn or tilt to one side (which would change the orientation of the axle).

These conditions are satisfied for many types of rotation (Fig. 10.17). Note that in general this moving axis of rotation is *not* at rest in an inertial frame of reference.

We can now solve dynamics problems involving a rigid body that undergoes translational and rotational motions at the same time, provided that the rotation axis satisfies the two conditions just mentioned. Problem-Solving Strategy 10.1

(Section 10.2) is equally useful here, and you should review it now. Keep in mind that when a body undergoes translational and rotational motions at the same time, we need two separate equations of motion *for the same body.* One of these, Eq. (10.12), describes the translational motion of the center of mass. The other equation of motion, Eq. (10.13), describes the rotational motion about the axis through the center of mass.

Example 10.6 Acceleration of a primitive yo-yo

For the primitive yo-yo in Example 10.4 (Fig. 10.18a), find the downward acceleration of the cylinder and the tension in the string.

SOLUTION

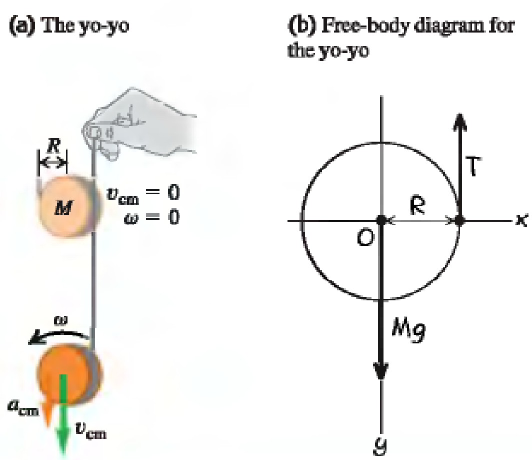
IDENTIFY: Figure 10.18b shows our free-body diagram for the yo-yo, including the choice of positive coordinate directions. With these coordinates, our target variables are $a_{\text{cm-}y}$ and T.

SET UP: We'll use Eqs. (10.12) and (10.13), along with the condition that the string does not slip on the cylinder.

EXECUTE: The equation for the translational motion of the center of mass is

$$\sum F_y = Mg + (-T) = Ma_{\text{cm-}y} \tag{10.14}$$

10.18 Dynamics of a primitive yo-yo (see Fig. 10.15).



The moment of inertia for an axis through the center of mass is $I_{\text{cm}} = \frac{1}{2}MR^2$. Only the tension force has a torque with respect to the axis through the center of mass, so the equation for rotational motion about this axis is

$$\sum \tau_z = TR = I_{\text{cm}}\alpha_z = \frac{1}{2}MR^2\alpha_z \tag{10.15}$$

The string unwinds without slipping, so $v_{\text{cm-}z} = R\omega_z$ from Eq. (10.11); the derivative of this relationship with respect to time is

$$a_{\text{cm-}y} = R\alpha_z \tag{10.16}$$

We now use Eq. (10.16) to eliminate α_z from Eq. (10.15) and then solve Eqs. (10.14) and (10.15) simultaneously for T and $a_{\text{cm-}y}$. The results are amazingly simple:

$$a_{\text{cm-}y} = \frac{2}{3}g \qquad T = \frac{1}{3}Mg$$

Using the constant-acceleration formula $v_{\text{cm-}y}{}^2 = v_{\text{cm-0}y}{}^2 + 2a_{\text{cm-}y}h$, you can show that the speed of the yo-yo after it has fallen a distance h is $v_{\text{cm}} = \sqrt{\frac{4}{3}gh}$, just as we found in Example 10.4.

EVALUATE: From the standpoint of dynamics, the tension force is essential; it causes the yo-yo's acceleration to be less than g, and its torque is what causes the yo-yo to turn. Yet when we analyzed this situation using energy methods in Example 10.4, we didn't have to consider the tension force at all! Because no mechanical energy was lost or gained, from the energy standpoint the string is merely a way to convert some of the gravitational potential energy into rotational kinetic energy.

Example 10.7 Acceleration of a rolling sphere

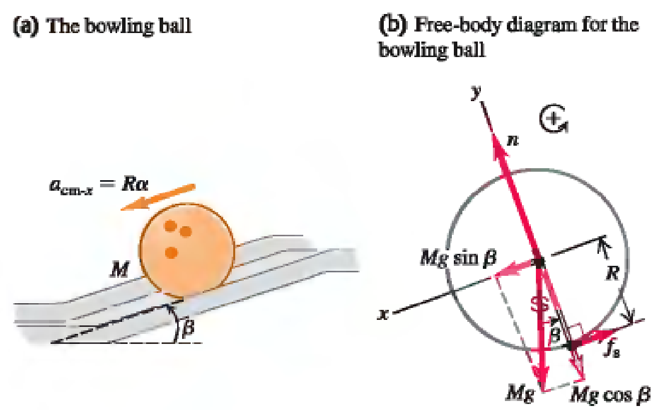
A solid bowling ball rolls without slipping down the return ramp at the side of the alley (Fig. 10.19a). The ramp is inclined at an angle β to the horizontal. What are the ball's acceleration and the magnitude of the friction force on the ball? Treat the ball as a uniform solid sphere, ignoring the finger holes.

SOLUTION

IDENTIFY: Our target variables are the acceleration of the ball's center of mass and the magnitude of the friction force. The free-body diagram in Fig. 10.19b shows that only the friction force exerts a torque about the center of mass.

SET UP: As in Example 10.6, we use Eq. (10.12) to describe the translational motion and Eq. (10.13) to describe the rotational motion.

10.19 A bowling ball rolling down a ramp.



Continued

EXECUTE: From Table 9.2 the moment of inertia of a solid sphere is $I_{cm} = \frac{2}{5}MR^2$. The equations of motion for translation and for rotation about the axis through the center of mass, respectively, are

$$\sum F_x = Mg\sin\beta + (-f) = Ma_{cm\text{-}x} \qquad (10.17)$$

$$\sum \tau_z = fR = I_{cm}\alpha_z = \left(\frac{2}{5}MR^2\right)\alpha_z \qquad (10.18)$$

If the ball rolls without slipping, we have the same kinematic relationship $a_{cm\text{-}x} = R\alpha_z$ as in Example 10.6. We use this to eliminate α_z from Eq. (10.18):

$$fR = \frac{2}{5}MRa_{cm\text{-}x}$$

This equation and Eq. (10.17) are two equations for two unknowns, $a_{cm\text{-}x}$ and f. We solve Eq. (10.17) for f, substitute the expression into the above equation to eliminate f, and then solve for $a_{cm\text{-}x}$ to obtain

$$a_{cm\text{-}x} = \frac{5}{7}g\sin\beta$$

The acceleration is just $\frac{5}{7}$ as large as it would be if the ball could *slide* without friction down the slope, like the toboggan in Example 5.10 (Section 5.2). Finally, we substitute this back into Eq. (10.17) and solve for f:

$$f = \frac{2}{7}Mg\sin\beta$$

EVALUATE: Because the ball does not slip at the instantaneous point of contact with the ramp, the friction force f is a *static* friction force; it prevents slipping and gives the ball its angular acceleration. We can derive an expression for the minimum coefficient of static friction μ_s needed to prevent slipping. The normal force is $n = Mg\cos\beta$. The maximum force of static friction equals $\mu_s n$, so the coefficient of friction must be at least as great as

$$\mu_s = \frac{f}{n} = \frac{\frac{2}{7}Mg\sin\beta}{Mg\cos\beta} = \frac{2}{7}\tan\beta$$

If the plane is tilted only a little, β is small, and only a small value of μ_s is needed to prevent slipping. But as the angle increases, the required value of μ_s increases, as we might expect intuitively. If the ball begins to slip, Eqs. (10.17) and (10.18) are both still valid, but it's no longer true that $v_{cm\text{-}x} = R\omega_z$ or $a_{cm\text{-}x} = R\alpha_z$; we have only two equations for three unknowns $(a_{cm\text{-}x}, \alpha_z, \text{and } f)$. To solve the problem of rolling *with* slipping requires taking *kinetic* friction into account (see Challenge Problem 10.101).

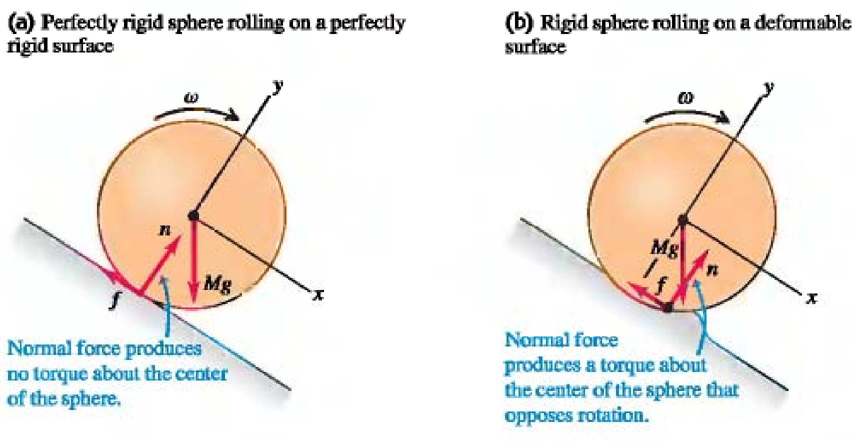
If the bowling ball descends a vertical distance h as it moves down the ramp, the displacement along the ramp is $h/\sin\beta$. You should be able to show that the speed of the ball at the bottom of the ramp would be $v_{cm} = \sqrt{\frac{10}{7}gh}$, which is just the result you found in Example 10.5 with $c = \frac{2}{5}$.

If the ball were rolling *uphill*, the force of friction would still be directed uphill as in Fig. 10.19b. Can you see why?

Rolling Friction

In Example 10.5 we said that we can ignore rolling friction if both the rolling body and the surface over which it rolls are perfectly rigid. In Fig. 10.20a a perfectly rigid sphere is rolling down a perfectly rigid incline. The line of action of the normal force passes through the center of the sphere, so its torque is zero; there is no sliding at the point of contact, so the friction force does no work. Figure 10.20b shows a more realistic situation, in which the surface "piles up" in front of the sphere and the sphere rides in a shallow trench. Because of these deformations, the contact forces on the sphere no longer act along a single point, but over an area; the forces are concentrated on the front of the sphere as shown. As a result, the normal force now exerts a torque that opposes the rotation. In addition, there is some sliding of the sphere over the surface due to the deformation, causing mechanical energy to be lost. The combination of these two effects is the phenomenon of *rolling friction*. Rolling friction also occurs if the rolling body is deformable, such as an automobile tire. Often the rolling body and the surface are rigid enough that rolling friction can be ignored, as we have assumed in all the examples in this section.

10.20 Rolling down **(a)** a perfectly rigid surface and **(b)** a deformable surface. The deformation in part (b) is greatly exaggerated.

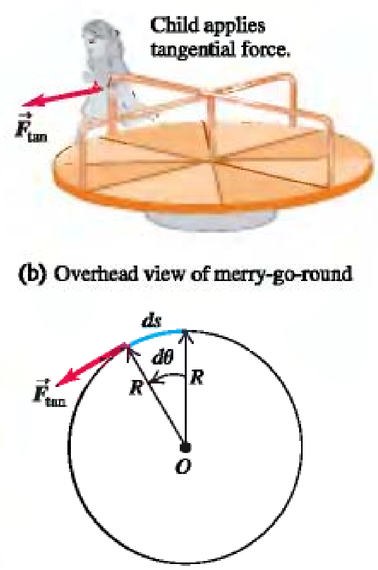


Test Your Understanding of Section 10.3 Suppose the solid cylinder used as a yo-yo in Example 10.6 is replaced by a hollow cylinder of the same mass and radius. (a) Will the acceleration of the yo-yo (i) increase, (ii) decrease, or (iii) remain the same? (b) Will the string tension (i) increase, (ii) decrease, or (iii) remain the same?

10.4 Work and Power in Rotational Motion

When you pedal a bicycle, you apply forces to a rotating body and do work on it. Similar things happen in many other real-life situations, such as a rotating motor shaft driving a power tool or a car engine propelling the vehicle. We can express this work in terms of torque and angular displacement.

10.21 A tangential force applied to a rotating body does work.



Suppose a tangential force $\vec{F}_{\tan}$ acts at the rim of a pivoted disk—for example, a child running while pushing on a playground merry-go-round (Fig. 10.21a). The disk rotates through an infinitesimal angle $d\theta$ about a fixed axis during an infinitesimal time interval dt (Fig. 10.21b). The work dW done by the force $\vec{F}_{\tan}$ while a point on the rim moves a distance ds is $dW = F_{\tan}\, ds$. If $d\theta$ is measured in radians, then $ds = R\, d\theta$ and

$$dW = F_{\tan} R\, d\theta$$

Now $F_{\tan}R$ is the *torque* τ_z due to the force $\vec{F}_{\tan}$, so

$$dW = \tau_z\, d\theta \qquad (10.19)$$

The total work W done by the torque during an angular displacement from θ_1 to θ_2 is

$$W = \int_{\theta_1}^{\theta_2} \tau_z\, d\theta \qquad \text{(work done by a torque)} \qquad (10.20)$$

If the torque remains *constant* while the angle changes by a finite amount $\Delta\theta = \theta_2 - \theta_1$, then

$$W = \tau_z(\theta_2 - \theta_1) = \tau_z \Delta\theta \qquad \text{(work done by a constant torque)} \quad (10.21)$$

The work done by a *constant* torque is the product of torque and the angular displacement. If torque is expressed in newton-meters $(\text{N}\cdot\text{m})$ and angular displacement in radians, the work is in joules. Equation (10.21) is the rotational analog of Eq. (6.1), $W = Fs$, and Eq. (10.20) is the analog of Eq. (6.7), $W = \int F_x\, dx$, for the work done by a force in a straight-line displacement.

If the force in Fig. 10.21 had an axial component (parallel to the rotation axis) or a radial component (directed toward or away from the axis), that component would do no work because the displacement of the point of application has only a tangential component. An axial or radial component of force would also make no contribution to the torque about the axis of rotation. So Eqs. (10.20) and (10.21) are correct for *any* force, no matter what its components.

When a torque does work on a rotating rigid body, the kinetic energy changes by an amount equal to the work done. We can prove this by using exactly the same procedure that we used in Eqs. (6.11) through (6.13) for the translational kinetic energy of a particle. Let τ_z represent the *net* torque on the body so that $\tau_z = I\alpha_z$ from Eq. (10.7), and assume that the body is rigid so that the moment of inertia I is constant. We then transform the integrand in Eq. (10.20) into an integrand with respect to ω_z as follows:

$$\tau_z\, d\theta = (I\alpha_z)\, d\theta = I\frac{d\omega_z}{dt}\, d\theta = I\frac{d\theta}{dt}\, d\omega_z = I\omega_z\, d\omega_z$$

10.22 The rotational kinetic energy of a wind turbine is equal to the total work done to set it spinning.

Since τ_z is the net torque, the integral in Eq. (10.20) is the *total* work done on the rotating rigid body. This equation then becomes

$$W_{tot} = \int_{\omega_1}^{\omega_2} I\omega_z \, d\omega_z = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2 \tag{10.22}$$

The change in the rotational kinetic energy of a *rigid* body equals the work done by forces exerted from outside the body (Fig. 10.22). This equation is analogous to Eq. (6.13), the work–energy theorem for a particle.

What about the *power* associated with work done by a torque acting on a rotating body? When we divide both sides of Eq. (10.19) by the time interval dt during which the angular displacement occurs, we find

$$\frac{dW}{dt} = \tau_z \frac{d\theta}{dt}$$

But dW/dt is the rate of doing work, or *power* P, and $d\theta/dt$ is angular velocity ω_z, so

$$P = \tau_z \omega_z \tag{10.23}$$

When a torque τ_z (with respect to the axis of rotation) acts on a body that rotates with angular velocity ω_z, its power (rate of doing work) is the product of τ_z and ω_z. This is the analog of the relationship $P = \vec{F} \cdot \vec{v}$ that we developed in Section 6.4 for particle motion.

Example 10.8 Engine power and torque

The power output of an automobile engine is advertised to be 200 hp at 6000 rpm. What is the corresponding torque?

SOLUTION

IDENTIFY: This example uses the relationship among power, angular velocity, and torque (the target variable).

SET UP: We are given the power output P and the angular velocity ω_z, so we can find the torque using Eq. (10.23).

EXECUTE: First we have to convert the power to watts and the angular velocity to rad/s:

$$P = 200\text{ hp} = 200\text{ hp}\left(\frac{746\text{ W}}{1\text{ hp}}\right) = 1.49 \times 10^5\text{ W}$$

$$\omega_z = 6000\text{ rev/min} = \left(\frac{6000\text{ rev}}{1\text{ min}}\right)\left(\frac{2\pi\text{ rad}}{1\text{ rev}}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right) = 628\text{ rad/s}$$

From Eq. (10.23),

$$\tau_z = \frac{P}{\omega_z} = \frac{1.49 \times 10^5\text{ N}\cdot\text{m/s}}{628\text{ rad/s}} = 237\text{ N}\cdot\text{m}$$

EVALUATE: You could apply this much torque by using a wrench 0.25 m long and applying a force of 948 N (213 lb) to the end of its handle. Could you do it?

Example 10.9 Calculating power from torque

An electric motor exerts a constant torque of 10 N · m on a grindstone mounted on its shaft. The moment of inertia of the grindstone about the shaft is 2.0 kg · m^2. If the system starts from rest, find the work done by the motor in 8.0 seconds and the kinetic energy at the end of this time. What was the average power delivered by the motor?

SOLUTION

IDENTIFY: Since the torque is constant, the grindstone has a constant angular acceleration α_z. If we can find the value of α_z, we can find the angle $\Delta\theta$ through which the grindstone turns in 8.0 s [which, through Eq. (10.21), tells us the work done W] and the angular velocity ω_z at that time (which tells us the kinetic energy K). We can find the average power P_{av} by dividing the work done by the time interval.

SET UP: We use the rotational version of Newton's second law, $\sum\tau_z = I\alpha_z$, to find the angular acceleration α_z. Given this we use the kinematics equations from Section 9.2 to calculate $\Delta\theta$ and ω_z and from these calculate W, K, and P_{av}.

EXECUTE: We have $\sum\tau_z = 10\text{ N}\cdot\text{m}$ (the only torque acting is that due to the motor) and $I = 2.0\text{ kg}\cdot\text{m}^2$, so from $\sum\tau_z = I\alpha_z$ the angular acceleration is 5.0 rad/s^2. From Eq. (9.11) the total angle through which the system turns in 8.0 s is

$$\Delta\theta = \frac{1}{2}\alpha_z t^2 = \frac{1}{2}(5.0\text{ rad/s}^2)(8.0\text{ s})^2 = 160\text{ rad}$$

and the total work done by the torque is

$$W = \tau_z \Delta\theta = (10\,\text{N}\cdot\text{m})(160\,\text{rad}) = 1600\,\text{J}$$

From Eqs. (9.7) and (9.17), the angular velocity and kinetic energy at $t = 8.0$ s are

$$\omega_z = \alpha_z t = (5.0\,\text{rad/s}^2)(8.0\,\text{s}) = 40\,\text{rad/s}$$

$$K = \frac{1}{2}I\omega_z^2 = \frac{1}{2}(2.0\,\text{kg}\cdot\text{m}^2)(40\,\text{rad/s})^2 = 1600\,\text{J}$$

The initial kinetic energy was zero, so the work done equals the increase in kinetic energy [see Eq. (10.22)].

The average power is

$$P_{av} = \frac{1600\,\text{J}}{8.0\,\text{s}} = 200\,\text{J/s} = 200\,\text{W}$$

EVALUATE: We can check our answer for *average* power by considering the *instantaneous* power $P = \tau_z \omega_z$. Because ω_z increases continuously, P increases continuously as well; its value is zero at $t = 0$ and increases to $(10\,\text{N}\cdot\text{m})(40\,\text{rad/s}) = 400$ W at $t = 8.0$ s. The angular velocity and the power increase uniformly with time, so the *average* power is just half this maximum value, or 200 W.

Test Your Understanding of Section 10.4 You apply equal torques to two different cylinders, one of which has a moment of inertia twice as large as the other cylinder. Each cylinder is initially at rest. After one complete rotation, which cylinder has the greater kinetic energy? (i) the cylinder with the larger moment of inertia; (ii) the cylinder with the smaller moment of inertia; (iii) both cylinders have the same kinetic energy.

10.5 Angular Momentum

Every rotational quantity that we have encountered in Chapters 9 and 10 is the analog of some quantity in the translational motion of a particle. The analog of *momentum* of a particle is **angular momentum,** a vector quantity denoted as $\vec{L}$. Its relationship to momentum $\vec{p}$ (which we will often call *linear momentum* for clarity) is exactly the same as the relationship of torque to force, $\vec{\tau} = \vec{r} \times \vec{F}$. For a particle with constant mass m, velocity $\vec{v}$, momentum $\vec{p} = m\vec{v}$, and position vector $\vec{r}$ relative to the origin O of an inertial frame, we define angular momentum $\vec{L}$ as

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \qquad \text{(angular momentum of a particle)} \qquad (10.24)$$

The value of $\vec{L}$ depends on the choice of origin O, since it involves the particle's position vector relative to O. The units of angular momentum are $\text{kg}\cdot\text{m}^2/\text{s}$.

In Fig. 10.23 a particle moves in the xy-plane; its position vector $\vec{r}$ and momentum $\vec{p} = m\vec{v}$ are shown. The angular momentum vector $\vec{L}$ is perpendicular to the xy-plane. The right-hand rule for vector products shows that its direction is along the $+z$-axis, and its magnitude is

$$L = mvr\sin\phi = mvl \qquad (10.25)$$

where l is the perpendicular distance from the line of $\vec{v}$ to O. This distance plays the role of "lever arm" for the momentum vector.

When a net force $\vec{F}$ acts on a particle, its velocity and momentum change, so its angular momentum may also change. We can show that the *rate of change* of angular momentum is equal to the torque of the net force. We take the time derivative of Eq. (10.24), using the rule for the derivative of a product:

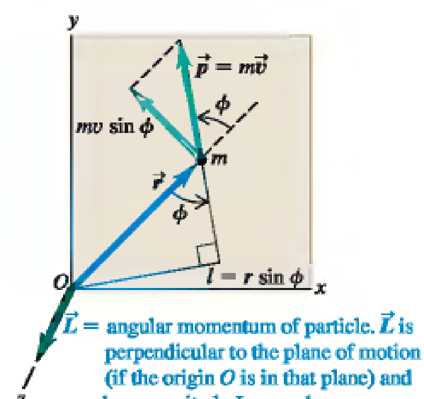
$$\frac{d\vec{L}}{dt} = \left(\frac{d\vec{r}}{dt} \times m\vec{v}\right) + \left(\vec{r} \times m\frac{d\vec{v}}{dt}\right) = (\vec{v} \times m\vec{v}) + (\vec{r} \times m\vec{a})$$

The first term is zero because it contains the vector product of the vector $\vec{v} = d\vec{r}/dt$ with itself. In the second term we replace $m\vec{a}$ with the net force $\vec{F}$, obtaining

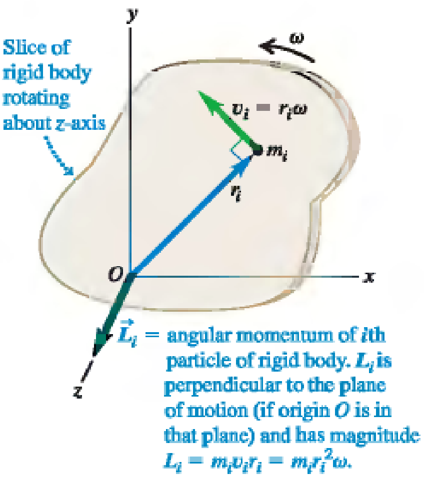
$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau} \qquad \text{(for a particle acted on by net force } \vec{F}\text{)} \qquad (10.26)$$

The rate of change of angular momentum of a particle equals the torque of the net force acting on it. Compare this result to Eq. (8.3), which states that the

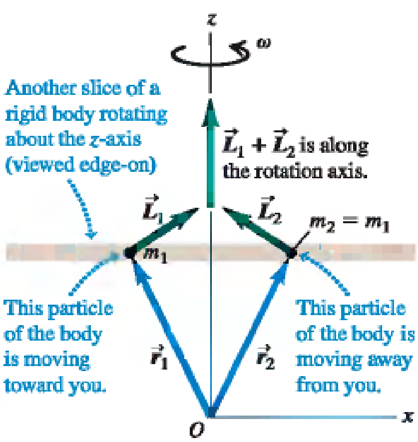
10.23 Calculating the angular momentum $\vec{L} = \vec{r} \times m\vec{v} = \vec{r} \times \vec{p}$ of a particle with mass m moving in the xy-plane.



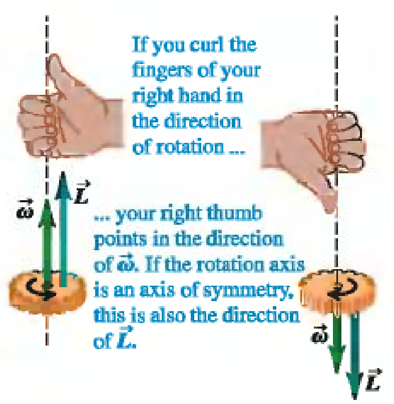
10.24 Calculating the angular momentum of a particle of mass m_i in a rigid body rotating at angular speed ω. (Compare Fig. 10.23.)



10.25 Two particles of the same mass located symmetrically on either side of the rotation axis of a rigid body. While the angular momentum vectors $\vec{L}_1$ and $\vec{L}_2$ of the two particles do not lie along the rotation axis, their vector sum $\vec{L}_1 + \vec{L}_2$ does.



10.26 For rotation about an axis of symmetry, $\vec{\omega}$ and $\vec{L}$ are parallel and along the axis. The directions of both vectors are given by the right-hand rule (compare Fig. 9.5).



rate of change $d\vec{p}/dt$ of the *linear* momentum of a particle equals the net force that acts on it.

Angular Momentum of a Rigid Body

We can use Eq. (10.25) to find the total angular momentum of a *rigid body* rotating about the z-axis with angular speed ω. First consider a thin slice of the body lying in the xy-plane (Fig. 10.24). Each particle in the slice moves in a circle centered at the origin, and at each instant its velocity $\vec{v}_i$ is perpendicular to its position vector $\vec{r}_i$, as shown. Hence in Eq. (10.25), $\phi = 90°$ for every particle. A particle with mass m_i at a distance r_i from O has a speed v_i equal to $r_i\omega$. From Eq. (10.25) the magnitude L_i of its angular momentum is

$$L_i = m_i(r_i\omega)r_i = m_i r_i^2 \omega \tag{10.27}$$

The direction of each particle's angular momentum, as given by the right-hand rule for the vector product, is along the $+z$-axis.

The *total* angular momentum of the slice of the body lying in the xy-plane is the sum $\sum L_i$ of the angular momenta L_i of the particles. Summing Eq. (10.27), we have

$$L = \sum L_i = \left(\sum m_i r_i^2\right)\omega = I\omega$$

where I is the moment of inertia of the slice about the z-axis.

We can do this same calculation for the other slices of the body, all parallel to the xy-plane. For points that do not lie in the xy-plane, a complication arises because the $\vec{r}$ vectors have components in the z-direction as well as the x- and y-directions; this gives the angular momentum of each particle a component perpendicular to the z-axis. But *if the z-axis is an axis of symmetry,* the perpendicular components for particles on opposite sides of this axis add up to zero (Fig. 10.25). So when a body rotates about an axis of symmetry, its angular momentum vector $\vec{L}$ lies along the symmetry axis, and its magnitude is $L = I\omega$.

The angular velocity vector $\vec{\omega}$ also lies along the rotation axis, as we discussed at the end of Section 9.1. Hence for a rigid body rotating around an axis of symmetry, $\vec{L}$ and $\vec{\omega}$ are in the same direction (Fig. 10.26). So we have the *vector* relationship

$$\vec{L} = I\vec{\omega} \qquad \text{(for a rigid body rotating around a symmetry axis)} \tag{10.28}$$

From Eq. (10.26) the rate of change of angular momentum of a particle equals the torque of the net force acting on the particle. For any system of particles (including both rigid and nonrigid bodies), the rate of change of the *total* angular momentum equals the sum of the torques of all forces acting on all the particles. The torques of the *internal* forces add to zero if these forces act along the line from one particle to another, as in Fig. 10.8, and so the sum of the torques includes only the torques of the *external* forces. (A similar cancellation occurred in our discussion of center-of-mass motion in Section 8.5.) If the total angular momentum of the system of particles is $\vec{L}$ and the sum of the external torques is $\sum \vec{\tau}$, then

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt} \qquad \text{(for any system of particles)} \tag{10.29}$$

Finally, if the system of particles is a rigid body rotating about a symmetry axis (the z-axis), then $L_z = I\omega_z$ and I is constant. If this axis has a fixed direction in space, then the vectors $\vec{L}$ and $\vec{\omega}$ change only in magnitude, not in direction. In that case, $dL_z/dt = I\,d\omega_z/dt = I\alpha_z$, or

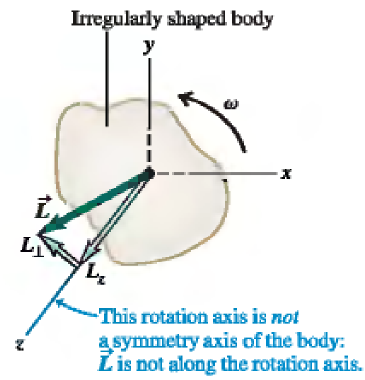
$$\sum \tau_z = I\alpha_z$$

which is again our basic relationship for the dynamics of rigid-body rotation. If the body is *not* rigid, I may change, and in that case, L changes even when ω is constant. For a nonrigid body, Eq. (10.29) is still valid, even though Eq. (10.7) is not.

When the axis of rotation is *not* a symmetry axis, the angular momentum is in general *not* parallel to the axis (Fig. 10.27). As the body turns, the angular momentum vector $\vec{L}$ traces out a cone around the rotation axis. Because $\vec{L}$ changes, there must be a net external torque acting on the body even though the angular velocity magnitude ω may be constant. If the body is an unbalanced wheel on a car, this torque is provided by friction in the bearings, which causes the bearings to wear out. "Balancing" a wheel means distributing the mass so that the rotation axis is an axis of symmetry; then $\vec{L}$ points along the rotation axis, and no net torque is required to keep the wheel turning.

In fixed-axis rotation we often use the term "angular momentum of the body" to refer to only the *component* of $\vec{L}$ along the rotation axis of the body (the z-axis in Fig. 10.27), with a positive or negative sign to indicate the sense of rotation just as with angular velocity.

10.27 If the rotation axis of a rigid body is not a symmetry axis, $\vec{L}$ does not in general lie along the rotation axis. Even if $\vec{\omega}$ is constant, the direction of $\vec{L}$ changes and a net torque is required to maintain rotation.



Example 10.10 Angular momentum and torque

A turbine fan in a jet engine has a moment of inertia of $2.5\ \text{kg}\cdot\text{m}^2$ about its axis of rotation. As the turbine is starting up, its angular velocity as a function of time is

$$\omega_z = (40\ \text{rad/s}^3)t^2$$

(a) Find the fan's angular momentum as a function of time, and find its value at time $t = 3.0$ s. (b) Find the net torque acting on the fan as a function of time, and find the torque at time $t = 3.0$ s.

SOLUTION

IDENTIFY: Just like an electric fan, the turbine fan rotates about an axis of symmetry (the z-axis). Hence the angular momentum vector has only a z-component L_z, which we can determine from the angular velocity ω_z. Since the direction of angular momentum is constant, the net torque likewise has only a component τ_z along the rotation axis; this is equal to the time derivative of L_z.

SET UP: We use Eq. (10.28) to find L_z from ω_z and Eq. (10.29) to find τ_z from the time derivative of L_z.

EXECUTE: (a) The component of angular momentum along the rotation (z) axis is

$$L_z = I\omega_z = (2.5\ \text{kg}\cdot\text{m}^2)(40\ \text{rad/s}^3)t^2 = (100\ \text{kg}\cdot\text{m}^2/\text{s}^3)t^2$$

(We dropped "rad" from the answer because a radian is a dimensionless quantity.) At time $t = 3.0$ s, $L_z = 900\ \text{kg}\cdot\text{m}^2/\text{s}$.

(b) From Eq. (10.29), the net torque component along the rotation axis is

$$\tau_z = \frac{dL_z}{dt} = (100\ \text{kg}\cdot\text{m}^2/\text{s}^3)(2t) = (200\ \text{kg}\cdot\text{m}^2/\text{s}^3)t$$

At time $t = 3.0$ s,

$$\tau_z = (200\ \text{kg}\cdot\text{m}^2/\text{s}^3)(3.0\ \text{s}) = 600\ \text{kg}\cdot\text{m}^2/\text{s}^2 = 600\ \text{N}\cdot\text{m}$$

EVALUATE: As a check on our result, note that the angular acceleration of the turbine fan is $\alpha_z = d\omega_z/dt = (40\ \text{rad/s}^3)(2t) = (80\ \text{rad/s}^2)t$. From the rotational equivalent of Newton's second law, the torque on the fan is $\tau_z = I\alpha_z = (2.5\ \text{kg}\cdot\text{m}^2)(80\ \text{rad/s}^2)t = (200\ \text{kg}\cdot\text{m}^2/\text{s}^3)t$, just as we calculated above.

Test Your Understanding of Section 10.5 A ball is attached to one end of a piece of string. You hold the other end of the string and whirl the ball in a circle around your hand. (a) If the ball moves at a constant speed, is its linear momentum $\vec{p}$ constant? Why or why not? (b) Is its angular momentum $\vec{L}$ constant? Why or why not?

10.6 Conservation of Angular Momentum

We have just seen that angular momentum can be used for an alternative statement of the basic dynamic principle for rotational motion. It also forms the basis for the **principle of conservation of angular momentum.** Like conservation of energy and of linear momentum, this principle is a universal conservation law, valid at all scales from atomic and nuclear systems to the motions of galaxies. This principle follows directly from Eq. (10.29): $\sum\vec{\tau} = d\vec{L}/dt$. If $\sum\vec{\tau} = \mathbf{0}$, then $d\vec{L}/dt = \mathbf{0}$, and $\vec{L}$ is constant.

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7.14 Ball Hits Bat

When the net external torque acting on a system is zero, the total angular momentum of the system is constant (conserved).

A circus acrobat, a diver, and an ice skater pirouetting on the toe of one skate all take advantage of this principle. Suppose an acrobat has just left a swing with arms and legs extended and rotating counterclockwise about her center of mass. When she pulls her arms and legs in, her moment of inertia I_{cm} with respect to her center of mass changes from a large value I_1 to a much smaller value I_2. The only external force acting on her is her weight, which has no torque with respect to an axis through her center of mass. So her angular momentum $L_z = I_{cm}\omega_z$ remains constant, and her angular velocity ω_z increases as I_{cm} decreases. That is,

$$I_1\omega_{1z} = I_2\omega_{2z} \qquad \text{(zero net external torque)} \qquad (10.30)$$

When a skater or ballerina spins with arms outstretched and then pulls her arms in, her angular velocity increases as her moment of inertia decreases. In each case there is conservation of angular momentum in a system in which the net external torque is zero.

When a system has several parts, the internal forces that the parts exert on each other cause changes in the angular momenta of the parts, but the *total* angular momentum doesn't change. Here's an example. Consider two bodies A and B that interact with each other but not with anything else, such as the astronauts we discussed in Section 8.2 (Fig. 8.8). Suppose body A exerts a force $\vec{F}_{A\text{ on }B}$ on body B; the corresponding torque (with respect to whatever point we choose) is $\vec{\tau}_{A\text{ on }B}$. According to Eq. (10.29), this torque is equal to the rate of change of angular momentum of B:

$$\vec{\tau}_{A\text{ on }B} = \frac{d\vec{L}_B}{dt}$$

At the same time, body B exerts a force $\vec{F}_{B\text{ on }A}$ on body A, with a corresponding torque $\vec{\tau}_{B\text{ on }A}$, and

$$\vec{\tau}_{B\text{ on }A} = \frac{d\vec{L}_A}{dt}$$

From Newton's third law, $\vec{F}_{B\text{ on }A} = -\vec{F}_{A\text{ on }B}$. Furthermore, if the forces act along the same line, as in Fig. 10.8, their lever arms with respect to the chosen axis are equal. Thus the *torques* of these two forces are equal and opposite, and $\vec{\tau}_{B\text{ on }A} = -\vec{\tau}_{A\text{ on }B}$. So if we add the two preceding equations, we find

$$\frac{d\vec{L}_A}{dt} + \frac{d\vec{L}_B}{dt} = 0$$

or, because $\vec{L}_A + \vec{L}_B$ is the *total* angular momentum $\vec{L}$ of the system,

$$\frac{d\vec{L}}{dt} = 0 \qquad \text{(zero net external torque)} \qquad (10.31)$$

That is, the total angular momentum of the system is constant. The torques of the internal forces can transfer angular momentum from one body to the other, but they can't change the *total* angular momentum of the system (Fig. 10.28).

10.28 A falling cat twists different parts of its body in different directions so that it lands feet first. At all times during this process the angular momentum of the cat as a whole remains zero.

Example 10.11 Anyone can be a ballerina

An acrobatic physics professor stands at the center of a turntable, holding his arms extended horizontally with a 5.0-kg dumbbell in each hand (Fig. 10.29). He is set rotating about a vertical axis, making one revolution in 2.0 s. Find the prof's new angular velocity if he pulls the dumbbells in to his stomach. His moment of inertia (without the dumbbells) is 3.0 kg · m^2 when his arms are outstretched, dropping to 2.2 kg · m^2 when his hands are at his stomach. The dumbbells are 1.0 m from the axis initially and 0.20 m from it at the end. Treat the dumbbells as particles.

SOLUTION

IDENTIFY: If we neglect friction in the turntable, no external torques act about the vertical (z) axis. Hence the angular momentum about this axis is constant.

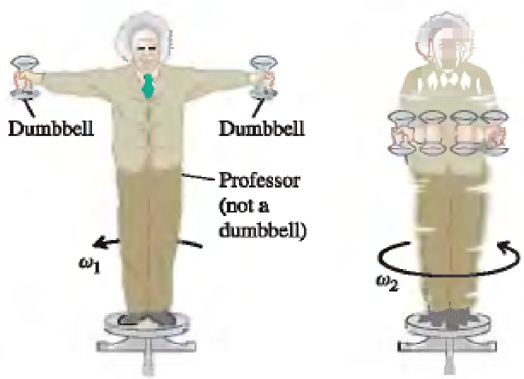
SET UP: We'll use Eq. (10.30) to find our target variable, the final angular velocity ω_{2z}.

EXECUTE: The moment of inertia of the system is $I = I_{\text{prof}} + I_{\text{dumbbells}}$. Each dumbbell of mass m contributes mr^2 to $I_{\text{dumbbells}}$, where r is the perpendicular distance from the rotation axis to the dumbbell. Initially we have

$$I_1 = 3.0\ \text{kg}\cdot\text{m}^2 + 2(5.0\ \text{kg})(1.0\ \text{m})^2 = 13\ \text{kg}\cdot\text{m}^2$$

$$\omega_{1z} = \frac{1\ \text{rev}}{2.0\ \text{s}} = 0.50\ \text{rev/s}$$

10.29 Fun with conservation of angular momentum.



The final moment of inertia is

$$I_2 = 2.2\ \text{kg}\cdot\text{m}^2 + 2(5.0\ \text{kg})(0.20\ \text{m})^2 = 2.6\ \text{kg}\cdot\text{m}^2$$

From Eq. (10.30), the final angular velocity is

$$\omega_{2z} = \frac{I_1}{I_2}\omega_{1z} = \frac{13\ \text{kg}\cdot\text{m}^2}{2.6\ \text{kg}\cdot\text{m}^2}(0.50\ \text{rev/s}) = 2.5\ \text{rev/s}$$

That is, the angular velocity increases by a factor of 5 while the angular momentum remains constant. Note that we didn't have to change "revolutions" to "radians" in this calculation. Why not?

EVALUATE: It's useful to examine how the kinetic energy changes in this process. To calculate the kinetic energy, we must express ω_1 and ω_2 in rad/s. (Why?) We have $\omega_{1z} = (0.50\ \text{rev/s})(2\pi\ \text{rad/rev}) = 3.14\ \text{rad/s}$ and $\omega_{2z} = (2.5\ \text{rev/s})(2\pi\ \text{rad/rev}) = 15.7\ \text{rad/s}$. The initial kinetic energy is

$$K_1 = \frac{1}{2}I_1\omega_{1z}^2 = \frac{1}{2}(13\ \text{kg}\cdot\text{m}^2)(3.14\ \text{rad/s})^2 = 64\ \text{J}$$

and the final kinetic energy is

$$K_2 = \frac{1}{2}I_2\omega_{2z}^2 = \frac{1}{2}(2.6\ \text{kg}\cdot\text{m}^2)(15.7\ \text{rad/s})^2 = 320\ \text{J}$$

The extra kinetic energy came from the work that the prof did to pull his arms and the dumbbells inward.

Example 10.12 A rotational "collision" I

Figure 10.30 shows two disks: one (A) an engine flywheel, and the other (B) a clutch plate attached to a transmission shaft. Their moments of inertia are I_A and I_B; initially, they are rotating with constant angular speeds ω_A and ω_B, respectively. We then push the disks together with forces acting along the axis, so as not to apply any torque on either disk. The disks rub against each other and eventually reach a common final angular speed ω. Derive an expression for ω.

SOLUTION

IDENTIFY: The only torque acting on either disk is the torque applied by the other disk; there are no external torques. Thus the total angular momentum of the system of two disks is the same before and after they are pushed together. At the end they rotate together as one body with total moment of inertia $I = I_A + I_B$ and angular speed ω, which is our target variable.

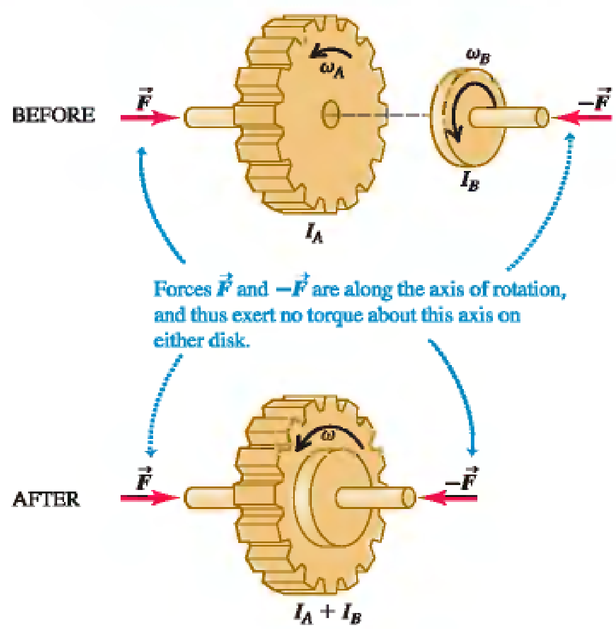
SET UP: Figure 10.30 shows that all of the angular velocities are in the same direction, so we can regard ω_A, ω_B, and ω as the components of angular velocity along the rotation axis.

EXECUTE: Conservation of angular momentum gives

$$I_A\omega_A + I_B\omega_B = (I_A + I_B)\omega$$

$$\omega = \frac{I_A\omega_A + I_B\omega_B}{I_A + I_B}$$

10.30 When the net external torque is zero, angular momentum is conserved.



Continued

EVALUATE: This "collision" between two disks is analogous to a completely inelastic collision (see Section 8.3). When two objects in translational motion along the same axis come together and stick, the linear momentum of the system is conserved. In the situation shown in Fig. 10.30, two objects in *rotational* motion along the same axis come together and stick, and the *angular* momentum is conserved. The kinetic energy of the system decreases in a completely inelastic collision; in the next example we'll see what becomes of the kinetic energy in the "collision" of two rotating disks.

Example 10.13 A rotational "collision" II

In Example 10.12, suppose flywheel A has a mass of 2.0 kg, a radius of 0.20 m, and an initial angular speed of 50 rad/s (about 500 rpm) and that clutch plate B has a mass of 4.0 kg, a radius of 0.10 m, and an initial angular speed of 200 rad/s. Find the common final angular speed ω after the disks are pushed into contact. What happens to the kinetic energy during this process?

SOLUTION

IDENTIFY: We need to calculate the rotational kinetic energy of each disk before the collision and their combined kinetic energy after the collision.

SET UP: We'll use the result of Example 10.12 and the expression $K = \frac{1}{2}I\omega^2$ for rotational kinetic energy.

EXECUTE: The moments of inertia of the two disks are

$$I_A = \frac{1}{2}m_A r_A^2 = \frac{1}{2}(2.0\ \text{kg})(0.20\ \text{m})^2 = 0.040\ \text{kg}\cdot\text{m}^2$$

$$I_B = \frac{1}{2}m_B r_B^2 = \frac{1}{2}(4.0\ \text{kg})(0.10\ \text{m})^2 = 0.020\ \text{kg}\cdot\text{m}^2$$

From Example 10.12 the final angular speed is

$$\begin{aligned}\omega &= \frac{I_A\omega_A + I_B\omega_B}{I_A + I_B}\\ &= \frac{(0.040\ \text{kg}\cdot\text{m}^2)(50\ \text{rad/s}) + (0.020\ \text{kg}\cdot\text{m}^2)(200\ \text{rad/s})}{0.040\ \text{kg}\cdot\text{m}^2 + 0.020\ \text{kg}\cdot\text{m}^2}\\ &= 100\ \text{rad/s}\end{aligned}$$

The kinetic energy before the collision is

$$\begin{aligned}K_1 &= \frac{1}{2}I_A\omega_A^2 + \frac{1}{2}I_B\omega_B^2\\ &= \frac{1}{2}(0.040\ \text{kg}\cdot\text{m}^2)(50\ \text{rad/s})^2\\ &\quad + \frac{1}{2}(0.020\ \text{kg}\cdot\text{m}^2)(200\ \text{rad/s})^2\\ &= 450\ \text{J}\end{aligned}$$

The kinetic energy after the collision is

$$\begin{aligned}K_2 &= \frac{1}{2}(I_A + I_B)\omega^2\\ &= \frac{1}{2}(0.040\ \text{kg}\cdot\text{m}^2 + 0.020\ \text{kg}\cdot\text{m}^2)(100\ \text{rad/s})^2 = 300\ \text{J}\end{aligned}$$

EVALUATE: One-third of the initial kinetic energy was lost during this "angular collision," the rotational analog of a completely inelastic collision. We shouldn't expect kinetic energy to be conserved, even though the net external force and torque are zero, because nonconservative (frictional) internal forces act while the two disks rub together and gradually approach a common angular velocity.

Example 10.14 Angular momentum in a crime bust

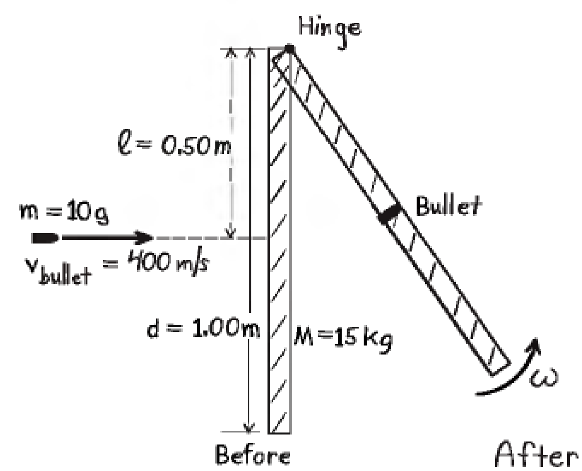
A door 1.00 m wide, of mass 15 kg, is hinged at one side so that it can rotate without friction about a vertical axis. It is unlatched. A police officer fires a bullet with a mass of 10 g and a speed of 400 m/s into the exact center of the door, in a direction perpendicular to the plane of the door. Find the angular speed of the door just after the bullet embeds itself in the door. Is kinetic energy conserved?

SOLUTION

IDENTIFY: We consider the door and bullet together as a system. There is no external torque about the axis defined by the hinges, so angular momentum about this axis is conserved.

SET UP: Figure 10.31 shows our sketch. The initial angular momentum is wholly in the bullet and is given by Eq. (10.25). The final angular momentum is that of a rigid body composed of the door and the embedded bullet. We'll set these two equal to each

10.31 Our sketch for this problem.



other and solve for the angular speed ω of the door and bullet just after the collision.

EXECUTE: The initial angular momentum of the bullet is:

$$L = mvl = (0.010\,\text{kg})(400\,\text{m/s})(0.50\,\text{m}) = 2.0\,\text{kg}\cdot\text{m}^2/\text{s}$$

The final angular momentum is $I\omega$, where $I = I_{\text{door}} + I_{\text{bullet}}$. From Table 9.2, for a door of width d,

$$I_{\text{door}} = \frac{Md^2}{3} = \frac{(15\,\text{kg})(1.0\,\text{m})^2}{3} = 5.0\,\text{kg}\cdot\text{m}^2$$

The moment of inertia of the bullet (with respect to the axis along the hinges) is

$$I_{\text{bullet}} = ml^2 = (0.010\,\text{kg})(0.50\,\text{m})^2 = 0.0025\,\text{kg}\cdot\text{m}^2$$

Conservation of angular momentum requires that $mvl = I\omega$, or

$$\omega = \frac{mvl}{I} = \frac{2.0\,\text{kg}\cdot\text{m}^2/\text{s}}{5.0\,\text{kg}\cdot\text{m}^2 + 0.0025\,\text{kg}\cdot\text{m}^2} = 0.40\,\text{rad/s}$$

The collision of bullet and door is inelastic because nonconservative friction forces act during the impact. Thus we do not expect kinetic energy to be conserved. To check, we calculate the initial and final kinetic energies:

$$K_1 = \frac{1}{2}mv^2 = \frac{1}{2}(0.010\,\text{kg})(400\,\text{m/s})^2 = 800\,\text{J}$$

$$K_2 = \frac{1}{2}I\omega^2 = \frac{1}{2}(5.0025\,\text{kg}\cdot\text{m}^2)(0.40\,\text{rad/s})^2 = 0.40\,\text{J}$$

The final kinetic energy is only 1/2000 of the initial value!

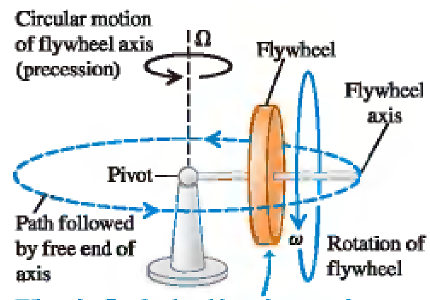
EVALUATE: The final angular speed of the door is quite slow: At 0.40 rad/s, the door takes 3.9 s to swing through 90° ($\pi/2$ radians). Can you see that the speed would double if the bullet were shot into the edge of the door near the doorknob?

Test Your Understanding of Section 10.6 If the polar ice caps were to completely melt due to global warming, the melted ice would redistribute itself over the earth. This change would cause the length of the day (the time needed for the earth to rotate once on its axis) to (i) increase; (ii) decrease; (iii) remain the same. (*Hint:* Use angular momentum ideas. Assume that the sun, moon, and planets exert negligibly small torques on the earth.)

10.7 Gyroscopes and Precession

In all the situations we've looked at so far in this chapter, the axis of rotation either has stayed fixed or has moved and kept the same direction (such as rolling without slipping). But a variety of new physical phenomena, some quite unexpected, can occur when the axis of rotation can change direction. For example, consider a toy gyroscope that's supported at one end (Fig. 10.32). If we hold it with the flywheel axis horizontal and let go, the free end of the axis simply drops owing to gravity—*if* the flywheel isn't spinning. But if the flywheel *is* spinning, what happens is quite different. One possible motion is a steady circular motion of the axis in a horizontal plane, combined with the spin motion of the flywheel about the axis. This surprising, nonintuitive motion of the axis is called **precession.** Precession is found in nature as well as in rotating machines such as gyroscopes. As you read these words, the earth itself is precessing; its spin axis (through the north and south poles) slowly changes direction, going through a complete cycle of precession every 26,000 years.

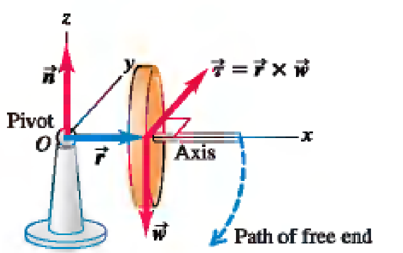
10.32 A gyroscope supported at one end. The horizontal circular motion of the flywheel and axis is called precession. The angular speed of precession is Ω.



To study this strange phenomenon of precession, we must remember that angular velocity, angular momentum, and torque are all *vector* quantities. In particular, we need the general relationship between the net torque $\sum\vec{\tau}$ that acts on a body and the rate of change of the body's angular momentum $\vec{L}$, given by Eq. (10.29), $\sum\vec{\tau} = d\vec{L}/dt$. Let's first apply this equation to the case in which the flywheel is *not* spinning (Fig. 10.33a). We take the origin O at the pivot and assume that the flywheel is symmetrical, with mass M and moment of inertia I about the flywheel axis. The flywheel axis is initially along the x-axis. The only external forces on the gyroscope are the normal force $\vec{n}$ acting at the pivot (assumed to be frictionless) and the weight $\vec{w}$ of the flywheel that acts at its center of mass, a distance r from the pivot. The normal force has zero torque with respect to the pivot, and the weight has a torque $\vec{\tau}$ in the y-direction, as shown in Fig. 10.33a. Initially, there is no rotation, and the initial angular momentum $\vec{L}_i$ is

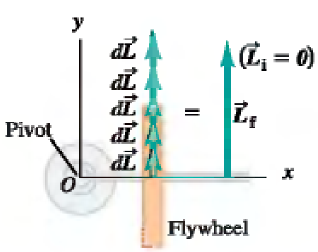
10.33 (a) If the flywheel in Fig. 10.32 is initially not spinning, its initial angular momentum is zero (b) In each successive time interval dt, the torque produces a change $d\vec{L} = \vec{\tau}\,dt$ in the angular momentum. The flywheel acquires an angular momentum $\vec{L}$ in the same direction as $\vec{\tau}$, and the flywheel axis falls.

(a) Nonrotating flywheel falls



When the flywheel is not rotating, its weight creates a torque around the pivot, causing it to fall along a circular path until its axis rests on the table surface.

(b) View from above as flywheel falls



In falling, the flywheel rotates about the pivot and thus acquires an angular momentum $\vec{L}$. The *direction* of $\vec{L}$ stays constant.

zero. From Eq. (10.29) the *change* $d\vec{L}$ in angular momentum in a short time interval dt following this is

$$d\vec{L} = \vec{\tau}\,dt \tag{10.32}$$

This change is in the y-direction because $\vec{\tau}$ is. As each additional time interval dt elapses, the angular momentum changes by additional increments $d\vec{L}$ in the y-direction because the direction of the torque is constant (Fig. 10.33b). The steadily increasing horizontal angular momentum means that the gyroscope rotates downward faster and faster around the y-axis until it hits either the stand or the table on which it sits.

Now let's see what happens if the flywheel *is* spinning initially, so the initial angular momentum $\vec{L}_i$ is not zero (Fig. 10.34a). Since the flywheel rotates around its symmetry axis, $\vec{L}_i$ lies along the axis. But each change in angular momentum $d\vec{L}$ is perpendicular to the axis because the torque $\vec{\tau} = \vec{r} \times \vec{w}$ is perpendicular to the axis (Fig. 10.34b). This causes the *direction* of $\vec{L}$ to change, but not its magnitude. The changes $d\vec{L}$ are always in the horizontal xy-plane, so the angular momentum vector and the flywheel axis with which it moves are always horizontal. In other words, the axis doesn't fall—it just precesses.

If this still seems mystifying to you, think about a ball attached to a string. If the ball is initially at rest and you pull the string toward you, the ball moves toward you also. But if the ball is initially moving and you continuously pull the string in a direction perpendicular to the ball's motion, the ball moves in a circle around your hand; it does not approach your hand at all. In the first case the ball has zero linear momentum $\vec{p}$ to start with; when you apply a force $\vec{F}$ toward you for a time dt, the ball acquires a momentum $d\vec{p} = \vec{F}\,dt$, which is also toward you. But if the ball already has linear momentum $\vec{p}$, a change in momentum $d\vec{p}$ that's perpendicular to $\vec{p}$ changes the direction of motion, not the speed. Replace $\vec{p}$ with $\vec{L}$ and $\vec{F}$ with $\vec{\tau}$ in this argument, and you'll see that precession is simply the rotational analog of uniform circular motion.

At the instant shown in Fig. 10.34a, the gyroscope has angular momentum $\vec{L}$. A short time interval dt later, the angular momentum is $\vec{L} + d\vec{L}$; the infinitesimal change in angular momentum is $d\vec{L} = \vec{\tau}\,dt$, which is perpendicular to $\vec{L}$. As the vector diagram in Fig. 10.35 shows, this means that the flywheel axis of the gyroscope has turned through a small angle $d\phi$ given by $d\phi = |d\vec{L}|/|\vec{L}|$. The rate at which the axis moves, $d\phi/dt$, is called the **precession angular speed;** denoting this quantity by Ω, we find

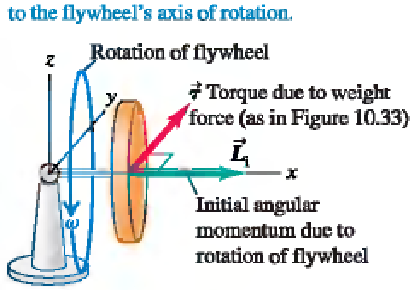
$$\Omega = \frac{d\phi}{dt} = \frac{|d\vec{L}|/|\vec{L}|}{dt} = \frac{\tau_z}{L_z} = \frac{wr}{I\omega} \tag{10.33}$$

Thus the precession angular speed is *inversely* proportional to the angular speed of spin about the axis. A rapidly spinning gyroscope precesses slowly; if friction

10.34 (a) The flywheel is spinning initially with angular momentum $\vec{L}_i$. The forces (not shown) are the same as those in Fig. 10.33a. (b) Because the initial angular momentum is not zero, each change $d\vec{L} = \vec{\tau}\,dt$ in angular momentum is perpendicular to $\vec{L}$. As a result, the magnitude of $\vec{L}$ remains the same but its direction changes continuously.

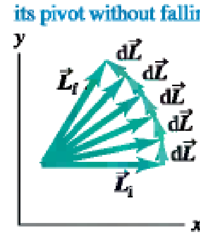
(a) Rotating flywheel

When the flywheel is rotating, the system starts with an angular momentum $\vec{L}_i$ parallel to the flywheel's axis of rotation.



(b) View from above

Now the effect of the torque is to cause the angular momentum to precess around the pivot. The gyroscope circles around its pivot without falling.

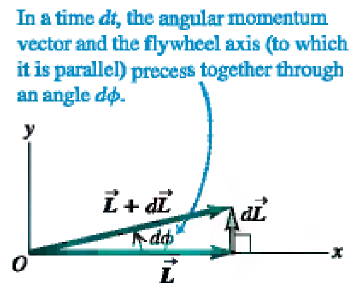


in its bearings causes the flywheel to slow down, the precession angular speed *increases!* The precession angular speed of the earth is very slow $(1 \text{ rev}/26{,}000 \text{ yr})$ because its spin angular momentum L_z is large and the torque τ_z, due to the gravitational influences of the moon and sun, is relatively small.

As a gyroscope precesses, its center of mass moves in a circle with radius r in a horizontal plane. Its vertical component of acceleration is zero, so the upward normal force $\vec{n}$ exerted by the pivot must be just equal in magnitude to the weight. The circular motion of the center of mass with angular speed Ω requires a force $\vec{F}$ directed toward the center of the circle, with magnitude $F = M\Omega^2 r$. This force must also be supplied by the pivot.

One key assumption that we made in our analysis of the gyroscope was that the angular momentum vector $\vec{L}$ is associated only with the spin of the flywheel and is purely horizontal. But there will also be a vertical component of angular momentum associated with the precessional motion of the gyroscope. By ignoring this, we've tacitly assumed that the precession is *slow*—that is, that the precession angular speed Ω is very much less than the spin angular speed ω. As Eq. (10.33) shows, a large value of ω automatically gives a small value of Ω, so this approximation is reasonable. When the precession is not slow, additional effects show up, including an up-and-down wobble or *nutation* of the flywheel axis that's superimposed on the precessional motion. You can see nutation occurring in a gyroscope as its spin slows down, so that Ω increases and the vertical component of $\vec{L}$ can no longer be ignored.

10.35 Detailed view of part of Fig. 10.34b.



Example 10.15 A precessing gyroscope

Figure 10.36a shows a top view of a cylindrical gyroscope wheel that has been set spinning by an electric motor. The pivot is at O, and the mass of the axle is negligible. (a) As seen from above, is the precession clockwise or counterclockwise? (b) If the gyro takes 4.0 s for one revolution of precession, at what angular speed does the wheel spin?

SOLUTION

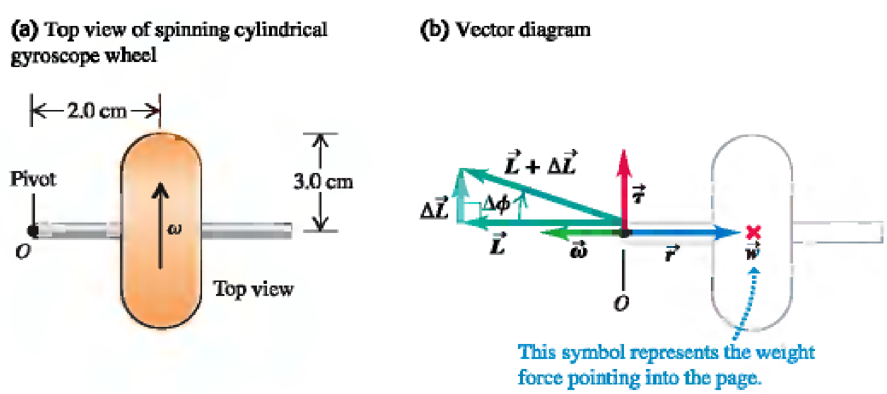
IDENTIFY: This situation is similar to the precessing flywheel shown in Fig. 10.34.

SET UP: We'll determine the direction of precession using the right-hand rule as in Fig. 10.34, which shows the same kind of gyroscope as Fig. 10.36. We'll use the relationship between precession angular speed Ω and spin angular speed ω, Eq. (10.33), to find the value of ω.

EXECUTE: (a) The right-hand rule shows that $\vec{\omega}$ and $\vec{L}$ are to the left (Fig. 10.36b). The weight $\vec{w}$ points into the page in this top view and acts at the center of mass (denoted by an $\times$); the torque $\vec{\tau} = \vec{r} \times \vec{w}$ is toward the top of the page; and $d\vec{L}/dt$ is also toward the top of the page. Adding a small $d\vec{L}$ to the $\vec{L}$ that we have initially changes the direction of $\vec{L}$ as shown, so the precession is clockwise as seen from above.

(b) Be careful not to confuse ω and Ω! We are given $\Omega = (1 \text{ rev})/(4.0 \text{ s}) = (2\pi \text{ rad})/(4.0 \text{ s}) = 1.57 \text{ rad/s}$. The weight is equal to mg, and the moment of inertia about its symmetry axis of a solid

10.36 In which direction and at what speed does this gyroscope precess?



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cylinder with radius R is $I = \frac{1}{2}mR^2$. Solving Eq. (10.33) for ω, we find

$$\omega = \frac{wr}{I\Omega} = \frac{mgr}{(mR^2/2)\Omega} = \frac{2gr}{R^2\Omega}$$

$$= \frac{2(9.8\ \mathrm{m/s^2})(2.0 \times 10^{-2}\ \mathrm{m})}{(3.0 \times 10^{-2}\ \mathrm{m})^2(1.57\ \mathrm{rad/s})} = 280\ \mathrm{rad/s} = 2600\ \mathrm{rev/min}$$

EVALUATE: The precession angular speed Ω is very much less than the spin angular speed ω, so this is an example of slow precession.

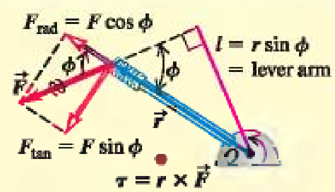
Test Your Understanding of Section 10.7 Suppose the mass of the flywheel in Fig. 10.34 were doubled but all other dimensions and the spin angular speed remained the same. What effect would this change have on the precession angular speed Ω? (i) Ω would increase by a factor of 4; (ii) Ω would double; (iii) Ω would be unaffected; (iv) Ω would be one-half as much; (v) Ω would be one-quarter as much.

CHAPTER 10 SUMMARY

Torque: When a force $\vec{F}$ acts on a body, the torque of that force with respect to a point O has a magnitude given by the product of the force magnitude F and the lever arm l. More generally, torque is a vector $\vec{\tau}$ equal to the vector product of $\vec{r}$ (the position vector of the point at which the force acts) and $\vec{F}$. (See Example 10.1.)

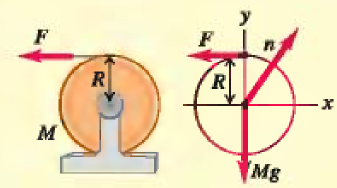
$$\tau = Fl \tag{10.2}$$

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{10.3}$$



Rotational dynamics: The rotational analog of Newton's second law says that the net torque acting on a body equals the product of the body's moment of inertia and its angular acceleration. (See Examples 10.2 and 10.3.)

$$\sum \tau_z = I\alpha_z \tag{10.7}$$



Combined translation and rotation: If a rigid body is both moving through space and rotating, its motion can be regarded as translational motion of the center of mass plus rotational motion about an axis through the center of mass. Thus the kinetic energy is a sum of translational and rotational kinetic energies. For dynamics, Newton's second law describes the motion of the center of mass, and the rotational equivalent of Newton's second law describes rotation about the center of mass. In the case of rolling without slipping, there is a special relationship between the motion of the center of mass and the rotational motion. (See Examples 10.4–10.7.)

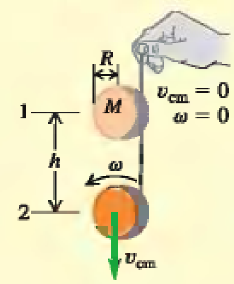
$$K = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 \tag{10.8}$$

$$\sum \vec{F}_{ext} = M\vec{a}_{cm} \tag{10.12}$$

$$\sum \tau_z = I_{cm}\alpha_z \tag{10.13}$$

$$v_{cm} = R\omega \tag{10.11}$$

(rolling without slipping)



Work done by a torque: A torque that acts on a rigid body as it rotates does work on that body. The work can be expressed as an integral of the torque. The work–energy theorem says that the total rotational work done on a rigid body is equal to the change in rotational kinetic energy. The power, or rate at which the torque does work, is the product of the torque and the angular velocity (See Examples 10.8 and 10.9.)

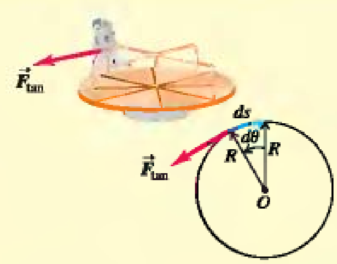
$$W = \int_{\theta_1}^{\theta_2} \tau_z \, d\theta \tag{10.20}$$

$$W = \tau_z(\theta_2 - \theta_1) = \tau_z\Delta\theta \tag{10.21}$$

(constant torque only)

$$W_{tot} = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2 \tag{10.22}$$

$$P = \tau_z\omega_z \tag{10.23}$$



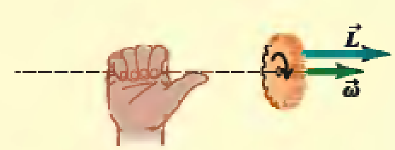
Angular momentum: The angular momentum of a particle with respect to point O is the vector product of the particle's position vector $\vec{r}$ relative to O and its momentum $\vec{p} = m\vec{v}$. When a symmetrical body rotates about a stationary axis of symmetry, its angular momentum is the product of its moment of inertia and its angular velocity vector $\vec{\omega}$. If the body is not symmetrical or the rotation (z) axis is not an axis of symmetry, the component of angular momentum along the rotation axis is $I\omega_z$. (See Example 10.10.)

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \tag{10.24}$$

(particle)

$$\vec{L} = I\vec{\omega} \tag{10.28}$$

(rigid body rotating about axis of symmetry)



Rotational dynamics and angular momentum: The net external torque on a system is equal to the rate of change of its angular momentum. If the net external torque on a system is zero, the total angular momentum of the system is constant (conserved). (See Examples 10.11–10.15.)

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt} \quad (10.29)$$

Key Terms

translational motion, *316*
line of action, *317*
lever arm (moment arm), *317*
torque, *317*
combined translation and rotation, *323*
rolling without slipping, *324*
angular momentum, *331*
principle of conservation of angular momentum, *333*
precession, *337*
precession angular speed, *338*

Answer to Chapter Opening Question ?

When the gymnast is in midair, no net torque acts about his center of mass. Hence the angular momentum of his body (the product of the moment of inertia I and the angular speed ω) around the center of mass remains constant. By moving his limbs outward, he increases I and hence ω decreases; if he pulls his limbs in, I decreases and ω increases.

Answers to Test Your Understanding Questions

10.1 Answer: (ii) The force P acts along a vertical line, so the lever arm is the horizontal distance from A to the line of action. This is the horizontal component of the distance L, which is $L\cos\theta$. Hence the magnitude of the torque is the product of the force magnitude P and the lever arm $L\cos\theta$, or $\tau = PL\cos\theta$.

10.2 Answer: (iii), (ii), (i) In order for the hanging object of mass m_2 to accelerate downward, the net force on it must be downward. Hence the magnitude m_2g of the downward weight force must be greater than the magnitude T_2 of the upward tension force. In order for the pulley to have a clockwise angular acceleration, the net torque on the pulley must be clockwise. The tension T_2 tends to rotate the pulley clockwise, while the tension T_1 tends to rotate the pulley counterclockwise. Both tension forces have the same lever arm R, so there is a clockwise torque T_2R and a counterclockwise torque T_1R. In order for the net torque to be clockwise, T_2 must be greater than T_1. Hence $m_2g > T_2 > T_1$.

10.3 Answers: (a) (ii), (b) (i) If you redo the calculation of Example 10.6 with a hollow cylinder (moment of inertia $I_{cm} = MR^2$ instead of a solid cylinder (moment of inertia $I_{cm} = \frac{1}{2}MR^2$), you will find $a_{cm\text{-}y} = \frac{1}{2}g$ and $T = \frac{1}{2}Mg$ (instead of $a_{cm\text{-}y} = \frac{2}{3}g$ and $T = \frac{1}{3}Mg$ for a solid cylinder). Hence the acceleration is less but the tension is greater. You can come to the same conclusion without doing the calculation. The greater moment of inertia means that the hollow cylinder will rotate more slowly and hence will roll downward more slowly. In order to slow the downward motion, a greater upward tension force is needed to oppose the downward force of gravity.

10.4 Answer: (iii) You apply the same torque over the same angular displacement to both cylinders. Hence, by Eq. (10.21), you do the same amount of work to both cylinders and impart the same kinetic energy to both. (The one with the smaller moment of inertia ends up with a greater angular speed, but that isn't what we are asked. Compare Conceptual Example 6.5 in Section 6.2.)

10.5 Answers: (a) no, (b) yes As the ball goes around the circle, the magnitude of $\vec{p} = m\vec{v}$ remains the same (the speed is constant) but its direction changes, so the linear momentum vector isn't constant. But $\vec{L} = \vec{r} \times \vec{p}$ *is* constant: The ball maintains a constant magnitude (the speed and the perpendicular distance from your hand to the ball are both constant) and a constant direction (along the rotation axis, perpendicular to the plane of the ball's motion). The linear momentum changes because there is a net *force* $\vec{F}$ on the ball (toward the center of the circle). The angular momentum remains constant because there is no net *torque;* the vector $\vec{r}$ points from your hand to the ball and the force $\vec{F}$ on the ball is directed toward your hand, so the vector product $\vec{\tau} = \vec{r} \times \vec{F}$ is zero.

10.6 Answer: (i) In the absence of any external torques, the earth's angular momentum $L_z = I\omega_z$ would remain constant. The melted ice would move from the poles toward the equator—that is, away from our planet's rotation axis—and the earth's moment of inertia I would increase slightly. Hence the angular velocity ω_z would decrease slightly and the day would be slightly longer.

10.7 Answer: (iii) Doubling the flywheel mass would double both its moment of inertia I and its weight w, so the ratio I/w would be unchanged. Equation (10.33) shows that the precession angular speed depends on this ratio, so there would be *no* effect on the value of Ω.

PROBLEMS

Discussion Questions

Q10.1. When cylinder-head bolts in an automobile engine are tightened, the critical quantity is the *torque* applied to the bolts. Why is the torque more important than the actual *force* applied to the wrench handle?

Q10.2. Can a single force applied to a body change both its translational and rotational motion? Explain.

Q10.3. Suppose you could use wheels of any type in the design of a soapbox-derby racer (an unpowered, four-wheel vehicle that coasts from rest down a hill). To conform to the rules on the total weight of the vehicle and rider, should you design with large massive wheels or small light wheels? Should you use solid wheels or wheels with most of the mass at the rim? Explain.

Q10.4. A four-wheel-drive car is accelerating forward from rest. Show the direction the car's wheels turn and how this causes a friction force due to the pavement that accelerates the car forward.

Q10.5. Serious bicyclists say that if you reduce the weight of a bike, it is more effective if you do so in the wheels rather than in the frame. Why would reducing weight in the wheels make it easier on the bicyclist than reducing the same amount in the frame?

Q10.6. The harder you hit the brakes while driving forward, the more the front end of your car will move down (and the rear end move up). Why? What happens when cars accelerate forward? Why do drag racers not use front-wheel drive only?

Q10.7. When an acrobat walks on a tightrope, she extends her arms straight out from her sides. She does this to make it easier for her to catch herself if she should tip to one side or the other. Explain how this works. [*Hint:* Think about Eq. (10.7).]

Q10.8. When you turn on an electric motor, it takes longer to come up to final speed if a grinding wheel is attached to the shaft. Why?

Q10.9. Experienced cooks can tell whether an egg is raw or hard-boiled by rolling it down a slope (taking care to catch it at the bottom). How is this possible? What are they looking for?

Q10.10. The work done by a force is the product of force and distance. The torque due to a force is the product of force and distance. Does this mean that torque and work are equivalent? Explain.

Q10.11. A valued client brings a treasured ball to your engineering firm, wanting to know whether the ball is solid or hollow. He has tried tapping on it, but that has given insufficient information. Design a simple, inexpensive experiment that you could perform quickly, without injuring the precious ball, to find out whether it is solid or hollow.

Q10.12. You make two versions of the same object out of the same material having uniform density. For one version, all the dimensions are exactly twice as great as for the other one. If the same torque acts on both versions, giving the smaller version angular acceleration α, what will be the angular acceleration of the larger version in terms of α?

Q10.13. Two identical masses are attached to frictionless pulleys by very light strings wrapped around the rim of the pulley and are released from rest. Both pulleys have the same mass and same diameter, but one is solid and the other is a hoop. As the masses fall, in which case is the tension in the string greater, or is it the same in both cases? Justify your answer.

Q10.14. The force of gravity acts on the baton in Fig. 10.11, and forces produce torques that cause a body's angular velocity to change. Why, then, is the angular velocity of the baton in the figure constant?

Q10.15. A certain solid uniform ball reaches a maximum height h_0 when it rolls up a hill without slipping. What maximum height (in terms of h_0) will it reach if you (a) double its diameter, (b) double its mass, (c) double both its diameter and mass, (d) double its angular speed at the bottom of the hill?

Q10.16. A wheel is rolling without slipping on a horizontal surface. In an inertial frame of reference in which the surface is at rest, is there any point on the wheel that has a velocity that is purely vertical? Is there any point that has a horizontal velocity component opposite to the velocity of the center of mass? Explain. Do your answers change if the wheel is slipping as it rolls? Why or why not?

Q10.17. Part of the kinetic energy of a moving automobile is in the rotational motion of its wheels. When the brakes are applied hard on an icy street, the wheels "lock" and the car starts to slide. What becomes of the rotational kinetic energy?

Q10.18. A hoop, a uniform solid cylinder, a spherical shell, and a uniform solid sphere are released from rest at the top of an incline. What is the order in which they arrive at the bottom of the incline? Does it matter whether or not the masses and radii of the objects are all the same? Explain.

Q10.19. A ball is rolling along at speed v without slipping on a horizontal surface when it comes to a hill that rises at a constant angle above the horizontal. In which case will it go higher up the hill: if the hill has enough friction to prevent slipping or if the hill is perfectly smooth? Justify your answers in both cases in terms of energy conservation and in terms of Newton's second law.

Q10.20. You are standing at the center of a large horizontal turntable in a carnival funhouse. The turntable is set rotating on frictionless bearings, and it rotates freely (that is, there is no motor driving the turntable). As you walk toward the edge of the turntable, what happens to the combined angular momentum of you and the turntable? What happens to the rotation speed of the turntable? Explain your answer.

Q10.21. Global Warming. As the earth's climate continues to warm, ice near the poles will melt and be added to the oceans. What effect will this have on the length of the day? (*Hint:* Consult a map to see where the oceans lie.)

Q10.22. A point particle travels in a straight line at constant speed, and the closest distance it comes to the origin of coordinates is a distance l. With respect to this origin, does the particle have nonzero angular momentum? As the particle moves along its straight-line path, does its angular momentum with respect to the origin change?

Q10.23. In Example 10.11 (Section 10.6) the angular speed ω changes, and this must mean that there is nonzero angular acceleration. But there is no torque about the rotation axis if the forces the professor applies to the weights are directly, radially inward. Then, by Eq. (10.7), α_z must be zero. Explain what is wrong with this reasoning that leads to this apparent contradiction.

Q10.24. In Example 10.11 (Section 10.6) the rotational kinetic energy of the professor and dumbbells increases. But since there are no external torques, no work is being done to change the rotational kinetic energy. Then, by Eq. (10.22), the kinetic energy must remain the same! Explain what is wrong with this reasoning that leads to this apparent contradiction. Where *does* the extra kinetic energy come from?

Q10.25. As discussed in Section 10.6, the angular momentum of a circus acrobat is conserved as she tumbles through the air. Is her *linear* momentum conserved? Why or why not?

Q10.26. If you stop a spinning raw egg for the shortest possible instant and then release it, the egg will start spinning again. If you do the same to a hard-boiled egg, it will remain stopped. Try it. Explain it.

Q10.27. A helicopter has a large main rotor that rotates in a horizontal plane and provides lift. There is also a small rotor on the tail that rotates in a vertical plane. What is the purpose of the tail rotor? (*Hint:* If there were no tail rotor, what would happen when the pilot changed the angular speed of the main rotor?) Some helicopters have no tail rotor, but instead have two large main rotors that rotate in a horizontal plane. Why is it important that the two main rotors rotate in opposite directions?

Q10.28. In a common design for a gyroscope, the flywheel and flywheel axis are enclosed in a light, spherical frame with the flywheel at the center of the frame. The gyroscope is then balanced on top of a pivot so that the flywheel is directly above the pivot. Does the gyroscope precess if it is released while the flywheel is spinning? Explain.

Q10.29. A gyroscope takes 3.8 s to precess 1.0 revolution about a vertical axis. Two minutes later, it takes only 1.9 s to precess 1.0 revolution. No one has touched the gyroscope. Explain.

Q10.30. A gyroscope is precessing as in Fig. 10.32. What happens if you gently add some weight to the end of the flywheel axis farthest from the pivot?

Q10.31. A bullet emerges from a rifle spinning on its axis. Explain how this prevents the bullet from tumbling and keeps the streamlined end pointed forward.

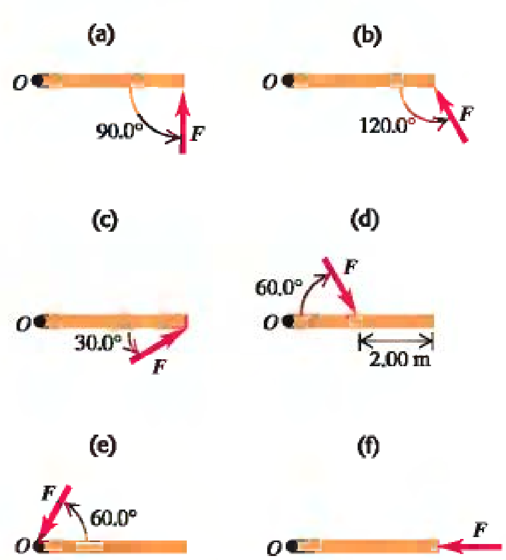
Q10.32. A certain uniform turntable of diameter D_0 has an angular momentum L_0. If you want to redesign it so it retains the same mass but has twice as much angular momentum at the same angular velocity as before, what should be its diameter in terms of D_0?

Exercises

Section 10.1 Torque

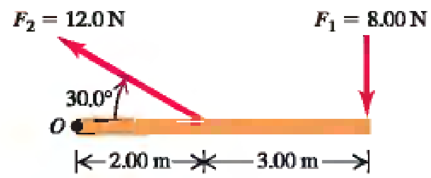
10.1. Calculate the torque (magnitude and direction) about point O due to the force $\vec{F}$ in each of the cases sketched in Fig. 10.37. In each case, the force $\vec{F}$ and the rod both lie in the plane of the page, the rod has length 4.00 m, and the force has magnitude $F = 10.0$ N.

Figure 10.37 Exercise 10.1.



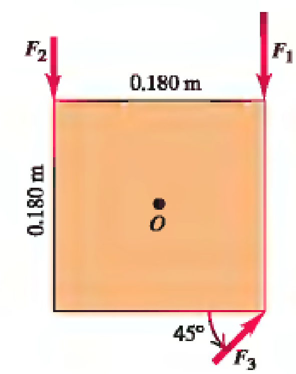
10.2. Calculate the net torque about point O for the two forces applied as in Fig. 10.38. The rod and both forces are in the plane of the page.

Figure 10.38 Exercise 10.2.



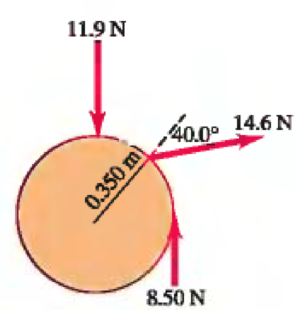
10.3. A square metal plate 0.180 m on each side is pivoted about an axis through point O at its center and perpendicular to the plate (Fig. 10.39). Calculate the net torque about this axis due to the three forces shown in the figure if the magnitudes of the forces are $F_1 = 18.0$ N, $F_2 = 26.0$ N, and $F_3 = 14.0$ N. The plate and all forces are in the plane of the page.

Figure 10.39 Exercise 10.3.



10.4. Three forces are applied to a wheel of radius 0.350 m, as shown in Fig. 10.40. One force is perpendicular to the rim, one is tangent to it, and the other one makes a 40.0° angle with the radius. What is the net torque on the wheel due to these three forces for an axis perpendicular to the wheel and passing through its center?

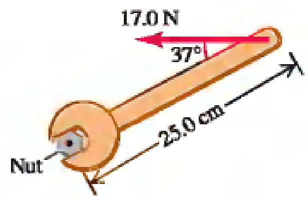
Figure 10.40 Exercise 10.4.



10.5. One force acting on a machine part is $\vec{F} = (-5.00\text{ N})\hat{\imath} + (4.00\text{ N})\hat{\jmath}$. The vector from the origin to the point where the force is applied is $\vec{r} = (-0.450\text{ m})\hat{\imath} + (0.150\text{ m})\hat{\jmath}$. (a) In a sketch, show $\vec{r}$, $\vec{F}$, and the origin. (b) Use the right-hand rule to determine the direction of the torque. (c) Calculate the vector torque produced by this force. Verify that the direction of the torque is the same as you obtained in part (b).

10.6. A machinist is using a wrench to loosen a nut. The wrench is 25.0 cm long, and he exerts a 17.0-N force at the end of the handle at 37° with the handle (Fig. 10.41). (a) What torque does the machinist exert about the center of the nut? (b) What is the maximum torque he could exert with this force, and how should the force be oriented?

Figure 10.41 Exercise 10.6.

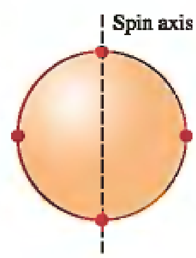


Section 10.2 Torque and Angular Acceleration for a Rigid Body

10.7. The flywheel of an engine has moment of inertia 2.50 kg · m^2 about its rotation axis. What constant torque is required to bring it up to an angular speed of 400 rev/min in 8.00 s, starting from rest?

10.8. A uniform, 8.40-kg, spherical shell 50.0 cm in diameter has four small 2.00-kg masses attached to its outer surface and equally spaced around it. This combination is spinning about an axis running through the center of the sphere and two of the small masses (Fig. 10.42). What friction torque is needed to reduce its angular speed from 75.0 rpm to 50.0 rpm in 30.0 s?

Figure 10.42 Exercise 10.8.



10.9. A machine part has the shape of a solid uniform sphere of mass 225 g and diameter 3.00 cm. It is spinning about a frictionless axle through its center, but at one point on its equator it is scraping against metal, resulting in a friction force of 0.0200 N at that point. (a) Find its angular acceleration. (b) How long will it take to decrease its rotational speed by 22.5 rad/s?

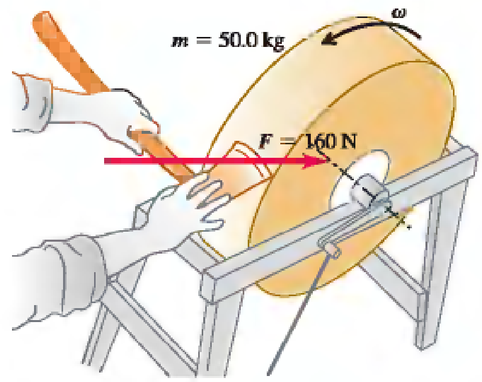
10.10. A cord is wrapped around the rim of a solid uniform wheel 0.250 m in radius and of mass 9.20 kg. A steady horizontal pull of 40.0 N to the right is exerted on the cord, pulling it off tangentially from the wheel. The wheel is mounted on frictionless bearings on a horizontal axle through its center. (a) Compute the angular acceleration of the wheel and the acceleration of the part of the cord that has already been pulled off the wheel. (b) Find the magnitude and direction of the force that the axle exerts on the wheel. (c) Which of the answers in parts (a) and (b) would change if the pull were upward instead of horizontal?

10.11. A solid, uniform cylinder with mass 8.25 kg and diameter 15.0 cm is spinning at 220 rpm on a thin, frictionless axle that passes along the cylinder axis. You design a simple friction brake to stop the cylinder by pressing the brake against the outer rim with a normal force. The coefficient of kinetic friction between the brake and rim is 0.333. What must the applied normal force be to bring the cylinder to rest after it has turned through 5.25 revolutions?

10.12. A stone is suspended from the free end of a wire that is wrapped around the outer rim of a pulley, similar to what is shown in Fig. 10.10. The pulley is a uniform disk with mass 10.0 kg and radius 50.0 cm and turns on frictionless bearings. You measure that the stone travels 12.6 m in the first 3.00 s starting from rest. Find (a) the mass of the stone and (b) the tension in the wire.

10.13. A grindstone in the shape of a solid disk with diameter 0.520 m, and a mass of 50.0 kg is rotating at 850 rev/min. You press an ax against the rim with a normal force of 160 N (Fig. 10.43), and the grindstone comes to rest in 7.50 s. Find the coefficient of friction between the ax and the grindstone. You can ignore friction in the bearings.

Figure 10.43 Exercise 10.13 and Problem 10.53.

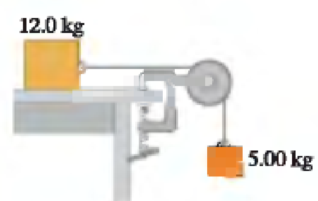


10.14. A 15.0-kg bucket of water is suspended by a very light rope wrapped around a solid uniform cylinder 0.300 m in diameter with mass 12.0 kg. The cylinder pivots on a frictionless axle through its center. The bucket is released from rest at the top of a well and falls 10.0 m to the water. (a) What is the tension in the rope while the bucket is falling? (b) With what speed does the bucket strike the water? (c) What is the time of fall? (d) While the bucket is falling, what is the force exerted on the cylinder by the axle?

10.15. A 2.00-kg textbook rests on a frictionless, horizontal surface. A cord attached to the book passes over a pulley whose diameter is 0.150 m, to a hanging book with mass 3.00 kg. The system is released from rest, and the books are observed to move 1.20 m in 0.800 s. (a) What is the tension in each part of the cord? (b) What is the moment of inertia of the pulley about its rotation axis?

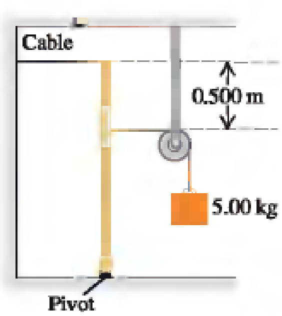
10.16. A 12.0-kg box resting on a horizontal, frictionless surface is attached to a 5.00-kg weight by a thin, light wire that passes over a frictionless pulley (Fig. 10.44). The pulley has the shape of a uniform solid disk of mass 2.00 kg and diameter 0.500 m. After the system is released, find (a) the tension in the wire on both sides of the pulley, (b) the acceleration of the box, and (c) the horizontal and vertical components of the force that the axle exerts on the pulley.

Figure 10.44 Exercise 10.16.



10.17. A thin, uniform, 15.0-kg post, 1.75 m long, is held vertically using a cable and is attached to a 5.00-kg mass and a pivot at its bottom end (Fig. 10.45). The string attached to the 5.00-kg mass passes over a massless, frictionless pulley and pulls perpendicular to the post. Suddenly the cable breaks. (a) Find the angular acceleration of the post about the pivot just after the cable breaks. (b) Will the angular acceleration in part (a) remain constant as the post falls (before it hits the pulley)? Why? (c) What is the acceleration of the

Figure 10.45 Exercise 10.17.



5.00-kg mass the instant after the cable breaks? Does this acceleration remain constant? Why?

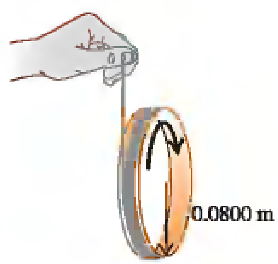
10.18. A thin, horizontal rod with length l and mass M pivots about a vertical axis at one end. A force with constant magnitude F is applied to the other end, causing the rod to rotate in a horizontal plane. The force is maintained perpendicular to the rod and to the axis of rotation. Calculate the magnitude of the angular acceleration of the rod.

Section 10.3 Rigid-Body Rotation About a Moving Axis

10.19. A 2.20-kg hoop 1.20 m in diameter is rolling to the right without slipping on a horizontal floor at a steady 3.00 rad/s. (a) How fast is its center moving? (b) What is the total kinetic energy of the hoop? (c) Find the velocity vector of each of the following points, as viewed by a person at rest on the ground: (i) the highest point on the hoop; (ii) the lowest point on the hoop, (iii) a point on the right side of the hoop, midway between the top and the bottom. (d) Find the velocity vector for each of the points in part (c), except as viewed by someone moving along with same velocity as the hoop.

10.20. A string is wrapped several times around the rim of a small hoop with radius 8.00 cm and mass 0.180 kg. The free end of the string is held in place and the hoop is released from rest (Fig. 10.46). After the hoop has descended 75.0 cm, calculate (a) the angular speed of the rotating hoop and (b) the speed of its center.

Figure 10.46 Exercise 10.20 and Problem 10.72.



10.21. What fraction of the total kinetic energy is rotational for the following objects rolling without slipping on a horizontal surface? (a) a uniform solid cylinder; (b) a uniform sphere; (c) a thin-walled, hollow sphere; (d) a hollow cylinder with outer radius R and inner radius $R/2$.

10.22. A hollow, spherical shell with mass 2.00 kg rolls without slipping down a 38.0° slope. (a) Find the acceleration, the friction force, and the minimum coefficient of friction needed to prevent slipping. (b) How would your answers to part (a) change if the mass were doubled to 4.00 kg?

10.23. A solid ball is released from rest and slides down a hillside that slopes downward at 65.0° from the horizontal. (a) What minimum value must the coefficient of static friction between the hill and ball surfaces have for no slipping to occur? (b) Would the coefficient of friction calculated in part (a) be sufficient to prevent a hollow ball (such as a soccer ball) from slipping? Justify your answer. (c) In part (a), why did we use the coefficient of static friction and not the coefficient of kinetic friction?

10.24. A uniform marble rolls down a symmetric bowl, starting from rest at the top of the left side. The top of each side is a distance h above the bottom of the bowl. The left half of the bowl is rough enough to cause the marble to roll without slipping, but the right half has no friction because it is coated with oil. (a) How far up the smooth side will the marble go, measured vertically from the bottom? (b) How high would the marble go if both sides were as rough as the left side? (c) How do you account for the fact that the marble goes *higher* with friction on the right side than without friction?

10.25. A 392-N wheel comes off a moving truck and rolls without slipping along a highway. At the bottom of a hill it is rotating at 25.0 rad/s. The radius of the wheel is 0.600 m, and its moment of inertia about its rotation axis is $0.800MR^2$. Friction does work on the wheel as it rolls up the hill to a stop, a height h above the bottom of the hill; this work has absolute value 3500 J. Calculate h.

10.26. A Ball Rolling Uphill. A bowling ball rolls without slipping up a ramp that slopes upward at an angle β to the horizontal (see Example 10.7 in Section 10.3). Treat the ball as a uniform, solid sphere, ignoring the finger holes. (a) Draw the free-body diagram for the ball. Explain why the friction force must be directed *uphill*. (b) What is the acceleration of the center of mass of the ball? (c) What minimum coefficient of static friction is needed to prevent slipping?

Section 10.4 Work and Power in Rotational Motion

10.27. A playground merry-go-round has radius 2.40 m and moment of inertia 2100 kg · m^2 about a vertical axle through its center, and it turns with negligible friction. (a) A child applies an 18.0-N force tangentially to the edge of the merry-go-round for 15.0 s. If the merry-go-round is initially at rest, what is its angular speed after this 15.0-s interval? (b) How much work did the child do on the merry-go-round? (c) What is the average power supplied by the child?

10.28. The engine delivers 175 hp to an aircraft propeller at 2400 rev/min. (a) How much torque does the aircraft engine provide? (b) How much work does the engine do in one revolution of the propeller?

10.29. A 1.50-kg grinding wheel is in the form of a solid cylinder of radius 0.100 m. (a) What constant torque will bring it from rest to an angular speed of 1200 rev/min in 2.5 s? (b) Through what angle has it turned during that time? (c) Use Eq. (10.21) to calculate the work done by the torque. (d) What is the grinding wheel's kinetic energy when it is rotating at 1200 rev/min? Compare your answer to the result in part (c).

10.30. An electric motor consumes 9.00 kJ of electrical energy in 1.00 min. If one-third of this energy goes into heat and other forms of internal energy of the motor, with the rest going to the motor output, how much torque will this engine develop if you run it at 2500 rpm?

10.31. The carbide tips of the cutting teeth of a circular saw are 8.6 cm from the axis of rotation. (a) The no-load speed of the saw, when it is not cutting anything, is 4800 rev/min. Why is its no-load power output negligible? (b) While the saw is cutting lumber, its angular speed slows to 2400 rev/min and the power output is 1.9 hp. What is the tangential force that the wood exerts on the carbide tips?

10.32. An airplane propeller is 2.08 m in length (from tip to tip) and has a mass of 117 kg. When the airplane's engine is first started, it applies a constant torque of 1950 N · m to the propeller, which starts from rest. (a) What is the angular acceleration of the propeller? Model the propeller as a slender rod and see Table 9.2. (b) What is the propeller's angular speed after making 5.00 revolutions? (c) How much work is done by the engine during the first 5.00 revolutions? (d) What is the average power output of the engine during the first 5.00 revolutions? (e) What is the instantaneous power output of the motor at the instant that the propeller has turned through 5.00 revolutions?

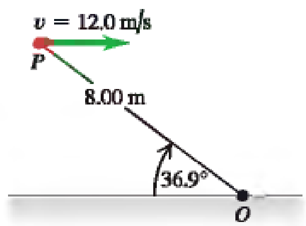
10.33. (a) Compute the torque developed by an industrial motor whose output is 150 kW at an angular speed of 4000 rev/min. (b) A drum with negligible mass, 0.400 m in diameter, is attached to the motor shaft, and the power output of the motor is used to raise a weight hanging from a rope wrapped around the drum. How heavy a weight can the motor lift at constant speed? (c) At what constant speed will the weight rise?

Section 10.5 Angular Momentum

10.34. A woman with mass 50 kg is standing on the rim of a large disk that is rotating at 0.50 rev/s about an axis through its center. The disk has mass 110 kg and radius 4.0 m. Calculate the magnitude of the total angular momentum of the woman-plus-disk system. (Assume that you can treat the woman as a point.)

10.35. A 2.00-kg rock has a horizontal velocity of magnitude 12.0 m/s when it is at point P in Fig. 10.47. (a) At this instant, what are the magnitude and direction of its angular momentum relative to point O? (b) If the only force acting on the rock is its weight, what is the rate of change (magnitude and direction) of its angular momentum at this instant?

Figure 10.47 Exercise 10.35.



10.36. (a) Calculate the magnitude of the angular momentum of the earth in a circular orbit around the sun. Is it reasonable to model it as a particle? (b) Calculate the magnitude of the angular momentum of the earth due to its rotation around an axis through the north and south poles, modeling it as a uniform sphere. Consult Appendix E and the astronomical data in Appendix F.

10.37. Find the magnitude of the angular momentum of the second hand on a clock about an axis through the center of the clock face. The clock hand has a length of 15.0 cm and a mass of 6.00 g. Take the second hand to be a slender rod rotating with constant angular velocity about one end.

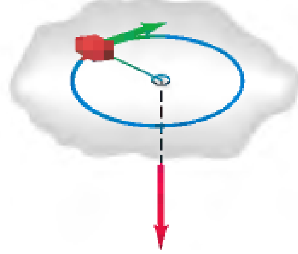
10.36. A hollow, thin-walled sphere of mass 12.0 kg and diameter 48.0 cm is rotating about an axle through its center. The angle (in radians) through which it turns as a function of time (in seconds) is given by $\theta(t) = At^2 + Bt^4$, where A has numerical value 1.50 and B has numerical value 1.10. (a) What are the units of the constants A and B? (b) At the time 3.00 s, find (i) the angular momentum of the sphere and (ii) the net torque on the sphere.

Section 10.6 Conservation of Angular Momentum

10.36. Under some circumstances, a star can collapse into an extremely dense object made mostly of neutrons and called a *neutron star.* The density of a neutron star is roughly 10^{14} times as great as that of ordinary solid matter. Suppose we represent the star as a uniform, solid, rigid sphere, both before and after the collapse. The star's initial radius was 7.0×10^5 km (comparable to our sun); its final radius is 16 km. If the original star rotated once in 30 days, find the angular speed of the neutron star.

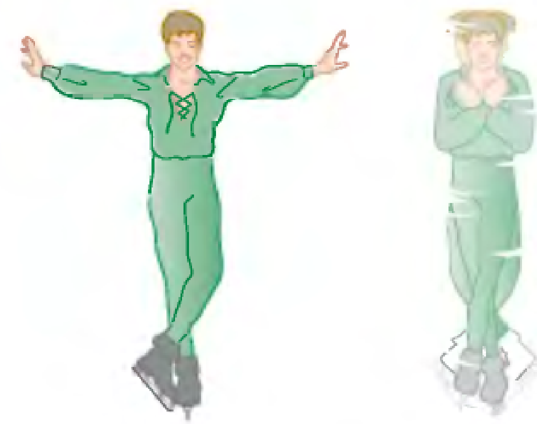
10.40. A small block on a frictionless, horizontal surface has a mass of 0.0250 kg. It is attached to a massless cord passing through a hole in the surface (Fig. 10.48). The block is originally revolving at a distance of 0.300 m from the hole with an angular speed of 1.75 rad/s. The cord is then pulled from below, shortening the radius of the circle in which the block revolves to 0.150 m. Model the block as a particle. (a) Is angular momentum of the block conserved? Why or why not? (b) What is the new angular speed? (c) Find the change in kinetic energy of the block. (d) How much work was done in pulling the cord?

Figure 10.48 Exercise 10.40, Problem 10.92, and Challenge Problem 10.103.



10.41. The Spinning Figure Skater. The outstretched hands and arms of a figure skater preparing for a spin can be considered a slender rod pivoting about an axis through its center (Fig. 10.49). When the skater's hands and arms are brought in and wrapped around his body to execute the spin, the hands and arms can be considered a thin-walled, hollow cylinder. His hands and arms have a combined mass 8.0 kg. When outstretched, they span 1.8 m; when wrapped, they form a cylinder of radius 25 cm. The moment of inertia about the rotation axis of the remainder of his body is constant and equal to 0.40 kg · m^2. If his original angular speed is 0.40 rev/s, what is his final angular speed?

Figure 10.49 Exercise 10.41.



10.42. A diver comes off a board with arms straight up and legs straight down, giving her a moment of inertia about her rotation axis of 18 kg · m^2. She then tucks into a small ball, decreasing this moment of inertia to 3.6 kg · m^2. While tucked, she makes two complete revolutions in 1.0 s. If she hadn't tucked at all, how many revolutions would she have made in the 1.5 s from board to water?

10.43. A large wooden turntable in the shape of a flat uniform disk has a radius of 2.00 m and a total mass of 120 kg. The turntable is initially rotating at 3.00 rad/s about a vertical axis through its center. Suddenly, a 70.0-kg parachutist makes a soft landing on the turntable at a point near the outer edge. (a) Find the angular speed of the turntable after the parachutist lands. (Assume that you can treat the parachutist as a particle.) (b) Compute the kinetic energy of the system before and after the parachutist lands. Why are these kinetic energies not equal?

10.44. A solid wood door 1.00 m wide and 2.00 m high is hinged along one side and has a total mass of 40.0 kg. Initially open and at rest, the door is struck at its center by a handful of sticky mud with mass 0.500 kg, traveling perpendicular to the door at 12.0 m/s just before impact. Find the final angular speed of the door. Does the mud make a significant contribution to the moment of inertia?

10.45. A small 10.0-g bug stands at one end of a thin uniform bar that is initially at rest on a smooth horizontal table. The other end of the bar pivots about a nail driven into the table and can rotate freely, without friction. The bar has mass 50.0 g and is 100 cm in length. The bug jumps off in the horizontal direction, perpendicular to the bar, with a speed of 20.0 cm/s relative to the table. (a) What is the angular speed of the bar just after the frisky insect leaps? (b) What is the total kinetic energy of the system just after the bug leaps? (c) Where does this energy come from?

10.46. Asteroid Collision! Suppose that an asteroid traveling straight toward the center of the earth were to collide with our planet at the equator and bury itself just below the surface. What would have to be the mass of this asteroid, in terms of the earth's mass M, for the day to become 25.0% longer than it presently is as a result of the collision? Assume that the asteroid is very small compared to the earth and that the earth is uniform throughout.

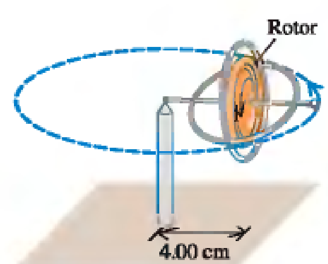
10.47. A thin, uniform metal bar, 2.00 m long and weighing 90.0 N, is hanging vertically from the ceiling by a frictionless pivot. Suddenly it is struck 1.50 m below the ceiling by a small 3.00-kg ball, initially traveling horizontally at 10.0 m/s. The ball rebounds in the opposite direction with a speed of 6.00 m/s. (a) Find the angular speed of the bar just after the collision. (b) During the collision, why is the angular momentum conserved but not the linear momentum?

Section 10.7 Gyroscopes and Precession

10.40. Draw a top view of the gyroscope shown in Fig. 10.32. (a) Draw labeled arrows on your sketch for $\vec{\omega}$, $\vec{L}$, and $\vec{\tau}$. Draw $d\vec{L}$ produced by $\vec{\tau}$. Draw $\vec{L} + d\vec{L}$. Determine the sense of the precession by examining the directions of $\vec{L}$ and $\vec{L} + d\vec{L}$. (b) Reverse the direction of the spin angular velocity of the rotor and repeat all the steps in part (a). (c) Move the pivot to the other end of the shaft, with the same direction of spin angular velocity as in part (b), and repeat all the steps. (d) Keeping the pivot as in part (c), reverse the spin angular velocity of the rotor and repeat all the steps.

10.40. The rotor (flywheel) of a toy gyroscope has mass 0.140 kg. Its moment of inertia about its axis is $1.20 \times 10^{-4}\ \mathrm{kg \cdot m^2}$. The mass of the frame is 0.0250 kg. The gyroscope is supported on a single pivot (Fig. 10.50) with its center of mass a horizontal distance of 4.00 cm from the pivot. The gyroscope is precessing in a horizontal plane at the rate of one revolution in 2.20 s. (a) Find the upward force exerted by the pivot. (b) Find the angular speed with which the rotor is spinning about its axis, expressed in rev/min. (c) Copy the diagram and draw vectors to show the angular momentum of the rotor and the torque acting on it.

Figure 10.50 Exercise 10.49.



10.50. A Gyroscope on the Moon. A certain gyroscope precesses at a rate of 0.50 rad/s when used on earth. If it were taken to a lunar base, where the acceleration due to gravity is $0.165g$, what would be its precession rate?

10.51. A gyroscope is precessing about a vertical axis. Describe what happens to the precession angular speed if the following changes in the variables are made, with all other variables remaining the same: (a) the angular speed of the spinning flywheel is doubled; (b) the total weight is doubled; (c) the moment of inertia about the axis of the spinning flywheel is doubled; (d) the distance from the pivot to the center of gravity is doubled. (e) What happens if all four of the variables in parts (a) through (d) are doubled?

10.52. The earth precesses once every 26,000 years and spins on its axis once a day. Estimate the magnitude of the torque that causes the precession of the earth. You may need some data from Appendix F. Make the estimate by assuming (i) the earth is a uniform sphere and (ii) the precession of the earth is like that of the gyroscope shown in Fig. 10.34. In this model, the precession axis and rotation axis are perpendicular. Actually, the angle between these two axes for the earth is only $23\frac{1}{2}°$; this affects the calculated torque by about a factor of 2.

Problems

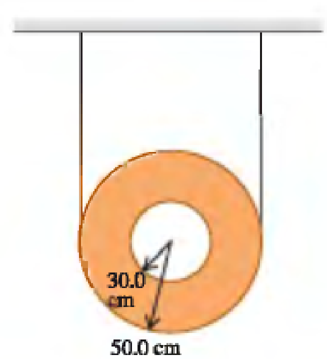
10.53. A 50.0-kg grindstone is a solid disk 0.520 m in diameter. You press an ax down on the rim with a normal force of 160 N (Fig. 10.43). The coefficient of kinetic friction between the blade and the stone is 0.60, and there is a constant friction torque of $6.50\ \mathrm{N \cdot m}$ between the axle of the stone and its bearings. (a) How much force must be applied tangentially at the end of a crank handle 0.500 m long to bring the stone from rest to 120 rev/min in 9.00 s? (b) After the grindstone attains an angular speed of 120 rev/min, what tangential force at the end of the handle is needed to maintain a constant angular speed of 120 rev/min? (c) How much time does it take the grindstone to come from 120 rev/min to rest if it is acted on by the axle friction alone?

10.54. An experimental bicycle wheel is placed on a test stand so that it is free to turn on its axle. If a constant net torque of $5.00\ \mathrm{N \cdot m}$ is applied to the tire for 2.00 s, the angular speed of the tire increases from 0 to 100 rev/min. The external torque is then removed, and the wheel is brought to rest by friction in its bearings in 125 s. Compute (a) the moment of inertia of the wheel about the rotation axis; (b) the friction torque; (c) the total number of revolutions made by the wheel in the 125-s time interval.

10.55. Speedometer. Your car's speedometer converts the angular speed of the wheels to the linear speed of the car, assuming standard-size tires and no slipping on the pavement. (a) If your car's standard tires are 24 inches in diameter, at what rate (in rpm) are your wheels rotating when you are driving at a freeway speed of 60 mi/h? (b) Suppose you put oversize, 30-inch-diameter tires on your car. How fast are you really going when your speedometer reads 60 mi/h? (c) If you now put on undersize, 20-inch-diameter tires, what will the speedometer read when you are actually traveling at 50 mi/h?

10.56. A uniform hollow disk has two pieces of thin light wire wrapped around its outer rim and is supported from the ceiling (Fig. 10.51). Suddenly one of the wires breaks, and the remaining wire does not slip as the disk rolls down. Use energy conservation to find the speed of the center of this disk after it has fallen a distance of 1.20 m.

Figure 10.51 Problem 10.56.



10.57. A thin, uniform 3.80-kg bar, 80.0 cm long, has very small 2.50-kg balls glued on at either end (Fig. 10.52). It is supported horizontally by a thin, horizontal, frictionless axle passing through its center and perpendicular to the bar. Suddenly the right-hand ball becomes detached and falls off, but the other ball remains glued to the bar. (a) Find the angular acceleration of the bar just after the ball falls off. (b) Will the angular acceleration remain constant as the bar continues to swing? If not, will it increase or decrease? (c) Find the angular velocity of the bar just as it swings through its vertical position.

Figure 10.52 Problem 10.57.

10.56. While exploring a castle, Exena the Exterminator is spotted by a dragon who chases her down a hallway. Exena runs into a room and attempts to swing the heavy door shut before the dragon gets her. The door is initially perpendicular to the wall, so it must be turned through 90° to close. The door is 3.00 m tall and 1.25 m wide, and it weighs 750 N. You can ignore the friction at the hinges. If Exena applies a force of 220 N at the edge of the door and perpendicular to it, how much time does it take her to close the door?

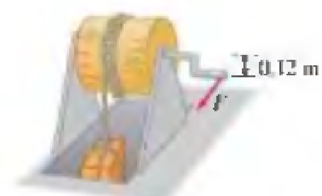
10.59. A thin rod of length l lies on the $+x$-axis with its left end at the origin. A string pulls on the rod with a force $\vec{F}$ directed toward a point P a distance h above the rod. Where along the rod should you attach the string to get the greatest torque about the origin if point P is (a) above the right end of the rod? (b) Above the left end of the rod? (c) Above the center of the rod?

10.60. Balancing Act. Attached to one end of a long, thin, uniform rod of length L and mass M is a small blob of clay of the same mass M. (a) Locate the position of the center of mass of the system of rod and clay. Note this position on a drawing of the rod. (b) You carefully balance the rod on a frictionless tabletop so that it is standing vertically, with the end without the clay touching the table. If the rod is now tipped so that it is a small angle θ away from the vertical, determine its angular acceleration at this instant. Assume that the end without the clay remains in contact with the tabletop. (*Hint:* See Table 9.2.) (c) You again balance the rod on the frictionless tabletop so that it is standing vertically, but now the end of the rod *with* the clay is touching the table. If the rod is again tipped so that it is a small angle θ away from the vertical, determine its angular acceleration at this instant. Assume that the end with the clay remains in contact with the tabletop. How does this compare to the angular acceleration in part (b)? (d) A pool cue is a tapered wooden rod that is thick at one end and thin at the other. You can easily balance a pool cue vertically on one finger if the thin end is in contact with your finger; this is quite a bit harder to do if the thick end is in contact with your finger. Explain why there is a difference.

10.61. You connect a light string to a point on the edge of a uniform vertical disk with radius R and mass M. The disk is free to rotate without friction about a stationary horizontal axis through its center. Initially, the disk is at rest with the string connection at the highest point on the disk. You pull the string with a constant horizontal force $\vec{F}$ until the wheel has made exactly one-quarter revolution about a horizontal axis through its center, and then you let go. (a) Use Eq. (10.20) to find the work done by the string. (b) Use Eq. (6.14) to find the work done by the string. Do you obtain the same result as in part (a)? (c) Find the final angular speed of the disk. (d) Find the maximum tangential acceleration of a point on the disk. (e) Find the maximum radial (centripetal) acceleration of a point on the disk.

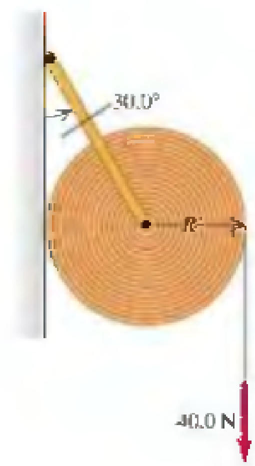
10.62. The mechanism shown in Fig. 10.53 is used to raise a crate of supplies from a ship's hold. The crate has total mass 50 kg. A rope is wrapped around a wooden cylinder that turns on a metal axle. The cylinder has radius 0.25 m and moment of inertia $I = 2.9\ \text{kg} \cdot \text{m}^2$ about the axle. The crate is suspended from the free end of the rope. One end of the axle pivots on frictionless bearings; a crank handle is attached to the other end. When the crank is turned, the end of the handle rotates about the axle in a vertical circle of radius 0.12 m, the cylinder turns, and the crate is raised. What magnitude of the force $\vec{F}$ applied tangentially to the rotating crank is required to raise the crate with an acceleration of $0.80\ \text{m/s}^2$? (You can ignore the mass of the rope as well as the moments of inertia of the axle and the crank.)

Figure 10.53 Problem 10.62.



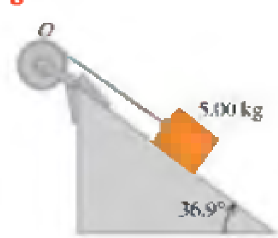
10.63. A large 16.0-kg roll of paper with radius $R = 18.0$ cm rests against the wall and is held in place by a bracket attached to a rod through the center of the roll (Fig. 10.54). The rod turns without friction in the bracket, and the moment of inertia of the paper and rod about the axis is $0.260\ \text{kg} \cdot \text{m}^2$. The other end of the bracket is attached by a frictionless hinge to the wall such that the bracket makes an angle of 30.0° with the wall. The weight of the bracket is negligible. The coefficient of kinetic friction between the paper and the wall is $\mu_k = 0.25$. A constant vertical force $F = 40.0$ N is applied to the paper, and the paper unrolls. (a) What is the magnitude of the force that the rod exerts on the paper as it unrolls? (b) What is the magnitude of the angular acceleration of the roll?

Figure 10.54 Problem 10.63.



10.64. A block with mass $m = 5.00$ kg slides down a surface inclined 36.9° to the horizontal (Fig. 10.55). The coefficient of kinetic friction is 0.25. A string attached to the block is wrapped around a flywheel on a fixed axis at O. The flywheel has mass 25.0 kg and moment of inertia $0.500\ \text{kg} \cdot \text{m}^2$ with respect to the axis of rotation. The string pulls without slipping at a perpendicular distance of 0.200 m from that axis. (a) What is the acceleration of the block down the plane? (b) What is the tension in the string?

Figure 10.55 Problem 10.64.

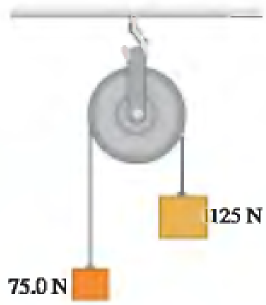


10.65. Two metal disks, one with radius $R_1 = 2.50$ cm and mass $M_1 = 0.80$ kg and the other with radius $R_2 = 5.00$ cm and mass $M_2 = 1.60$ kg, are welded together and mounted on a frictionless axis through their common center, as in Problem 9.89. (a) A light string is wrapped around the edge of the smaller disk, and a 1.50 kg block is suspended from the free end of the string. What is the magnitude of the downward acceleration of the block after it is released? (b) Repeat the calculation of part (a), this time with the string wrapped around the edge of the larger disk. In which case is the acceleration of the block greater? Does your answer make sense?

10.66. A lawn roller in the form of a thin-walled, hollow cylinder with mass M is pulled horizontally with a constant horizontal force F applied by a handle attached to the axle. If it rolls without slipping, find the acceleration and the friction force.

10.67. Two weights are connected by a very light flexible cord that passes over a 50.0-N frictionless pulley of radius 0.300 m. The pulley is a solid uniform disk and is supported by a hook connected to the ceiling (Fig. 10.56). What force does the ceiling exert on the hook?

Figure 10.56 Problem 10.67.

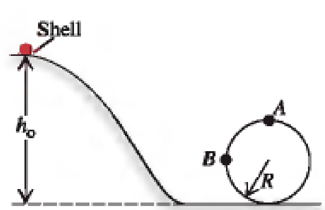


10.68. A solid disk is rolling without slipping on a level surface at a constant speed of 2.50 m/s. (a) If the disk rolls up a 30.0° ramp, how far along the ramp will it move before it stops? (b) Explain why your answer in part (a) does not depend on either the mass or the radius of the disk.

10.69. The Yo-yo. A yo-yo is made from two uniform disks, each with mass m and radius R, connected by a light axle of radius b. A light, thin string is wound several times around the axle and then held stationary while the yo-yo is released from rest, dropping as the string unwinds. Find the linear acceleration and angular acceleration of the yo-yo and the tension in the string.

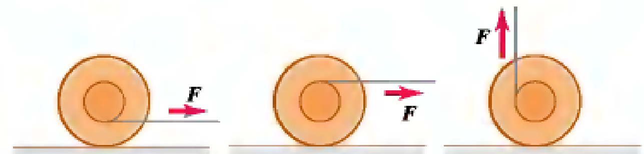
10.70. A thin-walled, hollow spherical shell of mass m and radius r starts from rest and rolls without slipping down the track shown in Fig. 10.57. Points A and B are on a circular part of the track having radius R. The diameter of the shell is very small compared to h_0 and R, and rolling friction is negligible. (a) What is the minimum height h_0 for which this shell will make a complete loop-the-loop on the circular part of the track? (b) How hard does the track push on the shell at point B, which is at the same level as the center of the circle? (c) Suppose that the track had no friction and the shell was released from the same height h_0 you found in part (a). Would it make a complete loop-the-loop? How do you know? (d) In part (c), how hard does the track push on the shell at point A, the top of the circle? How hard did it push on the shell in part (a)?

Figure 10.57 Problem 10.70.



10.71. Figure 10.58 shows three identical yo-yos initially at rest on a horizontal surface. For each yo-yo, the string is pulled in the direction shown. In each case, there is sufficient friction for the yo-yo to roll without slipping. Draw the free-body diagram for each yo-yo. In what direction will each yo-yo rotate? (Try it!) Explain your answers.

Figure 10.58 Problem 10.71.

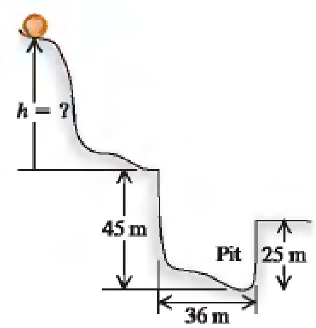


10.72. As shown in Fig. 10.46, a string is wrapped several times around the rim of a small hoop with radius 0.0800 m and mass 0.180 kg. The free end of the string is pulled upward in just the right way so that the hoop does not move vertically as the string unwinds. (a) Find the tension in the string as the string unwinds. (b) Find the angular acceleration of the hoop as the string unwinds. (c) Find the upward acceleration of the hand that pulls on the free end of the string. (d) How would your answers be different if the hoop were replaced by a solid disk of the same mass and radius?

10.73. Starting from rest, a constant force $F = 100$ N is applied to the free end of a 50-m cable wrapped around the outer rim of a uniform solid cylinder, similar to the situation shown in Fig. 10.9(a). The cylinder has mass 4.00 kg and diameter 30.0 cm and is free to turn about a fixed, frictionless axle through its center. (a) How long does it take to unwrap all the cable, and how fast is the cable moving just as the last bit comes off? (b) Now suppose that the cylinder is replaced by a uniform hoop, with all other quantities remaining unchanged. In this case, would the answers in part (a) be larger or smaller? Explain.

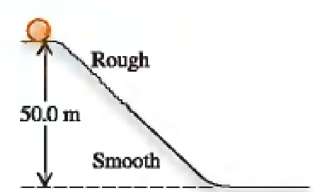
10.74. A uniform marble rolls without slipping down the path shown in Fig. 10.59, starting from rest. (a) Find the minimum height h required for the marble not to fall into the pit. (b) The moment of inertia of the marble depends on its radius. Explain why the answer to part (a) does not depend on the radius of the marble. (c) Solve part (a) for a block that slides without friction instead of the rolling marble. How does the minimum h in this case compare to the answer in part (a)?

Figure 10.59 Problem 10.74



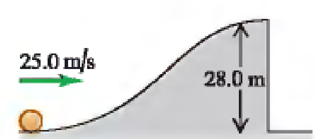
10.75. Rolling Stones. A solid, uniform, spherical boulder starts from rest and rolls down a 50.0-m-high hill, as shown in Fig. 10.60. The top half of the hill is rough enough to cause the boulder to roll without slipping, but the lower half is covered with ice and there is no friction. What is the translational speed of the boulder when it reaches the bottom of the hill?

Figure 10.60 Problem 10.75.



10.76. A solid, uniform ball rolls without slipping up a hill, as shown in Fig. 10.61. At the top of the hill, it is moving horizontally, and then it goes over the vertical cliff. (a) How far from the foot of the cliff does the ball land, and how fast is it

Figure 10.61 Problem 10.76.



moving just before it lands? (b) Notice that when the balls lands, it has a greater translational speed than when it was at the bottom of the hill. Does this mean that the ball somehow gained energy? Explain!

10.77. A 42.0-cm-diameter wheel, consisting of a rim and six spokes, is constructed from a thin, rigid plastic material having a linear mass density of 25.0 g/cm. This wheel is released from rest at the top of a hill 58.0 m high. (a) How fast is it rolling when it reaches the bottom of the hill? (b) How would your answer change if the linear mass density and the diameter of the wheel were each doubled?

10.78. A high-wheel antique bicycle has a large front wheel with the foot-powered crank mounted on its axle and a small rear wheel turning independently of the front wheel; there is no chain connecting the wheels. The radius of the front wheel is 65.5 cm, and the radius of the rear wheel is 22.0 cm. Your modern bike has a wheel diameter of 66.0 cm (26 inches) and front and rear sprockets with radii of 11.0 cm and 5.5 cm, respectively. The rear sprocket is rigidly attached to the axle of the rear wheel. You ride your modern bike and turn the front sprocket at 1.00 rev/s. The wheels of both bikes roll along the ground without slipping. (a) What is your linear speed when you ride your modern bike? (b) At what rate must you turn the crank of the antique bike in order to travel at the same speed as in part (a)? (c) What then is the angular speed (in rev/s) of the small rear wheel of the antique bike?

10.78. In a lab experiment you let a uniform ball roll down a curved track. The ball starts from rest and rolls without slipping. While on the track, the ball descends a vertical distance h. The lower end of the track is horizontal and extends over the edge of the lab table; the ball leaves the track traveling horizontally. While free falling after leaving the track, the ball moves a horizontal distance x and a vertical distance y. (a) Calculate x in terms of h and y, ignoring the work done by friction. (b) Would the answer to part (a) be any different on the moon? (c) Although you do the experiment very carefully, your measured value of x is consistently a bit smaller than the value calculated in part (a). Why? (d) What would x be for the same h and y as in part (a) if you let a silver dollar roll down the track? You can ignore the work done by friction.

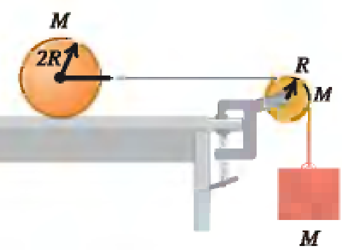
10.80. In a spring gun, a spring of force constant 400 N/m is compressed 0.15 m. When fired, 80.0% of the elastic potential energy stored in the spring is eventually converted into the kinetic energy of a 0.0590-kg uniform ball that is rolling without slipping at the base of a ramp. The ball continues to roll without slipping up the ramp with 90.0% of the kinetic energy at the bottom converted into an increase in gravitational potential energy at the instant it stops. (a) What is the speed of the ball's center of mass at the base of the ramp? (b) At this position, what is the speed of a point at the top of the ball? (c) At this position, what is the speed of a point at the bottom of the ball? (d) What maximum vertical height up the ramp does the ball move?

10.81. If a wheel rolls along a horizontal surface at constant speed, the coordinates of a certain point on the rim of the wheel are $x(t) = R[(2\pi t/T) - \sin(2\pi t/T)]$ and $y(t) = R[1 - \cos(2\pi t/T)]$, where R and T are constants. (a) Sketch the trajectory of the point from $t = 0$ to $t = 2T$. A curve with this shape is called a *cycloid*. (b) What are the meanings of the constants R and T? (c) Find the x- and y-components of the velocity and of the acceleration of the point at any time t. (d) Find the times at which the point is instantaneously at rest. What are the x- and y-components of the acceleration at these times? (e) Find the magnitude of the acceleration of the point. Does it depend on time? Compare to the magnitude of the acceleration of a particle in uniform circular motion, $a_{rad} = 4\pi^2R/T^2$. Explain your result for the magnitude of the acceleration of the point on the rolling wheel, using the idea that rolling is a combination of rotational and translational motion.

10.82. A child rolls a 0.600-kg basketball up a long ramp. The basketball can be considered a thin-walled, hollow sphere. When the child releases the basketball at the bottom of the ramp, it has a speed of 8.0 m/s. When the ball returns to her after rolling up the ramp and then rolling back down, it has a speed of 4.0 m/s. Assume the work done by friction on the basketball is the same when the ball moves up or down the ramp and that the basketball rolls without slipping. Find the maximum vertical height increase of the ball as it rolls up the ramp.

10.83. A uniform, solid cylinder with mass M and radius $2R$ rests on a horizontal tabletop. A string is attached by a yoke to a frictionless axle through the center of the cylinder so that the cylinder can rotate about the axle. The string runs over a disk-shaped pulley with mass M and radius R that is mounted on a frictionless axle through its center. A block of mass M is suspended from the free end of the string (Fig. 10.62). The string doesn't slip over the pulley surface, and the cylinder rolls without slipping on the tabletop. Find the magnitude of the acceleration of the block after the system is released from rest.

Figure 10.62 Problem 10.83.



10.84. A uniform drawbridge 8.00 m long is attached to the roadway by a frictionless hinge at one end, and it can be raised by a cable attached to the other end. The bridge is at rest, suspended at 60.0° above the horizontal, when the cable suddenly breaks. (a) Find the angular acceleration of the drawbridge just after the cable breaks. (Gravity behaves as though it all acts at the center of mass.) (b) Could you use the equation $\omega = \omega_0 + \alpha t$ to calculate the angular speed of the drawbridge at a later time? Explain why. (c) What is the angular speed of the drawbridge as it becomes horizontal?

10.85. A 5.00-kg ball is dropped from a height of 12.0 m above one end of a uniform bar that pivots at its center. The bar has mass 8.00 kg and is 4.00 m in length. At the other end of the bar sits another 5.00-kg ball, unattached to the bar. The dropped ball sticks to the bar after the collision. How high will the other ball go after the collision?

10.80. A uniform, 0.0300-kg rod of length 0.400 m rotates in a horizontal plane about a fixed axis through its center and perpendicular to the rod. Two small rings, each with mass 0.0200 kg, are mounted so that they can slide along the rod. They are initially held by catches at positions 0.0500 m on each side of the center of the rod, and the system is rotating at 30.0 rev/min. With no other changes in the system, the catches are released, and the rings slide outward along the rod and fly off at the ends. (a) What is the angular speed of the system at the instant when the rings reach the ends of the rod? (b) What is the angular speed of the rod after the rings leave it?

10.87. A uniform rod of length L rests on a frictionless horizontal surface. The rod pivots about a fixed frictionless axis at one end. The rod is initially at rest. A bullet traveling parallel to the horizontal surface and perpendicular to the rod with speed v strikes the rod at its center and becomes embedded in it. The mass of the bullet is one-fourth the mass of the rod. (a) What is the final angular speed of the rod? (b) What is the ratio of the kinetic energy of the system after the collision to the kinetic energy of the bullet before the collision?

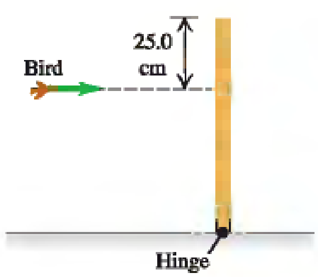
10.80. The solid wood door of a gymnasium is 1.00 m wide and 2.00 m high, has total mass 35.0 kg, and is hinged along one side. The door is open and at rest when a stray basketball hits the center of the door head-on, applying an average force of 1500 N to the door for 8.00 ms. Find the angular speed of the door after the impact. [*Hint:* Integrating Eq. (10.29) yields $\Delta L_z = \int_{t_1}^{t_2}(\sum \tau_z)\,dt = (\sum \tau_z)_{av}\Delta t$. The quantity $\int_{t_1}^{t_2}(\sum \tau_z)\,dt$ is called the angular impulse.]

10.80. A target in a shooting gallery consists of a vertical square wooden board, 0.250 m on a side and with mass 0.750 kg, that pivots on a horizontal axis along its top edge. The board is struck face-on at its center by a bullet with mass 1.90 g that is traveling at 360 m/s and that remains embedded in the board. (a) What is the angular speed of the board just after the bullet's impact? (b) What maximum height above the equilibrium position does the center of the board reach before starting to swing down again? (c) What minimum bullet speed would be required for the board to swing all the way over after impact?

10.90. Neutron Star Glitches. Occasionally, a rotating neutron star (see Exercise 10.39) undergoes a sudden and unexpected speedup called a *glitch*. One explanation is that a glitch occurs when the crust of the neutron star settles slightly, decreasing the moment of inertia about the rotation axis. A neutron star with angular speed $\omega_0 = 70.4$ rad/s underwent such a glitch in October 1975 that increased its angular speed to $\omega = \omega_0 + \Delta\omega$, where $\Delta\omega/\omega_0 = 2.01 \times 10^{-6}$. If the radius of the neutron star before the glitch was 11 km, by how much did its radius decrease in the starquake? Assume that the neutron star is a uniform sphere.

10.91. A 500.0-g bird is flying horizontally at 2.25 m/s, not paying much attention, when it suddenly flies into a stationary vertical bar, hitting it 25.0 cm below the top (Fig. 10.63). The bar is uniform, 0.750 m long, has a mass of 1.50 kg, and is hinged at its base. The collision stuns the bird so that it just drops to the ground afterward (but soon recovers to fly happily away). What is the angular velocity of the bar (a) just after it is hit by the bird, and (b) just as it reaches the ground?

Figure 10.63 Problem 10.91.



10.92. A small block with mass 0.250 kg is attached to a string passing through a hole in a frictionless, horizontal surface (see Fig. 10.48). The block is originally revolving in a circle with a radius of 0.800 m about the hole with a tangential speed of 4.00 m/s. The string is then pulled slowly from below, shortening the radius of the circle in which the block revolves. The breaking strength of the string is 30.0 N. What is the radius of the circle when the string breaks?

10.93. A horizontal plywood disk with mass 7.00 kg and diameter 1.00 m pivots on frictionless bearings about a vertical axis through its center. You attach a circular model-railroad track of negligible mass and average diameter 0.95 m to the disk. A 1.20-kg, battery-driven model train rests on the tracks. To demonstrate conservation of angular momentum, you switch on the train's engine. The train moves counterclockwise, soon attaining a constant speed of 0.600 m/s relative to the tracks. Find the magnitude and direction of the angular velocity of the disk relative to the earth.

10.94. A stiff uniform wire of mass M_0 and length L_0 is cut, bent, and the parts soldered together so that it forms a circular wheel having four identical spokes coming out from the center. None of the wire is wasted, and you can neglect the mass of the solder. (a) What is the moment of inertia of this wheel about an axle through its center perpendicular to the plane of the wheel? (b) If the wheel is given an initial spin with angular velocity ω_0, and stops uniformly in time T, what is the frictional torque at its axle?

10.95. In a physics laboratory you do the following ballistic pendulum experiment: You shoot a ball of mass m horizontally from a spring gun with a speed v. The ball is immediately caught a distance r below a frictionless pivot by a pivoted catcher assembly of mass M. The moment of inertia of this assembly about its rotation axis through the pivot is I. The distance r is much greater than the radius of the ball. (a) Use conservation of angular momentum to show that the angular speed of the ball and catcher just after the ball is caught is $\omega = mvr/(mr^2 + I)$. (b) After the ball is caught, the center of mass of the ball-catcher assembly system swings up with a maximum height increase h. Use conservation of energy to show that $\omega = \sqrt{2(M + m)gh/(mr^2 + I)}$. (c) Your lab partner says that linear momentum is conserved in the collision and derives the expression $mv = (m + M)V$, where V is the speed of the ball immediately after the collision. She then uses conservation of energy to derive that $V = \sqrt{2gh}$, so that $mv = (m + M)\sqrt{2gh}$. Use the results of parts (a) and (b) to show that this equation is satisfied only for the special case when r is given by $I = Mr^2$.

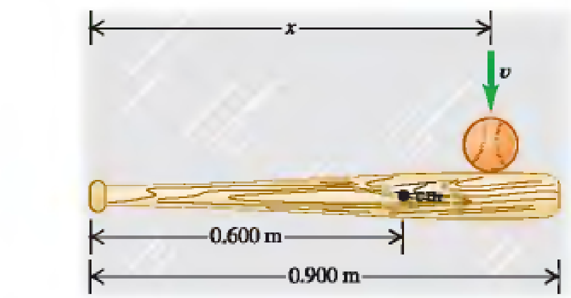
10.90. A 55-kg runner runs around the edge of a horizontal turntable mounted on a vertical, frictionless axis through its center. The runner's velocity relative to the earth has magnitude 2.8 m/s. The turntable is rotating in the opposite direction with an angular velocity of magnitude 0.20 rad/s relative to the earth. The radius of the turntable is 3.0 m, and its moment of inertia about the axis of rotation is 80 kg·m^2. Find the final angular velocity of the system if the runner comes to rest relative to the turntable. (You can model the runner as a particle.)

10.97. Recession of the Moon. Careful measurements of the earth–moon separation indicate that our satellite is presently moving away from us at approximately 3.0 cm per year. Neglect any angular momentum that the moon might be transferred from the earth to the moon. Calculate the rate of change (in rad/s per year) of the moon's angular velocity around the earth (consult Appendix E and the astronomical data in Appendix F). Is its angular velocity increasing or decreasing? (*Hint:* If L = constant, then $dL/dt = 0$.)

10.98. Center of Percussion. A baseball hat rests on a frictionless, horizontal surface. The bat has a length of 0.900 m, a mass of 0.800 kg, and its center of mass is 0.600 m from the handle end of the hat (Fig. 10.64). The moment of inertia of the hat about its center of mass is 0.0530 kg·m^2. The bat is struck by a baseball traveling perpendicular to the bat. The impact applies an impulse $J = \int_{t_1}^{t_2} F\,dt$ at a point a distance x from the handle end of the bat. What must x be so that the handle end of the bat remains at rest as the bat begins to move? [*Hint:* Consider the motion of the center of mass and the rotation about the center of mass. Find x so that these two motions combine to give $v = 0$ for the end of the bat just after the collision. Also, note that integration of Eq. (10.29) gives

$\Delta L = \int_{t_1}^{t_2}(\sum \tau)\,dt$ (see Problem 10.88).] The point on the bat you have located is called the *center of percussion*. Hitting a pitched ball at the center of percussion of the bat minimizes the "sting" the batter experiences on the hands.

Figure 10.64 Problem 10.98.



10.99. Consider a gyroscope with an axis that is not horizontal but is inclined from the horizontal by an angle β. Show that the precession angular frequency does not depend on the value of β but is given by Eq. (10.33).

Challenge Problems

10.100. A uniform ball of radius R rolls without slipping between two rails such that the horizontal distance is d between the two contact points of the rails to the ball. (a) In a sketch, show that at any instant $v_{cm} = \omega\sqrt{R^2 - d^2/4}$. Discuss this expression in the limits $d = 0$ and $d = 2R$. (b) For a uniform ball starting from rest and descending a vertical distance h while rolling without slipping down a ramp, $v_{cm} = \sqrt{10gh/7}$. Replacing the ramp with the two rails, show that

$$v_{cm} = \sqrt{\frac{10gh}{5 + 2/(1 - d^2/4R^2)}}$$

In each case, the work done by friction has been ignored. (c) Which speed in part (b) is smaller? Why? Answer in terms of how the loss of potential energy is shared between the gain in translational and rotational kinetic energies. (d) For which value of the ratio d/R do the two expressions for the speed in part (b) differ by 5.0%? By 0.50%?

10.101. When an object is rolling without slipping, the rolling friction force is much less than the friction force when the object is sliding; a silver dollar will roll on its edge much farther than it will slide on its flat side (see Section 5.3). When an object is rolling without slipping on a horizontal surface, we can approximate the friction force to be zero, so that a_x and α_z are approximately zero and v_x and ω_z are approximately constant. Rolling without slipping means $v_x = r\omega_z$ and $a_x = r\alpha_z$. If an object is set in motion on a surface *without* these equalities, sliding (kinetic) friction will act on the object as it slips until rolling without slipping is established. A solid cylinder with mass M and radius R, rotating with angular speed ω_0 about an axis through its center, is set on a horizontal surface for which the kinetic friction coefficient is μ_k. (a) Draw a free-body diagram for the cylinder on the surface. Think carefully about the direction of the kinetic friction force on the cylinder. Calculate the accelerations a_x of the center of mass and α_z of rotation about the center of mass. (b) The cylinder is initially slipping completely, so initially $\omega_z = \omega_0$ but $v_x = 0$. Rolling without slipping sets in when $v_x = R\omega_z$. Calculate the *distance* the cylinder rolls before slipping stops. (c) Calculate the work done by the friction force on the cylinder as it moves from where it was set down to where it begins to roll without slipping.

10.102. A demonstration gyroscope wheel is constructed by removing the tire from a bicycle wheel 0.650 m in diameter, wrapping lead wire around the rim, and taping it in place. The shaft projects 0.200 m at each side of the wheel, and a woman holds the ends of the shaft in her hands. The mass of the system is 8.00 kg; its entire mass may be assumed to be located at its rim. The shaft is horizontal, and the wheel is spinning about the shaft at 5.00 rev/s. Find the magnitude and direction of the force each hand exerts on the shaft (a) when the shaft is at rest; (b) when the shaft is rotating in a horizontal plane about its center at 0.050 rev/s; (c) when the shaft is rotating in a horizontal plane about its center at 0.300 rev/s. (d) At what rate must the shaft rotate in order that it may be supported at one end only?

10.103. A block with mass m is revolving with linear speed v_1 in a circle of radius r_1 on a frictionless horizontal surface (see Fig. 10.48). The string is slowly pulled from below until the radius of the circle in which the block is revolving is reduced to r_2. (a) Calculate the tension T in the string as a function of r, the distance of the block from the hole. Your answer will be in terms of the initial velocity v_1 and the radius r_1. (b) Use $W = \int_{r_1}^{r_2}\vec{T}(r)\cdot d\vec{r}$ to calculate the work done by $\vec{T}$ when r changes from r_1 to r_2. (c) Compare the results of part (b) to the change in the kinetic energy of the block.

11 EQUILIBRIUM AND ELASTICITY

LEARNING GOALS

By studying this chapter, you will learn:

- The conditions that must be satisfied for a body or structure to be in equilibrium.
- What is meant by the center of gravity of a body, and how it relates to the body's stability.
- How to solve problems that involve rigid bodies in equilibrium.
- How to analyze situations in which a body is deformed by tension, compression, pressure, or shear.
- What happens when a body is stretched so much that it deforms or breaks.

? This Roman aqueduct uses the principle of the arch to sustain the weight of the structure and the water it carries. Are the blocks that make up the arch being compressed, stretched, or a combination?

We've devoted a good deal of effort to understanding why and how bodies accelerate in response to the forces that act on them. But very often we're interested in making sure that bodies *don't* accelerate. Any building, from a multistory skyscraper to the humblest shed, must be designed so that it won't topple over. Similar concerns arise with a suspension bridge, a ladder leaning against a wall, or a crane hoisting a bucket full of concrete.

A body that can be modeled as a *particle* is in equilibrium whenever the vector sum of the forces acting on it is zero. But for the situations we've just described, that condition isn't enough. If forces act at different points on an extended body, an additional requirement must be satisfied to ensure that the body has no tendency to *rotate:* The sum of the *torques* about any point must be zero. This requirement is based on the principles of rotational dynamics developed in Chapter 10. We can compute the torque due to the weight of a body using the concept of center of gravity, which we introduce in this chapter.

Rigid bodies don't bend, stretch, or squash when forces act on them. But the rigid body is an idealization; all real materials are *elastic* and do deform to some extent. Elastic properties of materials are tremendously important. You want the wings of an airplane to be able to bend a little, but you'd rather not have them break off. The steel frame of an earthquake-resistant building has to be able to flex, but not too much. Many of the necessities of everyday life, from rubber bands to suspension bridges, depend on the elastic properties of materials. In this chapter we'll introduce the concepts of *stress, strain,* and *elastic modulus* and a simple principle called *Hooke's law* that helps us predict what deformations will occur when forces are applied to a real (not perfectly rigid) body.

11.1 Conditions for Equilibrium

We learned in Sections 4.2 and 5.1 that a particle is in ***equilibrium***—that is, the particle does not accelerate—in an inertial frame of reference if the vector sum of all the forces acting on the particle is zero, $\sum \vec{F} = \mathbf{0}$. For an *extended* body, the equivalent statement is that the center of mass of the body has zero acceleration if the vector sum of all external forces acting on the body is zero, as discussed in Section 8.5. This is often called the **first condition for equilibrium.** In vector and component forms,

$$\sum \vec{F} = \mathbf{0} \qquad \text{(first condition for equilibrium)} \qquad (11.1)$$
$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum F_z = 0$$

where the sum includes *external* forces only.

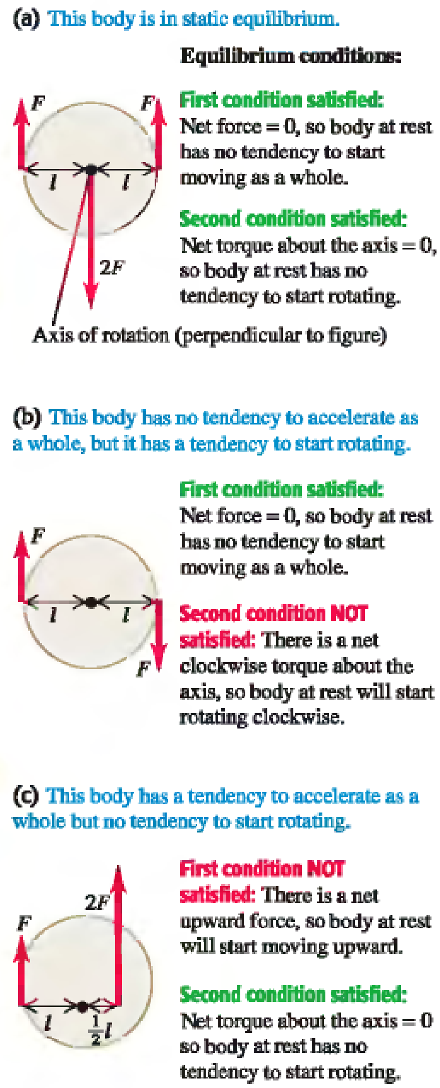
A second condition for an extended body to be in equilibrium is that the body must have no tendency to *rotate*. This condition is based on the dynamics of rotational motion in exactly the same way that the first condition is based on Newton's first law. A rigid body that, in an inertial frame, is not rotating about a certain point has zero angular momentum about that point. If it is not to start rotating about that point, the rate of change of angular momentum must *also* be zero. From the discussion in Section 10.5, particularly Eq. (10.29), this means that the sum of torques due to all the external forces acting on the body must be zero. A rigid body in equilibrium can't have any tendency to start rotating about *any* point, so the sum of external torques must be zero about any point. This is the **second condition for equilibrium:**

$$\sum \vec{\tau} = \mathbf{0} \quad \text{about any point} \qquad \text{(second condition for equilibrium)} \qquad (11.2)$$

The sum of the torques due to all external forces acting on the body, with respect to any specified point, must be zero.

In this chapter we will apply the first and second conditions for equilibrium to situations in which a rigid body is at rest (no translation or rotation). Such a body is said to be in **static equilibrium** (Fig. 11.1). But the same conditions apply to a rigid body in uniform *translational* motion (without rotation), such as an airplane in flight with constant speed, direction, and altitude. Such a body is in equilibrium but is not static.

11.1 To be in static equilibrium, a body at rest must satisfy *both* conditions for equilibrium: It can have no tendency to accelerate as a whole or to start rotating.



Test Your Understanding of Section 11.1 Which situation satisfies both the first and second conditions for equilibrium? (i) a seagull gliding at a constant angle below the horizontal and at a constant speed; (ii) an automobile crankshaft turning at an increasing angular speed in the engine of a parked car; (iii) a thrown baseball that does not rotate as it sails through the air.

11.2 Center of Gravity

In most equilibrium problems, one of the forces acting on the body is its weight. We need to be able to calculate the *torque* of this force. The weight doesn't act at a single point; it is distributed over the entire body. But we can always calculate the torque due to the body's weight by assuming that the entire force of gravity (weight) is concentrated at a point called the **center of gravity** (abbreviated "cg"). The acceleration due to gravity decreases with altitude; but if we can ignore this variation over the vertical dimension of the body, then the body's center of gravity is identical to its *center of mass* (abbreviated "cm"), which we defined in Section 8.5. We stated this result without proof in Section 10.2, and now we'll prove it.

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7.2 A Tilted Beam: Torques and Equilibrium

7.3 Arm Levers

First let's review the definition of the center of mass. For a collection of particles with masses $m_1, m_2, \ldots$ and coordinates (x_1, y_1, z_1), (x_2, y_2, z_2), ..., the coordinates x_{cm}, y_{cm}, and z_{cm} of the center of mass are given by

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

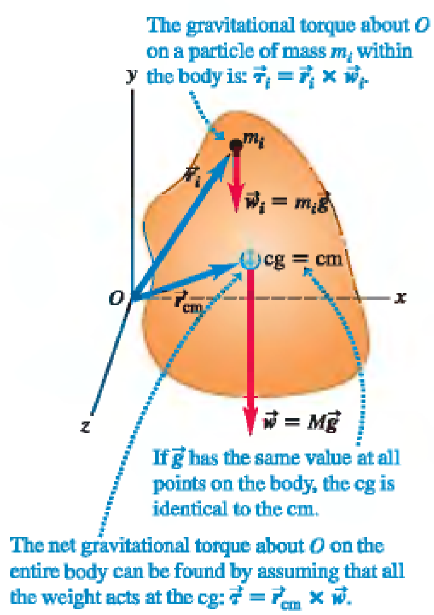
$$y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i y_i}{\sum_i m_i} \qquad \text{(center of mass)} \qquad (11.3)$$

$$z_{cm} = \frac{m_1z_1 + m_2z_2 + m_3z_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i z_i}{\sum_i m_i}$$

Also, x_{cm}, y_{cm}, and z_{cm} are the components of the position vector $\vec{r}_{cm}$ of the center of mass, so Eqs. (11.3) are equivalent to the vector equation

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} \qquad (11.4)$$

11.2 The center of gravity (cg) and center of mass (cm) of an extended body.



Now let's consider the gravitational torque on a body of arbitrary shape (Fig. 11.2). We assume that the acceleration due to gravity $\vec{g}$ has the same magnitude and direction at every point in the body. Every particle in the body experiences a gravitational force, and the total weight of the body is the vector sum of a large number of parallel forces. A typical particle has mass m_i and weight $\vec{w}_i = m_i\vec{g}$. If $\vec{r}_i$ is the position vector of this particle with respect to an arbitrary origin O, then the torque vector $\vec{\tau}_i$ of the weight $\vec{w}_i$ with respect to O is, from Eq. (10.3),

$$\vec{\tau}_i = \vec{r}_i \times \vec{w}_i = \vec{r}_i \times m_i\vec{g}$$

The *total* torque due to the gravitational forces on all the particles is

$$\begin{aligned}\vec{\tau} &= \sum_i \vec{\tau}_i = \vec{r}_1 \times m_1\vec{g} + \vec{r}_2 \times m_2\vec{g} + \cdots \\ &= (m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots) \times \vec{g} \\ &= \left(\sum_i m_i\vec{r}_i\right) \times \vec{g}\end{aligned}$$

When we multiply and divide this by the total mass of the body,

$$M = m_1 + m_2 + \cdots = \sum_i m_i$$

we get

$$\vec{\tau} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots}{m_1 + m_2 + \cdots} \times M\vec{g} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i} \times M\vec{g}$$

The fraction in this equation is just the position vector $\vec{r}_{cm}$ of the center of mass, with components x_{cm}, y_{cm}, and z_{cm}, as given by Eq. (11.4), and $M\vec{g}$ is equal to the total weight $\vec{w}$ of the body. Thus

$$\vec{\tau} = \vec{r}_{cm} \times M\vec{g} = \vec{r}_{cm} \times \vec{w} \qquad (11.5)$$

The total gravitational torque, given by Eq. (11.5), is the same as though the total weight $\vec{w}$ were acting on the position $\vec{r}_{cm}$ of the center of mass, which we also call the *center of gravity.* **If $\vec{g}$ has the same value at all points on a body, its center of gravity is identical to its center of mass.** Note, however, that the center of mass is defined independently of any gravitational effect.

While the value of $\vec{g}$ does vary somewhat with elevation, the variation is extremely slight (Fig. 11.3). Hence we will assume throughout this chapter that the center of gravity and center of mass are identical unless explicitly stated otherwise.

11.3 The acceleration due to gravity at the bottom of the 452-m-tall Petronas Towers in Malaysia is only 0.014% greater than at the top. The center of gravity of the towers is only about 2 cm below the center of mass.

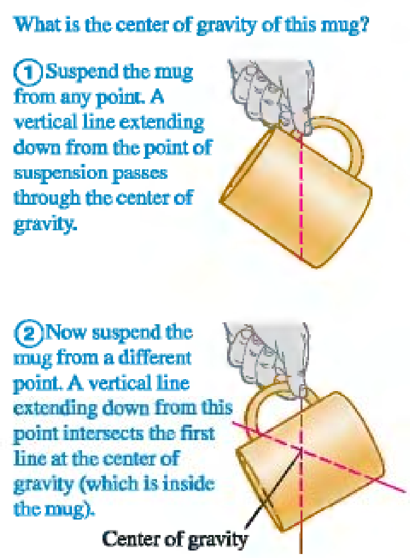
Finding and Using the Center of Gravity

We can often use symmetry considerations to locate the center of gravity of a body, just as we did for the center of mass. The center of gravity of a homogeneous sphere, cube, circular sheet, or rectangular plate is at its geometric center. The center of gravity of a right circular cylinder or cone is on its axis of symmetry.

For a body with a more complex shape, we can sometimes locate the center of gravity by thinking of the body as being made of symmetrical pieces. For example, we could approximate the human body as a collection of solid cylinders, with a sphere for the head. Then we can compute the coordinates of the center of gravity of the combination from Eqs. (11.3), letting $m_1, m_2, \ldots$ be the masses of the individual pieces and $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots$ be the coordinates of their centers of gravity.

When a body acted on by gravity is supported or suspended at a single point, the center of gravity is always at or directly above or below the point of suspension. If it were anywhere else, the weight would have a torque with respect to the point of suspension, and the body could not be in rotational equilibrium. Figure 11.4 shows how to use this fact to determine experimentally the location of the center of gravity of an irregular body.

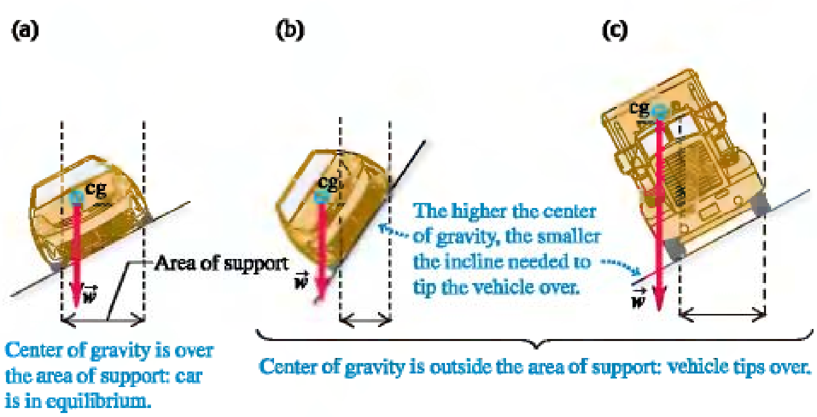
11.4 Finding the center of gravity of an irregularly shaped body—in this case, a coffee mug.



Using the same reasoning, we can see that a body supported at several points must have its center of gravity somewhere within the area bounded by the supports. This explains why a car can drive on a straight but slanted road if the slant angle is relatively small (Fig. 11.5a) but will tip over if the angle is too steep (Fig. 11.5b). The truck in Fig. 11.5c has a higher center of gravity than the car and will tip over on a shallower incline. When a truck overturns on a highway and blocks traffic for hours, it's the high center of gravity that's to blame.

The lower the center of gravity and the larger the area of support, the more difficult it is to overturn a body. Four-legged animals such as deer and horses have a large area of support bounded by their legs; hence they are naturally stable and need only small feet or hooves. Animals that walk erect on two legs, such as

11.5 In (a) the center of gravity is within the area bounded by the supports, and the car is in equilibrium. The car in (b) and the truck in (c) will tip over because their centers of gravity lie outside the area of support.



humans and birds, need relatively large feet to give them a reasonable area of support. If a two-legged animal holds its body approximately horizontal, like a chicken or the dinosaur *Tyrannosaurus rex,* it must perform a delicate balancing act as it walks to keep its center of gravity over the foot that is on the ground. A chicken does this by moving its head; *T. rex* probably did it by moving its massive tail.

Example 11.1 Walking the plank

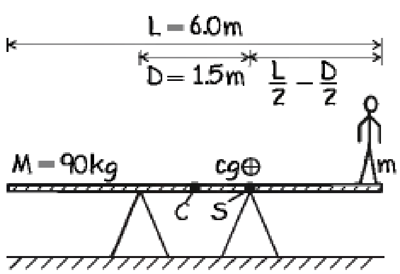
A uniform wooden plank of length $L = 6.0$ m and mass $M = 90$ kg rests on top of two sawhorses separated by $D = 1.5$ m, located equal distances from the center of the plank. Your cousin Throckmorton tries to stand on the right-hand end of the plank. If the plank is to remain at rest, how massive can Throckmorton be?

SOLUTION

IDENTIFY: If the system of plank and Throckmorton is just in balance, the center of gravity of this system will be directly over the right-hand sawhorse (just barely within the area bounded by the two supports). The target variable is Throcky's mass.

SET UP: Figure 11.6 shows our sketch. We take the origin at *C*, the geometric center and center of gravity of the uniform plank, and take the positive *x*-axis to point horizontally to the right. Then the *x*-coordinates of the centers of gravity of the plank (mass *M*) and Throcky (unknown mass *m*) are $x_P = 0$ and $x_T = L/2 = 3.0$ m, respectively. We will use Eqs. (11.3) to locate the center of gravity of the system of plank and Throcky.

11.6 Our sketch for this problem.



EXECUTE: From the first of Eqs. (11.3),

$$x_{cg} = \frac{M(0) + m(L/2)}{M + m} = \frac{m}{M + m}\frac{L}{2}$$

Setting this equal to $D/2$, the *x*-coordinate of the right-hand sawhorse, we have

$$\frac{m}{M + m}\frac{L}{2} = \frac{D}{2}$$
$$mL = (M + m)D$$
$$m = M\frac{D}{L - D} = (90\text{ kg})\frac{1.5\text{ m}}{6.0\text{ m} - 1.5\text{ m}}$$
$$= 30\text{ kg}$$

EVALUATE: To check our result, let's repeat the calculation with a different choice of origin. Now we take the origin to be at *S*, the position of the right-hand sawhorse, so that $x_{cg} = 0$. The centers of gravity of the plank and Throcky are now at $x_P = -D/2$ and $x_T = (L/2) - (D/2)$, respectively, so

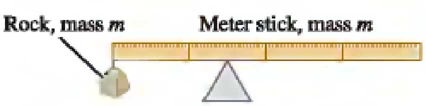
$$x_{cg} = \frac{M(-D/2) + m[(L/2) - (D/2)]}{M + m} = 0$$

$$m = \frac{MD/2}{(L/2) - (D/2)} = M\frac{D}{L - D} = 30\text{ kg}$$

The mass doesn't depend on our arbitrary choice of origin.

A 60-kg child could stand only halfway between the right-hand sawhorse and the end of the plank. Can you see why?

11.7 At what point will the meter stick with rock attached be in balance?



Test Your Understanding of Section 11.2 A rock is attached to the left end of a uniform meter stick that has the same mass as the rock. In order for the combination of rock and meter stick to balance atop the triangular object in Fig. 11.7, how far from the left end of the stick should the triangular object be placed? (i) less than 0.25 m; (ii) 0.25 m; (iii) between 0.25 m and 0.50 m; (iv) 0.50 m; (v) more than 0.50 m.

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7.4 Two Painters on a Beam
7.5 Lecturing from a Beam

11.3 Solving Rigid-Body Equilibrium Problems

There are just two key conditions for rigid-body equilibrium: The vector sum of the forces on the body must be zero, and the sum of the torques about any point must be zero. To keep things simple, we'll restrict our attention to situations in which we can treat all forces as acting in a single plane, which we'll call the *xy*-plane. Then we can ignore the condition $\sum F_z = 0$ in Eqs. (11.1), and in Eq. (11.2) we need consider only the z-components of torque (perpendicular to the plane). The first and second conditions for equilibrium are then

$$\sum F_x = 0 \quad \text{and} \quad \sum F_y = 0 \quad \text{(first condition for equilibrium, forces in } xy\text{-plane)}$$
$$\sum \tau_z = 0 \quad \text{(second condition for equilibrium, forces in } xy\text{-plane)} \tag{11.6}$$

CAUTION **Choosing the reference point for calculating torques** In equilibrium problems, the choice of reference point for calculating torques in $\sum \tau_z$ is completely arbitrary. But once you make your choice, you must use the *same* point to calculate *all* the torques on a body. It helps to pick the point so as to simplify the calculations as much as possible.

The challenge is to apply these simple conditions to specific problems. Problem-Solving Strategy 11.1 is very similar to the suggestions given in Section 5.2 for the equilibrium of a particle. You should compare it with Problem-Solving Strategy 10.1 (Section 10.2) for rotational dynamics problems.

Problem-Solving Strategy 11.1 Equilibrium of a Rigid Body

MP

IDENTIFY *the relevant concepts:* The first and second conditions for equilibrium are useful whenever there is a rigid body that is not rotating and not accelerating in space.

SET UP *the problem* using the following steps:
1. Draw a sketch of the physical situation, including dimensions, and select the body in equilibrium to be analyzed.
2. Draw a free-body diagram showing the forces acting *on* the selected body and no others. *Do not* include forces exerted *by* this body on other bodies. Be careful to show correctly the point at which each force acts; this is crucial for correct torque calculations. You can't represent a rigid body as a point.
3. Choose coordinate axes and specify a positive direction of rotation for torques. Represent forces in terms of their components with respect to the axes you have chosen; when you do this, cross out the original force so that you don't include it twice.
4. In choosing a point about which to compute torques, note that if a force has a line of action that goes *through* a particular point, the torque of the force with respect to that point is zero. You can often eliminate unknown forces or components from the torque equation by a clever choice of point for your calculation. The body doesn't actually have to be pivoted about an axis through the chosen point.

EXECUTE *the solution* as follows:
1. Write equations expressing the equilibrium conditions. Remember that $\sum F_x = 0$, $\sum F_y = 0$, and $\sum \tau_z = 0$ are always separate equations; *never* add x- and y-components in a single equation. Also remember that when a force is represented in terms of its components, you can compute the torque of that force by finding the torque of each component separately, each with its appropriate lever arm and sign, and adding the results. This is often easier than determining the lever arm of the original force.
2. You always need as many equations as you have unknowns. Depending on the number of unknowns, you may need to compute torques with respect to two or more axes to obtain enough equations. Often, there are several equally good sets of force and torque equations for a particular problem; there is usually no single "right" combination of equations.

EVALUATE *your answer:* A useful way to check your results is to rewrite the second condition for equilibrium, $\sum \tau_z = 0$, using a different choice of origin. If you've done everything correctly, you'll get the same answers using this new choice of origin as you did with your original choice.

Example 11.2 Weight distribution for a car

An auto magazine reports that a certain sports car has 53% of its weight on the front wheels and 47% on its rear wheels, with a 2.46-m wheelbase. This means that the total normal force on the front wheels is $0.53w$ and that on the rear wheels is $0.47w$, where w is the total weight. The wheelbase is the distance between the front and rear axles. How far in front of the rear axle is the car's center of gravity?

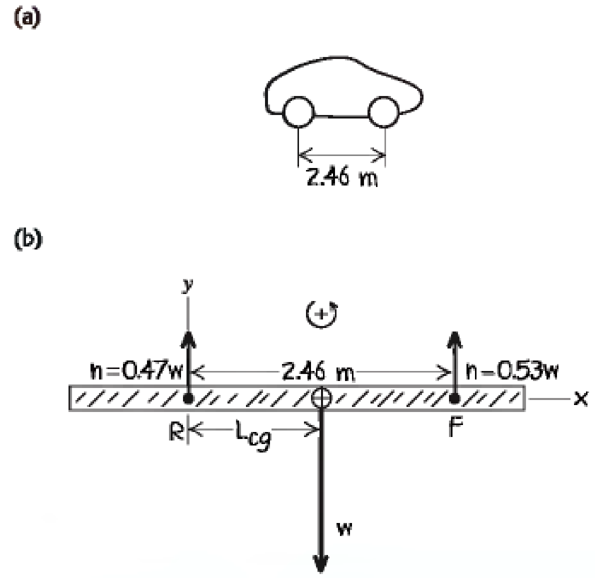
SOLUTION

IDENTIFY: We can use the two conditions for equilibrium, since the car is assumed to be at rest. The conditions also apply when the car is traveling in a straight line at constant speed, since the net force and net torque on the car are also zero in that situation. The target variable is the coordinate of the car's center of gravity.

Continued

SET UP: Figure 11.8 shows our sketch and a free-body diagram for the car, including x- and y-axes and our convention that counterclockwise torques are positive. The weight w acts at the center of gravity. The distance we want is L_{cg}; this is the lever arm of the weight with respect to the rear axle R, so it is reasonable to take torques with respect to R. The torque due to the weight is negative because it tends to cause a clockwise rotation about R. The torque due to the upward normal force at the front axle F is positive because it tends to cause a counterclockwise rotation about R.

11.8 Our sketches for this problem.



EXECUTE: You can see from Fig. 11.8b that the first condition for equilibrium is satisfied: $\sum F_x = 0$ because there aren't any x-components of force and $\sum F_y = 0$ because $0.47w + 0.53w + (-w) = 0$. The force equation doesn't involve the target variable L_{cg}, so we must solve for it using the torque equation for point R:

$$\sum \tau_R = 0.47w(0) - wL_{cg} + 0.53w(2.46\text{ m}) = 0$$
$$L_{cg} = 1.30\text{ m}$$

EVALUATE: Note that the cg is between the two supports, as it must be (see Section 11.2). You can check the numerical result for the cg position by writing the torque equation about the front axle F. You'll find that the cg is 1.16 m behind the front axle, or $(2.46\text{ m}) - (1.16\text{ m}) = 1.30$ m in front of the rear axle.

You can show that if f is the fraction of the weight on the front wheels and d is the wheelbase, the center of gravity is a distance fd in front of the rear wheels. The farther back the center of gravity, the smaller the value of fd and the smaller the fraction of weight on the front wheels. That's why owners of rear-wheel-drive vehicles put bags of sand in their trunks to improve traction on snow and ice. Would this strategy help with a front-wheel-drive car?

Example 11.3 A heroic rescue

Sir Lancelot is trying to rescue the Lady Elayne from Castle Von Doom by climbing a uniform ladder that is 5.0 m long and weighs 180 N. Lancelot, who weighs 800 N, stops a third of the way up the ladder (Fig. 11.9a). The bottom of the ladder rests on a horizontal stone ledge and leans across the moat in equilibrium against a vertical wall that is frictionless because of a thick layer of moss. The ladder makes an angle of 53.1° with the horizontal, conveniently forming a 3-4-5 right triangle. (a) Find the normal and friction forces on the ladder at its base. (b) Find the minimum coefficient of static friction needed to prevent slipping at the base. (c) Find the magnitude and direction of the contact force on the ladder at the base.

SOLUTION

IDENTIFY: The system of ladder and Lancelot is stationary, so we can use the two conditions for equilibrium to solve part (a). In part (b), we also need the relationship given in Section 5.3 among the static friction force, the coefficient of static friction, and the normal force. The contact force asked for in part (c) is the vector sum of the normal and friction forces acting at the base of the ladder, which we find in part (a).

SET UP: Figure 11.9b shows the free-body diagram for the system of the ladder and Lancelot. We choose the x- and y-directions as shown and take counterclockwise torques to be positive. The ladder is uniform, so its center of gravity is at its geometric center. Lancelot's 800-N weight acts at a point on the ladder one-third of the way from the base toward the wall.

The frictionless wall exerts only a normal force n_1 at the top of the ladder. The forces at the base are the upward normal force n_2 and the static friction force f_s, which must point to the right to prevent slipping; the magnitudes n_2 and f_s are the target variables in part (a). From Eq. (5.6), these magnitudes are related by $f_s \le \mu_s n_2$, where μ_s is the coefficient of static friction, the target variable in part (b).

EXECUTE: (a) From Eqs. (11.6), the first condition for equilibrium gives

$$\sum F_x = f_s + (-n_1) = 0$$
$$\sum F_y = n_2 + (-800\text{ N}) + (-180\text{ N}) = 0$$

These are two equations for the three unknowns n_1, n_2, and f_s. The first equation tells us that the two horizontal forces must be equal and opposite, and the second equation gives

$$n_2 = 980\text{ N}$$

The ground pushes up with a force of 980 N to balance the total (downward) weight $(800\text{ N} + 180\text{ N})$.

We don't yet have enough equations, but now we can use the second condition for equilibrium. We can take torques about any point we choose. The smart choice is point B, which gives us the fewest terms and fewest unknowns in the torque equation. That's because the two forces n_2 and f_s have no torque about that point. From Fig. 11.9b we see that the lever arm for the ladder's weight is 1.5 m, the lever arm for Lancelot's weight is 1.0 m, and the lever arm for n_1 is 4.0 m. The torque equation for point B is

$$\sum \tau_B = n_1(4.0\text{ m}) - (180\text{ N})(1.5\text{ m}) - (800\text{ N})(1.0\text{ m}) + n_2(0) + f_s(0) = 0$$

Solving for n_1, we get $n_1 = 268$ N. We now substitute this back into the $\sum F_x = 0$ equation to get

$$f_s = 268\text{ N}$$

(b) The static friction force f_s cannot exceed $\mu_s n_2$, so the *minimum* coefficient of static friction to prevent slipping is

$$(\mu_s)_{\min} = \frac{f_s}{n_2} = \frac{268\text{ N}}{980\text{ N}} = 0.27$$

(c) The components of the contact force $\vec{F}_B$ at the base are the static friction force f_s and the normal force n_2, so

$$\vec{F}_B = f_s\hat{\imath} + n_2\hat{\jmath} = (268\text{ N})\hat{\imath} + (980\text{ N})\hat{\jmath}$$

The magnitude and direction of $\vec{F}_B$ (Fig. 11.9c) are then

$$F_B = \sqrt{(268\text{ N})^2 + (980\text{ N})^2} = 1020\text{ N}$$

$$\theta = \arctan\frac{980\text{ N}}{268\text{ N}} = 75°$$

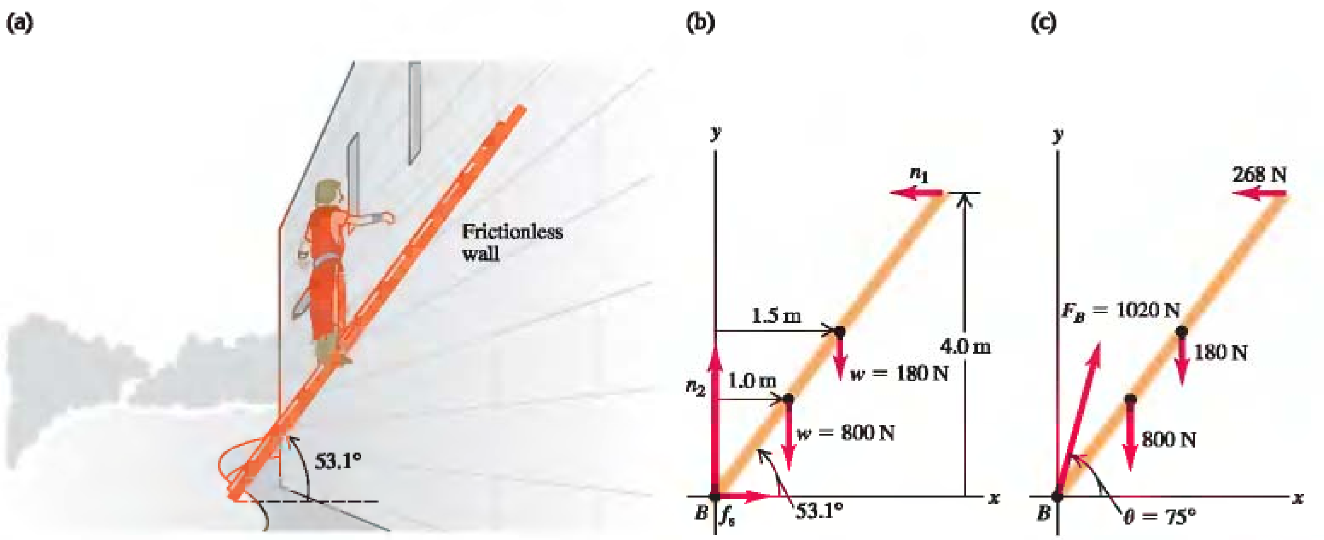
EVALUATE: As Fig. 11.9c shows, the contact force $\vec{F}_B$ is *not* directed along the length of the ladder. You may be surprised by this, but there's really no good reason the two directions should be the same. Can you show that if $\vec{F}_B$ were directed along the ladder, there would be a net counterclockwise torque with respect to the top of the ladder, and equilibrium would be impossible?

Here are a few final comments. First, as Lancelot climbs higher on the ladder, the lever arm and torque of his weight about B increase; this increases the values of n_1, f_s, and $(\mu_s)_{\min}$. At the top, his lever arm would be nearly 3 m, giving a minimum coefficient of static friction of nearly 0.7. The value of μ_s would not be this large for Lancelot's medieval ladder, so his ladder is likely to slip as he climbs. To prevent this, present-day ladders are usually equipped with nonslip rubber pads.

Second, a larger ladder angle would decrease the lever arms with respect to B of the weights of the ladder and Lancelot and increase the lever arm of n_1, all of which would decrease the required friction force. The R. D. Werner Ladder Co. recommends that its ladders be used at an angle of 75°. (Why not 90°?)

Finally, if we had assumed friction on the wall as well as on the floor, the problem would be impossible to solve by using the equilibrium conditions alone. (Try it!) Such a problem is said to be *statically indeterminate*. The difficulty is that it's no longer adequate to treat the body as being perfectly rigid. Another simple example of such a problem is a four-legged table; there is no way to use the equilibrium conditions alone to find the force on each separate leg.

11.9 (a) Sir Lancelot pauses a third of the way up the ladder, fearing it will slip. (b) Free-body diagram for the system of Sir Lancelot and the ladder. (c) The contact force at B is the superposition of the normal force and the static friction force.



Example 11.4 Equilibrium and pumping iron

Figure 11.10a shows a horizontal human arm lifting a dumbbell. The forearm is in equilibrium under the action of the weight w of the dumbbell, the tension T in the tendon connected to the biceps muscle, and the force E exerted on the forearm by the upper arm at the elbow joint. For clarity the point A where the tendon is attached is drawn farther away from the elbow than its actual position. The weight w and the angle θ between the tension force and the horizontal are given; we want to find the tendon tension and the two components of force at the elbow (three unknown scalar quantities in all). We neglect the weight of the forearm itself.

SOLUTION

IDENTIFY: The system is at rest, so once again we use the conditions for equilibrium.

SET UP: As Fig. 11.10b shows, we represent the tendon force in terms of its components T_x and T_y, using the given angle θ and the unknown magnitude T:

$$T_x = T\cos\theta \qquad T_y = T\sin\theta$$

We also represent the force at the elbow in terms of its components E_x and E_y. We'll guess that the directions of these components are

Continued

as shown in Fig. 11.10b; there's no need to agonize over this guess, since the results for E_x and E_y will tell us the actual directions. Our target variables are the magnitude T of the tendon tension and the components E_x and E_y of the force at the elbow.

EXECUTE: The simplest way to find the tension T is to take torques about the elbow joint. The resulting torque equation does not contain E_x, E_y, or T_x because the lines of action of all these forces pass through this point. The torque equation is then simply

$$\sum \tau_E = Lw - DT_y = 0$$

From this we find

$$T_y = \frac{Lw}{D} \qquad \text{and} \qquad T = \frac{Lw}{D\sin\theta}$$

To find E_x and E_y, we use the first conditions for equilibrium, $\sum F_x = 0$ and $\sum F_y = 0$:

$$\sum F_x = T_x + (-E_x) = 0$$

$$E_x = T_x = T\cos\theta = \frac{Lw}{D\sin\theta}\cos\theta = \frac{Lw}{D}\cot\theta$$

$$= \frac{Lw}{D}\frac{D}{h} = \frac{Lw}{h}$$

$$\sum F_y = T_y + E_y + (-w) = 0$$

$$E_y = w - \frac{Lw}{D} = -\frac{(L-D)w}{D}$$

The negative sign shows that our guess for the direction of E_y, shown in Fig. 11.10b, was wrong; it is actually vertically *downward*.

EVALUATE: We can check our results by finding E_x and E_y in a different way that uses two more torque equations. We take torques about the tendon attach point, A:

$$\sum \tau_A = (L-D)w + DE_y = 0 \qquad \text{and} \qquad E_y = -\frac{(L-D)w}{D}$$

Finally, we take torques about point B in the figure:

$$\sum \tau_B = Lw - hE_x = 0 \qquad \text{and} \qquad E_x = \frac{Lw}{h}$$

We chose points A and B because the tendon tension T has zero torque about either of these points. (Can you see why from Fig. 11.10b?) Notice how much we have simplified these calculations by choosing the point for calculating torques so as to eliminate one or more of the unknown quantities.

In our alternative determination of E_x and E_y, we didn't explicitly use the first condition for equilibrium (that the vector sum of the forces is zero). As a consistency check, you should compute $\sum F_x$ and $\sum F_y$ to verify that they really *are* zero!

As a specific example, suppose $w = 200\ \text{N}$, $D = 0.050\ \text{m}$, $L = 0.30\ \text{m}$, and $\theta = 80°$. Then from $\tan\theta = h/D$, we find

$$h = D\tan\theta = (0.050\ \text{m})(5.67) = 0.28\ \text{m}$$

From the previous general results we find

$$T = \frac{Lw}{D\sin\theta} = \frac{(0.30\ \text{m})(200\ \text{N})}{(0.050\ \text{m})(0.98)} = 1220\ \text{N}$$

$$E_y = -\frac{(L-D)w}{D} = -\frac{(0.30\ \text{m} - 0.050\ \text{m})(200\ \text{N})}{0.050\ \text{m}}$$

$$= -1000\ \text{N}$$

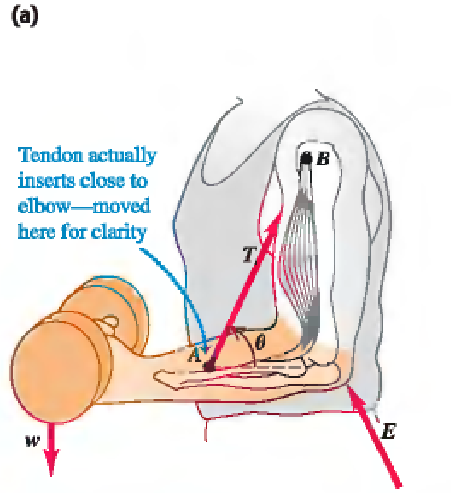
$$E_x = \frac{Lw}{h} = \frac{(0.30\ \text{m})(200\ \text{N})}{0.28\ \text{m}} = 210\ \text{N}$$

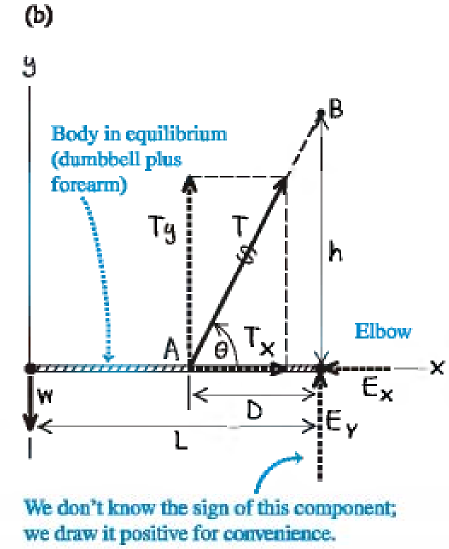
The magnitude of the force at the elbow is

$$E = \sqrt{E_x^2 + E_y^2} = 1020\ \text{N}$$

In view of the magnitudes of our results, neglecting the weight of the forearm itself, which may be 20 N or so, will cause only relatively small errors in our results.

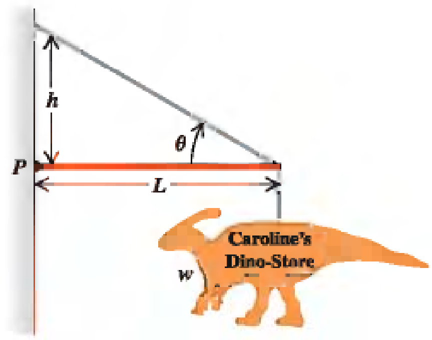
11.10 (a) The situation. (b) Our free-body diagram for the forearm. The weight of the forearm is neglected, and the distance D is greatly exaggerated for clarity.





Test Your Understanding of Section 11.3 A metal advertising sign (weight w) for a specialty shop is suspended from the end of a horizontal rod of length L and negligible mass (Fig. 11.11). The rod is supported by a cable at an angle θ from the horizontal and by a hinge at point P. Rank the following force magnitudes in order from greatest to smallest: (i) the weight w of the sign; (ii) the tension in the cable; (iii) the vertical component of force exerted on the rod by the hinge at P.

11.11 What are the tension in the diagonal cable and the force exerted by the hinge at P?



11.4 Stress, Strain, and Elastic Moduli

The rigid body is a useful idealized model, but the stretching, squeezing, and twisting of real bodies when forces are applied are often too important to ignore. Figure 11.12 shows three examples. We want to study the relationship between the forces and deformations for each case.

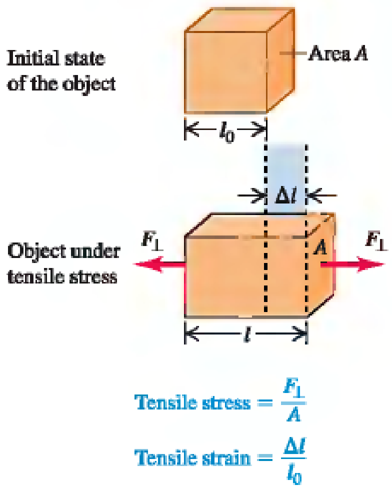
For each kind of deformation we will introduce a quantity called **stress** that characterizes the strength of the forces causing the deformation, on a "force per unit area" basis. Another quantity, **strain,** describes the resulting deformation. When the stress and strain are small enough, we often find that the two are directly proportional, and we call the proportionality constant an **elastic modulus.** The harder you pull on something, the more it stretches; the more you squeeze it, the more it compresses. We can express this relationship as an equation:

$$\frac{\text{Stress}}{\text{Strain}} = \text{Elastic modulus} \qquad \text{(Hooke's law)} \tag{11.7}$$

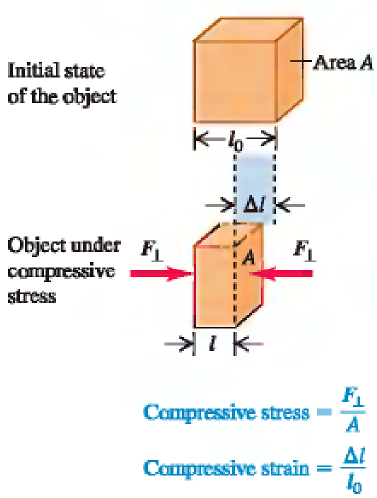
The proportionality of stress and strain (under certain conditions) is called **Hooke's law,** after Robert Hooke (1635–1703), a contemporary of Newton. We used one form of Hooke's law in Sections 6.3 and 7.2: The elongation of an ideal spring is proportional to the stretching force. Remember that Hooke's law is not really a general law but an experimental finding that is valid over only a limited range. The last section of this chapter discusses what this limited range is.

11.12 Three types of stress. **(a)** Bridge cables under *tensile stress,* being stretched by forces acting at their ends. **(b)** A diver under *bulk stress,* being squeezed from all sides by forces due to water pressure. **(c)** A ribbon under *shear stress,* being deformed and eventually cut by forces exerted by the scissors.

11.13 An object in tension. The net force on the object is zero, but the object deforms. The tensile stress (the ratio of the force to the cross-sectional area) produces a tensile strain (the elongation divided by the initial length). The elongation Δl is exaggerated for clarity.



11.14 An object in compression. The compressive stress and compressive strain are defined in the same way as tensile stress and strain (see Fig. 11.13), except that Δl now denotes the distance that the object contracts.



Tensile and Compressive Stress and Strain

The simplest elastic behavior to understand is the stretching of a bar, rod, or wire when its ends are pulled (Fig. 11.12a). Figure 11.13 shows an object that initially has uniform cross-sectional area A and length l_0. We then apply forces of equal magnitude $F_\perp$ but opposite directions at the ends (this ensures that the object has no tendency to move left or right). We say that the object is in **tension.** We've already talked a lot about tension in ropes and strings; it's the same concept here. The subscript $\perp$ is a reminder that the forces act perpendicular to the cross section.

We define the **tensile stress** at the cross section as the ratio of the force $F_\perp$ to the cross-sectional area A:

$$\text{Tensile stress} = \frac{F_\perp}{A} \tag{11.8}$$

This is a *scalar* quantity because $F_\perp$ is the *magnitude* of the force. The SI unit of stress is the **pascal** (abbreviated Pa and named for the 17th-century French scientist and philosopher Blaise Pascal). Equation (11.8) shows that 1 pascal equals 1 newton per square meter (N/m^2):

$$1\ \text{pascal} = 1\ \text{Pa} = 1\ \text{N/m}^2$$

In the British system the logical unit of stress would be the pound per square foot, but the pound per square inch (lb/in.^{2} or psi) is more commonly used. The conversion factors are

$$1\ \text{psi} = 6895\ \text{Pa} \qquad \text{and} \qquad 1\ \text{Pa} = 1.450 \times 10^{-4}\ \text{psi}$$

The units of stress are the same as those of *pressure,* which we will encounter often in later chapters. Air pressure in automobile tires is typically around $3 \times 10^5\ \text{Pa} = 300\ \text{kPa}$, and steel cables are commonly required to withstand tensile stresses of the order of 10^8 Pa.

The object shown in Fig. 11.13 stretches to a length $l = l_0 + \Delta l$ when under tension. The elongation Δl does not occur only at the ends; every part of the bar stretches in the same proportion. The **tensile strain** of the object is equal to the fractional change in length, which is the ratio of the elongation Δl to the original length l_0:

$$\text{Tensile strain} = \frac{l - l_0}{l_0} = \frac{\Delta l}{l_0} \tag{11.9}$$

Tensile strain is stretch per unit length. It is a ratio of two lengths, always measured in the same units, and so is a pure (dimensionless) number with no units.

Experiment shows that for a sufficiently small tensile stress, stress and strain are proportional, as in Eq. (11.7). The corresponding elastic modulus is called **Young's modulus,** denoted by Y:

$$Y = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{F_\perp/A}{\Delta l/l_0} = \frac{F_\perp}{A}\frac{l_0}{\Delta l} \qquad \text{(Young's modulus)} \tag{11.10}$$

Since strain is a pure number, the units of Young's modulus are the same as those of stress: force per unit area. Some typical values are listed in Table 11.1. (This table also gives values of two other elastic moduli that we will discuss later in this chapter.) A material with a large value of Y is relatively unstretchable; a large stress is required for a given strain. For example, the value of Y for cast steel $(2 \times 10^{11}\ \text{Pa})$ is much larger than that for rubber $(5 \times 10^8\ \text{Pa})$.

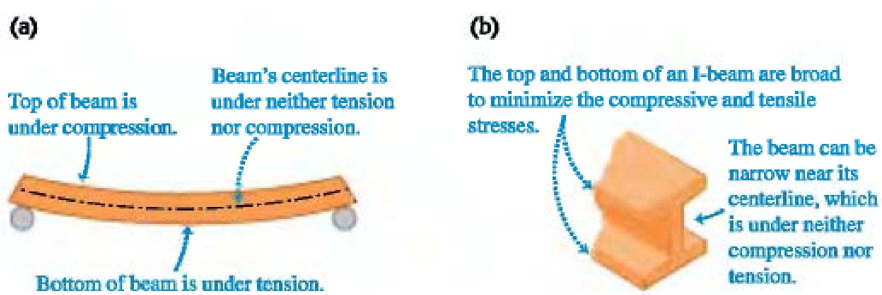
When the forces on the ends of a bar are pushes rather than pulls (Fig. 11.14), the bar is in **compression** and the stress is a **compressive stress.** The **compressive strain** of an object in compression is defined in the same way as the tensile strain, but Δl has the opposite direction. Hooke's law and Eq. (11.10) are valid for compression as well as tension if the compressive stress is not too great. For many materials, Young's modulus has the same value for both tensile and compressive stresses. Composite materials such as concrete and

Table 11.1 Approximate Elastic Moduli

Material	Young's Modulus, Y (Pa)	Bulk Modulus, B (Pa)	Shear Modulus, S (Pa)
Aluminum	7.0×10^{10}	7.5×10^{10}	2.5×10^{10}
Brass	9.0×10^{10}	6.0×10^{10}	3.5×10^{10}
Copper	11×10^{10}	14×10^{10}	4.4×10^{10}
Crown glass	6.0×10^{10}	5.0×10^{10}	2.5×10^{10}
Iron	21×10^{10}	16×10^{10}	7.7×10^{10}
Lead	1.6×10^{10}	4.1×10^{10}	0.6×10^{10}
Nickel	21×10^{10}	17×10^{10}	7.8×10^{10}
Steel	20×10^{10}	16×10^{10}	7.5×10^{10}

stone are an exception; they can withstand compressive stresses but fail under comparable tensile stresses. Stone was the primary building material used in ancient civilizations such as the Babylonians, Assyrians, and Romans, so their structures had to be designed to avoid tensile stresses. This explains why they made extensive use of arches in doorways and bridges, where the weight of the overlying material compresses the stones of the arch together and does not place them under tension.

In many situations, bodies can experience both tensile and compressive stresses at the same time. As an example, a horizontal beam supported at each end sags under its own weight. As a result, the top of the beam is under compression, while the bottom of the beam is under tension (Fig. 11.15a). To minimize the stress and hence the bending strain, the top and bottom of the beam are given a large cross-sectional area. There is neither compression nor tension along the centerline of the beam, so this part can have a small cross section; this helps to keep the weight of the bar to a minimum and further helps to reduce the stress. The result is an I-beam of the familiar shape used in building construction (Fig. 11.15b).



11.15 (a) A beam supported at both ends is under both compression and tension. (b) The cross-sectional shape of an I-beam minimizes both stress and weight.

Example 11.5 Tensile stress and strain

A steel rod 2.0 m long has a cross-sectional area of $0.30\ \text{cm}^2$. The rod is now hung by one end from a support structure, and a 550-kg milling machine is hung from the rod's lower end. Determine the stress, the strain, and the elongation of the rod.

SOLUTION

IDENTIFY: This example uses the definitions of stress, strain, and Young's modulus, which is the appropriate elastic modulus for an object under tension.

SET UP: We use Eqs. (11.8), (11.9), and (11.10) to find the tensile stress, the tensile strain, and the elongation Δl. We also use the value of Y for steel from Table 11.1.

EXECUTE: We find

$$\text{Stress} = \frac{F_\perp}{A} = \frac{(550\ \text{kg})(9.8\ \text{m/s}^2)}{3.0 \times 10^{-5}\ \text{m}^2} = 1.8 \times 10^{8}\ \text{Pa}$$

$$\text{Strain} = \frac{\Delta l}{l_0} = \frac{\text{Stress}}{Y} = \frac{1.8 \times 10^{8}\ \text{Pa}}{20 \times 10^{10}\ \text{Pa}} = 9.0 \times 10^{-4}$$

$$\text{Elongation} = \Delta l = (\text{Strain}) \times l_0 = (9.0 \times 10^{-4})(2.0\ \text{m})$$
$$= 0.0018\ \text{m} = 1.8\ \text{mm}$$

EVALUATE: The small size of this elongation, which results from a load of more than half a ton, is a testament to the stiffness of steel.

Bulk Stress and Strain

When a scuba diver plunges deep into the ocean, the water exerts nearly uniform pressure everywhere on his surface and squeezes him to a slightly smaller volume (Fig. 11.12b). This is a different situation from the tensile and compressive stresses and strains we have discussed. The stress is now a uniform pressure on all sides, and the resulting deformation is a volume change. We use the terms **bulk stress (or volume stress)** and **bulk strain** (or **volume strain)** to describe these quantities.

If an object is immersed in a fluid (liquid or gas) at rest, the fluid exerts a force on any part of the object's surface; this force is *perpendicular* to the surface. (If we tried to make the fluid exert a force parallel to the surface, the fluid would slip sideways to counteract the effort.) The force $F_\perp$ per unit area that the fluid exerts on the surface of an immersed object is called the **pressure** p in the fluid:

$$p = \frac{F_\perp}{A} \qquad \text{(pressure in a fluid)} \qquad (11.11)$$

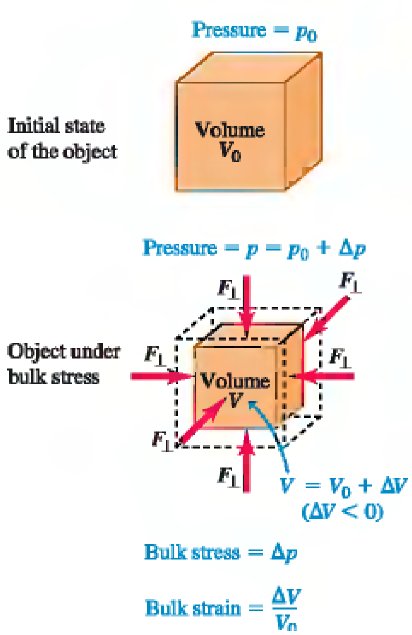
The pressure in a fluid increases with depth. For example, the pressure of the air is about 21% greater at sea level than in Denver (at an elevation of 1.6 km, or 1.0 mi). If an immersed object is relatively small, however, we can ignore pressure differences due to depth for the purpose of calculating bulk stress. Hence we will treat the pressure as having the same value at all points on an immersed object's surface.

Pressure has the same units as stress; commonly used units include 1 Pa $(=1\ \text{N/m}^2)$ and 1 lb/in.^{2} (1 psi). Also in common use is the **atmosphere,** abbreviated atm. One atmosphere is the approximate average pressure of the earth's atmosphere at sea level:

$$1\ \text{atmosphere} = 1\ \text{atm} = 1.013 \times 10^5\ \text{Pa} = 14.7\ \text{lb/in.}^2$$

CAUTION **Pressure vs. force** Unlike force, pressure has no intrinsic direction: The pressure on the surface of an immersed object is the same no matter how the surface is oriented. Hence pressure is a *scalar* quantity, not a vector quantity.

11.16 An object under bulk stress. Without the stress, the cube has volume V_0; when the stress is applied, the cube has a smaller volume V. The volume change ΔV is exaggerated for clarity.



Pressure plays the role of stress in a volume deformation. The corresponding strain is the fractional change in volume (Fig. 11.16)—that is, the ratio of the volume change ΔV to the original volume V_0:

$$\text{Bulk (volume) strain} = \frac{\Delta V}{V_0} \qquad (11.12)$$

Volume strain is the change in volume per unit volume. Like tensile or compressive strain, it is a pure number, without units.

When Hooke's law is obeyed, an increase in pressure (bulk stress) produces a *proportional* bulk strain (fractional change in volume). The corresponding elastic modulus (ratio of stress to strain) is called the **bulk modulus,** denoted by B. When the pressure on a body changes by a small amount Δp, from p_0 to $p_0 + \Delta p$, and the resulting bulk strain is $\Delta V/V_0$, Hooke's law takes the form

$$B = \frac{\text{Bulk stress}}{\text{Bulk strain}} = -\frac{\Delta p}{\Delta V/V_0} \qquad \text{(bulk modulus)} \qquad (11.13)$$

We include a minus sign in this equation because an *increase* of pressure always causes a *decrease* in volume. In other words, if Δp is positive, ΔV is negative. The bulk modulus B itself is a positive quantity.

For small pressure changes in a solid or a liquid, we consider B to be constant. The bulk modulus of a *gas*, however, depends on the initial pressure p_0. Table 11.1 includes values of the bulk modulus for several solid materials. Its units, force per unit area, are the same as those of pressure (and of tensile or compressive stress).

The reciprocal of the bulk modulus is called the **compressibility** and is denoted by k. From Eq. (11.13),

$$k = \frac{1}{B} = -\frac{\Delta V/V_0}{\Delta p} = -\frac{1}{V_0}\frac{\Delta V}{\Delta p} \quad \text{(compressibility)} \tag{11.14}$$

Compressibility is the fractional decrease in volume, $-\Delta V/V_0$, per unit increase Δp in pressure. The units of compressibility are those of *reciprocal pressure*, Pa^{-1} or atm^{-1}.

Table 11.2 lists the values of compressibility k for several liquids. For example, the compressibility of water is $46.4 \times 10^{-6}\ atm^{-1}$, which means that the volume of water decreases by 46.4 parts per million for each 1-atmosphere increase in pressure. Materials with small bulk modulus and large compressibility are easier to compress.

Table 11.2 Compressibilities of Liquids

	Compressibility, k	
Liquid	Pa^{-1}	atm^{-1}
Carbon disulfide	93×10^{-11}	94×10^{-6}
Ethyl alcohol	110×10^{-11}	111×10^{-6}
Glycerine	21×10^{-11}	21×10^{-6}
Mercury	3.7×10^{-11}	3.8×10^{-6}
Water	45.8×10^{-11}	46.4×10^{-6}

Example 11.6 Bulk stress and strain

A hydraulic press contains $0.25\ m^3$ (250 L) of oil. Find the decrease in the volume of the oil when it is subjected to a pressure increase $\Delta p = 1.6 \times 10^7$ Pa (about 160 atm or 2300 psi). The bulk modulus of the oil is $B = 5.0 \times 10^9$ Pa (about 5.0×10^4 atm), and its compressibility is $k = 1/B = 20 \times 10^{-6}\ atm^{-1}$.

SOLUTION

IDENTIFY: This example uses the ideas of bulk stress and strain. Our target variable is the volume change ΔV.

SET UP: We are given both the bulk modulus and the compressibility, so we can use either Eq. (11.13) or Eq. (11.14) to find ΔV.

EXECUTE: Solving Eq. (11.13) for ΔV, we find

$$\Delta V = -\frac{V_0\,\Delta p}{B} = -\frac{(0.25\ m^3)(1.6 \times 10^7\ Pa)}{5.0 \times 10^9\ Pa}$$
$$= -8.0 \times 10^{-4}\ m^3 = -0.80\ L$$

Alternatively, we can use Eq. (11.14). Solving for ΔV and using the approximate unit conversions given above, we get

$$\Delta V = -kV_0\,\Delta p = -(20 \times 10^{-6}\ atm^{-1})(0.25\ m^3)(160\ atm)$$
$$= -8.0 \times 10^{-4}\ m^3$$

EVALUATE: We get the same result for ΔV with either approach, as we should. Note that ΔV is negative, indicating that the volume decreases when the pressure increases. Even though the pressure increase is very large, the *fractional* change in volume is very small:

$$\frac{\Delta V}{V_0} = \frac{-8.0 \times 10^{-4}\ m^3}{0.25\ m^3} = -0.0032, \quad \text{or} \quad -0.32\%$$

Shear Stress and Strain

The third kind of stress-strain situation is called *shear.* The ribbon in Fig. 11.12c is under **shear stress:** One part of the ribbon is being pushed up while an adjacent part is being pushed down, producing a deformation of the ribbon. Figure 11.17 shows a body being deformed by a shear stress. In the figure, forces of equal magnitude but opposite direction act *tangent* to the surfaces of opposite ends of the object. We define the shear stress as the force $F_{\parallel}$ acting tangent to the surface, divided by the area A on which it acts:

$$\text{Shear stress} = \frac{F_{\parallel}}{A} \tag{11.15}$$

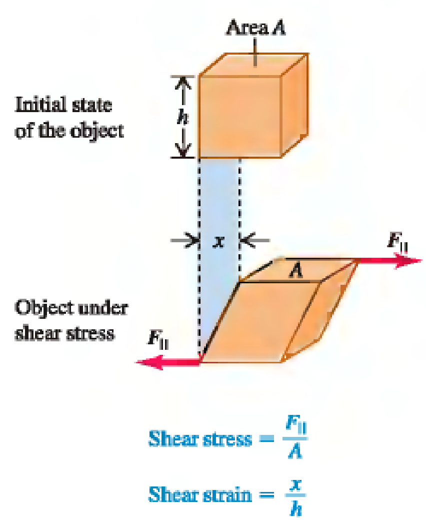
Shear stress, like the other two types of stress, is a force per unit area.

Figure 11.17 shows that one face of the object under shear stress is displaced by a distance x relative to the opposite face. We define **shear strain** as the ratio of the displacement x to the transverse dimension h:

$$\text{Shear strain} = \frac{x}{h} \tag{11.16}$$

In real-life situations, x is nearly always much smaller than h. Like all strains, shear strain is a dimensionless number; it is a ratio of two lengths.

11.17 An object under shear stress. Forces are applied tangent to opposite surfaces of the object (in contrast to the situation in Fig. 11.13, in which the forces act perpendicular to the surfaces). The deformation x is exaggerated for clarity.



If the forces are small enough that Hooke's law is obeyed, the shear strain is *proportional* to the shear stress. The corresponding elastic modulus (ratio of shear stress to shear strain) is called the **shear modulus,** denoted by S:

$$S = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{F_{\parallel}/A}{x/h} = \frac{F_{\parallel}}{A}\frac{h}{x} \quad \text{(shear modulus)} \tag{11.17}$$

with x and h defined as in Fig. 11.17.

Table 11.1 gives several values of shear modulus. For a given material, S is usually one-third to one-half as large as Young's modulus Y for tensile stress. Keep in mind that the concepts of shear stress, shear strain, and shear modulus apply to *solid* materials only. The reason is that the shear forces in Fig. 11.17 are required to deform the solid block, and the block tends to return to its original shape if the shear forces are removed. By contrast, gases and liquids do not have definite shapes.

Example 11.7 Shear stress and strain

Suppose the object in Fig. 11.17 is the brass base plate of an outdoor sculpture; it experiences shear forces as a result of an earthquake. The frame is 0.80 m square and 0.50 cm thick. How large a force must be exerted on each of its edges if the displacement x (see Fig. 11.17) is 0.16 mm?

SOLUTION

IDENTIFY: This example uses the relationship among shear stress, shear strain, and shear modulus. Our target variable is the force $F_{\parallel}$ exerted parallel to each edge, as shown in Fig. 11.17.

SET UP: We first find the shear strain using Eq. (11.16), and then determine the shear stress using Eq. (11.17). We can then solve for the target variable $F_{\parallel}$ using Eq. (11.15). The values of all the other quantities are given, including the shear modulus of brass (from Table 11.1, $S = 3.5 \times 10^{10}$ Pa). Note that h in Fig. 11.17 represents the 0.80-m length of each side of the square plate, and the area A is the product of the 0.80-m length and the 0.50-cm thickness.

EXECUTE: The shear strain is

$$\text{Shear strain} = \frac{x}{h} = \frac{1.6 \times 10^{-4}\ \text{m}}{0.80\ \text{m}} = 2.0 \times 10^{-4}$$

From Eq. (11.17) the shear stress equals the shear strain multiplied by the shear modulus S:

$$\begin{aligned}\text{Stress} &= (\text{Shear strain}) \times S\\ &= (2.0 \times 10^{-4})(3.5 \times 10^{10}\ \text{Pa}) = 7.0 \times 10^{6}\ \text{Pa}\end{aligned}$$

From Eq. (11.15), the force at each edge is the shear stress multiplied by the area of the edge:

$$\begin{aligned}F_{\parallel} &= (\text{Shear stress}) \times A\\ &= (7.0 \times 10^{6}\ \text{Pa})(0.80\ \text{m})(0.0050\ \text{m}) = 2.8 \times 10^{4}\ \text{N}\end{aligned}$$

EVALUATE: The required force is more than 3 tons! Brass has a large shear modulus, which means that it's intrinsically difficult to deform. Furthermore, the plate is relatively thick (0.50 cm), so the area A is relatively large and a large force $F_{\parallel}$ is needed to provide the necessary stress $F_{\parallel}/A$.

Test Your Understanding of Section 11.4 A copper rod of cross-sectional area 0.500 cm^2 and length 1.00 m is elongated by 2.00×10^{-2} mm, and a steel rod of the same cross-sectional area but 0.100 m in length is elongated by 2.00×10^{-3} mm. (a) Which rod has greater tensile *strain?* (i) the copper rod; (ii) the steel rod; (iii) the strain is the same for both. (b) Which rod is under greater tensile *stress?* (i) the copper rod; (ii) the steel rod; (iii) the stress is the same for both.

MP

11.5 Elasticity and Plasticity

Hooke's law—the proportionality of stress and strain in elastic deformations—has a limited range of validity. In the preceding section we used phrases such as "provided that the forces are small enough that Hooke's law is obeyed." Just what *are* the limitations of Hooke's law? We know that if you pull, squeeze, or twist *anything* hard enough, it will bend or break. Can we be more precise than that?

Let's look at tensile stress and strain again. Suppose we plot a graph of stress as a function of strain. If Hooke's law is obeyed, the graph is a straight line with a slope equal to Young's modulus. Figure 11.18 shows a typical stress-strain graph for a metal such as copper or soft iron. The strain is shown as the *percent* elongation; the horizontal scale is not uniform beyond the first portion of the curve, up to a strain of less than 1%. The first portion is a straight line, indicating Hooke's law behavior with stress directly proportional to strain. This straight-line portion ends at point a; the stress at this point is called the *proportional limit.*

From a to b, stress and strain are no longer proportional, and Hooke's law is *not* obeyed. If the load is gradually removed, starting at any point between O and b, the curve is retraced until the material returns to its original length. The deformation is *reversible,* and the forces are conservative; the energy put into the material to cause the deformation is recovered when the stress is removed. In region Ob we say that the material shows *elastic behavior.* Point b, the end of this region, is called the *yield point;* the stress at the yield point is called the *elastic limit.*

When we increase the stress beyond point b, the strain continues to increase. But now when we remove the load at some point beyond b, say c, the material does not come back to its original length. Instead, it follows the red line in Fig. 11.18. The length at zero stress is now greater than the original length; the material has undergone an irreversible deformation and has acquired what we call a *permanent set.* Further increase of load beyond c produces a large increase in strain for a relatively small increase in stress, until a point d is reached at which *fracture* takes place. The behavior of the material from b to d is called *plastic flow* or *plastic deformation.* A plastic deformation is irreversible; when the stress is removed, the material does not return to its original state.

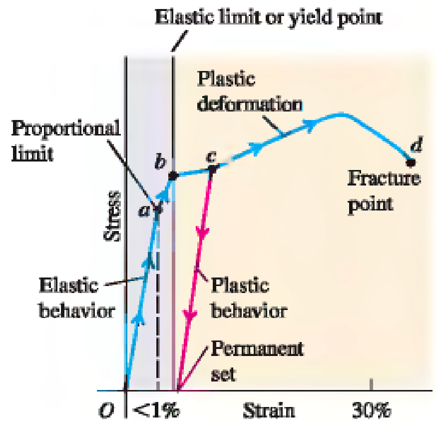
For some materials, such as the one whose properties are graphed in Fig. 11.18, a large amount of plastic deformation takes place between the elastic limit and the fracture point. Such a material is said to be *ductile.* But if fracture occurs soon after the elastic limit is passed, the material is said to be *brittle.* A soft iron wire that can have considerable permanent stretch without breaking is ductile, while a steel piano string that breaks soon after its elastic limit is reached is brittle.

Something very curious can happen when an object is stretched and then allowed to relax. An example is shown in Fig. 11.19, which is a stress-strain curve for vulcanized rubber that has been stretched by more than seven times its original length. The stress is not proportional to the strain, but the behavior is elastic because when the load is removed, the material returns to its original length. However, the material follows *different* curves for increasing and decreasing stress. This is called *elastic hysteresis.* The work done by the material when it returns to its original shape is less than the work required to deform it; there are nonconservative forces associated with internal friction. Rubber with large elastic hysteresis is very useful for absorbing vibrations, such as in engine mounts and shock-absorber bushings for cars.

The stress required to cause actual fracture of a material is called the *breaking stress,* the *ultimate strength,* or (for tensile stress) the *tensile strength.* Two materials, such as two types of steel, may have very similar elastic constants but vastly different breaking stresses. Table 11.3 gives typical values of breaking stress for several materials in tension. The conversion factor 6.9×10^8 Pa = 100,000 psi may help put these numbers in perspective. For example, if the breaking stress of a particular steel is 6.9×10^8 Pa, then a bar with a 1-in.^{2} cross section has a breaking strength of 100,000 lb.

Test Your Understanding of Section 11.5 While parking your car on a crowded street, you accidentally back into a steel post. You pull forward until the car no longer touches the post and then get out to inspect the damage. What does your rear bumper look like if the strain in the impact was (a) less than at the proportional limit; (b) greater than at the proportional limit, but less than at the yield point; (c) greater than at the yield point, but less than at the fracture point; and (d) greater than at the fracture point?

11.18 Typical stress-strain diagram for a ductile metal under tension.



11.19 Typical stress-strain diagram for vulcanized rubber. The curves are different for increasing and decreasing stress, a phenomenon called elastic hysteresis.

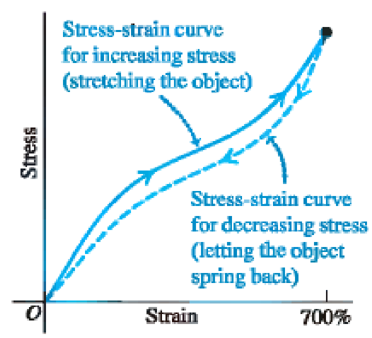


Table 11.3 Approximate Breaking Stresses

Material	Breaking Stress (Pa or N/m^2)
Aluminum	2.2×10^8
Brass	4.7×10^8
Glass	10×10^8
Iron	3.0×10^8
Phosphor bronze	5.6×10^8
Steel	$5 - 20 \times 10^8$

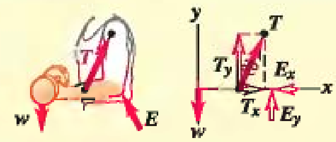
CHAPTER 11 SUMMARY

Conditions for equilibrium: For a rigid body to be in equilibrium, two conditions must be satisfied. First, the vector sum of forces must be zero. Second, the sum of torques about any point must be zero. The torque due to the weight of a body can be found by assuming the entire weight is concentrated at the center of gravity, which is at the same point as the center of mass if $\vec{g}$ has the same value at all points. (See Examples 11.1–11.4.)

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum F_z = 0 \tag{11.1}$$

$$\sum \vec{\tau} = 0 \quad \text{about } any \text{ point} \tag{11.2}$$

$$\vec{r}_{\text{cm}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \tag{11.4}$$

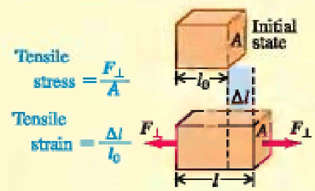


Stress, strain, and Hooke's law: Hooke's law states that in elastic deformations, stress (force per unit area) is proportional to strain (fractional deformation). The proportionality constant is called the elastic modulus.

$$\frac{\text{Stress}}{\text{Strain}} = \text{Elastic modulus} \tag{11.7}$$

Tensile and compressive stress: Tensile stress is tensile force per unit area, $F_\perp/A$. Tensile strain is fractional change in length, $\Delta l/l_0$. The elastic modulus is called Young's modulus Y. Compressive stress and strain are defined in the same way. (See Example 11.5.)

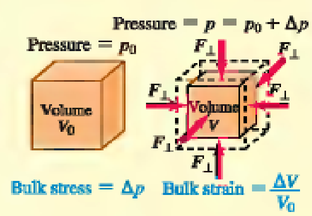
$$Y = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{F_\perp/A}{\Delta l/l_0} = \frac{F_\perp}{A}\frac{l_0}{\Delta l} \tag{11.10}$$



Bulk stress: Pressure in a fluid is force per unit area. Bulk stress is pressure change, Δp, and bulk strain is fractional volume change, $\Delta V/V_0$. The elastic modulus is called the bulk modulus, B. Compressibility, k, is the reciprocal of bulk modulus: $k = 1/B$. (See Example 11.6.)

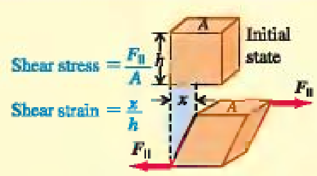
$$p = \frac{F_\perp}{A} \tag{11.11}$$

$$B = \frac{\text{Bulk stress}}{\text{Bulk strain}} = -\frac{\Delta p}{\Delta V/V_0} \tag{11.13}$$



Shear stress: Shear stress is force per unit area, $F_\parallel/A$, for a force applied tangent to a surface. Shear strain is the displacement x of one side divided by the transverse dimension h. The elastic modulus is called the shear modulus, S. (See Example 11.7.)

$$S = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{F_\parallel/A}{x/h} = \frac{F_\parallel}{A}\frac{h}{x} \tag{11.17}$$



The limits of Hooke's law: The proportional limit is the maximum stress for which stress and strain are proportional. Beyond the proportional limit, Hooke's law is not valid. The elastic limit is the stress beyond which irreversible deformation occurs. The breaking stress, or ultimate strength, is the stress at which the material breaks.

Key Terms

first condition for equilibrium, *355*
second condition for equilibrium, *355*
static equilibrium, *355*
center of gravity, *355*
stress, *363*
strain, *363*
elastic modulus, *363*
Hooke's law, *363*
tension, *364*
tensile stress, *364*
pascal, *364*
tensile strain, *364*
Young's modulus, *364*
compression, *364*
compressive stress, *364*
compressive strain, *364*
bulk stress (volume stress), *366*
bulk strain (volume strain), *366*
pressure, *366*
atmosphere, *366*
bulk modulus, *366*
compressibility, *367*
shear stress, *367*
shear strain, *367*
shear modulus, *368*

Answer to Chapter Opening Question ?

Each stone in the arch is under compression, not tension. This is because the forces on the stones tend to push them inward toward the center of the arch and thus squeeze them together. Compared to a solid supporting wall, a wall with arches is just as strong yet much more economical to build.

Answers to Test Your Understanding Questions

11.1 Answer: (i) Situation (i) satisfies both equilibrium conditions because the seagull has zero acceleration (so $\sum\vec{F} = \mathbf{0}$) and no tendency to start rotating (so $\sum\vec{\tau} = \mathbf{0}$). Situation (ii) satisfies the first condition because the crankshaft as a whole does not accelerate through space, but it does not satisfy the second condition; the crankshaft has an angular acceleration, so $\sum\vec{\tau}$ is not zero. Situation (iii) satisfies the second condition (there is no tendency to rotate) but not the first one; the baseball accelerates in its flight (due to gravity), so $\sum\vec{F}$ is not zero.

11.2 Answer: (ii) In equilibrium, the center of gravity must be at the point of support. Since the rock and meter stick have the same mass and hence the same weight, the center of gravity of the system is midway between their respective centers. The cg of the meter stick alone is 0.50 m from the left end (that is, at the middle of the meter stick), so the cg of the combination of rock and meter stick is 0.25 m from the left end.

11.3 Answer: (ii), (i), (iii) This is the same situation described in Example 11.4, with the rod replacing the forearm, the hinge replacing the elbow, and the cable replacing the tendon. The only difference is that the cable attachment point is at the end of the rod, so the distances D and L are identical. From Example 11.4, the tension is

$$T = \frac{Lw}{L\sin\theta} = \frac{w}{\sin\theta}$$

Since $\sin\theta$ is less than 1, the tension T is greater than the weight w. The vertical component of the force exerted by the hinge is

$$E_y = -\frac{(L - L)w}{L} = 0$$

In this situation, the hinge exerts *no* vertical force. You can see this easily if you calculate torques around the right end of the horizontal rod: The only force that exerts a torque around this point is the vertical component of the hinge force, so this force component must be zero.

11.4 Answers: (a) (iii), (b) (ii) In (a), the copper rod has 10 times the elongation Δl of the steel rod, but it also has 10 times the original length l_0. Hence the tensile strain $\Delta l/l_0$ is the same for both rods. In (b), the stress is equal to Young's modulus Y multiplied by the strain. From Table 11.1, steel has a larger value of Y, so a greater stress is required to produce the same strain.

11.5 In (a) and (b), the bumper will have sprung back to its original shape (although the paint may be scratched). In (c), the bumper will have a permanent dent or deformation. In (d), the bumper will be torn or broken.

PROBLEMS

For instructor-assigned homework, go to www.masteringphysics.com MP

Discussion Questions

Q11.1. Does a rigid object in uniform rotation about a fixed axis satisfy the first and second conditions for equilibrium? Why? Does it then follow that every particle in this object is in equilibrium? Explain.

Q11.2. (a) Is it possible for an object to be in translational equilibrium (the first condition) but *not* in rotational equilibrium (the second condition)? Illustrate your answer with a simple example. (b) Can an object be in rotational equilibrium yet *not* in translational equilibrium? Justify your answer with a simple example.

Q11.3. Car tires are sometimes "balanced" on a machine that pivots the tire and wheel about the center. Weights are placed around the wheel rim until it does not tip from the horizontal plane. Discuss this procedure in terms of the center of gravity.

Q11.4. Does the center of gravity of a solid body always lie within the material of the body? If not, give a counterexample.

Q11.5. In Section 11.2 we always assumed that the value of g was the same at all points on the body. This is *not* a good approximation if the dimensions of the body are great enough, because the value of g decreases with altitude. If this is taken into account, will the center of gravity of a long, vertical rod be above, below, or at its center of mass? Explain how this can be used to keep the long axis of an orbiting spacecraft pointed toward the earth. (This would be useful for a weather satellite that must always keep its camera lens trained on the earth.) The moon is not exactly spherical but is somewhat elongated. Explain why this same effect is responsible for keeping the same face of the moon pointed toward the earth at all times.

Q11.6. You are balancing a wrench by suspending it at a single point. Is the equilibrium stable, unstable, or neutral if the point is above, at, or below the wrench's center of gravity? In each case give the reasoning behind your answer. (For rotation, a rigid body is in *stable* equilibrium if a small rotation of the body produces a torque that tends to return the body to equilibrium; it is in *unstable* equilibrium if a small rotation produces a torque that tends to take the body farther from equilibrium; and it is in *neutral* equilibrium if a small rotation produces no torque.)

Q11.7. You can probably stand flatfooted on the floor and then rise up and balance on your tiptoes. Why are you unable do it if your toes are touching the wall of your room? (Try it!)

Q11.8. You freely pivot a horseshoe from a horizontal nail through one of its nail holes. You then hang a long string with a weight at its bottom from the same nail, so that the string hangs vertically in front of the horseshoe without touching it. How do you know that the horseshoe's center of gravity is along the line behind the string? How can you locate the center of gravity by repeating the process at another nail hole? Will the center of gravity be within the solid material of the horseshoe?

Q11.9. An object consists of a ball of weight W glued to the end of a uniform bar also of weight W. If you release it from rest, with the bar horizontal, what will be its behavior be as it falls if air resistance is negligible? Will it (a) remain horizontal; (b) rotate about its center of gravity; (c) rotate about the ball; or (d) rotate so that the ball swings downward? Explain your reasoning.

Q11.10. Suppose that the object in Question 11.9 is released from rest with the bar tilted at 60° above the horizontal with the ball at the upper end. As it is falling, will it (a) rotate about its center of gravity until it is horizontal; (b) rotate about its center of gravity until it is vertical with the ball at the bottom; (c) rotate about the ball until it is vertical with the ball at the bottom; or (d) remain at 60° above the horizontal?

Q11.11. Why must a water skier moving with constant velocity lean backward? What determines how far back she must lean? Draw a free-body diagram for the water skier to justify your answers.

Q11.12. In pioneer days, when a Conestoga wagon was stuck in the mud, people would grasp the wheel spokes and try to turn the wheels, rather than simply pushing the wagon. Why?

Q11.13. The mighty Zimbo claims to have leg muscles so strong that he can stand flat on his feet and lean forward to pick up an apple on the floor with his teeth. Should you pay to see him perform, or do you have any suspicions about his claim? Why?

Q11.14. Why is it easier to hold a 10-kg dumbbell in your hand at your side than it is to hold it with your arm extended horizontally?

Q11.15. Certain features of a person, such as height and mass, are fixed (at least over relatively long periods of time). Are the following features also fixed? (a) location of the center of gravity of the body; (b) moment of inertia of the body about an axis through the person's center of mass. Explain your reasoning.

Q11.16. During pregnancy, women often develop back pains from leaning backward while walking. Why do they have to walk this way?

Q11.17. Why is a tapered water glass with a narrow base easier to tip over than a glass with straight sides? Does it matter whether the glass is full or empty?

Q11.18. When a tall, heavy refrigerator is pushed across a rough floor, what factors determine whether it slides or tips?

Q11.19. If a metal wire has its length doubled and its diameter tripled, by what factor does its Young's modulns change?

Q11.20. Why is concrete with steel reinforcing rods embedded in it stronger than plain concrete?

Q11.21. A metal wire of diameter D stretches by 0.100 mm when supporting a weight W. If the same length wire is used to support a weight three times as heavy, what would its diameter have to be (in terms of D) so it still stretched only 0.100 mm?

Q11.22. Compare the mechanical properties of a steel cable, made by twisting many thin wires together, with the properties of a solid steel rod of the same diameter. What advantages does each have?

Q11.23. The material in human bones and elephant bones is essentially the same, bnt an elephant has much thicker legs. Explain why, in terms of breaking stress.

Q11.24. There is a small but appreciable amount of elastic hysteresis in the large tendon at the back of a horse's leg. Explain how this can cause damage to the tendon if a horse runs too hard for too long a time.

Q11.25. When rubber mounting blocks are used to absorb machine vibrations through elastic hysteresis, as mentioned in Section 11.5, what becomes of the energy associated with the vibrations?

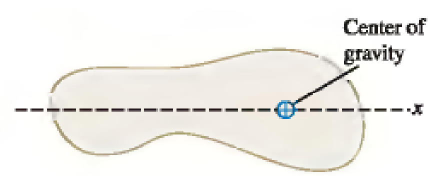
Exercises

Section 11.2 Center of Gravity

11.1. A 2.40-kg, 50.0-cm-long uniform bar has a small 1.10-kg mass glued to its left end and a small 2.20-kg mass glued to the other end. You want to balance this system horizontally on a fulcrum placed just under its center of gravity. How far from the left end should the fulcrum be placed?

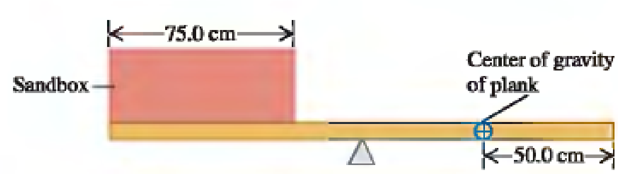
11.2. The center of gravity of an irregular object is shown in Fig. 11.20. You need to move the center of gravity 2.20 cm to the left by gluing on a tiny 1.50-kg mass, which will then be considered as part of the object. Where should you attach this additional mass?

Figure 11.20 Exercise 11.2.



11.3. A box of negligible mass rests at the left end of a 2.00-m, 25.0-kg plank (Fig. 11.21). The width of the box is 75.0 cm, and sand is to be distributed uniformly throughout it. The center of gravity of the nonuniform plank is 50.0 cm from the right end. What mass of sand should be put into the box so that the plank balances horizontally on a fulcrum placed just below its midpoint?

Figure 11.21 Exercise 11.3.

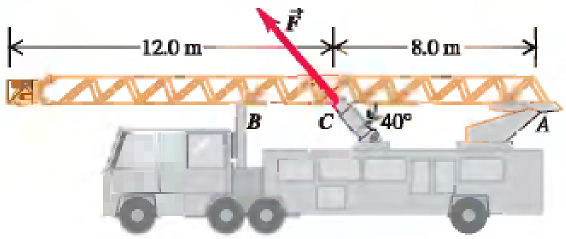


Section 11.3 Solving Rigid-Body Equilibrium Problems

11.4. A uniform 300-N trapdoor in a floor is hinged at one side. Find the net upward force needed to begin to open it and the total force exerted on the door by the hinges (a) if the upward force is applied at the center and (b) if the upward force is applied at the center of the edge opposite the hinges.

11.5. Raising a Ladder. A ladder carried by a fire truck is 20.0 m long. The ladder weighs 2800 N and its center of gravity is at its center. The ladder is pivoted at one end (A) about a pin (Fig. 11.22); you can ignore the friction torque at the pin. The ladder is raised into position by a force applied by a hydraulic piston at C. Point C is 8.0 m from A, and the force $\vec{F}$ exerted by the piston makes an angle of 40° with the ladder. What magnitude must $\vec{F}$ have to just lift the ladder off the support bracket at B? Start with a free-body diagram of the ladder.

Figure 11.22 Exercise 11.5.

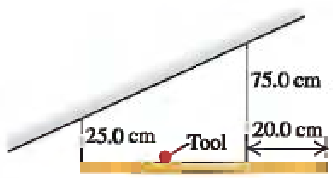


11.6. Two people are carrying a uniform wooden board that is 3.00 m long and weighs 160 N. If one person applies an upward force equal to 60 N at one end, at what point does the other person lift? Begin with a free-body diagram of the board.

11.7. Two people carry a heavy electric motor by placing it on a light board 2.00 m long. One person lifts at one end with a force of 400 N, and the other lifts the opposite end with a force of 600 N. (a) What is the weight of the motor, and where along the board is its center of gravity located? (b) Suppose the board is not light bnt weighs 200 N, with its center of gravity at its center, and the two people each exert the same forces as before. What is the weight of the motor in this case, and where is its center of gravity located?

11.8. A 60.0-cm, uniform, 50.0-N shelf is supported horizontally by two vertical wires attached to the sloping ceiling (Fig. 11.23). A very small 25.0-N tool is placed on the shelf midway between the points where the wires are attached to it. Find the tension in each wire. Begin by making a free-body diagram of the shelf.

Figure 11.23 Exercise 11.8.

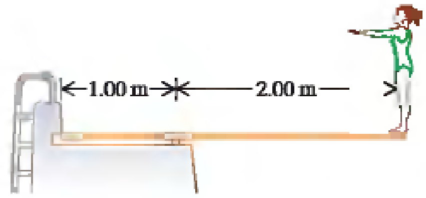


11.9. A 350-N, uniform, 1.50-m bar is suspended horizontally by two vertical cables at each end. Cable A can support a maximum tension of 500.0 N without breaking, and cable B can support up to 400.0 N. You want to place a small weight on this bar. (a) What is the heaviest weight you can put on without breaking either cable, and (b) where should you put this weight?

11.10. A uniform ladder 5.0 m long rests against a frictionless, vertical wall with its lower end 3.0 m from the wall. The ladder weighs 160 N. The coefficient of static friction between the foot of the ladder and the ground is 0.40. A man weighing 740 N climbs slowly up the ladder. Start by drawing a free-body diagram of the ladder. (a) What is the maximum frictional force that the ground can exert on the ladder at its lower end? (b) What is the actual frictional force when the man has climbed 1.0 m along the ladder? (c) How far along the ladder can the man climb before the ladder starts to slip?

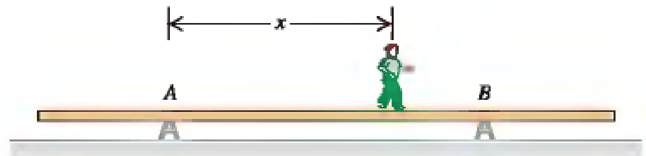
11.11. A diving board 3.00 m long is supported at a point 1.00 m from the end, and a diver weighing 500 N stands at the free end (Fig. 11.24). The diving board is of uniform cross section and weighs 280 N. Find (a) the force at the support point and (b) the force at the left-hand end.

Figure 11.24 Exercise 11.11.



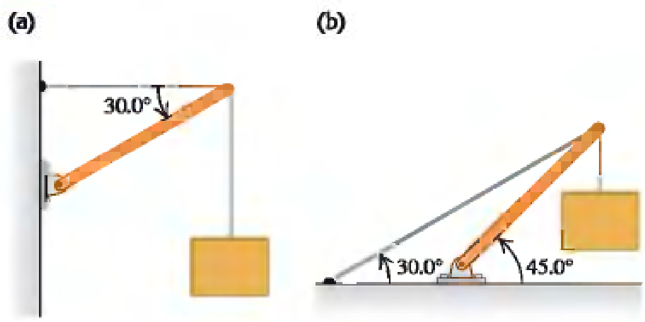
11.12. A uniform aluminum beam 9.00 m long, weighing 300 N, rests symmetrically on two supports 5.00 m apart (Fig. 11.25). A boy weighing 600 N starts at point A and walks toward the right. (a) In the same diagram construct two graphs showing the upward forces F_A and F_B exerted on the beam at points A and B, as functions of the coordinate x of the boy. Let 1 cm = 100 N vertically, and 1 cm = 1.00 m horizontally. (b) From your diagram, how far beyond point B can the boy walk before the beam tips? (c) How far from the right end of the beam should support B be placed so that the boy can walk just to the end of the beam without causing it to tip?

Figure 11.25 Exercise 11.12.



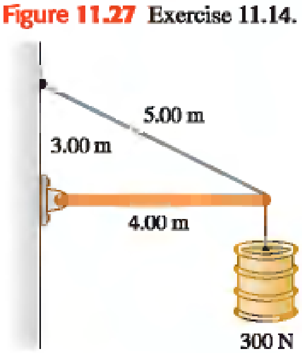
11.13. Find the tension T in each cable and the magnitude and direction of the force exerted on the strut by the pivot in each of the arrangements in Fig. 11.26. In each case let w be the weight of the suspended crate full of priceless art objects. The strut is uniform and also has weight w. Start each case with a free-body diagram of the strut.

Figure 11.26 Exercise 11.13.



11.14. The horizontal beam in Fig. 11.27 weighs 150 N, and its center of gravity is at its center. Find (a) the tension in the cable and (b) the horizontal and vertical components of the force exerted on the beam at the wall.

Figure 11.27 Exercise 11.14.

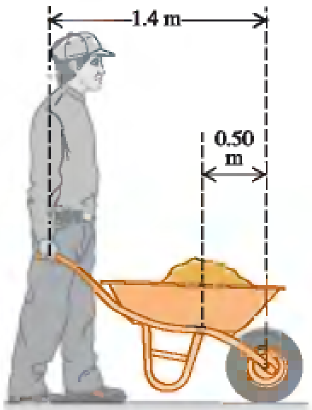


11.15. A door 1.00 m wide and 2.00 m high weighs 280 N and is supported by two hinges, one 0.50 m from the top and the other 0.50 m from the bottom.

Each hinge supports half the total weight of the door. Assuming that the door's center of gravity is at its center, find the horizontal components of force exerted on the door by each hinge.

11.16. Suppose that you can lift no more than 650 N (around 150 lb) unaided. (a) How much can you lift using a 1.4-m-long wheelbarrow that weighs 80.0 N and whose center of gravity is 0.50 m from the center of the wheel (Fig. 11.28)? The center of gravity of the load carried in the wheelbarrow is also 0.50 m from the center of the wheel. (b) Where does the force come from to enable you to lift more than 650 N using the wheelbarrow?

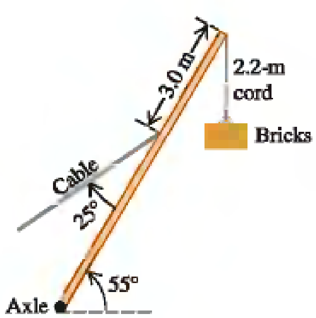
Figure 11.28 Exercise 11.16.



11.17. You take your dog Clea to the vet, and the doctor decides he must locate the little beast's center of gravity. It would be awkward to hang the pooch from the ceiling, so the vet must devise another method. He places Clea's front feet on one scale and her hind feet on another. The front scale reads 157 N, while the rear scale reads 89 N. The vet next measures Clea and finds that her rear feet are 0.95 m behind her front feet. How much does Clea weigh, and where is her center of gravity?

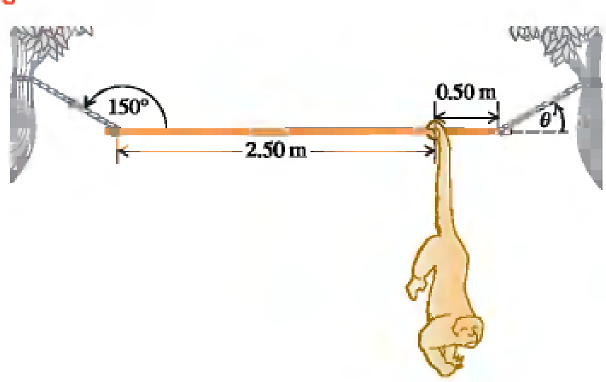
11.16. A 15,000-N crane pivots around a friction-free axle at its base and is supported by a cable making a 25° angle with the crane (Fig. 11.29). The crane is 16 m long and is not uniform, its center of gravity being 7.0 m from the axle as measured along the crane. The cable is attached 3.0 m from the upper end of the crane. When the crane is raised to 55° above the horizontal holding an 11,000-N pallet of bricks by a 2.2-m very light cord, find (a) the tension in the cable, and (b) the horizontal and vertical components of the force that the axle exerts on the crane. Start with a free-body diagram of the crane.

Figure 11.29 Exercise 11.18.



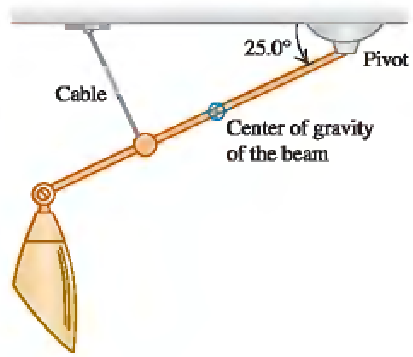
11.19. A 3.00-m-long, 240-N, uniform rod at the zoo is held in a horizontal position by two ropes at its ends (Fig. 11.30). The left rope makes an angle of 150° with the rod and the right rope makes an angle θ with the horizontal. A 90-N howler monkey (*Alouatta seniculus*) hangs motionless 0.50 m from the right end of the rod as he carefully studies you. Calculate the tensions in the two ropes and the angle θ. First make a free-body diagram of the rod.

Figure 11.30 Exercise 11.19.



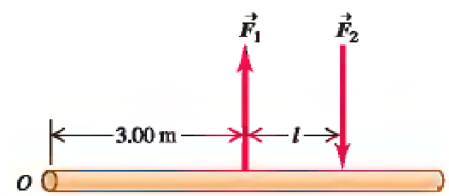
11.20. A nonuniform beam 4.50 m long and weighing 1.00 kN makes an angle of 25.0° below the horizontal. It is held in position by a frictionless pivot at its upper right end and by a cable 3.00 m farther down the beam and perpendicular to it (Fig. 11.31). The center of gravity of the beam is 2.00 m down the beam from the pivot. Lighting equipment exerts a 5.00-kN downward force on the lower left end of the beam. Find the tension T in the cable and the horizontal and vertical components of the force exerted on the beam by the pivot. Start by sketching a free-body diagram of the beam.

Figure 11.31 Exercise 11.20.



11.21. A Couple. Two forces equal in magnitude and opposite in direction, acting on an object at two different points, form what is called a *couple*. Two antiparallel forces with equal magnitudes $F_1 = F_2 = 8.00\,N$ are applied to a rod as shown in Fig. 11.32. (a) What should the distance l between the forces be if they are to provide a net torque of $6.40\,\mathrm{N \cdot m}$ about the left end of the rod? (b) Is the sense of this torque clockwise or counterclockwise? (c) Repeat parts (a) and (b) for a pivot at the point on the rod where $\vec{F}_2$ is applied.

Figure 11.32 Exercise 11.21.



Section 11.4 Stress, Strain, and Elastic Moduli

11.22. Biceps Muscle. A relaxed biceps muscle requires a force of 25.0 N for an elongation of 3.0 cm; the same muscle under maximum tension requires a force of 500 N for the same elongation. Find Young's modulus for the muscle tissue under each of these conditions if the muscle is assumed to be a uniform cylinder with length 0.200 m and cross-sectional area 50.0 $\mathrm{cm^2}$.

11.23. A circular steel wire 2.00 m long must stretch no more than 0.25 cm when a tensile force of 400 N is applied to each end of the wire. What minimum diameter is required for the wire?

11.24. Two circular rods, one steel and the other copper, are joined end to end. Each rod is 0.750 m long and 1.50 cm in diameter. The combination is subjected to a tensile force with magnitude 4000 N. For each rod, what are (a) the strain and (b) the elongation?

11.25. A metal rod that is 4.00 m long and 0.50 $\mathrm{cm^2}$ in cross-sectional area is found to stretch 0.20 cm under a tension of 5000 N. What is Young's modulus for this metal?

11.26. Stress on a Mountaineer's Rope. A nylon rope used by mountaineers elongates 1.10 m under the weight of a 65.0-kg climber. If the rope is 45.0 m in length and 7.0 mm in diameter, what is Young's modulus for nylon?

11.27. In constructing a large mobile, an artist hangs an aluminum sphere of mass 6.0 kg from a vertical steel wire 0.50 m long and $2.5 \times 10^{-3}\ \text{cm}^2$ in cross-sectional area. On the bottom of the sphere he attaches a similar steel wire, from which he hangs a brass cube of mass 10.0 kg. For each wire, compute (a) the tensile strain and (b) the elongation.

11.28. A vertical, solid steel post 25 cm in diameter and 2.50 m long is required to support a load of 8000 kg. You can ignore the weight of the post. What are (a) the stress in the post; (b) the strain in the post; and (c) the change in the post's length when the load is applied?

11.29. Outside a house 1.0 km from ground zero of a 100-kiloton nuclear bomb explosion, the pressure will rapidly rise to as high as 2.8 atm while the pressure inside the house remains 1.0 atm. If the front of the house measures 3.33 m high by 15.0 m wide, what is the resulting net force exerted by the air on the front of the house?

11.30. A solid gold bar is pulled up from the hold of the sunken RMS *Titanic.* (a) What happens to its volume as it goes from the pressure at the ship to the lower pressure at the ocean's surface? (b) The pressure difference is proportional to the depth. How many times greater would the volume change have been had the ship been twice as deep? (c) The bulk modulus of lead is one-fourth that of gold. Find the ratio of the volume change of a solid lead bar to that of a gold bar of equal volume for the same pressure change.

11.31. A petite young woman distributes her 500 N weight equally over the heels of her high-heeled shoes. Each heel has an area of $0.750\ \text{cm}^2$. (a) What pressure is exerted on the floor by each heel? (b) With the same pressure, how much weight could be supported by two flat-bottomed sandals, each of area $200\ \text{cm}^2$?

11.32. In the Challenger Deep of the Marianas Trench, the depth of seawater is 10.9 km and the pressure is 1.16×10^8 Pa (about 1.15×10^3 atm). (a) If a cubic meter of water is taken from the surface to this depth, what is the change in its volume? (Normal atmospheric pressure is about 1.0×10^5 Pa. Assume that k for seawater is the same as the freshwater value given in Table 11.2.) (b) What is the density of seawater at this depth? (At the surface, seawater has a density of $1.03 \times 10^3\ \text{kg/m}^3$.)

11.33. A specimen of oil having an initial volume of $600\ \text{cm}^3$ is subjected to a pressure increase of 3.6×10^6 Pa, and the volume is found to decrease by $0.45\ \text{cm}^3$. What is the bulk modulus of the material? The compressibility?

11.34. A square steel plate is 10.0 cm on a side and 0.500 cm thick. (a) Find the shear strain that results if a force of magnitude 9.0×10^5 N is applied to each of the four sides, parallel to the side. (b) Find the displacement x in centimeters.

11.35. A copper cube measures 6.00 cm on each side. The bottom face is held in place by very strong glue to a flat horizontal surface, while a horizontal force F is applied to the upper face parallel to one of the edges. (Consult Table 11.1.) (a) Show that the glue exerts a force F on the bottom face that is equal but opposite to the force on the top face. (b) How large must F be to cause the cube to deform by 0.250 mm? (c) If the same experiment were performed on a lead cube of the same size as the copper one, by what distance would it deform for the same force as in part (b)?

11.36. Shear forces are applied to a rectangular solid. The same forces are applied to another rectangular solid of the same material, but with three times each edge length. In each case the forces are small enough that Hooke's law is obeyed. What is the ratio of the shear strain for the larger object to that of the smaller object?

Section 11.5 Elasticity and Plasticity

11.37. In a materials testing laboratory, a metal wire made from a new alloy is found to break when a tensile force of 90.8 N is applied perpendicular to each end. If the diameter of the wire is 1.84 mm, what is the breaking stress of the alloy?

11.38. A 4.0-m-long steel wire has a cross-sectional area of $0.050\ \text{cm}^2$. Its proportional limit has a value of 0.0016 times its Young's modulus (see Table 11.1). Its breaking stress has a value of 0.0065 times its Young's modulus. The wire is fastened at its upper end and hangs vertically. (a) How great a weight can be hung from the wire without exceeding the proportional limit? (b) How much will the wire stretch under this load? (c) What is the maximum weight that the wire can support?

11.39. A steel cable with cross-sectional area $3.00\ \text{cm}^2$ has an elastic limit of 2.40×10^8 Pa. Find the maximum upward acceleration that can be given a 1200-kg elevator supported by the cable if the stress is not to exceed one-third of the elastic limit.

11.40. A brass wire is to withstand a tensile force of 350 N without breaking. What minimum diameter must the wire have?

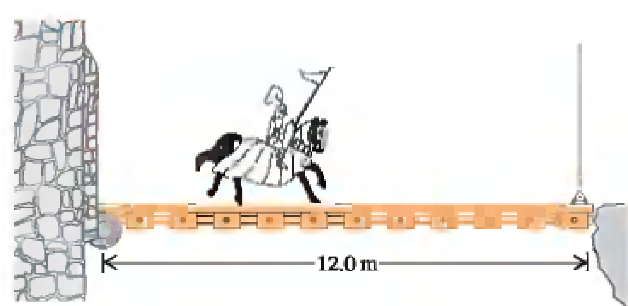
Problems

11.41. Mountain Climbing. Mountaineers often use a rope to lower themselves down the face of a cliff (this is called *rappelling*). They do this with their body nearly horizontal and their feet pushing against the cliff (Fig. 11.33). Suppose that an 82.0-kg climber, who is 1.90 m tall and has a center of gravity 1.1 m from his feet, rappels down a vertical cliff with his body raised 35.0° above the horizontal. He holds the rope 1.40 m from his feet, and it makes a 25.0° angle with the cliff face. (a) What tension does his rope need to support? (b) Find the horizontal and vertical components of the force that the cliff face exerts on the climber's feet. (c) What minimum coefficient of static friction is needed to prevent the climber's feet from slipping on the cliff face if he has one foot at a time against the cliff?

Figure 11.33 Problem 11.41.

11.42. Sir Lancelot rides slowly out of the castle at Camelot and onto the 12.0-m-long drawbridge that passes over the moat (Fig. 11.34). Unbeknownst to him, his enemies have partially

Figure 11.34 Problem 11.42.



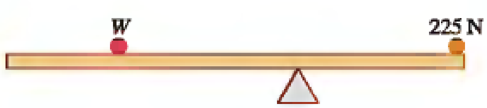
severed the vertical cable holding up the front end of the bridge so that it will break under a tension of 5.80×10^3 N. The bridge has mass 200 kg and its center of gravity is at its center. Lancelot, his lance, his armor, and his horse together have a combined mass of 600 kg. Will the cable break before Lancelot reaches the end of the drawbridge? If so, how far from the castle end of the bridge will the center of gravity of the horse plus rider be when the cable breaks?

11.43. Three vertical forces act on an airplane when it is flying at a constant altitude and with a constant velocity. These are the weight of the airplane, an aerodynamic force on the wing of the airplane, and an aerodynamic force on the airplane's horizontal tail. (The aerodynamic forces are exerted by the surrounding air, and are reactions to the forces that the wing and tail exert on the air as the airplane flies through it.) For a particular light airplane with a weight of 6700 N, the center of gravity is 0.30 m in front of the point where the wing's vertical aerodynamic force acts and 3.66 m in front of the point where the tail's vertical aerodynamic force acts. Determine the magnitude and direction (upward or downward) of each of the two vertical aerodynamic forces.

11.44. A pickup truck has a wheelbase of 3.00 m. Ordinarily, 10,780 N rests on the front wheels and 8820 N on the rear wheels when the truck is parked on a level road. (a) A box weighing 3600 N is now placed on the tailgate, 1.00 m behind the rear axle. How much total weight now rests on the front wheels? On the rear wheels? (b) How much weight would need to be placed on the tailgate to make the front wheels come off the ground?

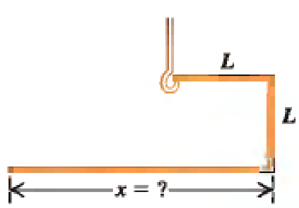
11.45. A uniform, 255-N rod that is 2.00 m long carries a 225-N weight at its right end and an unknown weight W toward the left end (Fig. 11.35). When W is placed 50.0 cm from the left end of the rod, the system just balances horizontally when the fulcrum is located 75.0 cm from the right end. (a) Find W. (b) If W is now moved 25.0 cm to the right, how far and in what direction must the fulcrum be moved to restore balance?

Figure 11.35 Problem 11.45.



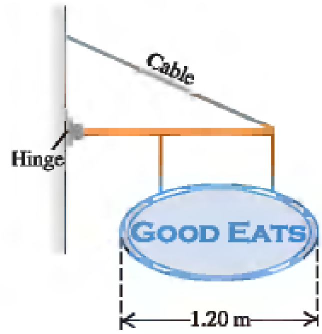
11.46. A thin uniform metal rod is bent into three perpendicular segments, two of which have length L. You want to determine what the length of the third segment should be so that the unit will hang with two segments horizontal when it is supported by a hook as shown in Fig. 11.36. Find x in terms of L.

Figure 11.36 Problem 11.46.



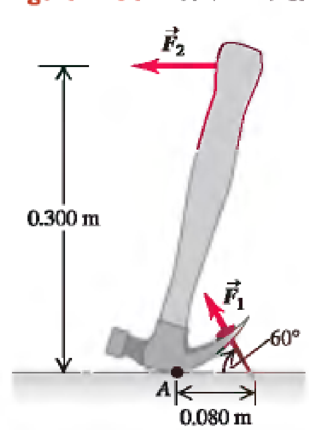
11.47. You open a restaurant and hope to entice customers by hanging out a sign (Fig. 11.37). The uniform horizontal beam supporting the sign is 1.50 m long, has a mass of 18.0 kg, and is hinged to the wall. The sign itself is uniform with a mass of 28.0 kg and overall length of 1.20 m. The two wires supporting the sign are each 32.0 cm long, are 90.0 cm apart, and are equally spaced from the middle of the sign. The cable supporting the beam is 2.00 m long. (a) What minimum tension must your cable be able to support without having your sign come crashing down? (b) What minimum vertical force must the hinge be able to support without pulling out of the wall?

Figure 11.37 Problem 11.47.



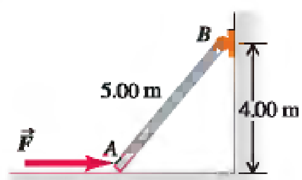
11.48. A claw hammer is used to pull a nail out of a board (Fig. 11.38). The nail is at an angle of 60° to the board, and a force $\vec{F}_1$ of magnitude 500 N applied to the nail is required to pull it from the board. The hammer head contacts the board at point A, which is 0.080 m from where the nail enters the board. A horizontal force $\vec{F}_2$ is applied to the hammer handle at a distance of 0.300 m above the board. What magnitude of force $\vec{F}_2$ is required to apply the required 500-N force (F_1) to the nail? (You can ignore the weight of the hammer.)

Figure 11.38 Problem 11.48.



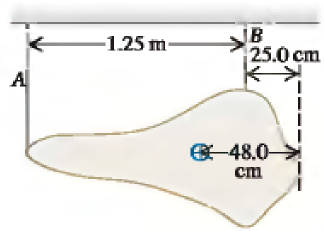
11.48. End A of the bar AB in Fig. 11.39 rests on a frictionless horizontal surface, and end B is hinged. A horizontal force $\vec{F}$ of magnitude 120 N is exerted on end A. You can ignore the weight of the bar. What are the horizontal and vertical components of the force exerted by the bar on the hinge at B?

Figure 11.39 Problem 11.49.



11.50. A museum of modern art is displaying an irregular 358-N sculptore by hanging it from two thin vertical wires, A and B, that are 1.25 m apart (Fig. 11.40). The center of gravity of this piece of art is located 48.0 cm from its extreme right tip. Find the tension in each wire.

Figure 11.40 Problem 11.50.

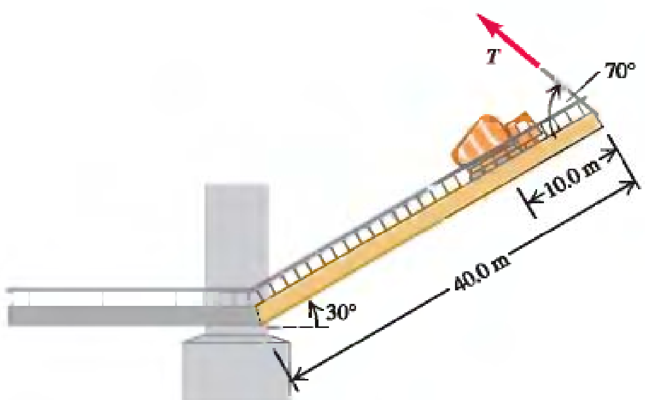


11.51. A beam of mass M and length L is supported horizontally at its ends by two cables making angles θ and ϕ with the horizontal ceiling (Fig. 11.41). (a) Show that if the beam is uniform, these two angles must be equal and the tensions in the cables must also be equal. (b) Suppose now that the center of gravity is $3L/4$ from the left end of the beam. Show that the angles are not completely independent but must obey the equation $\tan\theta = 3\tan\phi$.

Figure 11.41 Problem 11.51.

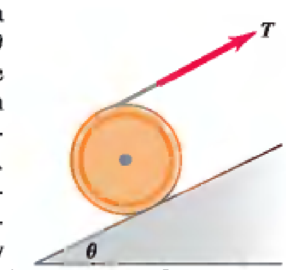
11.52. A Truck on a Drawbridge. A loaded cement mixer drives onto an old drawbridge, where it stalls with its center of gravity three-quarters of the way across the span. The truck driver radios for help, sets the handbrake, and waits. Meanwhile, a boat approaches, so the drawbridge is raised by means of a cable attached to the end opposite the hinge (Fig. 11.42). The drawbridge is 40.0 m long and has a mass of 12,000 kg; its center of gravity is at its midpoint. The cement mixer, with driver, has mass 30,000 kg. When the drawbridge has been raised to an angle of 30° above the horizontal, the cable makes an angle of 70° with the surface of the bridge. (a) What is the tension T in the cable when the drawbridge is held in this position? (b) What are the horizontal and vertical components of the force the hinge exerts on the span?

Figure 11.42 Problem 11.52.



11.53. A uniform solid cylinder of mass M is supported on a ramp that rises at an angle θ above the horizontal by a wire that is wrapped around its rim and pulls on it tangentially parallel to the ramp (Fig. 11.43). (a) Show that there *must* be friction on the surface for the cylinder to balance this way. (b) Show that the tension in the wire must be equal to the friction force, and find this tension.

Figure 11.43 Problem 11.53.

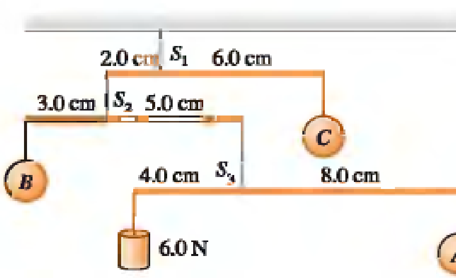


11.54. A nonuniform fire escape ladder is 6.0 m long when extended to the icy alley below. It is held at the top by a frictionless pivot, and there is negligible frictional force from the icy surface at the bottom. The ladder weighs 250 N, and its center of gravity is 2.0 m along the ladder from its bottom. A mother and child of total weight 750 N are on the ladder 1.5 m from the pivot. The ladder makes an angle θ with the horizontal. Find the magnitude and direction of (a) the force exerted by the icy alley on the ladder and (b) the force exerted by the ladder on the pivot. (c) Do your answers in parts (a) and (b) depend on the angle θ?

11.55. A uniform strut of mass m makes an angle θ with the horizontal. It is supported by a frictionless pivot located at one-third its length from its lower left end and a horizontal rope at its upper right end. A cable and package of total weight w hang from its upper right end. (a) Find the vertical and horizontal components V and H of the pivot's force on the strut as well as the tension T in the rope. (b) If the maximum safe tension in the rope is 700 N and the mass of the strut is 20.0 kg, find the maximum safe weight of the cable and package when the strut makes an angle of 55.0° with the horizontal. (c) For what angle θ can no weight be safely suspended from the right end of the strut?

11.56. You are asked to design the decorative mobile shown in Fig. 11.44. The strings and rods have negligible weight, and the rods are to hang horizontally. (a) Draw a free-body diagram for each rod. (b) Find the weights of the balls A, B, and C. Find the tensions in the strings S_1, S_2, and S_3. (c) What can you say about the horizontal location of the mobile's center of gravity? Explain.

Figure 11.44 Problem 11.56.

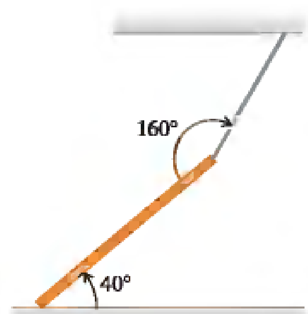


11.57. A uniform, 7.5-m-long beam weighing 9000 N is hinged to a wall and supported by a thin cable attached 1.5 m from the free end of the beam. The cable runs between the beam and the wall and makes a 40° angle with the beam. What is the tension in the cable when the beam is at an angle of 30° above the horizontal?

11.58. A uniform drawbridge must be held at a 37° angle above the horizontal to allow ships to pass underneath. The drawbridge weighs 45,000 N and is 14.0 m long. A cable is connected 3.5 m from the hinge where the bridge pivots (measured along the bridge) and pulls horizontally on the bridge to hold it in place. (a) What is the tension in the cable? (b) Find the magnitude and direction of the force the hinge exerts on the bridge.

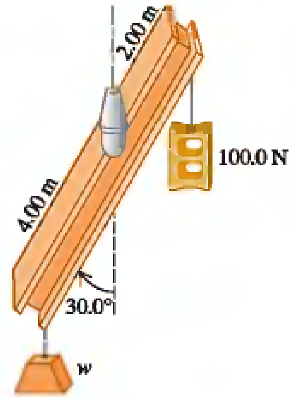
11.58. A uniform, 250-kg beam is supported by a cable connected to the ceiling, as shown in Fig. 11.45. The lower end of the beam rests on the floor. (a) What is the tension in the cable? (b) What is the minimum coefficient of static friction between the beam and the floor required for the beam to remain in this position?

Figure 11.45 Problem 11.59.



11.60. (a) In Fig. 11.46 a 6.00-m-long, uniform beam is hanging from a point 1.00 m to the right of its center. The beam weighs 140 N and makes an angle of 30.0° with the vertical. At the right-hand end of the beam a 100-N weight is hung; an unknown weight w hangs at the left end. If the system is in equilibrium, what is w? You can ignore the thickness of the beam. (b) If the beam makes, instead, an angle of 45.0° with the vertical, what is w?

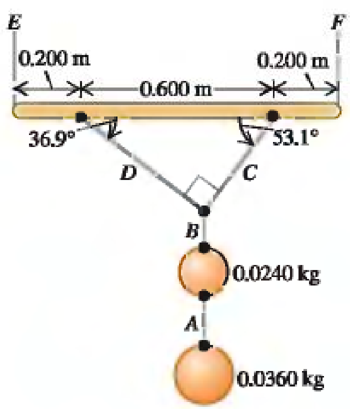
Figure 11.46 Problem 11.60.



11.61. A uniform, horizontal flagpole 5.00 m long with a weight of 200 N is hinged to a vertical wall at one end. A 600-N stuntwoman hangs from its other end. The flagpole is supported by a guy wire running from its outer end to a point on the wall directly above the pole. (a) If the tension in this wire is not to exceed 1000 N, what is the minimum height above the pole at which it may be fastened to the wall? (b) If the flagpole remains horizontal, by how many newtons would the tension be increased if the wire were fastened 0.50 m below this point?

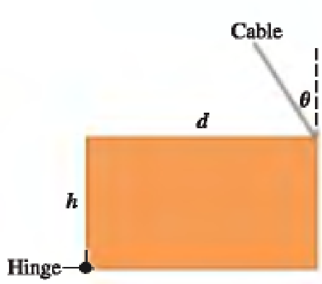
11.62. A holiday decoration consists of two shiny glass spheres with masses 0.0240 kg and 0.0360 kg suspended, as shown in Fig. 11.47, from a uniform rod with mass 0.120 kg and length 1.00 m. The rod is suspended from the ceiling by a vertical cord at each end, so that it is horizontal. Calculate the tension in each of the cords A through F.

Figure 11.47 Problem 11.62.



11.63. A uniform rectangular plate of width d, height h, and weight W is supported with its top and bottom edges horizontal (Fig. 11.48). At the lower left corner there is a hinge, and at the upper right corner there is a cable. (a) For what angle θ with the vertical will the tension in the cable be the least, and what is that tension? (b) Under the conditions of part (a), find the horizontal and vertical components of the force that the hinge exerts on the plate.

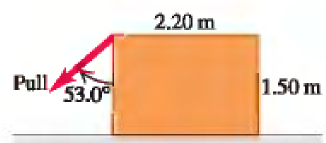
Figure 11.48 Problem 11.63.



11.64. When you stretch a wire, rope, or rubber band, it gets thinner as well as longer. When Hooke's law holds, the fractional decrease in width is proportional to the tensile strain. If w_0 is the original width and Δw is the change in width, then $\Delta w/w_0 = -\sigma \Delta l/l_0$, where the minus sign reminds us that width decreases when length increases. The dimensionless constant σ, different for different materials, is called *Poisson's ratio.* (a) If the steel rod of Example 11.5 (Section 11.4) has a circular cross section and a Poisson's ratio of 0.23, what is its change in diameter when the milling machine is hung from it? (b) A cylinder made of nickel (Poisson's ratio = 0.42) has radius 2.0 cm. What tensile force $F_{\perp}$ must be applied perpendicular to each end of the cylinder to cause its radius to decrease by 0.10 mm? Assume that the breaking stress and proportional limit for the metal are extremely large and are not exceeded.

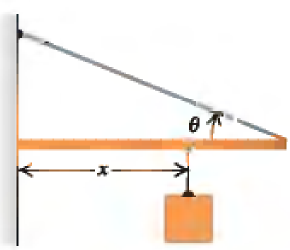
11.65. A worker wants to turn over a uniform 1250-N rectangular crate by pulling at 53.0° on one of its vertical sides (Fig. 11.49). The floor is rough enough to prevent the crate from slipping. (a) What pull is needed to just start the crate to tip? (b) How hard does the floor push on the crate? (c) Find the friction force on the crate. (d) What is the minimum coefficient of static friction needed to prevent the crate from slipping on the floor?

Figure 11.49 Problem 11.65.



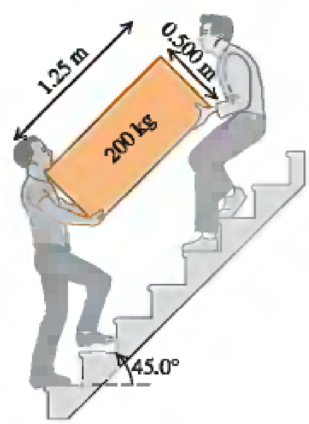
11.66. One end of a uniform meter stick is placed against a vertical wall (Fig. 11.50). The other end is held by a lightweight cord that makes an angle θ with the stick. The coefficient of static friction between the end of the meter stick and the wall is 0.40. (a) What is the maximum value the angle θ can have if the stick is to remain in equilibrium? (b) Let the angle θ be 15°. A block of the same weight as the meter stick is suspended from the stick, as shown, at a distance x from the wall. What is the minimum value of x for which the stick will remain in equilibrium? (c) When $\theta = 15°$, how large must the coefficient of static friction be so that the block can be attached 10 cm from the left end of the stick without causing it to slip?

Figure 11.50 Problem 11.66.



11.67. Two friends are carrying a 200-kg crate up a flight of stairs. The crate is 1.25 m long and 0.500 m high, and its center of gravity is at its center. The stairs make a 45.0° angle with respect to the floor. The crate also is carried at a 45.0° angle, so that its bottom side is parallel to the slope of the stairs (Fig. 11.51). If the force each person applies is

Figure 11.51 Problem 11.67.



vertical, what is the magnitude of each of these forces? Is it better to be the person above or below on the stairs?

11.60. Forearm. In the human arm, the forearm and hand pivot about the elbow joint. Consider a simplified model in which the biceps muscle is attached to the forearm 3.80 cm from the elbow joint. Assume that the person's hand and forearm together weigh 15.0 N and that their center of gravity is 15.0 cm from the elbow (not quite halfway to the hand). The forearm is held horizontally at a right angle to the upper arm, with the biceps muscle exerting its force perpendicular to the forearm. (a) Draw a free-body diagram for the forearm, and find the force exerted by the biceps when the hand is empty. (b) Now the person holds a 80.0-N weight in his hand, with the forearm still horizontal. Assume that the center of gravity of this weight is 33.0 cm from the elbow. Construct a free-body diagram for the forearm, and find the force now exerted by the biceps. Explain why the biceps muscle needs to be very strong. (c) Under the conditions of part (b), find the magnitude and direction of the force that the elbow joint exerts on the forearm. (d) While holding the 80.0-N weight, the person raises his forearm until it is at an angle of 53.0° above the horizontal. If the biceps muscle continues to exert its force perpendicular to the forearm, what is this force when the forearm is in this position? Has the force increased or decreased from its value in part (b)? Explain why this is so, and test your answer by actually doing this with your own arm.

11.60. Refer to the discussion of holding a dumbbell in Example 11.4 (Section 11.3). The maximum weight that can be held in this way is limited by the maximum allowable tendon tension T (determined by the strength of the tendons) and by the distance D from the elbow to where the tendon attaches to the forearm. (a) Let T_{max} represent the maximum value of the tendon tension. Use the results of Example 11.4 to express w_{max} (the maximum weight that can be held) in terms of T_{max}, L, D, and h. Your expression should *not* include the angle θ. (b) The tendons of different primates are attached to the forearm at different values of D. Calculate the derivative of w_{max} with respect to D, and determine whether the derivative is positive or negative. (c) A chimpanzee tendon is attached to the forearm at a point farther from the elbow than for humans. Use this to explain why chimpanzees have stronger arms than humans. (The disadvantage is that chimpanzees have less flexible arms than do humans.)

11.70. A uniform, 90.0-N table is 3.6 m long, 1.0 m high, and 1.2 m wide. A 1500-N weight is placed 0.50 m from one end of the table, a distance of 0.60 m from each of the two legs at that end. Draw a free-body diagram for the table and find the force that each of the four legs exerts on the floor.

11.71. Flying Buttress. (a) A symmetric building has a roof sloping upward at 35.0° above the horizontal on each side. If each side of the uniform roof weighs 10,000 N, find the horizontal force that this roof exerts at the top of the wall, which tends to push out the walls. Which type of building would be more in danger of collapsing: one with tall walls or one with short walls? Explain. (b) As you saw in part (a), tall walls are in danger of collapsing from the weight of the roof. This problem plagued the ancient builders of large structures. A solution used in the great Gothic cathedrals during the 1200s was the flying buttress, a stone support running between the walls and the ground that helped to hold in the walls. A Gothic church has a uniform roof weighing a total of 20,000 N and rising at 40° above the horizontal at each wall. The walls are 40 m tall, and a flying buttress meets each wall 10 m below the base of the roof. What horizontal force must this flying buttress apply to the wall?

11.72. You are trying to raise a bicycle wheel of mass m and radius R up over a curb of height h. To do this, you apply a horizontal force $\vec{F}$ (Fig. 11.52). What is the smallest magnitude of the force $\vec{F}$ that will succeed in raising the wheel onto the curb when the force is applied (a) at the center of the wheel, and (b) at the top of the wheel? (c) In which case is less force required?

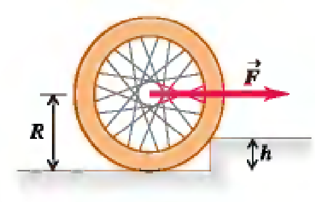


Figure 11.52 Problem 11.72.

11.73. The Farmyard Gate. A gate 4.00 m wide and 2.00 m high weighs 500 N. Its center of gravity is at its center, and it is hinged at A and B. To relieve the strain on the top hinge, a wire CD is connected as shown in Fig. 11.53. The tension in CD is increased until the horizontal force at hinge A is zero. (a) What is the tension in the wire CD? (b) What is the magnitude of the horizontal component of the force at hinge B? (c) What is the combined vertical force exerted by hinges A and B?

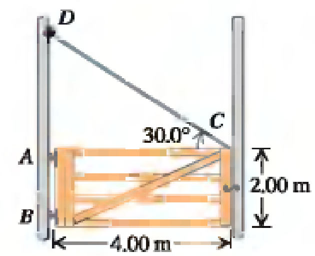
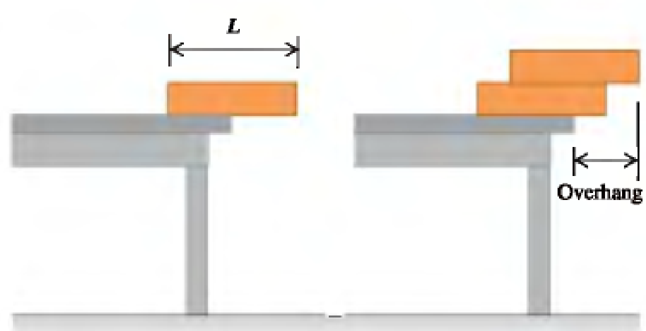


Figure 11.53 Problem 11.73.

11.74. If you put a uniform block at the edge of a table, the center of the block must be over the table for the block not to fall off. (a) If you stack two identical blocks at the table edge, the center of the top block must be over the bottom block, and the center of gravity of the two blocks together must be over the table. In terms of the length L of each block, what is the maximum overhang possible (Fig. 11.54)? (b) Repeat part (a) for three identical blocks and for four identical blocks. (c) Is it possible to make a stack of blocks such that the uppermost block is not directly over the table at all? How many blocks would it take to do this? (Try this with your friends using copies of this book.)

Figure 11.54 Problem 11.74.



11.75. Two uniform 75.0-g marbles 2.00 cm in diameter are stacked as shown in Fig. 11.55 in a container that is 3.00 cm wide. (a) Find the force that the container exerts on the marbles at the points of contact A, B, and C. (b) What force does each marble exert on the other?

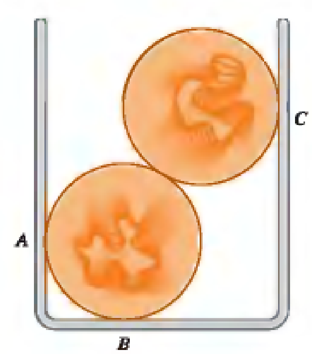
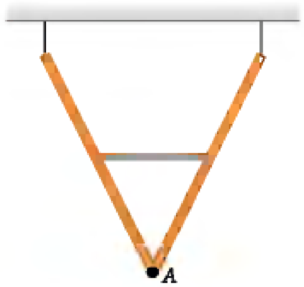


Figure 11.55 Problem 11.75.

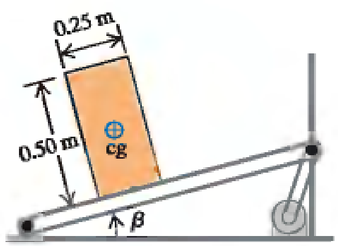
11.76. Two identical, uniform beams weighing 260 N each are connected at one end by a frictionless hinge. A light horizontal crossbar attached at the midpoints of the beams maintains an angle of 53.0° between the beams. The beams are suspended from the ceiling by vertical wires such that they form a "V," as shown in Fig. 11.56. (a) What force does the crossbar exert on each beam? (b) Is the crossbar under tension or compression? (c) What force (magnitude and direction) does the hinge at point A exert on each beam?

Figure 11.56 Problem 11.76.



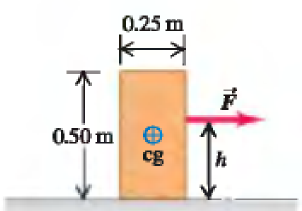
11.77. An engineer is designing a conveyor system for loading hay bales into a wagon (Fig. 11.57). Each bale is 0.25 m wide, 0.50 m high, and 0.80 m long (the dimension perpendicular to the plane of the figure), with mass 30.0 kg. The center of gravity of each bale is at its geometrical center. The coefficient of static friction between a bale and the conveyor belt is 0.60, and the belt moves with constant speed. (a) The angle β of the conveyor is slowly increased. At some critical angle a bale will tip (if it doesn't slip first), and at some different critical angle it will slip (if it doesn't tip first). Find the two critical angles and determine which happens at the smaller angle. (b) Would the outcome of part (a) be different if the coefficient of friction were 0.40?

Figure 11.57 Problem 11.77.



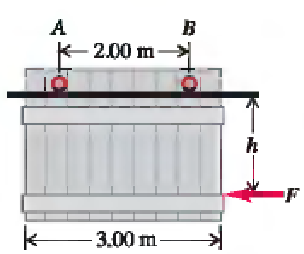
11.70. The hay bale of Problem 11.77 is dragged along a horizontal surface with constant speed by a force $\vec{F}$ (Fig. 11.58). The coefficient of kinetic friction is 0.35. (a) Find the magnitude of the force $\vec{F}$. (b) Find the value of h at which the bale just begins to tip.

Figure 11.58 Problem 11.78.



11.79. A garage door is mounted on an overhead rail (Fig. 11.59). The wheels at A and B have rusted so that they do not roll, but rather slide along the track. The coefficient of kinetic friction is 0.52. The distance between the wheels is 2.00 m, and each is 0.50 m from the vertical sides of the door. The door is uniform and weighs 950 N. It is pushed to the left at constant speed by a horizontal force $\vec{F}$. (a) If the distance h is 1.60 m, what is the vertical component of the force exerted on each wheel by the track? (b) Find the maximum value h can have without causing one wheel to leave the track.

Figure 11.59 Problem 11.79.

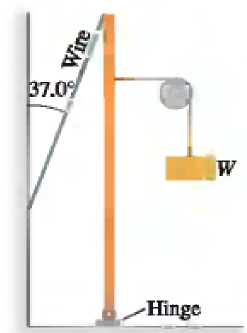


11.80. A horizontal boom is supported at its left end by a frictionless pivot. It is held in place by a cable attached to the right-hand end of the boom. A chain and crate of total weight w hang from somewhere along the boom. The boom's weight w_b cannot be ignored and the boom may or may not be uniform. (a) Show that the tension in the cable is the same whether the cable makes an angle θ or an angle $180° - \theta$ with the horizontal, and that the horizontal force component exerted on the boom by the pivot has equal magnitude bnt opposite direction for the two angles. (b) Show that the cable cannot be horizontal. (c) Show that the tension in the cable is a minimum when the cable is vertical, pulling upward on the right end of the boom. (d) Show that when the cable is vertical, the force exerted by the pivot on the boom is vertical.

11.61. Prior to being placed in its hole, a 5700-N, 9.0-m-long, uniform utility pole makes some nonzero angle with the vertical. A vertical cable attached 2.0 m below its upper end holds it in place while its lower end rests on the ground. (a) Find the tension in the cable and the magnitude and direction of the force exerted by the ground on the pole. (b) Why don't we need to know the angle the pole makes with the vertical, as long as it is not zero?

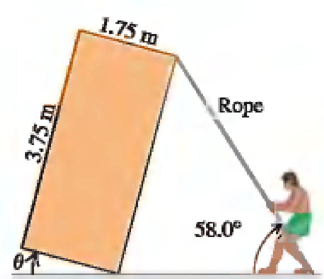
11.82. A weight W is supported by attaching it to a vertical uniform metal pole by a thin cord passing over a pulley having negligible mass and friction. The cord is attached to the pole 40.0 cm below the top and pulls horizontally on it (Fig. 11.60). The pole is pivoted about a hinge at its base, is 1.75 m tall, and weighs 55.0 N. A thin wire connects the top of the pole to a vertical wall. The nail that holds this wire to the wall will pull out if an *outward* force greater than 22.0 N acts on it. (a) What is the greatest weight W that can be supported this way without pulling out the nail? (b) What is the *magnitude* of the force that the hinge exerts on the pole?

Figure 11.60 Problem 11.82.



11.83. Pyramid Builders. Ancient pyramid builders are balancing a uniform rectangular slab of stone tipped at an angle θ above the horizontal using a rope (Fig. 11.61). The rope is held by five workers who share the force equally. (a) If $\theta = 20.0°$, what force does each worker exert on the rope? (b) As θ increases, does each worker have to exert more or less force than in part (a), assuming they do not change the angle of the rope? Why? (c) At what angle do the workers need to exert *no force* to balance the slab? What happens if θ exceeds this value?

Figure 11.61 Problem 11.83.



11.84. Hooke's Law for a Wire. A wire of length l_0 and cross-sectional area A supports a hanging weight W. (a) Show that if the wire obeys Equation (11.7), it behaves like a spring of force constant AY/l_0, where Y is Young's modulus for the material of which the wire is made. (b) What would the force constant be for a 75.0-cm length of 16-gauge $(\text{diameter} = 1.291\text{ mm})$ copper wire? See Table 11.1. (c) What would W have to be to stretch the wire in part (b) by 1.25 mm?

11.85. A 12.0-kg mass, fastened to the end of an aluminum wire with an unstretched length of 0.50 m, is whirled in a vertical circle with a constant angular speed of 120 rev/min. The cross-sectional area of the wire is 0.014 cm^2. Calculate the elongation of the wire when the mass is (a) at the lowest point of the path and (b) at the highest point of its path.

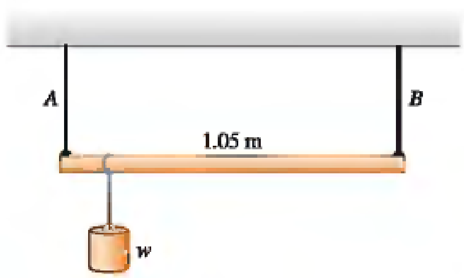
11.60. A metal wire 3.50 m long and 0.70 mm in diameter was given the following test. A load weighing 20 N was originally hung from the wire to keep it taut. The position of the lower end of the wire was read on a scale as load was added.

Added Load (N)	Scale Reading (cm)
0	3.02
10	3.07
20	3.12
30	3.17
40	3.22
50	3.27
60	3.32
70	4.27

(a) Graph these values, plotting the increase in length horizontally and the added load vertically. (b) Calculate the value of Young's modulus. (c) The proportional limit occurred at a scale reading of 3.34 cm. What was the stress at this point?

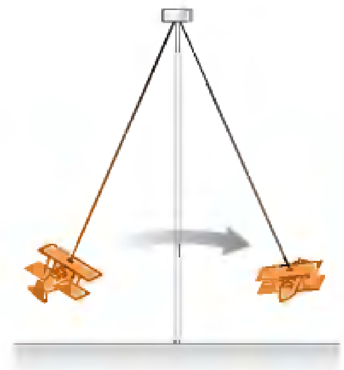
11.87. A 1.05-m-long rod of negligible weight is supported at its ends by wires A and B of equal length (Fig. 11.62). The cross-sectional area of A is 2.00 mm^2 and that of B is 4.00 mm^2. Young's modulus for wire A is 1.80×10^{11} Pa; that for B is 1.20×10^{11} Pa. At what point along the rod should a weight w be suspended to produce (a) equal stresses in A and B, and (b) equal strains in A and B?

Figure 11.62 Problem 11.87.



11.89. An amusement park ride consists of airplane-shaped cars attached to steel rods (Fig. 11.63). Each rod has a length of 15.0 m and a cross-sectional area of 8.00 cm^2. (a) How much is the rod stretched when the ride is at rest? (Assume that each car plus two people seated in it has a total weight of 1900 N.) (b) When operating, the ride has a maximum angular speed of 8.0 rev/min. How much is the rod stretched then?

Figure 11.63 Problem 11.88.



11.89. A brass rod with a length of 1.40 m and a cross-sectional area of 2.00 cm^2 is fastened end to end to a nickel rod with length L and cross-sectional area 1.00 cm^2. The compound rod is subjected to equal and opposite pulls of magnitude 4.00×10^4 N at its ends. (a) Find the length L of the nickel rod if the elongations of the two rods are equal. (b) What is the stress in each rod? (c) What is the strain in each rod?

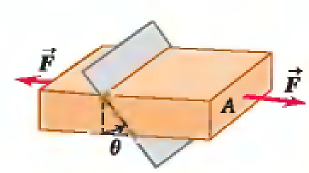
11.90. Stress on the Shin Bone. Compressive strength of our bones is important in everyday life. Young's modulus for bone is about 1.4×10^{10} Pa. Bone can take only about a 1.0% change in its length before fracturing. (a) What is the maximum force that can be applied to a bone whose minimum cross-sectional area is 3.0 cm^2? (This is approximately the cross-sectional area of a tibia, or shin bone, at its narrowest point.) (b) Estimate the maximum height from which a 70-kg man could jump and not fracture the tibia. Take the time between when he first touches the floor and when he has stopped to be 0.030 s, and assume that the stress is distributed equally between his legs.

11.91. You hang a floodlamp from the end of a vertical steel wire. The floodlamp stretches the wire 0.18 mm and the stress is proportional to the strain. How much would it have stretched (a) if the wire were twice as long? (b) If the wire had the same length but twice the diameter? (c) For a copper wire of the original length and diameter?

11.92. A moonshiner produces pure ethanol (ethyl alcohol) late at night and stores it in a stainless steel tank in the form of a cylinder 0.300 m in diameter with a tight-fitting piston at the top. The total volume of the tank is 250 L (0.250 m^3). In an attempt to squeeze a little more into the tank, the moonshiner piles 1420 kg of lead bricks on top of the piston. What additional volume of ethanol can the moonshiner squeeze into the tank? (Assume that the wall of the tank is perfectly rigid.)

11.93. A bar with cross-sectional area A is subjected to equal and opposite tensile forces $\vec{F}$ at its ends. Consider a plane through the bar making an angle θ with a plane at right angles to the bar (Fig. 11.64). (a) What is the tensile (normal) stress at this plane in terms of F, A, and θ? (b) What is the shear (tangential) stress at the plane in terms of F, A, and θ? (c) For what value of θ is the tensile stress a maximum? (d) For what value of θ is the shear stress a maximum?

Figure 11.64 Problem 11.93.



11.84. A horizontal, uniform, copper rod has an original length l_0, cross-sectional area A, Young's modulus Y, and mass m. It is supported by a frictionless pivot at its right end and by a cable at its left end. Both pivot and cable are attached so that they exert their forces uniformly over the rod's cross section. The cable makes an angle θ with the rod and compresses it. (a) Find the stress exerted by the cable and pivot on the rod. (b) Find the change in length of the rod due to this stress. (c) The mass of the rod equals $\rho A l_0$, where ρ is the density. Show that the answers to parts (a) and (b) are independent of the cross-sectional area of the rod. (d) The density of copper is $8900\ \text{kg/m}^3$. Take Y for compression as given for copper in Table 11.1. Find the stress and change in length for an original length of 1.8 m and an angle of 30°. (e) By how much would you multiply the answers of part (d) if the rod were twice as long?

Challenge Problems

11.85. A bookcase weighing 1500 N rests on a horizontal surface for which the coefficient of static friction is $\mu_s = 0.40$. The bookcase is 1.80 m tall and 2.00 m wide; its center of gravity is at its geometrical center. The bookcase rests on four short legs that are each 0.10 m from the edge of the bookcase. A person pulls on a rope attached to an upper corner of the bookcase with a force $\vec{F}$ that makes an angle θ with the bookcase (Fig. 11.65). (a) If $\theta = 90°$, so $\vec{F}$ is horizontal, show that as F is increased from zero, the bookcase will start to slide before it tips, and calculate the magnitude of $\vec{F}$ that will start the bookcase sliding. (b) If $\theta = 0°$, so $\vec{F}$ is vertical, show that the bookcase will tip over rather than slide, and calculate the magnitude of $\vec{F}$ that will cause the bookcase to start to tip. (c) Calculate as a function of θ the magnitude of $\vec{F}$ that will cause the bookcase to start to slide and the magnitude that will cause it to start to tip. What is the smallest value that θ can have so that the bookcase will still start to slide before it starts to tip?

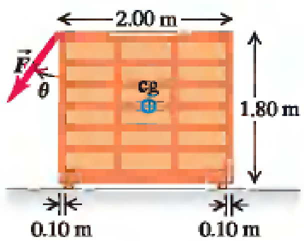


Figure 11.65 Challenge Problem 11.95.

11.96. Knocking Over a Post. One end of a post weighing 400 N and with height h rests on a rough horizontal surface with $\mu_s = 0.30$. The upper end is held by a rope fastened to the surface and making an angle of 36.9° with the post (Fig. 11.66). A horizontal force $\vec{F}$ is exerted on the post as shown. (a) If the force $\vec{F}$ is applied at the midpoint of the post, what is the largest value it can have without causing the post to slip? (b) How large can the force be without causing the post to slip if its point of application is $\frac{6}{10}$ of the way from the ground to the top of the post? (c) Show that if the point of application of the force is too high, the post cannot be made to slip, no matter how great the force. Find the critical height for the point of application.

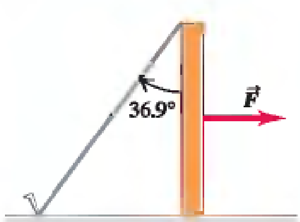
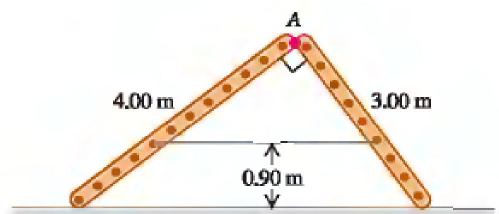


Figure 11.66 Challenge Problem 11.96.

11.97. Minimizing the Tension. A heavy horizontal girder of length L has several objects suspended from it. It is supported by a frictionless pivot at its left end and a cable of negligible weight that is attached to an I-beam at a point a distance h directly above the girder's center. Where should the other end of the cable be attached to the girder so that the cable's tension is a minimum? (*Hint:* In evaluating and presenting your answer, don't forget that the maximum distance of the point of attachment from the pivot is the length L of the beam.)

11.98. Two ladders, 4.00 m and 3.00 m long, are hinged at point A and tied together by a horizontal rope 0.90 m above the floor (Fig. 11.67). The ladders weigh 480 N and 360 N, respectively, and the center of gravity of each is at its center. Assume that the floor is freshly waxed and frictionless. (a) Find the upward force at the bottom of each ladder. (b) Find the tension in the rope. (c) Find the magnitude of the force one ladder exerts on the other at point A. (d) If an 800-N painter stands at point A, find the tension in the horizontal rope.

Figure 11.67 Challenge Problem 11.98.



11.99. A device for measuring compressibility consists of a cylinder filled with oil and fitted with a piston at one end. A block of sodium is immersed in the oil, and a force is applied to the piston. Assume that the piston and walls of the cylinder are perfectly rigid and that there are no friction and no oil leak. Compute the compressibility of the sodium in terms of the applied force F, the piston displacement x, the piston area A, the initial volume of the oil V_O, the initial volume of the sodium V_S, and the compressibility of the oil k_O.

11.100. Bulk Modulus of an Ideal Gas. The equation of state (the equation relating pressure, volume, and temperature) for an ideal gas is $pV = nRT$, where n and R are constants. (a) Show that if the gas is compressed while the temperature T is held constant, the bulk modulus is equal to the pressure. (b) When an ideal gas is compressed without the transfer of any heat into or out of it, the pressure and volume are related by $pV^\gamma = \text{constant}$, where γ is a constant having different values for different gases. Show that, in this case, the bulk modulus is given by $B = \gamma p$.

11.101. An angler hangs a 4.50-kg fish from a vertical steel wire 1.50 m long and $5.00 \times 10^{-3}\ \text{cm}^2$ in cross-sectional area. The upper end of the wire is securely fastened to a support. (a) Calculate the amount the wire is stretched by the hanging fish. The angler now applies a force $\vec{F}$ to the fish, pulling it very slowly downward by 0.500 mm from its equilibrium position. For this downward motion, calculate (b) the work done by gravity; (c) the work done by the force $\vec{F}$; (d) the work done by the force the wire exerts on the fish; and (e) the change in the elastic potential energy (the potential energy associated with the tensile stress in the wire). Compare the answers in parts (d) and (e).

12 GRAVITATION

?The rings of Saturn are made of countless individual orbiting particles. Do all the ring particles orbit at the same speed, or do the inner particles orbit faster or slower than the outer ones?

LEARNING GOALS

By studying this chapter, you will learn:

- How to calculate the gravitational forces that any two bodies exert on each other.
- How to relate the weight of an object to the general expression for gravitational force.
- How to use and interpret the generalized expression for gravitational potential energy.
- How to relate the speed, orbital period, and mechanical energy of a satellite in a circular orbit.
- The laws that describe the motions of planets, and how to work with these laws.
- What black holes are, how to calculate their properties, and how they are discovered.

Some of the earliest investigations in physical science started with questions that people asked about the night sky. Why doesn't the moon fall to earth? Why do the planets move across the sky? Why doesn't the earth fly off into space rather than remaining in orbit around the sun? The study of gravitation provides the answers to these and many related questions.

As we remarked in Chapter 5, gravitation is one of the four classes of interactions found in nature, and it was the earliest of the four to be studied extensively. Newton discovered in the 17th century that the same interaction that makes an apple fall out of a tree also keeps the planets in their orbits around the sun. This was the beginning of *celestial mechanics,* the study of the dynamics of objects in space. Today, our knowledge of celestial mechanics allows us to determine how to put a satellite into any desired orbit around the earth or to choose just the right trajectory to send a spacecraft to another planet.

In this chapter you will learn the basic law that governs gravitational interactions. This law is *universal:* Gravity acts in the same fundamental way between the earth and your body, between the sun and a planet, and between a planet and one of its moons. We'll apply the law of gravitation to phenomena such as the variation of weight with altitude, the orbits of satellites around the earth, and the orbits of planets around the sun.

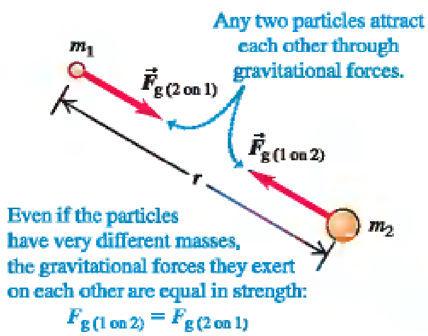
12.1 Newton's Law of Gravitation

The example of gravitational attraction that's probably most familiar to you is your *weight,* the force that attracts you toward the earth. During his study of the motions of the planets and of the moon, Newton discovered the fundamental character of the gravitational attraction between *any* two bodies. Along with his three laws of motion, Newton published the **law of gravitation** in 1687. It may be stated as follows:

Every particle of matter in the universe attracts every other particle with a force that is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them.

Translating this into an equation, we have

$$F_g = \frac{Gm_1m_2}{r^2} \qquad \text{(law of gravitation)} \qquad (12.1)$$



12.1 The gravitational forces between two particles of masses m_1 and m_2.

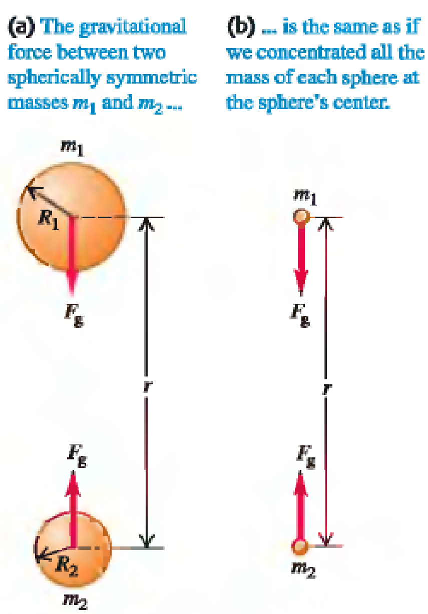
where F_g is the magnitude of the gravitational force on either particle, m_1 and m_2 are their masses, r is the distance between them (Fig. 12.1), and G is a fundamental physical constant called the **gravitational constant.** The numerical value of G depends on the system of units used.

Equation (12.1) tells us that the gravitational force between two particles decreases with increasing distance r: If the distance is doubled, the force is only one-fourth as great, and so on. Although many of the stars in the night sky are far more massive than the sun, they are so far away that their gravitational force on the earth is negligibly small.

CAUTION **Don't confuse g and G** Because the symbols g and G are so similar, it's common to confuse the two very different gravitational quantities that these symbols represent. Lowercase g is the acceleration due to gravity, which relates the weight w of a body to its mass m: $w = mg$. The value of g is different at different locations on the earth's surface and on the surfaces of different planets. By contrast, capital G relates the gravitational force between any two bodies to their masses and the distance between them. We call G a *universal* constant because it has the same value for any two bodies, no matter where in space they are located. In the next section we'll see how the values of g and G are related.

Gravitational forces always act along the line joining the two particles, and they form an action–reaction pair. Even when the masses of the particles are different, the two interaction forces have equal magnitude (Fig. 12.1). The attractive force that your body exerts on the earth has the same magnitude as the force that the earth exerts on you. When you fall from a diving board into a swimming pool, the entire earth rises up to meet you! (You don't notice this because the earth's mass is greater than yours by a factor of about 10^{23}. Hence the earth's acceleration is only 10^{-23} as great as yours.)

12.2 The gravitational effect *outside* any spherically symmetric mass distribution is the same as though all of the mass were concentrated at its center.



Gravitation and Spherically Symmetric Bodies

We have stated the law of gravitation in terms of the interaction between two *particles*. It turns out that the gravitational interaction of any two bodies having *spherically symmetric* mass distributions (such as solid spheres or spherical shells) is the same as though we concentrated all the mass of each at its center, as in Fig. 12.2. Thus, if we model the earth as a spherically symmetric body with mass m_E, the force it exerts on a particle or a spherically symmetric body with mass m, at a distance r between centers, is

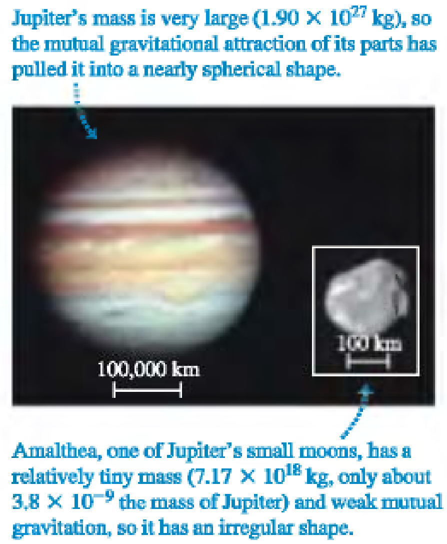
$$F_g = \frac{Gm_Em}{r^2} \qquad (12.2)$$

provided that the body lies outside the earth. A force of the same magnitude is exerted *on* the earth by the body. (We will prove these statements in Section 12.6.)

At points *inside* the earth the situation is different. If we could drill a hole to the center of the earth and measure the gravitational force on a body at various depths, we would find that toward the center of the earth the force *decreases,* rather than increasing as $1/r^2$. As the body enters the interior of the earth (or other spherical body), some of the earth's mass is on the side of the body opposite from the center and pulls in the opposite direction. Exactly at the center, the earth's gravitational force on the body is zero.

Spherically symmetric bodies are an important case because moons, planets, and stars all tend to be spherical. Since all particles in a body gravitationally attract each other, the particles tend to move to minimize the distance between them. As a result, the body naturally tends to assume a spherical shape, just as a lump of clay forms into a sphere if you squeeze it with equal forces on all sides. This effect is greatly reduced in celestial bodies of low mass, since the gravitational attraction is less, and these bodies tend *not* to be spherical (Fig. 12.3).

12.3 Spherical and nonspherical bodies: the planet Jupiter and one of Jupiter's small moons, Amalthea.



Determining the Value of G

To determine the value of the gravitational constant G, we have to *measure* the gravitational force between two bodies of known masses m_1 and m_2 at a known distance r. The force is extremely small for bodies that are small enough to be brought into the laboratory, but it can be measured with an instrument called a *torsion balance*, which Sir Henry Cavendish used in 1798 to determine G.

A modern version of the Cavendish torsion balance is shown in Fig. 12.4. A light, rigid rod shaped like an inverted T is supported by a very thin, vertical quartz fiber. Two small spheres, each of mass m_1, are mounted at the ends of the horizontal arms of the T. When we bring two large spheres, each of mass m_2, to the positions shown, the attractive gravitational forces twist the T through a small angle. To measure this angle, we shine a beam of light on a mirror fastened to the T. The reflected beam strikes a scale, and as the T twists, the reflected beam moves along the scale.

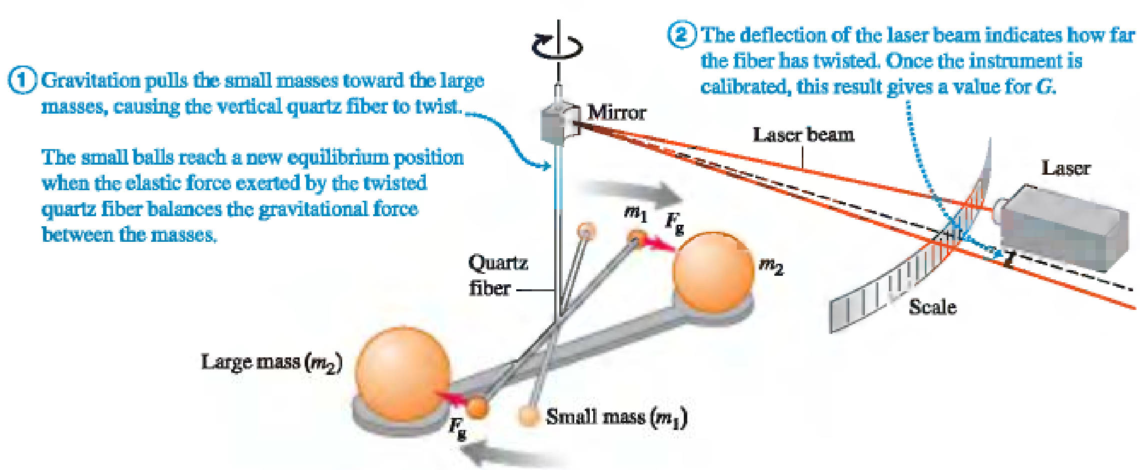
After calibrating the Cavendish balance, we can measure gravitational forces and thus determine G. The presently accepted value (in SI units) is

$$G = 6.6742(10) \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2$$

To three significant figures, $G = 6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2$. Because $1\ \text{N} = 1\ \text{kg} \cdot \text{m/s}^2$, the units of G can also be expressed (in fundamental SI units) as $\text{m}^3/(\text{kg} \cdot \text{s}^2)$.

Gravitational forces combine vectorially. If each of two masses exerts a force on a third, the *total* force on the third mass is the vector sum of the individual forces of the first two. Example 12.3 makes use of this property, which is often called *superposition of forces*.

12.4 The principle of the Cavendish balance, used for determining the value of G. The angle of deflection has been exaggerated here for clarity.



Example 12.1 Calculating gravitational force

The mass m_1 of one of the small spheres of a Cavendish balance is 0.0100 kg, the mass m_2 of one of the large spheres is 0.500 kg, and the center-to-center distance between each large sphere and the nearer small one is 0.0500 m. Find the gravitational force F_g on each sphere due to the nearest other sphere.

SOLUTION

IDENTIFY: Because the 0.0100-kg and 0.500-kg objects are spherically symmetric, we can calculate the gravitational force of one on the other by assuming that they are particles separated by 0.0500 m. Each sphere experiences the *same* magnitude of force from the other sphere, even though their masses are very different.

SET UP: We use the law of gravitation, Eq. (12.1), to determine F_g.

EXECUTE: The magnitude of the force that one sphere exerts on the other is

$$F_g = \frac{(6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2)(0.0100\ \text{kg})(0.500\ \text{kg})}{(0.0500\ \text{m})^2}$$
$$= 1.33 \times 10^{-10}\ \text{N}$$

EVALUATE: This is a very small force, which is what we expect: We don't experience noticeable gravitational pulls from ordinary low-mass objects in our environment. It takes a truly massive object such as the earth to exert a substantial gravitational force.

Example 12.2 Acceleration due to gravitational attraction

Suppose one large sphere and one small sphere are detached from the apparatus in Example 12.1 and placed 0.0500 m (between centers) from each other at a point in space far removed from all other bodies. What is the magnitude of the acceleration of each, relative to an inertial system?

SOLUTION

IDENTIFY: The gravitational forces that the two spheres exert on each other have the same magnitude. (The system of two spheres is so distant from other bodies that we can neglect any other forces.) But the *accelerations* of the two spheres are different because their masses are different.

SET UP: We found the magnitude of the force on each sphere in Example 12.1. To determine the magnitude of each sphere's acceleration, we'll use Newton's second law.

EXECUTE: The acceleration of the smaller sphere has magnitude

$$a_1 = \frac{F_g}{m_1} = \frac{1.33 \times 10^{-10}\ \text{N}}{0.0100\ \text{kg}} = 1.33 \times 10^{-8}\ \text{m/s}^2$$

The acceleration of the larger sphere has magnitude

$$a_2 = \frac{F_g}{m_2} = \frac{1.33 \times 10^{-10}\ \text{N}}{0.500\ \text{kg}} = 2.66 \times 10^{-10}\ \text{m/s}^2$$

EVALUATE: The larger sphere has 50 times the mass of the smaller one and hence has 1/50 the acceleration. Note that the accelerations are *not* constant; the gravitational forces increase as the spheres move toward each other.

Example 12.3 Superposition of gravitational forces

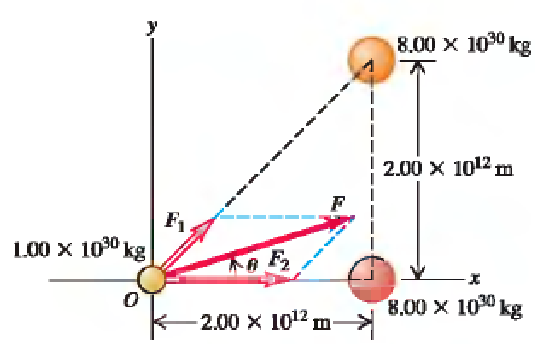
Many stars in the sky are actually systems of two or more stars held together by their mutual gravitational attraction. Figure 12.5 shows a three-star system at an instant when the stars are at the vertices of a 45° right triangle. Find the magnitude and direction of the total gravitational force exerted on the small star by the two large ones.

SOLUTION

IDENTIFY: We use the principle of superposition: The total force on the small star is the vector sum of the forces due to each large star.

SET UP: We assume that the stars are spheres so that we can use the law of gravitation for each force, as in Fig. 12.2. We first calculate the magnitude of each force using Eq. (12.1) and then compute the vector sum using components along the axes shown in Fig. 12.5.

12.5 The total gravitational force on the small star (at O) is the vector sum of the forces exerted on it by the two larger stars. (For comparison, the mass of the sun—a rather ordinary star—is 1.99×10^{30} kg and the earth–sun distance is 1.50×10^{11} m.)



EXECUTE: The magnitude F_1 of the force on the small star due to the upper large one is

$$F_1 = \frac{\left[\begin{array}{c}(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}) \\ \times (8.00 \times 10^{30}\ \mathrm{kg})(1.00 \times 10^{30}\ \mathrm{kg})\end{array}\right]}{(2.00 \times 10^{12}\ \mathrm{m})^2 + (2.00 \times 10^{12}\ \mathrm{m})^2}$$
$$= 6.67 \times 10^{25}\ \mathrm{N}$$

The magnitude F_2 of the force due to the lower large star is

$$F_2 = \frac{\left[\begin{array}{c}(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}) \\ \times (8.00 \times 10^{30}\ \mathrm{kg})(1.00 \times 10^{30}\ \mathrm{kg})\end{array}\right]}{(2.00 \times 10^{12}\ \mathrm{m})^2}$$
$$= 1.33 \times 10^{26}\ \mathrm{N}$$

The x- and y-components of these forces are

$$F_{1x} = (6.67 \times 10^{25}\ \mathrm{N})(\cos 45°) = 4.72 \times 10^{25}\ \mathrm{N}$$
$$F_{1y} = (6.67 \times 10^{25}\ \mathrm{N})(\sin 45°) = 4.72 \times 10^{25}\ \mathrm{N}$$
$$F_{2x} = 1.33 \times 10^{26}\ \mathrm{N}$$
$$F_{2y} = 0$$

The components of the total force on the small star are

$$F_x = F_{1x} + F_{2x} = 1.81 \times 10^{26}\ \mathrm{N}$$
$$F_y = F_{1y} + F_{2y} = 4.72 \times 10^{25}\ \mathrm{N}$$

The magnitude of this force is

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{(1.81 \times 10^{26}\ \mathrm{N})^2 + (4.72 \times 10^{25}\ \mathrm{N})^2}$$
$$= 1.87 \times 10^{26}\ \mathrm{N}$$

and its direction relative to the x-axis is

$$\theta = \arctan\frac{F_y}{F_x} = \arctan\frac{4.72 \times 10^{25}\ \mathrm{N}}{1.81 \times 10^{26}\ \mathrm{N}} = 14.6°$$

EVALUATE: While the total force on the small star is tremendous, the magnitude of the resulting acceleration is not: $a = F/m = (1.87 \times 10^{26}\ \mathrm{N})/(1.00 \times 10^{30}\ \mathrm{kg}) = 1.87 \times 10^{-4}\ \mathrm{m/s^2}$.

Can you show that the total force on the small star is *not* directed toward the center of mass of the two large stars? (See Problem 12.51.)

Why Gravitational Forces Are Important

Comparing Examples 12.1 and 12.3 shows that gravitational forces are negligible between ordinary household-sized objects, but very substantial between objects that are the size of stars. Indeed, gravitation is *the* most important force on the scale of planets, stars, and galaxies (Fig. 12.6). It is responsible for holding our earth together and for keeping the planets in orbit about the sun. The mutual gravitational attraction between different parts of the sun compresses material at the sun's core to very high densities and temperatures, making it possible for nuclear reactions to take place there. These reactions generate the sun's energy output, which makes it possible for life to exist on earth and for you to read these words.

The gravitational force is so important on the cosmic scale because it acts *at a distance,* without any direct contact between bodies. Electric and magnetic forces have this same remarkable property, but they are less important on astronomical scales because large accumulations of matter are electrically neutral; that is, they contain equal amounts of positive and negative charge. As a result, the electric and magnetic forces between stars or planets are very small or zero. The strong and weak interactions that we discussed in Section 5.5 also act at a distance, but their influence is negligible at distances much greater than the diameter of an atomic nucleus (about 10^{-14} m).

A useful way to describe forces that act at a distance is in terms of a *field.* One body sets up a disturbance or field at all points in space, and the force that acts on a second body at a particular point is its response to the first body's field at that point. There is a field associated with each force that acts at a distance, and so we refer to gravitational fields, electric fields, magnetic fields, and so on. We won't need the field concept for our study of gravitation in this chapter, so we won't discuss it further here. But in later chapters we'll find that the field concept is an extraordinarily powerful tool for describing electric and magnetic interactions.

12.6 Our solar system is part of a spiral galaxy like this one, which contains roughly 10^{11} stars as well as gas, dust, and other matter. The entire assemblage is held together by the mutual gravitational attraction of all the matter in the galaxy.

Test Your Understanding of Section 12.1 The planet Saturn has about 100 times the mass of the earth and is about 10 times farther from the sun than the earth is. Compared to the acceleration of the earth caused by the sun's gravitational pull, how great is the acceleration of Saturn due to the sun's gravitation? (i) 100 times greater; (ii) 10 times greater; (iii) the same; (iv) 1/10 as great; (v) 1/100 as great.

12.2 Weight

We defined the *weight* of a body in Section 4.4 as the attractive gravitational force exerted on it by the earth. We can now broaden our definition:

The weight of a body is the total gravitational force exerted on the body by all other bodies in the universe.

When the body is near the surface of the earth, we can neglect all other gravitational forces and consider the weight as just the earth's gravitational attraction. At the surface of the *moon* we consider a body's weight to be the gravitational attraction of the moon, and so on.

If we again model the earth as a spherically symmetric body with radius R_E and mass m_E, the weight w of a small body of mass m at the earth's surface (a distance R_E from its center) is

$$w = F_g = \frac{Gm_E m}{R_E^2} \quad \text{(weight of a body of mass } m \text{ at the earth's surface)} \tag{12.3}$$

But we also know from Section 4.4 that the weight w of a body is the force that causes the acceleration g of free fall, so by Newton's second law, $w = mg$. Equating this with Eq. (12.3) and dividing by m, we find

$$g = \frac{Gm_E}{R_E^2} \quad \text{(acceleration due to gravity at the earth's surface)} \tag{12.4}$$

The acceleration due to gravity g is independent of the mass m of the body because m doesn't appear in this equation. We already knew that, but we can now see how it follows from the law of gravitation.

We can *measure* all the quantities in Eq. (12.4) except for m_E, so this relationship allows us to compute the mass of the earth. Solving Eq. (12.4) for m_E and using $R_E = 6380\text{ km} = 6.38 \times 10^6\text{ m}$ and $g = 9.80\text{ m/s}^2$, we find

$$m_E = \frac{gR_E^2}{G} = 5.98 \times 10^{24}\text{ kg}$$

This is very close to the currently accepted value of 5.974×10^{24} kg. Once Cavendish had measured G, he computed the mass of the earth in just this way.

At a point above the earth's surface a distance r from the center of the earth (a distance $r - R_E$ above the surface), the weight of a body is given by Eq. (12.3) with R_E replaced by r:

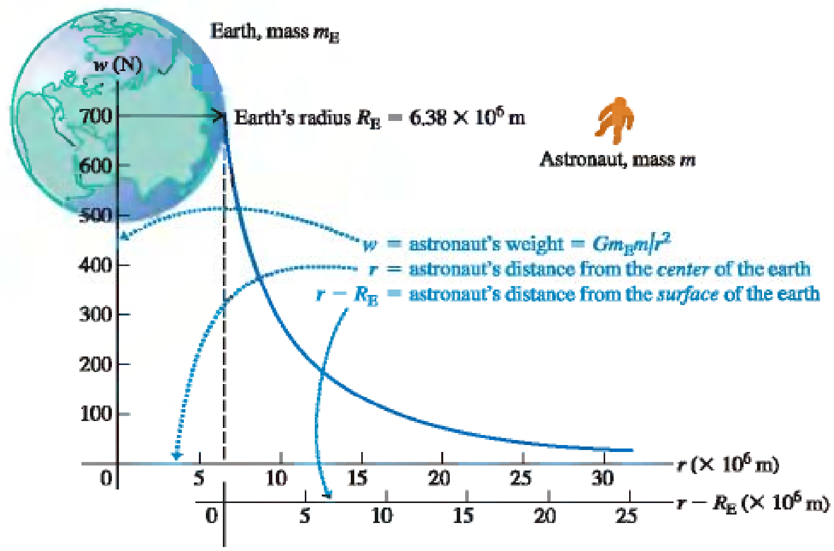
$$w = F_g = \frac{Gm_E m}{r^2} \tag{12.5}$$

The weight of a body decreases inversely with the square of its distance from the earth's center (Fig. 12.7). Figure 12.8 shows how the weight varies with height above the earth for an astronaut who weighs 700 N at the earth's surface.

The *apparent* weight of a body on earth differs slightly from the earth's gravitational force because the earth rotates and is therefore not precisely an inertial frame of reference. We have ignored this effect in our earlier discussion and have assumed that the earth *is* an inertial system. We will return to the effect of the earth's rotation in Section 12.7.

In our discussion of weight, we've used the fact that the earth is an approximately spherically symmetric distribution of mass. But this does *not* mean that the earth is uniform. To demonstrate that it cannot be uniform, let's first calculate

12.7 In an airliner at high altitude, you are farther from the center of the earth than when on the ground and hence weigh slightly less. Can you show that at an altitude of 10 km above the surface, you weigh 0.3% less than you do on the ground?



12.6 An astronaut who weighs 700 N at the earth's surface experiences less gravitational attraction when above the surface. The relevant distance r is from the astronaut to the *center* of the earth (*not* from the astronaut to the earth's surface).

the average *density*, or mass per unit volume, of the earth. If we assume a spherical earth, the volume is

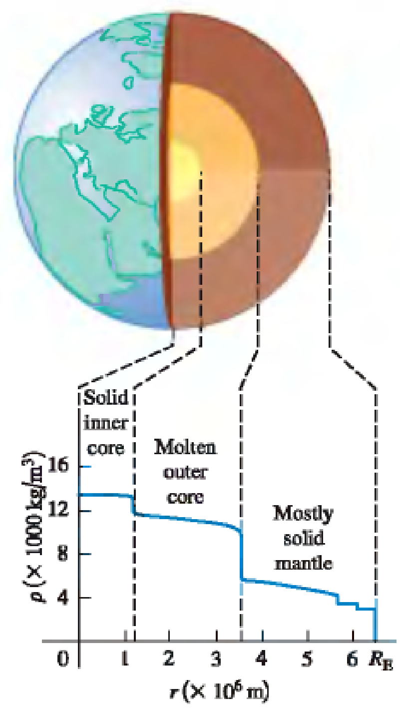
$$V_E = \frac{4}{3}\pi R_E^3 = \frac{4}{3}\pi(6.38 \times 10^6 \text{ m})^3 = 1.09 \times 10^{21} \text{ m}^3$$

The average density ρ (the Greek letter rho) of the earth is the total mass divided by the total volume:

$$\rho = \frac{m_E}{V_E} = \frac{5.97 \times 10^{24} \text{ kg}}{1.09 \times 10^{21} \text{ m}^3}$$
$$= 5500 \text{ kg/m}^3 = 5.5 \text{ g/cm}^3$$

(For comparison, the density of water is $1000 \text{ kg/m}^3 = 1.00 \text{ g/cm}^3$.) If the earth were uniform, we would expect the density of individual rocks near the earth's surface to have this same value. In fact, the density of surface rocks is substantially lower, ranging from about $2000 \text{ kg/m}^3 = 2 \text{ g/cm}^3$ for sedimentary rocks to about $3300 \text{ kg/m}^3 = 3.3 \text{ g/cm}^3$ for basalt. So the earth *cannot* be uniform, and the interior of the earth must be much more dense than the surface in order that the *average* density be $5500 \text{ kg/m}^3 = 5.5 \text{ g/cm}^3$. According to geophysical models of the earth's interior, the maximum density at the center is about $13{,}000 \text{ kg/m}^3 = 13 \text{ g/cm}^3$. Figure 12.9 is a graph of density as a function of distance from the center.

12.9 The density of the earth decreases with increasing distance from its center.



Example 12.4 Gravity on Mars

An unmanned lander is sent to the surface of the planet Mars, which has radius $R_M = 3.40 \times 10^6$ m and mass $m_M = 6.42 \times 10^{23}$ kg. The earth weight of the Mars lander is 3920 N. Calculate its weight F_g and the acceleration g_M due to the gravity of Mars: (a) 6.0×10^6 m above the surface of Mars (the distance at which the moon Phobos orbits Mars); and (b) at the surface of Mars. Neglect the gravitational effects of the (very small) moons of Mars.

SOLUTION

IDENTIFY: We need to find the lander weight F_g and the gravitational acceleration g_M at two different distances from the center of Mars.

SET UP: We find the weight F_g using Eq. (12.5) with m_E (the mass of the earth) replaced with m_M (the mass of Mars). Note that the

Continued

value of G is the same everywhere in the universe; it is a fundamental physical constant. We then find the acceleration g_M using $F_g = mg_M$, where m is the mass of the lander. We're not given the value of this mass, but we can determine it from the lander's weight on earth.

EXECUTE: (a) The distance r from the *center* of Mars is

$$r = (6.0 \times 10^6\ \text{m}) + (3.40 \times 10^6\ \text{m}) = 9.4 \times 10^6\ \text{m}$$

The mass m of the lander is its earth weight w divided by the acceleration of gravity g on earth:

$$m = \frac{w}{g} = \frac{3920\ \text{N}}{9.8\ \text{m/s}^2} = 400\ \text{kg}$$

The mass is the same whether the lander is on the earth, on Mars, or in between. From Eq. (12.5),

$$\begin{aligned} F_g &= \frac{Gm_M m}{r^2} \\ &= \frac{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(6.42 \times 10^{23}\ \text{kg})(400\ \text{kg})}{(9.4 \times 10^6\ \text{m})^2} \\ &= 194\ \text{N} \end{aligned}$$

The acceleration due to the gravity of Mars at this point is

$$g_M = \frac{F_g}{m} = \frac{194\ \text{N}}{400\ \text{kg}} = 0.48\ \text{m/s}^2$$

This is also the acceleration experienced by Phobos in its orbit, 6.0×10^6 m above the surface of Mars. (b) To find F_g and g_M at the surface, we repeat the calculations in part (a), replacing $r = 9.4 \times 10^6$ m with $R_M = 3.40 \times 10^6$ m. Alternatively, because F_g and g_M are inversely proportional to $1/r^2$ (at any point outside the planet), we can multiply the results of part (a) by the factor

$$\left(\frac{9.4 \times 10^6\ \text{m}}{3.40 \times 10^6\ \text{m}}\right)^2$$

You should use both methods to show that at the surface $F_g = 1500$ N and $g_M = 3.7\ \text{m/s}^2$.

EVALUATE: The results for part (b) show that an object's weight and the acceleration due to gravity are roughly 40% as large on the surface of Mars as they are on the earth's surface. Science-fiction films and stories set on Mars commonly describe the planet's lower temperatures and thinner atmosphere, but they seldom focus on the experience of being in a low-gravity environment.

Test Your Understanding of Section 12.2 Rank the following hypothetical planets in order from highest to lowest surface gravity: (i) mass = 2 times the mass of the earth, radius = 2 times the radius of the earth; (ii) mass = 4 times the mass of the earth, radius = 4 times the radius of the earth; (iii) mass = 4 times the mass of the earth, radius = 2 times the radius of the earth; (iv) mass = 2 times the mass of the earth, radius = 4 times the radius of the earth.

12.3 Gravitational Potential Energy

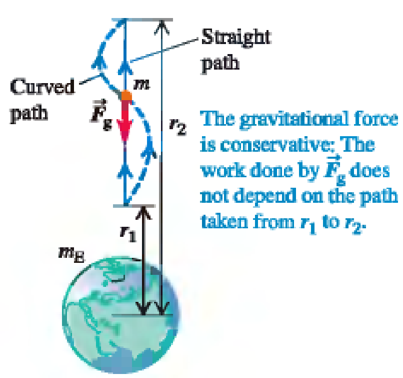
When we first developed the concept of gravitational potential energy in Section 7.1, we assumed that the gravitational force on a body is constant in magnitude and direction. This led to the expression $U = mgy$. But we now know that the earth's gravitational force on a body of mass m at any point outside the earth is given more generally by Eq. (12.2), $F_g = Gm_E m/r^2$, where m_E is the mass of the earth and r is the distance of the body from the earth's center. For problems in which r changes enough that the gravitational force can't be considered constant, we need a more general expression for gravitational potential energy.

To find this expression, we follow the same basic sequence of steps as in Section 7.1. We consider a body of mass m outside the earth, and first compute the work W_{grav} done by the gravitational force when the body moves directly away from or toward the center of the earth from $r = r_1$ to $r = r_2$, as in Fig. 12.10. This work is given by

$$W_{grav} = \int_{r_1}^{r_2} F_r\, dr \tag{12.6}$$

where F_r is the radial component of the gravitational force $\vec{F}$—that is, the component in the direction *outward* from the center of the earth. Because $\vec{F}$ points

12.10 Calculating the work done on a body by the gravitational force as the body moves from radial coordinate r_1 to r_2.



directly *inward* toward the center of the earth, F_r is negative. It differs from Eq. (12.2), the magnitude of the gravitational force, by a minus sign:

$$F_r = -\frac{Gm_E m}{r^2} \tag{12.7}$$

Substituting Eq. (12.7) into Eq. (12.6), we see that W_{grav} is given by

$$W_{grav} = -Gm_E m \int_{r_1}^{r_2} \frac{dr}{r^2} = \frac{Gm_E m}{r_2} - \frac{Gm_E m}{r_1} \tag{12.8}$$

The path doesn't have to be a straight line; it could also be a curve like the one in Fig. 12.10. By an argument similar to that in Section 7.1, this work depends only on the initial and final values of r, not on the path taken. This also proves that the gravitational force is always *conservative*.

We now define the corresponding potential energy U so that $W_{grav} = U_1 - U_2$, as in Eq. (7.3). Comparing this with Eq. (12.8), we see that the appropriate definition for **gravitational potential energy** is

$$U = -\frac{Gm_E m}{r} \quad \text{(gravitational potential energy)} \tag{12.9}$$

Figure 12.11 shows how the gravitational potential energy depends on the distance r between the body of mass m and the center of the earth. When the body moves away from the earth, r increases, the gravitational force does negative work, and U increases (i.e., becomes less negative). When the body "falls" toward earth, r decreases, the gravitational work is positive, and the potential energy decreases (i.e., becomes more negative).

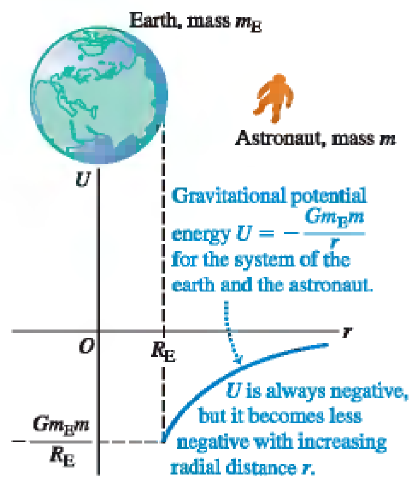
You may be troubled by Eq. (12.9) because it states that gravitational potential energy is always negative. But in fact you've seen negative values of U before. In using the formula $U = mgy$ in Section 7.1, we found that U was negative whenever the body of mass m was at a value of y below the arbitrary height we chose to be $y = 0$—that is, whenever the body and the earth were closer together than some certain arbitrary distance. (See, for instance, Example 7.2 in Section 7.1.) In defining U by Eq. (12.9), we have chosen U to be zero when the body of mass m is infinitely far from the earth $(r = \infty)$. As the body moves toward the earth, gravitational potential energy decreases and so becomes negative.

If we wanted, we could make $U = 0$ at the surface of the earth, where $r = R_E$, by simply adding the quantity $Gm_E m/R_E$ to Eq. (12.9). This would make U positive when $r > R_E$. We won't do this for two reasons: One, it would make the expression for U more complicated; and two, the added term would not affect the *difference* in potential energy between any two points, which is the only physically significant quantity.

CAUTION **Gravitational force vs. gravitational potential energy** Be careful not to confuse the expressions for gravitational force, Eq. (12.7), and gravitational potential energy, Eq. (12.9). The force F_r is proportional to $1/r^2$, while potential energy U is proportional to $1/r$.

Armed with Eq. (12.9), we can now use general energy relationships for problems in which the $1/r^2$ behavior of the earth's gravitational force has to be included. If the gravitational force on the body is the only force that does work, the total mechanical energy of the system is constant, or *conserved*. In the following example we'll use this principle to calculate **escape speed**, the speed required for a body to escape completely from a planet.

12.11 A graph of the gravitational potential energy U for the system of the earth (mass m_E) and an astronaut (mass m) versus the astronaut's distance r from the center of the earth.



Example 12.5 "From the earth to the moon"

In Jules Verne's 1865 story with this title, three men were sent to the moon in a shell fired from a giant cannon sunk in the earth in Florida. (a) Find the muzzle speed needed to shoot the shell straight up to a height above the earth equal to the earth's radius. (b) Find the *escape speed*—that is, the muzzle speed that would allow the shell to escape from the earth completely. Neglect air resistance, the earth's rotation, and the gravitational pull of the moon. The earth's radius is $R_E = 6380\text{ km} = 6.38 \times 10^6\text{ m}$, and its mass is $m_E = 5.97 \times 10^{24}\text{ kg}$ (see Appendix F).

SOLUTION

IDENTIFY: Once the shell leaves the muzzle of the cannon, only the (conservative) gravitational force does work and mechanical energy is conserved. We use this fact to find the speed at which the shell must leave the muzzle into order to (a) come to a halt at a distance of two earth radii from the planet's center and (b) come to a halt at an infinite distance from earth.

SET UP: In both parts (a) and (b) we use the equation for energy conservation, $K_1 + U_1 = K_2 + U_2$, where the potential energy U is given by Eq. (12.9). Figure 12.12 shows our sketches. Point 1 is where the shell leaves the cannon with speed v_1 (the target variable). At this point the distance from the center of the earth is $r_1 = R_E$, the earth's radius. Point 2 is where the shell reaches its maximum height; in part (a) it is at $r_2 = 2R_E$ (Fig. 12.12a), and in part (b) it is infinitely far from the earth at $r_2 = \infty$ (Fig 12.12b). In either case the shell is at rest at point 2, so $v_2 = 0$ and $K_2 = 0$. Let m be the mass of the shell (with passengers).

EXECUTE: (a) We can determine v_1 from the energy-conservation equation

$$K_1 + U_1 = K_2 + U_2$$

$$\frac{1}{2}mv_1^2 + \left(-\frac{Gm_Em}{R_E}\right) = 0 + \left(-\frac{Gm_Em}{2R_E}\right)$$

Rearranging this, we find that

$$v_1 = \sqrt{\frac{Gm_E}{R_E}}$$

$$= \sqrt{\frac{(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(5.97 \times 10^{24}\text{ kg})}{6.38 \times 10^6\text{ m}}}$$

$$= 7900\text{ m/s}(= 28{,}400\text{ km/h} = 17{,}700\text{ mi/h})$$

(b) We want the shell barely to be able to "reach" point 2 at $r_2 = \infty$, with no kinetic energy left over. Hence $K_2 = 0$ and $U_2 = 0$ (the potential energy goes to zero at infinity; see Fig. 12.11). The total energy is therefore zero, and when the shell is fired its positive kinetic energy K_1 and negative potential energy U_1 must also add to zero:

$$\frac{1}{2}mv_1^2 + \left(-\frac{Gm_Em}{R_E}\right) = 0 + 0$$

$$v_1 = \sqrt{\frac{2Gm_E}{R_E}}$$

$$= \sqrt{\frac{2(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(5.97 \times 10^{24}\text{ kg})}{6.38 \times 10^6\text{ m}}}$$

$$= 1.12 \times 10^4\text{ m/s}(= 40{,}200\text{ km/h} = 25{,}000\text{ mi/h})$$

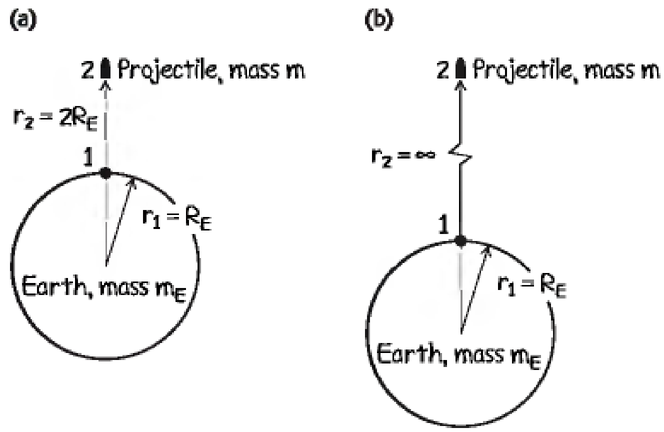
EVALUATE: This result does not depend on the mass of the shell, nor does it depend on the direction in which the shell is launched. Modern spacecraft launched from Florida must attain essentially the speed found in part (b) to escape the earth. A spacecraft on the ground at Cape Canaveral is already moving at 410 m/s to the east because of the earth's rotation; by launching to the east, the spacecraft takes advantage of this "free" contribution toward escape speed.

To generalize our result, the initial speed v_1 needed for a body to escape from the surface of a spherical mass M with radius R (ignoring air resistance) is

$$v_1 = \sqrt{\frac{2GM}{R}} \qquad \text{(escape speed)}$$

You can use this result to compute the escape speed for other bodies. You will find 5.02×10^3 m/s for Mars, 5.95×10^4 m/s for Jupiter, and 6.18×10^5 m/s for the sun.

12.12 Our sketches for this problem.



More on Gravitational Potential Energy

As a final note, let's show that when we are close to the earth's surface, Eq. (12.9) reduces to the familiar $U = mgy$ from Chapter 7. We first rewrite Eq. (12.8) as

$$W_{grav} = Gm_Em\frac{r_1 - r_2}{r_1 r_2}$$

If the body stays close to the earth, then in the denominator we may replace r_1 and r_2 by R_E, the earth's radius, so

$$W_{grav} = Gm_E m \frac{r_1 - r_2}{R_E^2}$$

According to Eq. (12.4), $g = Gm_E/R_E^2$, so

$$W_{grav} = mg(r_1 - r_2)$$

If we replace the r's by y's, this is just Eq. (7.1) for the work done by a constant gravitational force. In Section 7.1 we used this equation to derive Eq. (7.2), $U = mgy$, so we may consider this expression for gravitational potential energy to be a special case of the more general Eq. (12.9).

Test Your Understanding of Section 12.3 Is it possible for a planet to have the same surface gravity as the earth (that is, the same value of g at the surface) and yet have a greater escape speed?

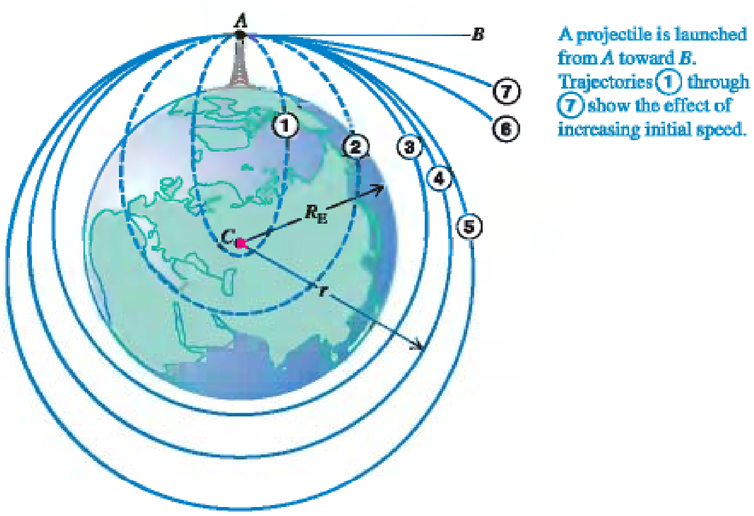
12.4 The Motion of Satellites

Artificial satellites orbiting the earth are a familiar part of modern technology (Fig. 12.13). But how do they stay in orbit, and what determines the properties of their orbits? We can use Newton's laws and the law of gravitation to provide the answers. We'll see in the next section that the motion of planets can be analyzed in the same way.

To begin, think back to the discussion of projectile motion in Section 3.3. In Example 3.6 a motorcycle rider rides horizontally off the edge of a cliff, launching himself into a parabolic path that ends on the flat ground at the base of the cliff. If he survives and repeats the experiment with increased launch speed, he will land farther from the starting point. We can imagine him launching himself with great enough speed that the earth's curvature becomes significant. As he falls, the earth curves away beneath him. If he is going fast enough, and if his launch point is high enough that he clears the mountaintops, he may be able to go right on around the earth without ever landing.

Figure 12.14 shows a variation on this theme. We launch a projectile from point A in the direction AB, tangent to the earth's surface. Trajectories 1 through 7 show the effect of increasing the initial speed. In trajectories 3 through 5 the

12.13 With a length of 13.2 m and a mass of 11,000 kg, the Hubble Space Telescope is among the largest satellites placed in orbit.



12.14 Trajectories of a projectile launched from a great height (ignoring air resistance). Orbits 1 and 2 would be completed as shown if the earth were a point mass at C. (This illustration is based on one in Isaac Newton's *Principia*.)

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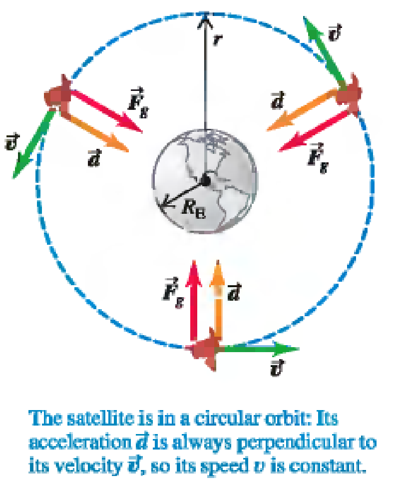
4.6 Satellites Orbit

projectile misses the earth and becomes a satellite. If there is no retarding force, the projectile's speed when it returns to point A is the same as its initial speed and it repeats its motion indefinitely.

Trajectories 1 through 5 close on themselves and are called **closed orbits.** All closed orbits are ellipses or segments of ellipses; trajectory 4 is a circle, a special case of an ellipse. (We'll discuss the properties of an ellipse in Section 12.5.) Trajectories 6 and 7 are **open orbits.** For these paths the projectile never returns to its starting point but travels ever farther away from the earth.

Satellites: Circular Orbits

12.15 The force $\vec{F}_g$ due to the earth's gravitational attraction provides the centripetal acceleration that keeps a satellite in orbit. Compare to Fig. 5.28.



A *circular* orbit, like trajectory 4 in Fig. 12.14, is the simplest case. It is also an important case, since many artificial satellites have nearly circular orbits and the orbits of the planets around the sun are also fairly circular. The only force acting on a satellite in circular orbit around the earth is the earth's gravitational attraction, which is directed toward the center of the earth and hence toward the center of the orbit (Fig. 12.15). As we discussed in Section 5.4, this means that the satellite is in *uniform* circular motion and its speed is constant. The satellite isn't falling *toward* the earth; rather, it's constantly falling *around* the earth. In a circular orbit the speed is just right to keep the distance from the satellite to the center of the earth constant.

Let's see how to find the constant speed v of a satellite in a circular orbit. ? The radius of the orbit is r, measured from the *center* of the earth; the acceleration of the satellite has magnitude $a_{rad} = v^2/r$ and is always directed toward the center of the circle. By the law of gravitation, the net force (gravitational force) on the satellite of mass m has magnitude $F_g = Gm_E m/r^2$ and is in the same direction as the acceleration. Newton's second law $(\sum\vec{F} = m\vec{a})$ then tells us that

$$\frac{Gm_E m}{r^2} = \frac{mv^2}{r}$$

Solving this for v, we find

$$v = \sqrt{\frac{Gm_E}{r}} \quad \text{(circular orbit)} \tag{12.10}$$

This relationship shows that we can't choose the orbit radius r and the speed v independently; for a given radius r, the speed v for a circular orbit is determined.

The satellite's mass m doesn't appear in Eq. (12.10), which shows that the motion of a satellite does not depend on its mass. If we could cut a satellite in half without changing its speed, each half would continue on with the original motion. An astronaut on board a space shuttle is herself a satellite of the earth, held by the earth's gravitational attraction in the same orbit as the shuttle. The astronaut has the same velocity and acceleration as the shuttle, so nothing is pushing her against the floor or walls of the shuttle. She is in a state of *apparent weightlessness,* as in a freely falling elevator; see the discussion following Example 5.9 in Section 5.2. (*True* weightlessness would occur only if the astronaut were infinitely far from any other masses, so that the gravitational force on her would be zero.) Indeed, every part of her body is apparently weightless; she feels nothing pushing her stomach against her intestines or her head against her shoulders (Fig. 12.16).

12.16 These space shuttle astronauts are in a state of apparent weightlessness. Which are right side up and which are upside down?

Apparent weightlessness is not just a feature of circular orbits; it occurs whenever gravity is the only force acting on a spacecraft. Hence it occurs for orbits of any shape, including open orbits such as trajectories (6) and (7) in Fig. 12.14.

12.17 Both the International Space Station and the moon are satellites of the earth. The moon orbits much farther from the center of the earth than does the Space Station, so it has a slower orbital speed and a longer orbital period.

International Space Station
Distance from center of earth = 6800 km (400 km above the surface)
Orbital speed = 7.7 km/s
Orbital period = 93 min

Moon
Distance from center of earth = 384,000 km
Orbital speed = 1.0 km/s
Orbital period = 27.3 days

We can derive a relationship between the radius r of a circular orbit and the period T, the time for one revolution. The speed v is the distance $2\pi r$ traveled in one revolution, divided by the period:

$$v = \frac{2\pi r}{T} \tag{12.11}$$

To get an expression for T, we solve Eq. (12.11) for T and substitute v from Eq. (12.10):

$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{Gm_E}} = \frac{2\pi r^{3/2}}{\sqrt{Gm_E}} \qquad \text{(circular orbit)} \tag{12.12}$$

Equations (12.10) and (12.12) show that larger orbits correspond to slower speeds and longer periods (Fig. 12.17).

It's interesting to compare Eq. (12.10) to the calculation of escape speed in Example 12.5. We see that the escape speed from a spherical body with radius R is $\sqrt{2}$ times greater than the speed of a satellite in a circular orbit at that radius. If our spacecraft is in circular orbit around *any* planet, we have to multiply our speed by a factor of $\sqrt{2}$ to escape to infinity, regardless of the planet's mass.

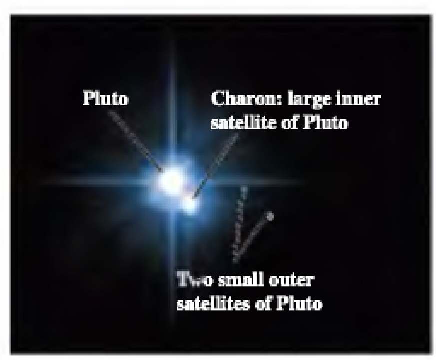
Since the speed v in a circular orbit is determined by Eq. (12.10) for a given orbit radius r, the total mechanical energy $E = K + U$ is determined as well. Using Eqs. (12.9) and (12.10), we have

$$E = K + U = \frac{1}{2}mv^2 + \left(-\frac{Gm_Em}{r}\right) = \frac{1}{2}m\left(\frac{Gm_E}{r}\right) - \frac{Gm_Em}{r}$$
$$E = -\frac{Gm_Em}{2r} \qquad \text{(circular orbit)} \tag{12.13}$$

The total mechanical energy in a circular orbit is negative and equal to one-half the potential energy. Increasing the orbit radius r means increasing the mechanical energy (that is, making E less negative). If the satellite is in a relatively low orbit that encounters the outer fringes of earth's atmosphere, mechanical energy decreases due to negative work done by the force of air resistance; as a result, the orbit radius decreases until the satellite hits the ground or burns up in the atmosphere.

We have talked mostly about earth satellites, but we can apply the same analysis to the circular motion of *any* body under its gravitational attraction to a stationary body. Other examples include the earth's moon and the moons of other worlds (Fig. 12.18).

12.18 The two small satellites of Pluto were discovered in 2005. In accordance with Eq. (12.12), the larger the satellite's orbit, the longer it takes to complete one orbit around Pluto.



Example 12.6 A satellite orbit

Suppose you want to place a 1000-kg weather satellite into a circular orbit 300 km above the earth's surface. (a) What speed, period, and radial acceleration must it have? (b) How much work has to be done to place this satellite in orbit? (c) How much additional work would have to be done to make this satellite escape the earth? The earth's radius is $R_E = 6380$ km and its mass is $m_E = 5.97 \times 10^{24}$ kg.

SOLUTION

IDENTIFY: The satellite is in a circular orbit, so we can use the equations derived in this section.

SET UP: In part (a), we first find the radius r of the satellite's orbit from its altitude. We then calculate the speed v and period T using Eqs. (12.10) and (12.12). The acceleration in a circular orbit is given by the familiar formula from Chapter 3, $a_{rad} = v^2/r$. In parts (b) and (c), the work required is the difference between the initial and final mechanical energy, which for a circular orbit is given by Eq. (12.13).

EXECUTE: (a) The radius of the satellite's orbit is

$$r = 6380\text{ km} + 300\text{ km} = 6680\text{ km} = 6.68 \times 10^6\text{ m}$$

From Eq. (12.10), the orbital speed is

$$v = \sqrt{\frac{Gm_E}{r}} = \sqrt{\frac{(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(5.97 \times 10^{24}\text{ kg})}{6.68 \times 10^6\text{ m}}}$$
$$= 7720\text{ m/s}$$

We find the orbital period from Eq. (12.12):

$$T = \frac{2\pi r}{v} = \frac{2\pi(6.68 \times 10^6\text{ m})}{7720\text{ m/s}}$$
$$= 5440\text{ s} = 90.6\text{ min}$$

The radial acceleration is

$$a_{rad} = \frac{v^2}{r} = \frac{(7720\text{ m/s})^2}{6.68 \times 10^6\text{ m}}$$
$$= 8.92\text{ m/s}^2$$

This is the value of g at a height of 300 km above the earth's surface; it is somewhat less than the value of g at the surface.

(b) The work required is the difference between E_2, the total mechanical energy when the satellite is in orbit, and E_1, the original mechanical energy when the satellite was at rest on the launch pad back on earth. From Eq. (12.13), the energy in orbit is

$$E_2 = -\frac{Gm_E m}{2r}$$
$$= -\frac{(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(5.97 \times 10^{24}\text{ kg})(1000\text{ kg})}{2(6.38 \times 10^6\text{ m})}$$
$$= -2.99 \times 10^{10}\text{ J}$$

At rest on the earth's surface $(r = R_E)$, the kinetic energy is zero:

$$E_1 = K_1 + U_1 = 0 + \left(-\frac{Gm_E m}{R_E}\right)$$
$$= -\frac{(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(5.97 \times 10^{24}\text{ kg})(1000\text{ kg})}{6.38 \times 10^6\text{ m}}$$
$$= -6.25 \times 10^{10}\text{ J}$$

and so

$$W_{required} = E_2 - E_1 = -2.99 \times 10^{10}\text{ J} - (-6.25 \times 10^{10}\text{ J})$$
$$= 3.26 \times 10^{10}\text{ J}$$

(c) We saw in part (b) of Example 12.5 that for a satellite to escape to infinity, the total mechanical energy must be zero. The total mechanical energy in the circular orbit is $E_2 = -2.99 \times 10^{10}$ J; to increase this to zero, an amount of work equal to 2.99×10^{10} J would have to be done. This extra energy could be supplied by rocket engines attached to the satellite.

EVALUATE: In part (b) we ignored the satellite's initial kinetic energy (while it was still on the launch pad) due to the rotation of the earth. You should check to see how much difference this makes (see Example 12.5 for useful data).

Test Your Understanding of Section 12.4 Your personal spacecraft is in a low-altitude circular orbit around the earth. Air resistance from the outer regions of the atmosphere does negative work on the spacecraft, causing the orbital radius to decrease slightly. Does the speed of the spacecraft (i) remain the same, (ii) increase, or (iii) decrease?

12.5 Kepler's Laws and the Motion of Planets

The name *planet* comes from a Greek word meaning "wanderer," and indeed the planets continuously change their positions in the sky relative to the background of stars. One of the great intellectual accomplishments of the 16th and 17th centuries was the threefold realization that the earth is also a planet, that all planets orbit the sun, and that the apparent motions of the planets as seen from the earth can be used to precisely determine their orbits.

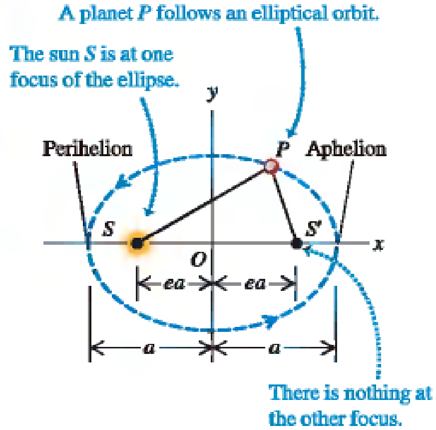
The first and second of these ideas were published by Nicolaus Copernicus in Poland in 1543. The nature of planetary orbits was deduced between 1601 and

1619 by the German astronomer and mathematician Johannes Kepler, using a voluminous set of precise data on apparent planetary motions compiled by his mentor, the Danish astronomer Tycho Brahe. By trial and error, Kepler discovered three empirical laws that accurately described the motions of the planets:

1. **Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse.**
2. **A line from the sun to a given planet sweeps out equal areas in equal times.**
3. **The periods of the planets are proportional to the $\frac{3}{2}$ powers of the major axis lengths of their orbits.**

Kepler did not know *why* the planets moved in this way. Three generations later, when Newton turned his attention to the motion of the planets, he discovered that each of Kepler's laws can be *derived;* they are consequences of Newton's laws of motion and the law of gravitation. Let's see how each of Kepler's laws arises.

12.19 Geometry of an ellipse. The sum of the distances SP and $S'P$ is the same for every point on the curve. The sizes of the sun (S) and planet (P) are exaggerated for clarity.



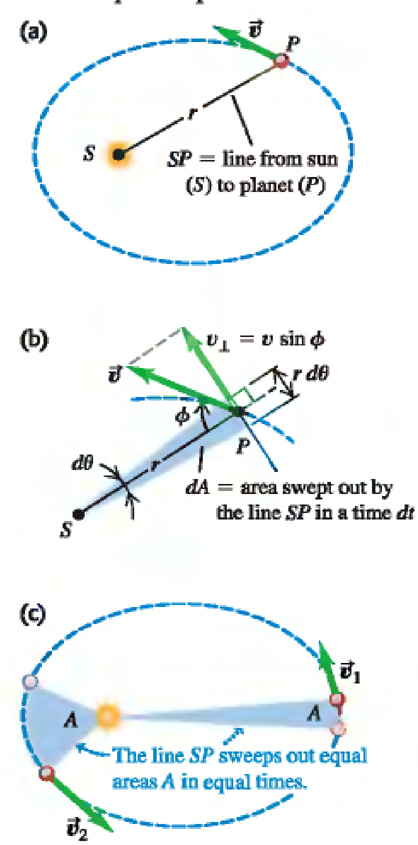
Kepler's First Law

First consider the elliptical orbits described in Kepler's first law. Figure 12.19 shows the geometry of an ellipse. The longest dimension is the *major axis,* with half-length a; this half-length is called the **semi-major axis.** The sum of the distances from S to P and from S' to P is the same for all points on the curve. S and S' are the *foci* (plural of *focus*). The sun is at S, and the planet is at P; we think of them both as points because the size of each is very small in comparison to the distance between them. There is nothing at the other focus S'.

The distance of each focus from the center of the ellipse is ea, where e is a dimensionless number between 0 and 1 called the **eccentricity.** If $e = 0$, the ellipse is a circle. The actual orbits of the planets are fairly circular; their eccentricities range from 0.007 for Venus to 0.206 for Mercury. (The earth's orbit has $e = 0.017$.) The point in the planet's orbit closest to the sun is the *perihelion,* and the point most distant from the sun is the *aphelion.*

Newton was able to show that for a body acted on by an attractive force proportional to $1/r^2$, the only possible closed orbits are a circle or an ellipse; he also showed that open orbits (trajectories 6 and 7 in Fig. 12.14) must be parabolas or hyperbolas. These results can be derived by a straightforward application of Newton's laws and the law of gravitation, together with a lot more differential equations than we're ready for.

12.20 (a) The planet (P) moves about the sun (S) in an elliptical orbit. (b) In a time dt the line SP sweeps out an area $dA = \frac{1}{2}(r\,d\theta)r = \frac{1}{2}r^2\,d\theta$. (c) The planet's speed varies so that the line SP sweeps out the same area A in a given time t regardless of the planet's position in its orbit.



Kepler's Second Law

Figure 12.20 shows Kepler's second law. In a small time interval dt, the line from the sun S to the planet P turns through an angle $d\theta$. The area swept out is the colored triangle with height r, base length $r\,d\theta$, and area $dA = \frac{1}{2}r^2\,d\theta$ (Fig. 12.20b). The rate at which area is swept out, dA/dt, is called the *sector velocity:*

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} \tag{12.14}$$

The essence of Kepler's second law is that the sector velocity has the same value at all points in the orbit. When the planet is close to the sun, r is small and $d\theta/dt$ is large; when the planet is far from the sun, r is large and $d\theta/dt$ is small.

To see how Kepler's second law follows from Newton's laws, we express dA/dt in terms of the velocity vector $\vec{v}$ of the planet P. The component of $\vec{v}$ perpendicular to the radial line is $v_\perp = v\sin\phi$. From Fig. 12.20b the displacement along the direction of $v_\perp$ during time dt is $r\,d\theta$, so we also have $v_\perp = r\,d\theta/dt$. Using this relationship in Eq. (12.14), we find

$$\frac{dA}{dt} = \frac{1}{2}rv\sin\phi \qquad \text{(sector velocity)} \tag{12.15}$$

Now $rv\sin\phi$ is the magnitude of the vector product $\vec{r} \times \vec{v}$, which in turn is $1/m$ times the angular momentum $\vec{L} = \vec{r} \times m\vec{v}$ of the planet with respect to the sun. So we have

$$\frac{dA}{dt} = \frac{1}{2m}|\vec{r} \times m\vec{v}| = \frac{L}{2m} \tag{12.16}$$

Thus Kepler's second law—that sector velocity is constant—means that angular momentum is constant!

It is easy to see why the angular momentum of the planet *must* be constant. According to Eq. (10.26), the rate of change of $\vec{L}$ equals the torque of the gravitational force $\vec{F}$ acting on the planet:

$$\frac{d\vec{L}}{dt} = \vec{\tau} = \vec{r} \times \vec{F}$$

In our situation, $\vec{r}$ is the vector from the sun to the planet, and the force $\vec{F}$ is directed from the planet to the sun. So these vectors always lie along the same line, and their vector product $\vec{r} \times \vec{F}$ is zero. Hence $d\vec{L}/dt = \mathbf{0}$. This conclusion does not depend on the $1/r^2$ behavior of the force; angular momentum is conserved for *any* force that acts always along the line joining the particle to a fixed point. Such a force is called a *central force.* (Kepler's first and third laws are valid *only* for a $1/r^2$ force.)

Conservation of angular momentum also explains why the orbit lies in a plane. The vector $\vec{L} = \vec{r} \times m\vec{v}$ is always perpendicular to the plane of the vectors $\vec{r}$ and $\vec{v}$; since $\vec{L}$ is constant in magnitude *and* direction, $\vec{r}$ and $\vec{v}$ always lie in the same plane, which is just the plane of the planet's orbit.

Kepler's Third Law

We have already derived Kepler's third law for the particular case of circular orbits. Equation (12.12) shows that the period of a satellite or planet in a circular orbit is proportional to the $\frac{3}{2}$ power of the orbit radius. Newton was able to show that this same relationship holds for an *elliptical* orbit, with the orbit radius r replaced by the semi-major axis a:

$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_S}} \quad \text{(elliptical orbit around the sun)} \tag{12.17}$$

Since the planet orbits the sun, not the earth, we have replaced the earth's mass m_E in Eq. (12.12) with the sun's mass m_S. Note that the period does not depend on the eccentricity e. An asteroid in an elongated elliptical orbit with semi-major axis a will have the same orbital period as a planet in a circular orbit of radius a. The key difference is that the asteroid moves at different speeds at different points in its elliptical orbit (Fig. 12.20c), while the planet's speed is constant around its circular orbit.

Conceptual Example 12.7 Orbital speeds

At what point in an elliptical orbit (Fig. 12.19) does a planet have the greatest speed?

SOLUTION

Mechanical energy is conserved as the planet moves around its orbit. The planet's kinetic energy $K = \frac{1}{2}mv^2$ is maximum when the potential energy $U = -Gm_S m/r$ is minimum (that is, most negative; see Fig. 12.11), which occurs when r is a minimum. Hence the speed v is maximum at perihelion.

Your intuition about falling bodies is helpful here. As the planet falls inward toward the sun, it picks up speed, and its speed is maximum when closest to the sun. By the same reasoning, the planet slows down as it moves away from the sun, and its speed is minimum at aphelion.

Example 12.8 Kepler's third law

The asteroid Pallas has an orbital period of 4.62 years and an orbital eccentricity of 0.233. Find the semi-major axis of its orbit.

SOLUTION

IDENTIFY: This example uses Kepler's third law, which relates the period T and the semi-major axis a for an object (like an asteroid) that orbits.

SET UP: We use Eq. (12.17) to determine a from the given value of T. Note that we don't need the value of the eccentricity.

EXECUTE: From Eq. (12.17), $a^{3/2} = \left(\sqrt{Gm_S}\,T\right)/2\pi$. To solve for a, we raise this expression to the $\frac{2}{3}$ power:

$$a = \left(\frac{Gm_S T^2}{4\pi^2}\right)^{1/3}$$

Since $G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$ and $m_S = 1.99 \times 10^{30}$ kg (the mass of the sun from Appendix F) are given in SI units, we must express the period T in seconds rather than years using a conversion factor from Appendix E: $T = (4.62\ \mathrm{yr})(3.156 \times 10^7\ \mathrm{s/yr}) = 1.46 \times 10^8$ s. Using this value, we find $a = 4.15 \times 10^{11}$ m. (Plug in the numbers yourself to check.)

EVALUATE: Our result is intermediate between the semi-major axes of Mars and Jupiter (see Appendix F). Indeed, most known asteroids orbit in an "asteroid belt" between the orbits of these two planets.

As a historical note, Pallas wasn't discovered until 1802, almost two centuries after the publication of Kepler's third law. While Kepler deduced his three laws from the motions of the five planets (other than the earth) known in his time, these laws have proven to apply equally well to all of the planets, asteroids, and comets subsequently discovered to be orbiting the sun.

Example 12.9 Comet Halley

Comet Halley moves in an elongated elliptical orbit around the sun (Fig. 12.21). At perihelion, the comet is 8.75×10^7 km from the sun; at aphelion, it is 5.26×10^9 km from the sun. Find the semi-major axis, eccentricity, and period of the orbit.

SOLUTION

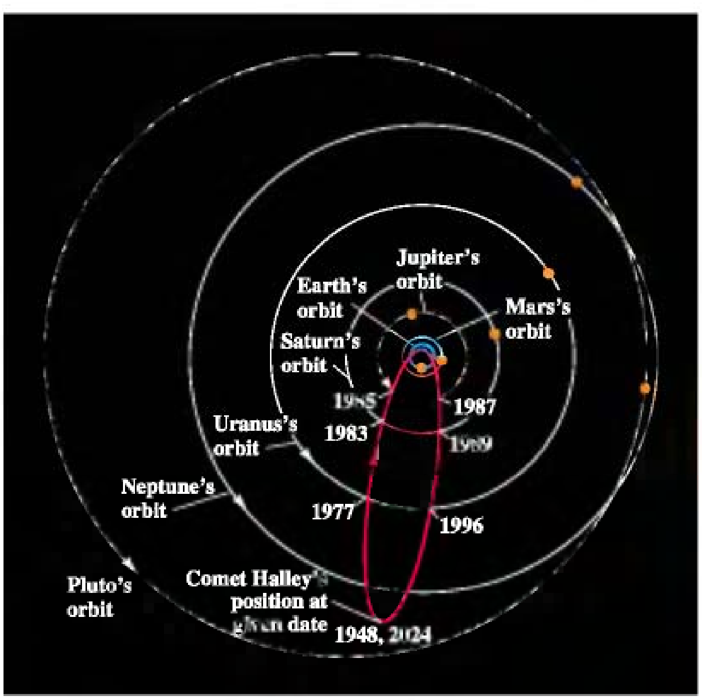
IDENTIFY: We are given the perihelion and aphelion distances, and we are to find the semi-major axis a, eccentricity e, and orbital period T (which is related to the semi-major axis by Kepler's third law).

SET UP: Figure 12.19 shows us how to find a and e from the perihelion and aphelion distances. Once we know the value of a, we can find the orbital period from Eq. (12.17).

EXECUTE: From Fig. 12.19 the length of the major axis equals the sum of the comet–sun distance at perihelion and the comet–sun distance at aphelion. The length of the major axis is $2a$, so

$$a = \frac{8.75 \times 10^7\ \mathrm{km} + 5.26 \times 10^9\ \mathrm{km}}{2} = 2.67 \times 10^9\ \mathrm{km}$$

(a)



(b)

12.21 (a) The orbit of Comet Halley. (b) Comet Halley as it appeared in 1986. At the heart of the comet is an icy body, called the nucleus, that is about 10 km across. When the comet's orbit carries it close to the sun, the heat of sunlight causes the nucleus to partially evaporate. The evaporated material forms the tail, which can be tens of millions of kilometers long.

Continued

Further inspection of Fig. 12.19 shows that the comet–sun distance at perihelion is

$$a - ea = a(1 - e)$$

Since we are given that this distance is 8.75×10^7 km, the eccentricity is

$$e = 1 - \frac{8.75 \times 10^7\ \text{km}}{a} = 1 - \frac{8.75 \times 10^7\ \text{km}}{2.67 \times 10^9\ \text{km}} = 0.967$$

The period is given by Eq. (12.17):

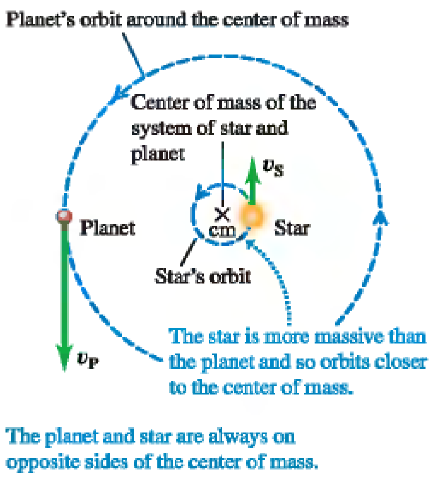
$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_S}} = \frac{2\pi(2.67 \times 10^{12}\ \text{m})^{3/2}}{\sqrt{(6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2)(1.99 \times 10^{30}\ \text{kg})}}$$
$$= 2.38 \times 10^9\ \text{s} = 75.5\ \text{years}$$

EVALUATE: The eccentricity is very close to 1, so the comet has a very elongated orbit (see Fig. 12.21a). Comet Halley was at perihelion in early 1986; it will next reach perihelion one period later, in 2061.

Planetary Motions and the Center of Mass

We have assumed that as a planet or comet orbits the sun, the sun remains absolutely stationary. Of course, this can't be correct; because the sun exerts a gravitational force on the planet, the planet exerts a gravitational force on the sun of the same magnitude but opposite direction. In fact, *both* the sun and the planet orbit around their common center of mass (Fig. 12.22). We've made only a small error by ignoring this effect, however; the sun's mass is about 750 times the total mass of all the planets combined, so the center of mass of the solar system is not far from the center of the sun. Remarkably, astronomers have used this effect to detect the presence of planets orbiting other stars. Sensitive telescopes are able to detect the apparent "wobble" of a star as it orbits the common center of mass of the star and an unseen companion planet. (The planets are too faint to observe directly.) By analyzing these "wobbles," astronomers have discovered planets in orbit around more than a hundred other stars.

12.22 A star and its planet both orbit about their common center of mass.



Newton's analysis of planetary motions is used on a daily basis by modern-day astronomers. But the most remarkable result of Newton's work is that the motions of bodies in the heavens obey the *same* laws of motion as do bodies on the earth. This *Newtonian synthesis,* as it has come to be called, is one of the great unifying principles of science. It has had profound effects on the way that humanity looks at the universe—not as a realm of impenetrable mystery, but as a direct extension of our everyday world, subject to scientific study and calculation.

Test Your Understanding of Section 12.5 The orbit of Comet X has a semi-major axis that is four times larger than the semi-major axis of Comet Y. What is the ratio of the orbital period of X to the orbital period of Y? (i) 2; (ii) 4; (iii) 8; (iv) 16; (v) 32; (vi) 64.

*12.6 Spherical Mass Distributions

We have stated without proof that the gravitational interaction between two spherically symmetric mass distributions is the same as though all the mass of each were concentrated at its center. Now we're ready to prove this statement. Newton searched for a proof for several years, and he delayed publication of the law of gravitation until he found one.

Here's our program. Rather than starting with two spherically symmetric masses, we'll tackle the simpler problem of a point mass m interacting with a thin spherical shell with total mass M. We will show that when m is outside the sphere, the *potential energy* associated with this gravitational interaction is the same as though M were all concentrated at the center of the sphere. We learned in Section 7.4 that the force is the negative derivative of the potential energy, so the *force* on m is also the same as for a point mass M. Any spherically symmetric mass distribution can be thought of as being made up of many concentric spherical shells, so our result will also hold for *any* spherically symmetric M.

A Point Mass Outside a Spherical Shell

We start by considering a ring on the surface of the shell (Fig. 12.23a), centered on the line from the center of the shell to m. We do this because all of the particles that make up the ring are the same distance s from the point mass m. From Eq. (12.9) the potential energy of interaction between the earth (mass m_E) and a point mass m, separated by a distance r, is $U = -Gm_E m/r$. By changing notation in this expression, we see that in the situation shown in Fig. 12.23a, the potential energy of interaction between the point mass m and a particle of mass m_i within the ring is given by

$$U_i = -\frac{Gmm_i}{s}$$

To find the potential energy of interaction between m and the entire ring of mass $dM = \sum_i m_i$, we sum this expression for U_i over all particles in the ring. Calling this potential energy dU, we find

$$dU = \sum_i U_i = \sum_i \left(-\frac{Gmm_i}{s}\right) = -\frac{Gm}{s}\sum_i m_i = -\frac{Gm\,dM}{s} \qquad (12.18)$$

To proceed, we need to know the mass dM of the ring. We can find this with the aid of a little geometry. The radius of the shell is R, so in terms of the angle ϕ shown in the figure, the radius of the ring is $R\sin\phi$, and its circumference is $2\pi R\sin\phi$. The width of the ring is $R\,d\phi$, and its area dA is approximately equal to its width times its circumference:

$$dA = 2\pi R^2 \sin\phi\, d\phi$$

The ratio of the ring mass dM to the total mass M of the shell is equal to the ratio of the area dA of the ring to the total area $A = 4\pi R^2$ of the shell:

$$\frac{dM}{M} = \frac{2\pi R^2 \sin\phi\, d\phi}{4\pi R^2} = \frac{1}{2}\sin\phi\, d\phi \qquad (12.19)$$

Now we solve Eq. (12.19) for dM and substitute the result into Eq. (12.18) to find the potential energy of interaction between the point mass m and the ring:

$$dU = -\frac{GMm\sin\phi\, d\phi}{2s} \qquad (12.20)$$

The total potential energy of interaction between the point mass and the *shell* is the integral of Eq. (12.20) over the whole sphere as ϕ varies from 0 to π (*not* 2π!) and s varies from $r - R$ to $r + R$. To carry out the integration, we have to express the integrand in terms of a single variable; we choose s. To express ϕ and $d\phi$ in terms of s, we have to do a little more geometry. Figure 12.23b shows that s is the hypotenuse of a right triangle with sides $(r - R\cos\phi)$ and $R\sin\phi$, so the Pythagorean theorem gives

$$\begin{aligned} s^2 &= (r - R\cos\phi)^2 + (R\sin\phi)^2 \\ &= r^2 - 2rR\cos\phi + R^2 \end{aligned} \qquad (12.21)$$

We take differentials of both sides:

$$2s\,ds = 2rR\sin\phi\, d\phi$$

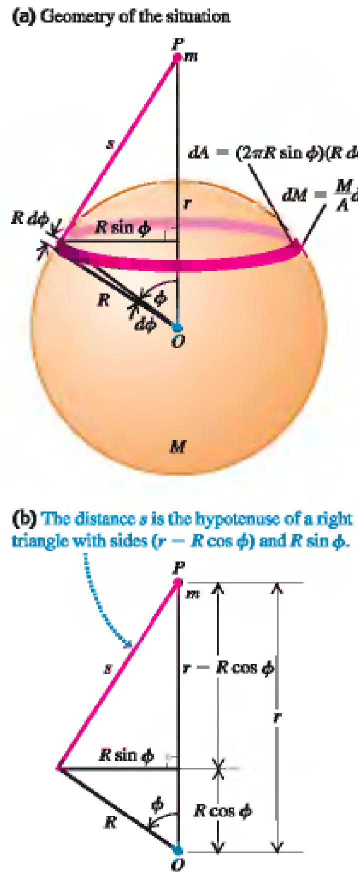
Next we divide this by $2rR$ and substitute the result into Eq. (12.20):

$$dU = -\frac{GMm}{2s}\frac{s\,ds}{rR} = -\frac{GMm}{2rR}ds \qquad (12.22)$$

We can now integrate Eq. (12.22), recalling that s varies from $r - R$ to $r + R$:

$$U = -\frac{GMm}{2rR}\int_{r-R}^{r+R} ds = -\frac{GMm}{2rR}[(r + R) - (r - R)] \qquad (12.23)$$

12.23 Calculating the gravitational potential energy of interaction between a point mass m outside a spherical shell and a ring on the surface of the shell.



Finally, we have

$$U = -\frac{GMm}{r} \quad \text{(point mass } m \text{ outside spherical shell } M\text{)} \qquad (12.24)$$

This is equal to the potential energy of two point masses m and M at a distance r. So we have proved that the gravitational potential energy of the spherical shell M and the point mass m at any distance r is the same as though they were point masses. Because the force is given by $F_r = -dU/dr$, the force is also the same.

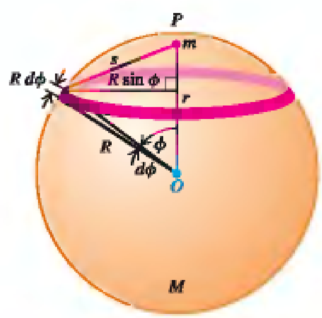
The Gravitational Force Between Spherical Mass Distributions

Any spherically symmetric mass distribution can be thought of as a combination of concentric spherical shells. Because of the principle of superposition of forces, what is true of one shell is also true of the combination. So we have proved half of what we set out to prove: that the gravitational interaction between any spherically symmetric mass distribution and a point mass is the same as though all the mass of the spherically symmetric distribution were concentrated at its center.

The other half is to prove that *two* spherically symmetric mass distributions interact as though they were both points. That's easier. In Fig. 12.23a the forces the two bodies exert on each other are an action–reaction pair, and they obey Newton's third law. So we have also proved that the force that m exerts *on* the sphere M is the same as though M were a point. But now if we replace m with a spherically symmetric mass distribution centered at m's location, the resulting gravitational force on any part of M is the same as before, and so is the total force. This completes our proof.

A Point Mass Inside a Spherical Shell

12.24 When a point mass m is *inside* a uniform spherical shell of mass M, the potential energy is the same no matter where inside the shell the point mass is located. The force from the masses' mutual gravitational interaction is zero.



We assumed at the beginning that the point mass m was outside the spherical shell, so our proof is valid only when m is outside a spherically symmetric mass distribution. When m is *inside* a spherical shell, the geometry is as shown in Fig. 12.24. The entire analysis goes just as before; Eqs. (12.18) through (12.22) are still valid. But when we get to Eq. (12.23), the limits of integration have to be changed to $R - r$ and $R + r$. We then have

$$U = -\frac{GMm}{2rR}\int_{R-r}^{R+r} ds = -\frac{GMm}{2rR}[(R + r) - (R - r)] \qquad (12.25)$$

and the final result is

$$U = -\frac{GMm}{R} \quad \text{(point mass } m \text{ inside spherical shell } M\text{)} \qquad (12.26)$$

Compare this result to Eq. (12.24): Instead of having r, the distance between m and the center of M, in the denominator, we have R, the radius of the shell. This means that U in Eq. (12.26) doesn't depend on r and thus has the same value everywhere inside the shell. When m moves around inside the shell, no work is done on it, so the force on m at any point inside the shell must be zero.

More generally, at any point in the interior of any spherically symmetric mass distribution (not necessarily a shell), at a distance r from its center, the gravitational force on a point mass m is the same as though we removed all the mass at points farther than r from the center and concentrated all the remaining mass at the center.

Example 12.10 "Journey to the center of the earth"

Suppose we drill a hole through the earth (radius R_E, mass m_E) along a diameter and drop a mail pouch (mass m) down the hole. Derive an expression for the gravitational force on the pouch as a function of its distance r from the center. Assume that the density of the earth is uniform (not a very realistic model; see Fig. 12.9).

SOLUTION

IDENTIFY: According to the statements above, the gravitational force at a distance r from the center is determined only by the mass M within a spherical region of radius r (Fig. 12.25). The mass outside this radius has no effect on the mail pouch.

SET UP: The gravitational force on the mail pouch is the same as if all the mass M within radius r were concentrated at the center of the earth. The mass of a uniform sphere is proportional to the volume of the sphere, which is $\frac{4}{3}\pi r^3$ for the sphere of radius r and $\frac{4}{3}\pi R_E^3$ for the entire earth.

EXECUTE: The ratio of the mass M of the sphere of radius r to the mass of the earth, m_E, is

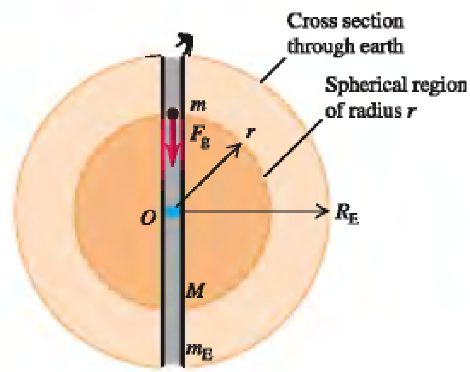
$$\frac{M}{m_E} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R_E^3} = \frac{r^3}{R_E^3}, \qquad \text{so} \qquad M = m_E \frac{r^3}{R_E^3}$$

The magnitude of the gravitational force on m is given by

$$F_g = \frac{GMm}{r^2} = \frac{Gm}{r^2}\left(m_E \frac{r^3}{R_E^3}\right) = \frac{Gm_E m}{R_E^3} r$$

EVALUATE: At points inside this uniform-density sphere, F_g is *directly proportional* to the distance r from the center, rather than proportional to $1/r^2$ as it is outside the sphere. Right at the surface, where $r = R_E$, the above expression gives $F_g = Gm_E m/R_E^2$, as we should expect. In the next chapter we'll learn how to compute the time it would take for the mail pouch to emerge on the other side of the earth under the assumption of uniform density.

12.25 A hole through the center of the earth (assumed to be uniform). When an object is a distance r from the center, only the mass inside a sphere of radius r exerts a net gravitational force on it.



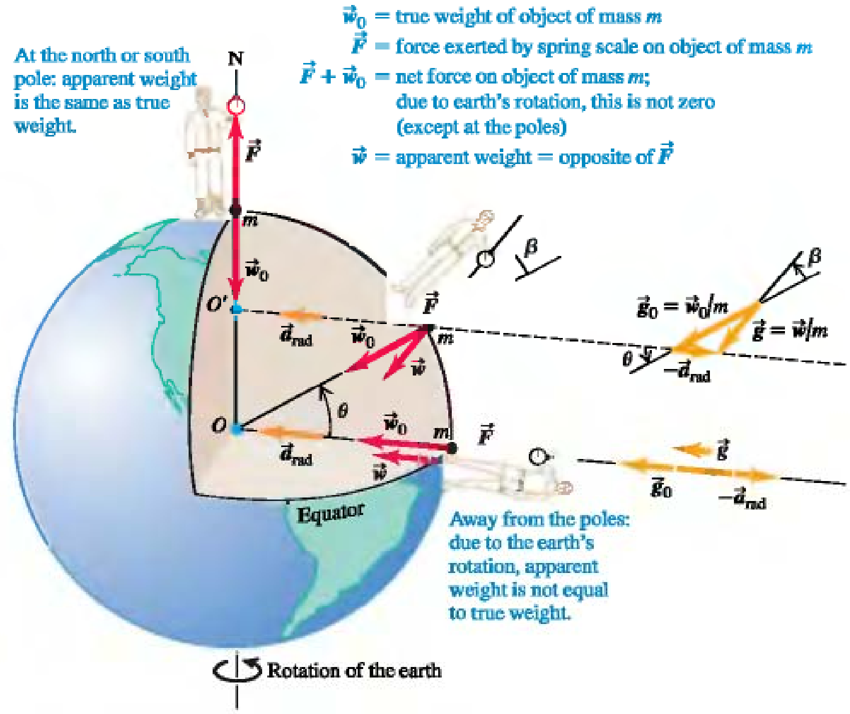
Test Your Understanding of Section 12.6 In the classic 1913 science-fiction novel *At the Earth's Core* by Edgar Rice Burroughs, explorers discover that the earth is a hollow sphere and that an entire civilization lives on the inside of the sphere. Would it be possible to stand and walk on the inner surface of a hollow, nonrotating planet?

*12.7 Apparent Weight and the Earth's Rotation

Because the earth rotates on its axis, it is not precisely an inertial frame of reference. For this reason the apparent weight of a body on earth is not precisely equal to the earth's gravitational attraction, which we will call the **true weight** $\vec{w}_0$ of the body. Figure 12.26 is a cutaway view of the earth, showing three observers. Each one holds a spring scale with a body of mass m hanging from it. Each scale applies a tension force $\vec{F}$ to the body hanging from it, and the reading on each scale is the magnitude F of this force. If the observers are unaware of the earth's rotation, each one *thinks* that the scale reading equals the weight of the body because he thinks the body on his spring scale is in equilibrium. So each observer thinks that the tension $\vec{F}$ must be opposed by an equal and opposite force $\vec{w}$, which we call the **apparent weight.** But if the bodies are rotating with the earth, they are *not* precisely in equilibrium. Our problem is to find the relationship between the apparent weight $\vec{w}$ and the true weight $\vec{w}_0$.

If we assume that the earth is spherically symmetric, then the true weight $\vec{w}_0$ has magnitude $Gm_E m/R_E^2$, where m_E and R_E are the mass and radius of the earth. This value is the same for all points on the earth's surface. If the center of the earth can be taken as the origin of an inertial coordinate system, then the body at the north pole really *is* in equilibrium in an inertial system, and the reading on that observer's spring scale is equal to w_0. But the body at the equator is moving in a circle of radius R_E with speed v, and there must be a net inward force equal to the mass times the centripetal acceleration:

$$w_0 - F = \frac{mv^2}{R_E}$$



12.26 Except at the poles, the reading for an object being weighed on a scale (the *apparent weight*) is less than the gravitational force of attraction on the object (the *true weight*). The reason is that a net force is needed to provide a centripetal acceleration as the object rotates with the earth. For clarity, the illustration greatly exaggerates the angle β between the true and apparent weight vectors.

So the magnitude of the apparent weight (equal to the magnitude of F) is

$$w = w_0 - \frac{mv^2}{R_E} \qquad \text{(at the equator)} \tag{12.27}$$

If the earth were not rotating, the body when released would have a free-fall acceleration $g_0 = w_0/m$. Since the earth *is* rotating, the falling body's actual acceleration relative to the observer at the equator is $g = w/m$. Dividing Eq. (12.27) by m and using these relationships, we find

$$g = g_0 - \frac{v^2}{R_E} \qquad \text{(at the equator)}$$

To evaluate v^2/R_E, we note that in 86,164 s a point on the equator moves a distance equal to the earth's circumference, $2\pi R_E = 2\pi(6.38 \times 10^6\ \text{m})$. (The solar day, 86,400 s, is $\frac{1}{365}$ longer than this because in one day the earth also completes $\frac{1}{365}$ of its orbit around the sun.) Thus we find

$$v = \frac{2\pi(6.38 \times 10^6\ \text{m})}{86{,}164\ \text{s}} = 465\ \text{m/s}$$

$$\frac{v^2}{R_E} = \frac{(465\ \text{m/s})^2}{6.38 \times 10^6\ \text{m}} = 0.0339\ \text{m/s}^2$$

So for a spherically symmetric earth the acceleration due to gravity should be about 0.03 m/s^2 less at the equator than at the poles.

At locations intermediate between the equator and the poles, the true weight $\vec{w}_0$ and the centripetal acceleration are not along the same line, and we need to write a vector equation corresponding to Eq. (12.27). From Fig. 12.26 we see that the appropriate equation is

$$\vec{w} = \vec{w}_0 - m\vec{a}_{rad} = m\vec{g}_0 - m\vec{a}_{rad} \tag{12.28}$$

The difference in the magnitudes of g and g_0 lies between zero and 0.0339 m/s^2. As shown in Fig. 12.26, the *direction* of the apparent weight differs from the

Table 12.1 Variations of g with Latitude and Elevation

Station	North Latitude	Elevation (m)	$g\,(\mathrm{m/s^2})$
Canal Zone	09°	0	9.78243
Jamaica	18°	0	9.78591
Bermuda	32°	0	9.79806
Denver, Co	40°	1638	9.79609
Pittsburgh, PA	40.5°	235	9.80118
Cambridge, MA	42°	0	9.80398
Greenland	70°	0	9.82534

direction toward the center of the earth by a small angle β, which is 0.1° or less.

Table 12.1 gives the values of g at several locations, showing variations with latitude. There are also small additional variations due to the lack of perfect spherical symmetry of the earth, local variations in density, and differences in elevation.

12.27 This orbiting astronaut is acted on by the earth's gravity, but he *feels* weightless because his acceleration is equal to $\vec{g}$.

Apparent Weight and Apparent Weightlessness

Our discussion of apparent weight can also be applied to the phenomenon of apparent weightlessness in orbiting spacecraft, which we described in Section 12.4. Bodies in an orbiting spacecraft are *not* weightless; the earth's gravitational attraction continues to act on them just as though they were at rest relative to the earth. The apparent weight of a body in a spacecraft is again given by Eq. (12.28):

$$\vec{w} = \vec{w}_0 - m\vec{a}_{\mathrm{rad}} = m\vec{g}_0 - m\vec{a}_{\mathrm{rad}}$$

But for a spacecraft in orbit, as well as any body inside the spacecraft, the acceleration $\vec{a}_{\mathrm{rad}}$ toward the earth's center is equal to the value of the acceleration of gravity $\vec{g}_0$ at the position of the spacecraft. Hence

$$\vec{g}_0 = \vec{a}_{\mathrm{rad}}$$

and the apparent weight is

$$\vec{w} = 0$$

This is what we mean when we say that an astronaut or other body in the spacecraft is apparently weightless. Note that we didn't make any assumptions about the shape of the orbit. As we mentioned in Section 12.4, an astronaut will be apparently weightless no matter what the orbit (Fig. 12.27).

Test Your Understanding of Section 12.7 Imagine a planet that has the same mass and radius as the earth, but that makes 10 rotations during the time the earth makes one rotation. What would be the difference between the acceleration due to gravity at the planet's equator and the acceleration due to gravity at its poles? (i) 0.00339 $\mathrm{m/s^2}$; (ii) 0.0339 $\mathrm{m/s^2}$; (iii) 0.339 $\mathrm{m/s^2}$; (iv) 3.39 $\mathrm{m/s^2}$.

12.8 Black Holes

The concept of a black hole is one of the most interesting and startling products of modern gravitational theory, yet the basic idea can be understood on the basis of Newtonian principles.

The Escape Speed from a Star

Think first about the properties of our own sun. Its mass $M = 1.99 \times 10^{30}$ kg and radius $R = 6.96 \times 10^{8}$ m are much larger than those of any planet, but compared to other stars, our sun is not exceptionally massive. You can find the sun's average density ρ in the same way we found the average density of the earth in Section 12.2:

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{1.99 \times 10^{30}\,\text{kg}}{\frac{4}{3}\pi(6.96 \times 10^8\,\text{m})^3}$$
$$= 1410\,\text{kg/m}^3$$

The sun's temperatures range from 5800 K (about 5500°C, or 10,000°F) at the surface up to 1.5×10^7 K (about 2.7×10^7°F) in the interior, so it surely contains no solids or liquids. Yet gravitational attraction pulls the sun's gas atoms together until the sun is, on average, 41% denser than water and about 1200 times as dense as the air we breathe.

Now think about the escape speed for a body at the surface of the sun. In Example 12.5 (Section 12.3) we found that the escape speed from the surface of a spherical mass M with radius R is $v = \sqrt{2GM/R}$. We can relate this to the average density. Substituting $M = \rho V = \rho\left(\frac{4}{3}\pi R^3\right)$ into the expression for escape speed gives

$$v = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{8\pi G\rho}{3}}R \qquad (12.29)$$

Using either form of this equation, you can show that the escape speed for a body at the surface of our sun is $v = 6.18 \times 10^5$ m/s (about 2.2 million km/h, or 1.4 million mi/h). This value, roughly 1/500 the speed of light, is independent of the mass of the escaping body; it depends on only the mass and radius (or average density and radius) of the sun.

Now consider various stars with the same average density ρ and different radii R. Equation (12.29) shows that for a given value of density ρ, the escape speed v is directly proportional to R. In 1783 the Rev. John Mitchell, an amateur astronomer, noted that if a body with the same average density as the sun had about 500 times the radius of the sun, its escape speed would be greater than the speed of light c. With his statement that "all light emitted from such a body would be made to return toward it," Mitchell became the first person to suggest the existence of what we now call a **black hole**—an object that exerts a gravitational force on other bodies, but cannot emit any light of its own.

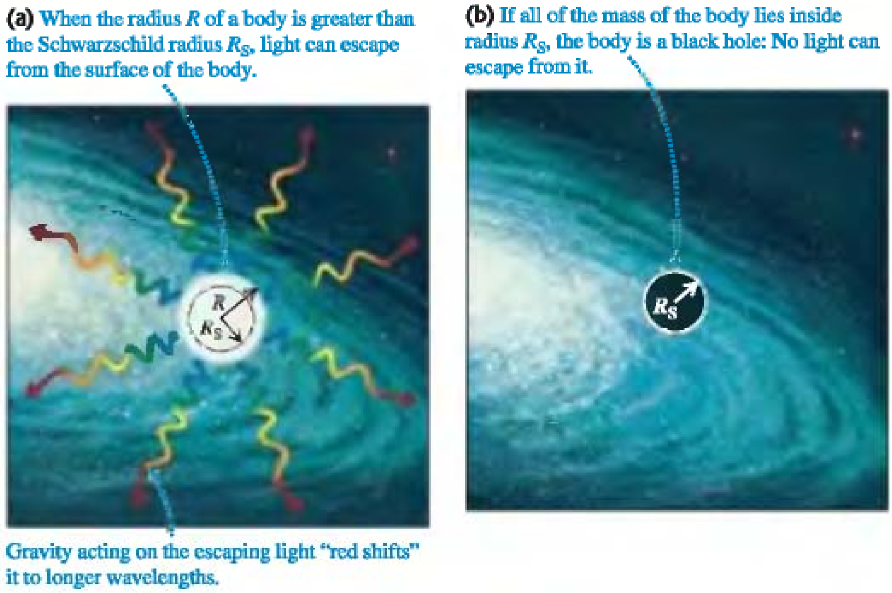
Black Holes, the Schwarzschild Radius, and the Event Horizon

The first expression for escape speed in Eq. (12.29) suggests that a body of mass M will act as a black hole if its radius R is less than or equal to a certain critical radius. How can we determine this critical radius? You might think that you can find the answer by simply setting $v = c$ in Eq. (12.29). As a matter of fact, this does give the correct result, but only because of two compensating errors. The kinetic energy of light is *not* $mc^2/2$, and the gravitational potential energy near a black hole is *not* given by Eq. (12.9). In 1916, Karl Schwarzschild used Einstein's general theory of relativity (in part a generalization and extension of Newtonian gravitation theory) to derive an expression for the critical radius R_S, now called the **Schwarzschild radius.** The result turns out to be the same as though we had set $v = c$ in Eq. (12.29), so

$$c = \sqrt{\frac{2GM}{R_S}}$$

Solving for the Schwarzschild radius R_S, we find

$$R_S = \frac{2GM}{c^2} \qquad \text{(Schwarzschild radius)} \qquad (12.30)$$



12.28 (a) A body with a radius R greater than the Schwarzschild radius R_S. (b) If the body collapses to a radius smaller than R_S, it is a black hole with an escape speed greater than the speed of light. The surface of the sphere of radius R_S is called the event horizon of the black hole.

If a spherical, nonrotating body with mass M has a radius less than R_S, then *nothing* (not even light) can escape from the surface of the body, and the body is a black hole (Fig. 12.28). In this case, any other body within a distance R_S of the center of the black hole is trapped by the gravitational attraction of the black hole and cannot escape from it.

The surface of the sphere with radius R_S surrounding a black hole is called the **event horizon:** Since light can't escape from within that sphere, we can't see events occurring inside. All that an observer outside the event horizon can know about a black hole is its mass (from its gravitational effects on other bodies), its electric charge (from the electric forces it exerts on other charged bodies), and its angular momentum (because a rotating black hole tends to drag space—and everything in that space—around with it). All other information about the body is irretrievably lost when it collapses inside its event horizon.

Example 12.11 Black hole calculations

Astrophysical theory suggests that a burned-out star will collapse under its own gravity to form a black hole when its mass is at least three solar masses. If it does, what is the radius of its event horizon?

SOLUTION

IDENTIFY: The radius in question is the Schwarzschild radius.

SET UP: We use Eq. (12.30) with a value of M equal to three solar masses, or $M = 3(1.99 \times 10^{30}\ \text{kg}) = 6.0 \times 10^{30}\ \text{kg}$.

EXECUTE: From Eq. (12.30),

$$R_S = \frac{2GM}{c^2} = \frac{2(6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2)(6.0 \times 10^{30}\ \text{kg})}{(3.00 \times 10^8\ \text{m/s})^2}$$
$$= 8.9 \times 10^3\ \text{m} = 8.9\ \text{km}$$

or less than 6 miles.

EVALUATE: If the radius of such an object is just equal to the Schwarzschild radius, the average density has the incredibly large value

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{6.0 \times 10^{30}\ \text{kg}}{\frac{4}{3}\pi(8.9 \times 10^3\ \text{m})^3}$$
$$= 2.0 \times 10^{18}\ \text{kg/m}^3$$

This is about 10^{15} times as great as the density of familiar matter on earth and is comparable to the densities of atomic nuclei. In fact, once the body collapses to a radius of R_S, nothing can prevent it from collapsing further. All of the mass ends up being crushed down to a single point called a *singularity* at the center of the event horizon. This point has zero volume and so has *infinite* density.

A Visit to a Black Hole

At points far from a black hole, its gravitational effects are the same as those of any normal body with the same mass. If the sun collapsed to form a black hole, the orbits of the planets would be unaffected. But things get dramatically different close to the black hole. If you decided to become a martyr for science and jump into a black hole, the friends you left behind would notice several odd effects as you moved toward the event horizon, most of them associated with effects of general relativity.

If you carried a radio transmitter to send back your comments on what was happening, your friends would have to retune their receiver continuously to lower and lower frequencies, an effect called the *gravitational red shift.* Consistent with this shift, they would observe that your clocks (electronic or biological) would appear to run more and more slowly, an effect called *time dilation.* In fact, during their lifetimes they would never see you make it to the event horizon.

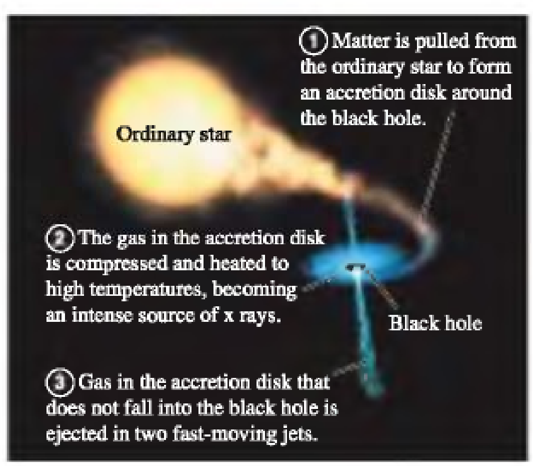
In your frame of reference, you would make it to the event horizon in a rather short time but in a rather disquieting way. As you fell feet first into the black hole, the gravitational pull on your feet would be greater than that on your head, which would be slightly farther away from the black hole. The *differences* in gravitational force on different parts of your body would be great enough to stretch you along the direction toward the black hole and compress you perpendicular to it. These effects (called *tidal forces*) would rip you to atoms, and then rip your atoms apart, before you reached the event horizon.

Detecting Black Holes

If light cannot escape from a black hole and if black holes are as small as Example 12.11 suggests, how can we know that such things exist? The answer is that any gas or dust near the black hole tends to be pulled into an *accretion disk* that swirls around and into the black hole, rather like a whirlpool (Fig. 12.29). Friction within the accretion disk's material causes it to lose mechanical energy and spiral into the black hole; as it moves inward, it is compressed together. This causes heating of the material, just as air compressed in a bicycle pump gets hotter. Temperatures in excess of 10^6 K can occur in the accretion disk, so hot that the disk emits not just visible light (as do bodies that are "red-hot" or "white-hot") but x rays. Astronomers look for these x rays (emitted by the material *before* it crosses the event horizon) to signal the presence of a black hole. Several promising candidates have been found, and astronomers now express considerable confidence in the existence of black holes.

Black holes in binary star systems like the one depicted in Fig. 12.29 have masses a few times greater than the sun's mass. There is also mounting evidence for the existence of much larger *supermassive black holes.* One example is thought to lie at the center of our Milky Way galaxy, some 26,000 light-years from earth in the direction of the constellation Sagittarius. High-resolution images of the galactic center reveal stars moving at speeds greater than 1500 km/s about an unseen object that lies at the position of a source of radio

12.29 A binary star system in which an ordinary star and a black hole orbit each other. The black hole itself cannot be seen, but the x rays from its accretion disk can be detected.



waves called Sgr A* (Fig. 12.30). By analyzing these motions, astronomers can infer the period T and semi-major axis a of each star's orbit. The mass m_X of the unseen object can then be calculated using Kepler's third law in the form given in Eq. (12.17), with the mass of the sun m_S replaced by m_X:

$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_X}} \qquad \text{so} \qquad m_X = \frac{4\pi^2 a^3}{GT^2}$$

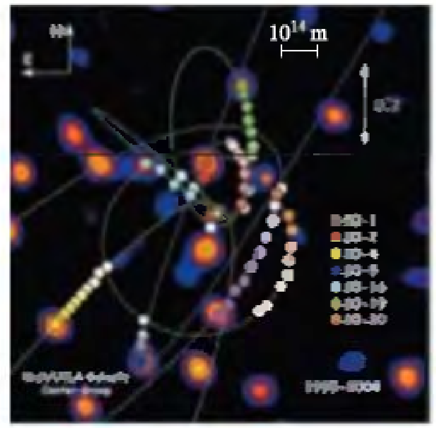
The conclusion is that the mysterious dark object at the galactic center has a mass of 7.3×10^{36} kg, or 3.7 *million* times the mass of the sun. Yet observations with radio telescopes show that it has a radius no more than about 10^{11} m, comparable to the distance from the earth to the sun. These observations suggest that this massive, compact object is a black hole with a Schwarzschild radius of 1.1×10^{10} m. Astronomers hope to improve the resolution of their observations so that they can actually see the event horizon of this black hole.

Other lines of research suggest that even larger black holes, in excess of 10^9 times the mass of the sun, lie at the centers of other galaxies. Observational and theoretical studies of black holes of all sizes continue to be an exciting area of research in both physics and astronomy.

Test Your Understanding of Section 12.8 If the sun somehow collapsed to form a black hole, what effect would this event have on the orbit of the earth? (i) The orbit would shrink; (ii) the orbit would expand; (iii) the orbit would remain the same size.

MP

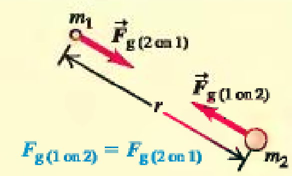
12.30 This false-color image shows the motions of stars at the center of our galaxy over a nine-year period. Analyzing these orbits using Kepler's third law indicates that the stars are moving about an unseen object that is some 3.7×10^6 times the mass of the sun. The scale bar indicates a length of 10^{14} m (670 times the distance from the earth to the sun) at the distance of the galactic center.



CHAPTER 12 SUMMARY

Newton's law of gravitation: *Any* two bodies with masses m_1 and m_2, a distance r apart, attract each other with forces inversely proportional to r^2. These forces form an action–reaction pair and obey Newton's third law. When two or more bodies exert gravitational forces on a particular body, the total gravitational force on that individual body is the vector sum of the forces exerted by the other bodies. The gravitational interaction between spherical mass distributions, such as planets or stars, is the same as if all the mass of each distribution were concentrated at the center. (See Examples 12.1–12.3 and 12.10.)

$$F_g = \frac{Gm_1m_2}{r^2} \tag{12.1}$$

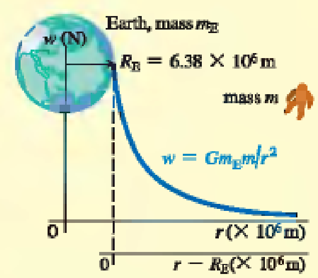


Gravitational force, weight, and gravitational potential energy: The weight w of a body is the total gravitational force exerted on it by all other bodies in the universe. Near the surface of the earth (mass m_E and radius R_E), the weight is essentially equal to the gravitational force of the earth alone. The gravitational potential energy U of two masses m and m_E separated by a distance r is inversely proportional to r. The potential energy is never positive; it is zero only when the two bodies are infinitely far apart. (See Examples 12.4 and 12.5.)

$$w = F_g = \frac{Gm_Em}{R_E^2} \tag{12.3}$$
(weight at earth's surface)

$$g = \frac{Gm_E}{R_E^2} \tag{12.4}$$
(acceleration due to gravity at earth's surface)

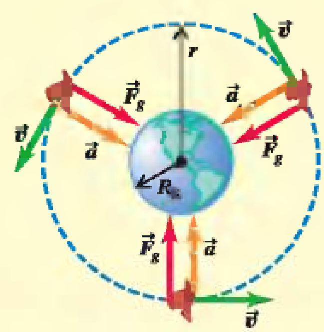
$$U = -\frac{Gm_Em}{r} \tag{12.9}$$



Orbits: When a satellite moves in a circular orbit, the centripetal acceleration is provided by the gravitational attraction of the earth. Kepler's three laws describe the more general case: an elliptical orbit of a planet around the sun or a satellite around a planet. (See Examples 12.6–12.9.)

$$v = \sqrt{\frac{Gm_E}{r}} \tag{12.10}$$
(speed in circular orbit)

$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{Gm_E}} = \frac{2\pi r^{3/2}}{\sqrt{Gm_E}} \tag{12.12}$$
(period in circular orbit)



Black holes: If a nonrotating spherical mass distribution with total mass M has a radius less than its Schwarzschild radius R_S, it is called a black hole. The gravitational interaction prevents anything, including light, from escaping from within a sphere with radius R_S. (See Example 12.11.)

$$R_S = \frac{2GM}{c^2} \tag{12.30}$$
(Schwarzschild radius)

If all of the body is inside its Schwarzschild radius $R_S = 2GM/c^2$, the body is a black hole.

Key Terms

law of gravitation, *384*
gravitational constant, *384*
gravitational potential energy, *391*
escape speed, *391*
closed orbit, *394*
open orbit, *394*
semi-major axis, *397*
eccentricity, *397*
true weight, *403*
apparent weight, *403*
black hole, *406*
Schwarzschild radius, *406*
event horizon, *407*

Answer to Chapter Opening Question ?

The smaller the orbital radius r of a satellite, the faster its orbital speed v see [Eq. (12.10)]. Hence a particle near the inner edge of Saturn's rings has a faster speed than a particle near the outer edge of the rings.

Answers to Test Your Understanding Questions

12.1 Answer: (v) From Eq. (12.1), the gravitational force of the sun (mass m_1) on a planet (mass m_2) a distance r away has magnitude $F_g = Gm_1m_2/r^2$. Compared to the earth, Saturn has a value of r^2 that is $10^2 = 100$ times greater and a value of m_2 that is also 100 times greater. Hence the *force* that the sun exerts on Saturn has the same magnitude as the force that the sun exerts on earth. The *acceleration* of a planet equals the net force divided by the planet's mass: Since Saturn has 100 times more mass than the earth, its acceleration is $1/100$ as great as that of the earth.

12.2 Answer: (iii), (i), (ii), (iv) From Eq. (12.4), the acceleration due to gravity at the surface of a planet of mass m_P and radius R_P is $g_P = Gm_P/R_P^2$. That is, g_P is directly proportional to the planet's mass and inversely proportional to the square of its radius. It follows that compared to the value of g at the earth's surface, the value of g_P on each planet is (i) $2/2^2 = 1/2$ as great; (ii) $4/4^2 = 1/4$ as great; (iii) $4/2^2 = 1$ time as great—that is, the same as on earth; and (iv) $2/4^2 = 1/8$ as great.

12.3 Answer: yes This is possible because surface gravity and escape speed depend in different ways on the planet's mass m_P and radius R_P: The value of g at the surface is Gm_P/R_P^2, while the escape speed is $\sqrt{2Gm_P/R_P}$. For the planet Saturn, for example, m_P is about 100 times the earth's mass and R_P is about 10 times the earth's radius. The value of g is different than on earth by a factor $(100)/(10)^2 = 1$ (i.e., it is the same as on earth), while the escape speed is greater by a factor $\sqrt{100/10} = 3.2$. It may help to remember that the surface gravity tells you about conditions right next to the planet's surface, while the escape speed (which tells you how fast you must travel to escape to infinity) depends on conditions at *all* points between the planet's surface and infinity. Because Saturn has so much more mass than the earth, its gravitational effects are appreciable at much greater distances and its escape speed is higher.

12.4 Answer: (ii) Equation (12.10) shows that in a smaller-radius orbit, the spacecraft has a faster speed. The negative work done by air resistance decreases the *total* mechanical energy $E = K + U$; the kinetic energy K increases (becomes more positive), but the gravitational potential energy U decreases (becomes more negative) by a greater amount.

12.5 Answer: (iii) Equation (12.17) shows that the orbital period T is proportional to the $\frac{3}{2}$ power of the semi-major axis a. Hence the orbital period of Comet X is longer than that of Comet Y by a factor of $4^{3/2} = 8$.

12.6 Answer: no Our analysis shows that there is *zero* gravitational force inside a hollow spherical shell. Hence visitors to the interior of a hollow planet would find themselves weightless, and they could not stand or walk on the planet's inner surface.

12.7 Answer: (iv) The discussion following Eq. (12.27) shows that the difference between the acceleration due to gravity at the equator and at the poles is v^2/R_E. Since this planet has the same radius and hence the same circumference as the earth, the speed v at its equator must be 10 times the speed of the earth's equator. Hence v^2/R_E is $10^2 = 100$ times greater than for the earth, or $100(0.0339\ \text{m/s}^2) = 3.39\ \text{m/s}^2$. The acceleration due to gravity at the poles is $9.80\ \text{m/s}^2$, while at the equator it is dramatically less, $9.80\ \text{m/s}^2 - 3.39\ \text{m/s}^2 = 6.41\ \text{m/s}^2$. You can show that if this planet were to rotate 17.0 times faster than the earth, the acceleration due to gravity at the equator would be *zero* and loose objects would fly off the equator's surface!

12.8 Answer: (iii) If the sun collapsed into a black hole (which, according to our understanding of stars, it cannot do), it would have the same mass but a much smaller radius. Because the gravitational attraction of the sun on the earth does not depend on the sun's radius, the earth's orbit would be unaffected.

PROBLEMS

For instructor-assigned homework, go to www.masteringphysics.com MP

Discussion Questions

Q12.1. A student wrote: "The only reason an apple falls downward to meet the earth instead of the earth rising upward to meet the apple is that the earth is much more massive and so exerts a much greater pull." Please comment.

Q12.2. A planet makes a circular orbit with period T around a star. If it were to orbit, at the same distance, a star with three times the mass of the original star, would the new period (in terms of T) would be (a) $3T$, (b) $T\sqrt{3}$, (c) T, (d) $T/\sqrt{3}$, or (e) $T/3$?

Q12.3. If all planets had the same average density, how would the acceleration due to gravity at the surface of a planet depend on its radius?

Q12.4. Is a pound of butter on the earth the same amount as a pound of butter on Mars? What about a kilogram of butter? Explain.

Q12.5. Example 12.2 (Section 12.1) shows that the acceleration of each sphere caused by the gravitational force is inversely proportional to the mass of that sphere. So why does the force of gravity give all masses the same acceleration when they are dropped near the surface of the earth?

Q12.6. When will you attract the sun more: today at noon, or tonight at midnight? Explain.

Q12.7. Since the moon is constantly attracted toward the earth by the gravitational interaction, why doesn't it crash into the earth?

Q12.8. A planet makes a circular orbit with period T around a star. If the planet were to orbit at the same distance around this star, but had three times as much mass, what would the new period (in terms of T) would be: (a) $3T$, (b) $T\sqrt{3}$, (c) T, (d) $T/\sqrt{3}$, or (e) $T/3$?

Q12.9. The sun pulls on the moon with a force that is more than twice the magnitude of the force with which the earth attracts the moon. Why, then, doesn't the sun take the moon away from the earth?

Q12.10. As defined in Chapter 7, gravitational potential energy is $U = mgy$ and is positive for a body of mass m above the earth's surface (which is at $y = 0$). But in this chapter, gravitational potential energy is $U = -Gm_E m/r$, which is *negative* for a body of mass m above the earth's surface (which is at $r = R_E$). How can you reconcile these seemingly incompatible descriptions of gravitational potential energy?

Q12.11. A planet is moving at constant speed in a circular orbit around a star. In one complete orbit, what is the net amount of work done on the planet by the star's gravitational force: positive, negative, or zero? What if the planet's orbit is an ellipse, so that the speed is not constant? Explain your answers.

Q12.12. Does the escape speed for an object at the earth's surface depend on the direction in which it is launched? Explain. Does your answer depend on whether or not you include the effects of air resistance?

Q12.13. If a projectile is fired straight up from the earth's surface, what would happen if the total mechanical energy (kinetic plus potential) is (a) less than zero, and (b) greater than zero? In each case, ignore air resistance and the gravitational effects of the sun, the moon, and the other planets.

Q12.14. Discuss whether this statement is correct: "In the absence of air resistance, the trajectory of a projectile thrown near the earth's surface is an *ellipse,* not a parabola."

Q12.15. The earth is closer to the sun in November than in May. In which of these months does it move faster in its orbit? Explain why.

Q12.16. A communications firm wants to place a satellite in orbit so that it is always directly above the earth's 45th parallel (latitude 45° north). This means that the plane of the orbit will not pass through the center of the earth. Is such an orbit possible? Why or why not?

Q12.17. At what point in an elliptical orbit is the acceleration maximum? At what point is it minimum? Justify your answers.

Q12.18. Which takes more fuel: a voyage from the earth to the moon or from the moon to the earth? Explain.

Q12.19. What would Kepler's third law be for circular orbits if an amendment to Newton's law of gravitation made the gravitational force inversely proportional to r^3? Would this change affect Kepler's other two laws? Explain.

Q12.20. In the elliptical orbit of Comet Halley shown in Fig. 12.21a, the sun's gravity is responsible for making the comet fall inward from aphelion to perihelion. But what is responsible for making the comet move from perihelion back outward to aphelion?

Q12.21. Many people believe that orbiting astronauts feel weightless because they are "beyond the pull of the earth's gravity." How far from the earth would a spacecraft have to travel to be truly beyond the earth's gravitational influence? If a spacecraft were really unaffected by the earth's gravity, would it remain in orbit? Explain. What is the real reason astronauts in orbit feel weightless?

Q12.22. As part of their training before going into orbit, astronauts ride in an airliner that is flown along the same parabolic trajectory as a freely falling projectile. Explain why this gives the same experience of apparent weightlessness as being in orbit.

Exercises

Section 12.1 Newton's Law of Gravitation

12.1. What is the ratio of the gravitational pull of the sun on the moon to that of the earth on the moon? (Assume the distance of the moon from the sun can be approximated by the distance of the earth from the sun.) Use the data in Appendix F. Is it more accurate to say that the moon orbits the earth, or that the moon orbits the sun?

12.2. Cavendish Experiment. In the Cavendish balance apparatus shown in Fig. 12.4, suppose that $m_1 = 1.10$ kg, $m_2 = 25.0$ kg, and the rod connecting the m_1 pairs is 30.0 cm long. If, in each pair, m_1 and m_2 are 12.0 cm apart center-to-center, find (a) the net force and (b) the net torque (about the rotation axis) on the rotating part of the apparatus. (c) Does it seem that the torque in part (b) would be enough to easily rotate the rod? Suggest some ways to improve the sensitivity of this experiment.

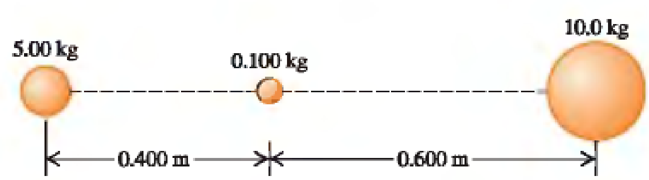
12.3. How far from a very small 100-kg ball would a particle have to be placed so that the ball pulled on the particle just as hard as the earth does? Is it reasonable that you could actually set up this as an experiment? Why?

12.4. Two uniform spheres, each with mass M and radius R, touch each other. What is the magnitude of their gravitational force of attraction?

12.5. An interplanetary spaceship passes through the point in space where the gravitational forces from the sun and the earth on the ship exactly cancel. (a) How far from the center of the earth is it? Use the data in Appendix F. (b) Once it reached the point found in part (a), could the spaceship turn off its engines and just hover there indefinitely? Explain.

12.6. (a) In Fig. 12.31 what are the magnitude and direction of the net gravitational force exerted on the 0.100-kg uniform sphere by the other two uniform spheres? The centers of all three spheres are on the same line. (b) According to Newton's third law, does the 0.100-kg sphere exert forces of the same magnitude as your answer to part (a), but in the opposite direction, on *each* of the other two spheres?

Figure 12.31 Exercise 12.6.



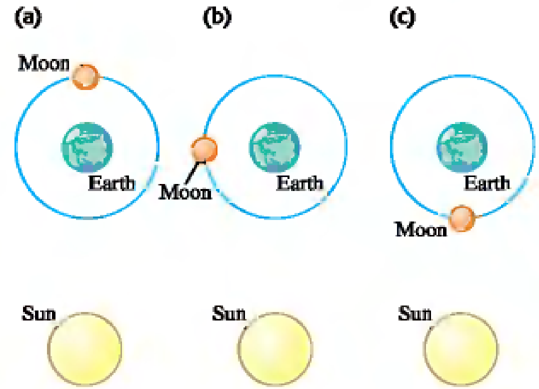
12.7. A typical adult human has a mass of about 70 kg. (a) What force does a full moon exert on such a human when it is directly overhead with its center 378,000 km away? (b) Compare this force with the force exerted on the human by the earth.

12.8. An 8.00-kg point mass and a 15.0-kg point mass are held in place 50.0 cm apart. A particle of mass m is released from a point between the two masses 20.0 cm from the 8.00-kg mass along the line connecting the two fixed masses. Find the magnitude and direction of the acceleration of the particle.

12.9. Calculate the magnitude and direction of the net gravitational force on the moon due to the earth and the sun when the moon is in each of the positions shown in Fig. 12.32. (Note that the figure is *not* drawn to scale. Assume that the sun is in the plane of the earth-moon orbit, even though this is not actually the case.) Use the data in Appendix F.

12.10. Four identical masses of 800 kg each are placed at the corners of a square whose side length is 10.0 cm. What is the net gravitational force (magnitude and direction) on one of the masses, due to the other three?

Figure 12.32 Exercise 12.9.

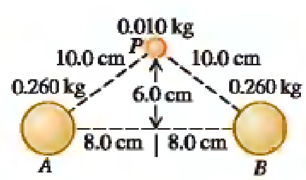


12.11. A particle of mass $3m$ is located 1.00 m from a particle of mass m. (a) Where should you put a third mass M so that the net gravitational force on M due to the two masses is exactly zero? (b) Is the equilibrium of M at this point stable or unstable (i) for points along the line connecting m and $3m$, and (ii) for points along the line passing through M and perpendicular to the line connecting m and $3m$?

12.12. The point masses m and $2m$ lie along the x-axis, with m at the origin and $2m$ at $x = L$. A third point mass M is moved along the x-axis. (a) At what point is the net gravitational force on M due to the other two masses equal to zero? (b) Sketch the x-component of the net force on M due to m and $2m$, taking quantities to the right as positive. Include the regions $x < 0$, $0 < x < L$, and $x > L$. Be especially careful to show the behavior of the graph on either side of $x = 0$ and $x = L$.

12.13. Two uniform spheres, each of mass 0.260 kg, are fixed at points A and B (Fig. 12.32). Find the magnitude and direction of the initial acceleration of a uniform sphere with mass 0.010 kg if released from rest at point P and acted on only by forces of gravitational attraction of the spheres at A and B.

Figure 12.33 Exercise 12.13.



Section 12.2 Weight

12.14. Use the mass and radius of the dwarf planet Pluto given in Appendix F to calculate the acceleration due to gravity at the surface of Pluto.

12.15. At what distance above the surface of the earth is the acceleration due to the earth's gravity 0.980 m/s^2 if the acceleration due to gravity at the surface has magnitude 9.80 m/s^2?

12.16. The mass of Venus is 81.5% that of the earth, and its radius is 94.9% that of the earth. (a) Compute the acceleration due to gravity on the surface of Venus from these data. (b) If a rock weighs 75.0 N on earth, what would it weigh at the surface of Venus?

12.17. Titania, the largest moon of the planet Uranus, has $\frac{1}{8}$ the radius of the earth and $\frac{1}{1700}$ the mass of the earth. (a) What is the acceleration due to gravity at the surface of Titania? (b) What is the average density of Titania? (This is less than the density of rock, which is one piece of evidence that Titania is made primarily of ice.)

12.16. Rhea, one of Saturn's moons, has a radius of 765 km and an acceleration due to gravity of 0.278 m/s^2 at its surface. Calculate its mass and average density.

12.19. Calculate the earth's gravity force on a 75-kg astronaut who is repairing the Hubble Space Telescope 600 km above the earth's surface, and then compare this value with his weight at the earth's surface. In view of your result, explain why we say astronauts are weightless when they orbit the earth in a satellite such as a space shuttle. Is it because the gravitational pull of the earth is negligibly small?

12.20. Neutron stars, such as the one at the center of the Crab Nebula, have about the same mass as our sun but have a *much* smaller diameter. If you weigh 675 N on the earth, what would you weigh at the surface of a neutron star that has the same mass as our sun and a diameter of 20 km?

12.21. An experiment using the Cavendish balance to measure the gravitational constant G found that a uniform 0.400-kg sphere attracts another uniform 0.00300-kg sphere with a force of 8.00×10^{-10} N, when the distance between the centers of the spheres is 0.0100 m. The acceleration due to gravity at the earth's surface is 9.80 m/s^2, and the radius of the earth is 6380 km. Compute the mass of the earth from these data.

12.22. Exploring Europa. There is strong evidence that Europa, a satellite of Jupiter, has a liquid ocean beneath its icy surface. Many scientists think we should land a vehicle there to search for life. Before launching it, we would want to test such a lander under the gravity conditions at the surface of Europa. One way to do this is to put the lander at the end of a rotating arm in an orbiting earth satellite. If the arm is 4.25 m long and pivots about one end, at what angular speed (in rpm) should it spin so that the acceleration of the lander is the same as the acceleration due to gravity at the surface of Europa? The mass of Europa is 4.8×10^{22} kg and its diameter is 3138 km.

Section 12.3 Gravitational Potential Energy

12.23. The asteroid Dactyl, discovered in 1993, has a radius of only about 700 m and a mass of about 3.6×10^{12} kg. Use the results of Example 12.5 (Section 12.3) to calculate the escape speed for an object at the surface of Dactyl. Could a person reach this speed just by walking?

12.24. Mass of a Comet. On July 4, 2005, the NASA spacecraft Deep Impact fired a projectile onto the surface of Comet Tempel 1. This comet is about 9.0 km across. Observations of surface debris released by the impact showed that dust with a speed as low as 1.0 m/s was able to escape the comet. (a) Assuming a spherical shape, what is the mass of this comet? (*Hint:* See Example 12.5 in Section 12.3.) (b) How far from the comet's center will this debris be when it has lost (i) 90.0% of its initial kinetic energy at the surface; and (ii) all of its kinetic energy at the surface?

12.25. Use the results of Example 12.5 (Section 12.3) to calculate the escape speed for a spacecraft (a) from the surface of Mars; and (b) from the surface of Jupiter. Use the data in Appendix F. (c) Why is the escape speed for a spacecraft independent of the spacecraft's mass?

12.26. Ten days after it was launched toward Mars in December 1998, the *Mars Climate Orbiter* spacecraft (mass 629 kg) was 2.87×10^6 km from the earth and traveling at 1.20×10^4 km/h relative to the earth. At this time, what were (a) the spacecraft's kinetic energy relative to the earth and (b) the potential energy of the earth-spacecraft system?

Section 12.4 The Motion of Satellites

12.27. For a satellite to be in a circular orbit 780 km above the surface of the earth, (a) what orbital speed must it be given, and (b) what is the period of the orbit (in hours)?

12.28. Aura Mission. On July 15, 2004, NASA launched the Aura spacecraft to study the earth's climate and atmosphere. This satellite was injected into an orbit 705 km above the earth's surface, and we shall assume a circular orbit. (a) How many hours does it take this satellite to make one orbit? (b) How fast (in km/s) is the Aura spacecraft moving?

12.29. Assume that the earth's orbit around the sun is circular. Use the earth's orbital radius and orbital period given in Appendix F to calculate the mass of the sun.

12.30. International Space Station. The International Space Station makes 15.65 revolutions per day in its orbit around the earth. Assuming a circular orbit, how high is this satellite above the surface of the earth?

12.31. Deimos, a moon of Mars, is about 12 km in diameter with mass 2.0×10^{15} kg. Suppose you are stranded alone on Deimos and want to play a one-person game of baseball. You would be the pitcher, and you would be the batter! (a) With what speed would you have to throw a baseball so that it would go into a circular orbit just above the surface and return to you so you could hit it? Do you think you could actually throw it at this speed? (b) How long (in hours) after throwing the ball should you be ready to hit it? Would this be an action-packed baseball game?

Section 12.5 Kepler's Laws and the Motion of Planets

12.32. Planet Vulcan. Suppose that a planet were discovered between the sun and Mercury, with a circular orbit of radius equal to $\frac{2}{3}$ of the average orbit radius of Mercury. What would be the orbital period of such a planet? (Such a planet was once postulated, in part to explain the precession of Mercury's orbit. It was even given the name Vulcan, although we now have no evidence that it actually exists. Mercury's precession has been explained by general relativity.)

12.33. The star Rho^1 Cancri is 57 light-years from the earth and has a mass 0.85 times that of our sun. A planet has been detected in a circular orbit around Rho^1 Cancri with an orbital radius equal to 0.11 times the radius of the earth's orbit around the sun. What are (a) the orbital speed and (b) the orbital period of the planet of Rho^1 Cancri?

12.34. In March 2006, two small satellites were discovered orbiting Pluto, one at a distance of 48,000 km and the other at 64,000 km. Pluto already was known to have a large satellite Charon, orbiting at 19,600 km with an orbital period of 6.39 days. Assuming that the satellites do not affect each other, find the orbital periods of the two small satellites *without* using the mass of Pluto.

12.35. (a) Use Fig. 12.19 to show that the sun-planet distance at perihelion is $(1 - e)a$, the sun-planet distance at aphelion is $(1 + e)a$, and therefore the sum of these two distances is $2a$. (b) When the dwarf planet Pluto was at perihelion in 1989, it was almost 100 million km closer to the sun than Neptune. The semimajor axes of the orbits of Pluto and Neptune are 5.92×10^{12} m and 4.50×10^{12} m, respectively, and the eccentricities are 0.248 and 0.010. Find Pluto's closest distance and Neptune's farthest distance from the sun. (c) How many years after being at perihelion in 1989 will Pluto again be at perihelion?

12.36. Hot Jupiters. In 2004 astronomers reported the discovery of a large Jupiter-sized planet orbiting very close to the star HD 179949 (hence the term "hot Jupiter"). The orbit was just $\frac{1}{9}$ the distance of Mercury from our sun, and it takes the planet only 3.09 days to make one orbit (assumed to be circular). (a) What is the mass of the star? Express your answer in kilograms and as a multiple of our sun's mass. (b) How fast (in km/s) is this planet moving?

12.37. The *Helios B* spacecraft had a speed of 71 km/s when it was 4.3×10^7 km from the sun. (a) Prove that it was not in a circular orbit about the sun. (b) Prove that its orbit about the sun was closed and therefore elliptical.

*Section 12.6 Spherical Mass Distributions

12.38. A uniform, spherical, 1000.0-kg shell has a radius of 5.00 m. (a) Find the gravitational force this shell exerts on a 2.00-kg point mass placed at the following distances from the center of the shell: (i) 5.01 m, (ii) 4.99 m, (iii) 2.72 m. (b) Sketch a qualitative graph of the magnitude of the gravitational force this sphere exerts on a point mass m as a function of the distance r of m from the center of the sphere. Include the region from $r = 0$ to $r \rightarrow \infty$.

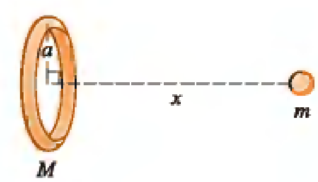
12.39. A uniform, solid, 1000.0-kg sphere has a radius of 5.00 m. (a) Find the gravitational force this sphere exerts on a 2.00-kg point mass placed at the following distances from the center of the sphere: (i) 5.01 m, and (ii) 2.50 m. (b) Sketch a qualitative graph of the magnitude of the gravitational force this sphere exerts on a point mass m as a function of the distance r of m from the center of the sphere. Include the region from $r = 0$ to $r \rightarrow \infty$.

12.40. A thin, uniform rod has length L and mass M. A small uniform sphere of mass m is placed a distance x from one end of the rod, along the axis of the rod (Fig. 12.34). (a) Calculate the gravitational potential energy of the rod–sphere system. Take the potential energy to be zero when the rod and sphere are infinitely far apart. Show that your answer reduces to the expected result when x is much larger than L. (*Hint:* Use the power series expansion for $\ln(1 + x)$ given in Appendix B.) (b) Use $F_x = -dU/dx$ to find the magnitude and direction of the gravitational force exerted on the sphere by the rod (see Section 7.4). Show that your answer reduces to the expected result when x is much larger than L.

Figure 12.34 Exercise 12.40 and Problem 12.84.

12.41. Consider the ring-shaped body of Fig. 12.35. A particle with mass m is placed a distance x from the center of the ring, along the line through the center of the ring and perpendicular to its plane. (a) Calculate the gravitational potential energy U of this system. Take the potential energy to be zero when the two objects are far apart. (b) Show that your answer to part (a) reduces to the expected result when x is much larger than the radius a of the ring. (c) Use $F_x = -dU/dx$ to find the magnitude and direction of the force on the particle (see Section 7.4). (d) Show that your answer to part (c) reduces to the expected result when x is much larger than a. (e) What are the values of U and F_x when $x = 0$? Explain why these results make sense.

Figure 12.35 Exercise 12.41 and Problem 12.83.



*Section 12.7 Apparent Weight and the Earth's Rotation

12.42. The weight of Santa Claus at the North Pole, as determined by a spring balance, is 875 N. What would this spring balance read for his weight at the equator, assuming that the earth is spherically symmetric?

12.43. The acceleration due to gravity at the north pole of Neptune is approximately 10.7 m/s^2. Neptune has mass 1.0×10^{26} kg and radius 2.5×10^4 km and rotates once around its axis in about 16 h.

(a) What is the gravitational force on a 5.0-kg object at the north pole of Neptune? (b) What is the apparent weight of this same object at Neptune's equator? (Note that Neptune's "surface" is gaseous, not solid, so it is impossible to stand on it.)

*Section 12.8 Black Holes

12.44. Mini Black Holes. Cosmologists have speculated that black holes the size of a proton could have formed during the early days of the Big Bang when the universe began. If we take the diameter of a proton to be 1.0×10^{-15} m, what would be the mass of a mini black hole?

12.45. To what fraction of its current radius would the earth have to be compressed to become a black hole?

12.46. (a) Show that a black hole attracts an object of mass m with a force of $mc^2R_S/(2r^2)$, where r is the distance between the object and the center of the black hole. (b) Calculate the magnitude of the gravitational force exerted by a black hole of Schwarzschild radius 14.0 mm on a 5.00-kg mass 3000 km from it. (c) What is the mass of this black hole?

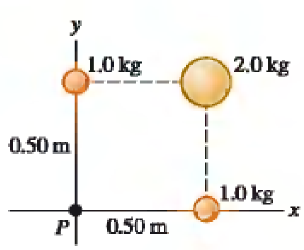
12.47. At the Galaxy's Core. Astronomers have observed a small, massive object at the center of our Milky Way galaxy (see Section 12.8). A ring of material orbits this massive object; the ring has a diameter of about 15 light-years and an orbital speed of about 200 km/s. (a) Determine the mass of the object at the center of the Milky Way galaxy. Give your answer both in kilograms and in solar masses (one solar mass is the mass of the sun). (b) Observations of stars, as well as theories of the structure of stars, suggest that it is impossible for a single star to have a mass of more than about 50 solar masses. Can this massive object be a single, ordinary star? (c) Many astronomers believe that the massive object at the center of the Milky Way galaxy is a black hole. If so, what must the Schwarzschild radius of this black hole be? Would a black hole of this size fit inside the earth's orbit around the sun?

12.46. In 2005 astronomers announced the discovery of a large black hole in the galaxy Markarian 766 having clumps of matter orbiting around once every 27 hours and moving at 30,000 km/s. (a) How far are these clumps from the center of the black hole? (b) What is the mass of this black hole, assuming circular orbits? Express your answer in kilograms and as a multiple of our sun's mass. (c) What is the radius of its event horizon?

Problems

12.46. Three uniform spheres are fixed at the positions shown in Fig. 12.36. (a) What are the magnitude and direction of the force on a 0.0150-kg particle placed at P? (b) If the spheres are in deep outer space and a 0.0150-kg particle is released from rest 300 m from the origin along a line 45° below the $-x$-axis, what will the particle's speed be when it reaches the origin?

Figure 12.36 Problem 12.49.



12.50. A uniform sphere with mass 60.0 kg is held with its center at the origin, and a second uniform sphere with mass 80.0 kg is held with its center at the point $x = 0$, $y = 3.00$ m. (a) What are the magnitude and direction of the net gravitational force due to these objects on a third uniform sphere with mass 0.500 kg placed at the point $x = 4.00$ m, $y = 0$? (b) Where, other than infinitely far away, could the third sphere be placed such that the net gravitational force acting on it from the other two spheres is equal to zero?

12.51. (a) Show that the gravitational force on the small star due to the two large stars in Example 12.3 (Section 12.1) is *not* directed toward the point midway between the two large masses. (b) Consider the two large stars as making up a single, rigid body, as if they were joined by a rod of negligible mass. Calculate the torque exerted by the small star on the rigid body for a pivot at its center of mass. (c) Explain how the result in part (b) shows that the center of mass does not coincide with the center of gravity. Why is this the case in this situation?

12.52. At a certain instant, the earth, the moon, and a stationary 1250-kg spacecraft lie at the vertices of an equilateral triangle whose sides are 3.84×10^5 km in length. (a) Find the magnitude and direction of the net gravitational force exerted on the spacecraft by the earth and moon. State the direction as an angle measured from a line connecting the earth and the spacecraft. In a sketch, show the earth, the moon, the spacecraft, and the force vector. (b) What is the minimum amount of work that you would have to do to move the spacecraft to a point far from the earth and moon? You can ignore any gravitational effects due to the other planets or the sun.

12.53. An experiment is performed in deep space with two uniform spheres, one with mass 25.0 kg and the other with mass 100.0 kg. They have equal radii, $r = 0.20$ m. The spheres are released from rest with their centers 40.0 m apart. They accelerate toward each other because of their mutual gravitational attraction. You can ignore all gravitational forces other than that between the two spheres. (a) Explain why linear momentum is conserved. (b) When their centers are 20.0 m apart, find (i) the speed of each sphere and (ii) the magnitude of the relative velocity with which one sphere is approaching the other. (c) How far from the initial position of the center of the 25.0-kg sphere do the surfaces of the two spheres collide?

12.54. Assume that the moon orbits the earth in a circular orbit. From the observed orbital period of 27.3 days, calculate the distance of the moon from the center of the earth. Assume that the moon's motion is determined solely by the gravitational force exerted on it by the earth, and use the mass of the earth given in Appendix F.

12.55. Geosynchronous Satellites. Many satellites are moving in a circle in the earth's equatorial plane. They are at such a height above the earth's surface that they always remain above the same point. (a) Find the altitude of these satellites above the earth's surface. (Such an orbit is said to be *geosynchronous*.) (b) Explain, with a sketch, why the radio signals from these satellites cannot directly reach receivers on earth that are north of 81.3° N latitude.

12.56. A landing craft with mass 12,500 kg is in a circular orbit 5.75×10^5 m above the surface of a planet. The period of the orbit is 5800 s. The astronauts in the lander measure the diameter of the planet to be 9.60×10^6 m. The lander sets down at the north pole of the planet. What is the weight of a 85.6-kg astronaut as he steps out onto the planet's surface?

12.57. What is the escape speed from a 300-km-diameter asteroid with a density of 2500 kg/m^3?

12.56. (a) Asteroids have average densities of about 2500 kg/m^3 and radii from 470 km down to less than a kilometer. Assuming that the asteroid has a spherically symmetric mass distribution, estimate the radius of the largest asteroid from which you could escape simply by jumping off. (*Hint:* You can estimate your jump speed by relating it to the maximum height that you can jump on earth.) (b) Europa, one of Jupiter's four large moons, has a radius of 1570 km. The acceleration due to gravity at its surface is 1.33 m/s^2. Calculate its average density.

12.56. (a) Suppose you are at the earth's equator and observe a satellite passing directly overhead and moving from west to east in the sky. Exactly 12.0 hours later, you again observe this satellite to be directly overhead. How far above the earth's surface is the satellite's orbit? (b) You observe another satellite directly overhead and traveling east to west. This satellite is again overhead in 12.0 hours. How far is this satellite's orbit above the surface of the earth?

12.60. Planet X rotates in the same manner as the earth, around an axis through its north and south poles, and is perfectly spherical. An astronaut who weighs 943.0 N on the earth weighs 915.0 N at the north pole of Planet X and only 850.0 N at its equator. The distance from the north pole to the equator is 18,850 km, measured along the surface of Planet X. (a) How long is the day on Planet X? (b) If a 45,000-kg satellite is placed in a circular orbit 2000 km above the surface of Planet X, what will be its orbital period?

12.61. There are two equations from which a change in the gravitational potential energy U of the system of a mass m and the earth can be calculated. One is $U = mgy$ (Eq. 7.2). The other is $U = -Gm_Em/r$ (Eq. 12.9). As shown in Section 12.3, the first equation is correct only if the gravitational force is a constant over the change in height Δy. The second is always correct. Actually, the gravitational force is never exactly constant over any change in height, but if the variation is small, we can ignore it. Consider the difference in U between a mass at the earth's surface and a distance h above it using both equations, and find the value of h for which Eq. (7.2) is in error by 1%. Express this value of h as a fraction of the earth's radius, and also obtain a numerical value for it.

12.62. Your starship, the *Aimless Wanderer*, lands on the mysterious planet Mongo. As chief scientist-engineer, you make the following measurements: A 2.50-kg stone thrown upward from the ground at 12.0 m/s returns to the ground in 8.00 s; the circumference of Mongo at the equator is 2.00×10^5 km; and there is no appreciable atmosphere on Mongo. The starship commander, Captain Confusion, asks for the following information: (a) What is the mass of Mongo? (b) If the *Aimless Wanderer* goes into a circular orbit 30,000 km above the surface of Mongo, how many hours will it take the ship to complete one orbit?

12.63. Calculate the percent difference between your weight in Sacramento, near sea level, and at the top of Mount Everest, which is 8800 m above sea level.

12.64. In Example 12.5 (Section 12.3) we ignored the gravitational effects of the moon on a spacecraft en route from the earth to the moon. In fact, we must include the gravitational potential energy due to the moon as well. For this problem, you can ignore the motion of the earth and moon. (a) If the moon has radius R_M and the distance between the centers of the earth and the moon is R_{EM}, find the total gravitational potential energy of the particle-earth and particle-moon systems when a particle with mass m is between the earth and the moon, and a distance r from the center of the earth. Take the gravitational potential energy to be zero when the objects are far from each other. (b) There is a point along a line between the earth and the moon where the net gravitational force is zero. Use the expression derived in part (a) and numerical values from Appendix F to find the distance of this point from the center of the earth. With what speed must a spacecraft be launched from the surface of the earth just barely to reach this point? (c) If a spacecraft were launched from the earth's surface toward the moon with an initial speed of 11.2 km/s, with what speed would it impact the moon?

12.65. An unmanned spacecraft is in a circular orbit around the moon, observing the lunar surface from an altitude of 50.0 km (see Appendix F). To the dismay of scientists on earth, an electrical fault causes an on-board thruster to fire, decreasing the speed of the spacecraft by 20.0 m/s. If nothing is done to correct its orbit, with what speed (in km/h) will the spacecraft crash into the lunar surface?

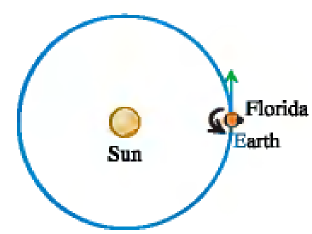
***12.66.** What would be the length of a day (that is, the time required for one rotation of the earth on its axis) if the rate of rotation of the earth were such that $g = 0$ at the equator?

12.67. Falling Hammer. A hammer with mass m is dropped from rest from a height h above the earth's surface. This height is not necessarily small compared with the radius R_E of the earth. If you ignore air resistance, derive an expression for the speed v of the hammer when it reaches the surface of the earth. Your expression should involve h, R_E, and m_E, the mass of the earth.

12.68. (a) Calculate how much work is required to launch a spacecraft of mass m from the surface of the earth (mass m_E, radius R_E) and place it in a circular *low earth orbit*—that is, an orbit whose altitude above the earth's surface is much less than R_E. (As an example, the International Space Station is in low earth orbit at an altitude of about 400 km, much less than $R_E = 6380$ km.) You can ignore the kinetic energy that the spacecraft has on the ground due to the earth's rotation. (b) Calculate the minimum amount of additional work required to move the spacecraft from low earth orbit to a very great distance from the earth. You can ignore the gravitational effects of the sun, the moon, and the other planets. (c) Justify the statement: "In terms of energy, low earth orbit is halfway to the edge of the universe."

12.68. A spacecraft is to be launched from the surface of the earth so that it will escape from the solar system altogether. (a) Find the speed relative to the center of the earth with which the spacecraft must be launched. Take into consideration the gravitational effects of both the earth and the sun, and include the effects of the earth's orbital speed, but ignore air resistance. (b) The rotation of the earth can help this spacecraft achieve escape speed. Find the speed that the spacecraft must have relative to the earth's *surface* if the spacecraft is launched from Florida at the point shown in Fig. 12.37. The rotation and orbital motions of the earth are in the same direction. The launch facilities in Florida are 28.5° north of the equator. (c) The European Space Agency (ESA) uses launch facilities in French Guiana (immediately north of Brazil), 5.15° north of the equator. What speed relative to the earth's surface would a spacecraft need to escape the solar system if launched from French Guiana?

Figure 12.37 Problem 12.69.



***12.70. Gravity Inside the Earth.** Find the gravitational force that the earth exerts on a 10.0-kg mass if it is placed at the following locations. Consult Fig. 12.9, and assume a constant density through each of the interior regions (mantle, outer core, inner core), but *not* the same density in each of these regions. Use the graph to estimate the average density for each region. (a) at the surface of the earth; (b) at the outer surface of the molten outer core; (c) at the surface of the solid inner core; (d) at the center of the earth.

12.71. Kirkwood Gaps. Hundreds of thousands of asteroids orbit the sun within the *asteroid belt*, which extends from about 3×10^8 km to about 5×10^8 km from the sun. (a) Find the orbital period (in years) of (i) an asteroid at the inside of the belt and (ii) an asteroid at the outside of the belt. Assume circular orbits. (b) In 1867 the American astronomer Daniel Kirkwood pointed out that several gaps exist in the asteroid belt where relatively few asteroids are found. It is now understood that these *Kirkwood gaps* are caused by the gravitational attraction of Jupiter, the largest planet, which orbits the sun once every 11.86 years. As an example, if an asteroid has an orbital period half that of Jupiter, or 5.93 years, on every other orbit this asteroid would be at its closest to Jupiter and feel a strong attraction toward the planet. This attraction, acting over and over on successive orbits, could sweep asteroids out of the Kirkwood gap. Use this hypothesis to determine the orbital radius for this Kirkwood gap. (c) One of several other Kirkwood gaps appears at a distance from the sun where the orbital period is 0.400 that of Jupiter. Explain why this happens, and find the orbital radius for this Kirkwood gap.

12.72. If a satellite is in a sufficiently low orbit, it will encounter air drag from the earth's atmosphere. Since air drag does negative work (the force of air drag is directed opposite the motion), the mechanical energy will decrease. According to Eq. (12.13), if E decreases (becomes more negative), the radius r of the orbit will decrease. If air drag is relatively small, the satellite can be considered to be in a circular orbit of continually decreasing radius. (a) According to Eq. (12.10), if the radius of a satellite's circular orbit decreases, the satellite's orbital speed v *increases.* How can you reconcile this with the statement that the mechanical energy *decreases?* (*Hint:* Is air drag the only force that does work on the satellite as the orbital radius decreases?) (b) Due to air drag, the radius of a satellite's circular orbit decreases from r to $r - \Delta r$, where the positive quantity Δr is much less than r. The mass of the satellite is m. Show that the increase in orbital speed is $\Delta v = +(\Delta r/2)\sqrt{Gm_E/r^3}$; that the change in kinetic energy is $\Delta K = +(Gm_E m/2r^2)\,\Delta r$; that the change in gravitational potential energy is $\Delta U = -2\,\Delta K = -(Gm_E m/r^2)\,\Delta r$; and that the amount of work done by the force of air drag is $W = -(Gm_E m/2r^2)\,\Delta r$. Interpret these results in light of your comments in part (a). (c) A satellite with mass 3000 kg is initially in a circular orbit 300 km above the earth's surface. Due to air drag, the satellite's altitude decreases to 250 km. Calculate the initial orbital speed; the increase in orbital speed; the initial mechanical energy; the change in kinetic energy; the change in gravitational potential energy; the change in mechanical energy; and the work done by the force of air drag. (d) Eventually a satellite will descend to a low enough altitude in the atmosphere that the satellite burns up and the debris falls to the earth. What becomes of the initial mechanical energy?

12.73. Binary Star—Equal Masses. Two identical stars with mass M orbit around their center of mass. Each orbit is circular and has radius R, so that the two stars are always on opposite sides of the circle. (a) Find the gravitational force of one star on the other. (b) Find the orbital speed of each star and the period of the orbit. (c) How much energy would be required to separate the two stars to infinity?

12.74. Binary Star—Different Masses. Two stars, with masses M_1 and M_2, are in circular orbits around their center of mass. The star with mass M_1 has an orbit of radius R_1; the star with mass M_2 has an orbit of radius R_2. (a) Show that the ratio of the orbital radii of the two stars equals the reciprocal of the ratio of their masses, that is, $R_1/R_2 = M_2/M_1$. (b) Explain why the two stars have the same orbital period, and show that the period T is given by $T = 2\pi(R_1 + R_2)^{3/2}/\sqrt{G(M_1 + M_2)}$. (c) The two stars in a certain binary star system move in circular orbits. The first star, Alpha, has an orbital speed of 36.0 km/s. The second star, Beta, has an orbital speed of 12.0 km/s. The orbital period is 137 d. What are the masses of each of the two stars? (d) One of the best candidates for a black hole is found in the binary system called A0620-0090. The two objects in the binary system are an orange star, V616 Monocerotis, and a compact object believed to be a black hole (Fig. 12.22). The orbital period of A0620-0090 is 7.75 hours, the mass of V616 Monocerotis is estimated to be 0.67 times the mass of the sun, and the mass of the black hole is estimated to be 3.8 times the mass of the sun. Assuming that the orbits are circular, find the radius of each object's orbit and the orbital speed of each object. Compare these answers to the orbital radius and orbital speed of the earth in its orbit around the sun.

12.75. Comets travel around the sun in elliptical orbits with large eccentricities. If a comet has speed 2.0×10^4 m/s when at a distance of 2.5×10^{11} m from the center of the sun, what is its speed when at a distance of 5.0×10^{10} m?

12.76. As Mars orbits the sun in its elliptical orbit, its distance of closest approach to the center of the sun (at perihelion) is 2.067×10^{11} m, and its maximum distance from the center of the sun (at aphelion) is 2.492×10^{11} m. If the orbital speed of Mars at aphelion is 2.198×10^4 m/s, what is its orbital speed at perihelion? (You can ignore the influence of the other planets.)

12.77. Consider a spacecraft in an elliptical orbit around the earth. At the low point, or perigee, of its orbit, it is 400 km above the earth's surface; at the high point, or apogee, it is 4000 km above the earth's surface. (a) What is the period of the spacecraft's orbit? (b) Using conservation of angular momentum, find the ratio of the spacecraft's speed at perigee to its speed at apogee. (c) Using conservation of energy, find the speed at perigee and the speed at apogee. (d) It is necessary to have the spacecraft escape from the earth completely. If the spacecraft's rockets are fired at perigee, by how much would the speed have to be increased to achieve this? What if the rockets were fired at apogee? Which point in the orbit is more efficient to use?

12.70. The planet Uranus has a radius of 25,560 km and a surface acceleration due to gravity of 11.1 m/s^2 at its poles. Its moon Miranda (discovered by Kuiper in 1948) is in a circular orbit about Uranus at an altitude of 104,000 km above the planet's surface. Miranda has a mass of 6.6×10^{19} kg and a radius of 235 km. (a) Calculate the mass of Uranus from the given data. (b) Calculate the magnitude of Miranda's acceleration due to its orbital motion about Uranus. (c) Calculate the acceleration due to Miranda's gravity at the surface of Miranda. (d) Do the answers to parts (b) and (c) mean that an object released 1 m above Miranda's surface on the side toward Uranus will fall *up* relative to Miranda? Explain.

12.79. A 3000-kg spacecraft is in a circular orbit 2000 km above the surface of Mars. How much work must the spacecraft engines perform to move the spacecraft to a circular orbit that is 4000 km above the surface?

12.68. One of the brightest comets of the 20th century was Comet Hyakutake, which passed close to the sun in early 1996. The orbital period of this comet is estimated to be about 30,000 years. Find the semi-major axis of this comet's orbit. Compare it to the average sun-Pluto distance and to the distance to Alpha Centauri, the nearest star to the sun, which is 4.3 light-years distant.

12.61. Planets are not uniform inside. Normally, they are densest at the center and have decreasing density outward toward the surface. Model a spherically symmetric planet, with the same radius as the earth, as having a density that decreases linearly with distance from the center. Let the density be 15.0×10^3 kg/m^3 at the center and 2.0×10^3 kg/m^3 at the surface. What is the acceleration due to gravity at the surface of this planet?

12.62. A uniform wire with mass M and length L is bent into a semicircle. Find the magnitude and direction of the gravitational force this wire exerts on a point with mass m placed at the center of curvature of the semicircle.

***12.83.** An object in the shape of a thin ring has radius a and mass M. A uniform sphere with mass m and radius R is placed with its center at a distance x to the right of the center of the ring, along a line through the center of the ring, and perpendicular to its plane (Fig. 12.35). What is the gravitational force that the sphere exerts on the ring-shaped object? Show that your result reduces to the expected result when x is much larger than a.

***12.84.** A thin, uniform rod has length L and mass M. Calculate the magnitude of the gravitational force the rod exerts on a particle with mass m that is at a point along the axis of the rod a distance x from one end (Fig. 12.34). Show that your result reduces to the expected result when x is much larger than L.

***12.85.** A shaft is drilled from the surface to the center of the earth (Fig. 12.25). As in Example 12.10 (Section 12.6), make the unrealistic assumption that the density of the earth is uniform. With this approximation, the gravitational force on an object with mass m, that is inside the earth at a distance r from the center, has magnitude $F_g = Gm_E mr/R_E^3$ (as shown in Example 12.10) and points toward the center of the earth. (a) Derive an expression for the gravitational potential energy $U(r)$ of the object–earth system as a function of the object's distance from the center of the earth. Take the potential energy to be zero when the object is at the center of the earth. (b) If an object is released in the shaft at the earth's surface, what speed will it have when it reaches the center of the earth?

Challenge Problems

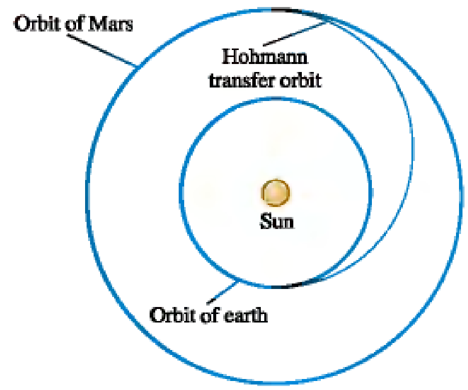
12.68. (a) When an object is in a circular orbit of radius r around the earth (mass m_E), the period of the orbit is T, given by Eq. (12.12), and the orbital speed is v, given by Eq. (12.10). Show that when the object is moved into a circular orbit of slightly larger radius $r + \Delta r$, where $\Delta r \ll r$, its new period is $T + \Delta T$ and its new orbital speed is $v - \Delta v$, where Δr, ΔT, and Δv are all positive quantities and

$$\Delta T = \frac{3\pi \Delta r}{v} \qquad \text{and} \qquad \Delta v = \frac{\pi\, \Delta r}{T}$$

(*Hint:* Use the expression $(1 + x)^n \approx 1 + nx$, valid for $|x| \ll 1$.) (b) The International Space Station (ISS) is in a nearly circular orbit at an altitude of 398.00 km above the surface of the earth. A maintenance crew is about to arrive on the space shuttle that is also in a circular orbit in the same orbital plane as the ISS, but with an altitude of 398.10 km. The crew has come to remove a faulty 125-m electrical cable, one end of which is attached to the ISS and the other end of which is floating free in space. The plan is for the shuttle to snag the free end just at the moment that the shuttle, the ISS, and the center of the earth all lie along the same line. The cable will then break free from the ISS when it becomes taut. How long after the free end is caught by the space shuttle will it detach from the ISS? Give your answer in minutes. (c) If the shuttle misses catching the cable, show that the crew must wait a time $t \approx T^2/\Delta T$ before they have a second chance. Find the numerical value of t and explain whether it would be worth the wait.

12.67. Interplanetary Navigation. The most efficient way to send a spacecraft from the earth to another planet is by using a *Hohmann transfer orbit* (Fig. 12.38). If the orbits of the departure and destination planets are circular, the Hohmann transfer orbit is an elliptical orbit whose perihelion and aphelion are tangent to the orbits of the two planets. The rockets are fired briefly at the departure planet to put the spacecraft into the transfer orbit; the spacecraft then coasts until it reaches the destination planet. The rockets are then fired again to put the spacecraft into the same orbit about the sun as the destination planet. (a) For a flight from earth to Mars, in what direction must the rockets be fired at the earth and at Mars: in the direction of motion, or opposite the direction of motion? What about from a flight from Mars to the earth? (b) How long does a one-way trip from the the earth to Mars take, between the firings of the rockets? (c) To reach Mars from the earth, the launch must be timed so that Mars will be at the right spot when the spacecraft reaches Mars's orbit around the sun. At launch, what must the angle between a sun-Mars line and a sun-earth line be? Use data from Appendix F.

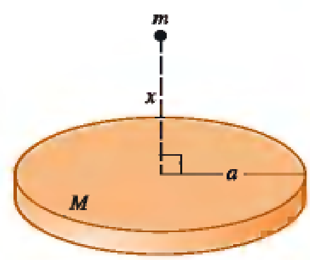
Figure 12.38 Challenge Problem 12.87.



12.68. Tidal Forces near a Black Hole. An astronaut inside a spacecraft, which protects her from harmful radiation, is orbiting a black hole at a distance of 120 km from its center. The black hole is 5.00 times the mass of the sun and has a Schwarzschild radius of 15.0 km. The astronaut is positioned inside the spaceship such that one of her 0.030-kg ears is 6.0 cm farther from the black hole than the center of mass of the spacecraft and the other ear is 6.0 cm closer. (a) What is the tension between her ears? Would the astronaut find it difficult to keep from being torn apart by the gravitational forces? (Since her whole body orbits with the same angular velocity, one ear is moving too slowly for the radius of its orbit and the other is moving too fast. Hence her head must exert forces on her ears to keep them in their orbits.) (b) Is the center of gravity of her head at the same point as the center of mass? Explain.

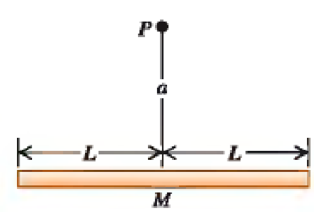
***12.89.** Mass M is distributed uniformly over a disk of radius a. Find the gravitational force (magnitude and direction) between this disk-shaped mass and a particle with mass m located a distance x above the center of the disk (Fig. 12.39). Does your result reduce to the correct expression as x becomes very large? (*Hint:* Divide the disk into infinitesimally thin concentric rings, use the expression derived in Exercise 12.41 for the gravitational force due to each ring, and integrate to find the total force.)

Figure 12.39 Challenge Problem 12.89.



***12.90.** Mass M is distributed uniformly along a line of length $2L$. A particle with mass m is at a point that is a distance a above the center of the line on its perpendicular bisector (point P in Fig. 12.40). For the gravitational force that the line exerts on the particle, calculate the components perpendicular and parallel to the line. Does your result reduce to the correct expression as a becomes very large?

Figure 12.40 Challenge Problem 12.90.



PERIODIC MOTION

13

?Suppose you doubled the mass of a clock's pendulum (including the rod and the weight at the end) while keeping its dimensions the same. Would the clock run fast or slow?

LEARNING GOALS

By studying this chapter, you will learn:

- How to describe oscillations in terms of amplitude, period, frequency, and angular frequency.
- How to do calculations with simple harmonic motion, an important type of oscillation.
- How to use energy concepts to analyze simple harmonic motion.
- How to apply the ideas of simple harmonic motion to different physical situations.
- How to analyze the motions of a simple pendulum.
- What a physical pendulum is, and how to calculate the properties of its motion.
- What determines how rapidly an oscillation dies out.
- How a driving force applied to an oscillator at the right frequency can cause a very large response, or resonance.

Many kinds of motion repeat themselves over and over: the vibration of a quartz crystal in a watch, the swinging pendulum of a grandfather clock, the sound vibrations produced by a clarinet or an organ pipe, and the back-and-forth motion of the pistons in a car engine. This kind of motion, called **periodic motion** or **oscillation,** is the subject of this chapter. Understanding periodic motion will be essential for our later study of waves, sound, alternating electric currents, and light.

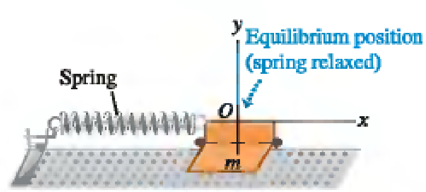
A body that undergoes periodic motion always has a stable equilibrium position. When it is moved away from this position and released, a force or torque comes into play to pull it back toward equilibrium. But by the time it gets there, it has picked up some kinetic energy, so it overshoots, stopping somewhere on the other side, and is again pulled back toward equilibrium. Picture a ball rolling back and forth in a round bowl or a pendulum that swings back and forth past its straight-down position.

In this chapter we will concentrate on two simple examples of systems that can undergo periodic motions: spring-mass systems and pendulums. We will also study why oscillations often tend to die out with time and why some oscillations can build up to greater and greater displacements from equilibrium when periodically varying forces act.

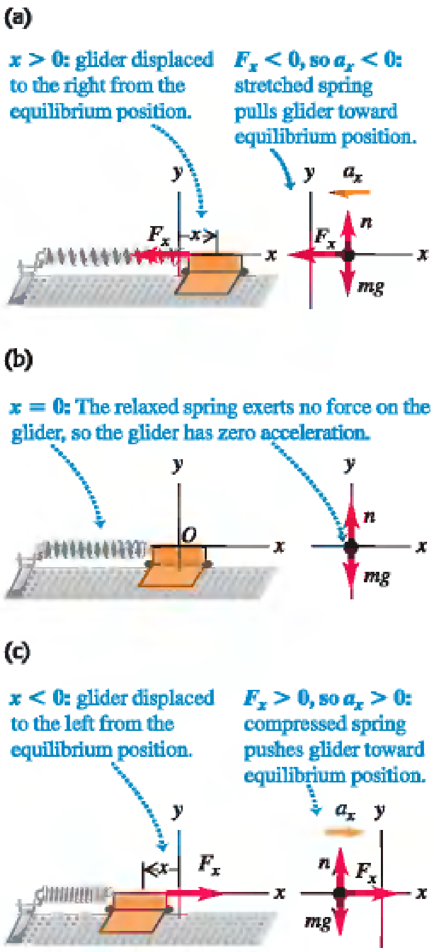
13.1 Describing Oscillation

Figure 13.1 shows one of the simplest systems that can have periodic motion. A body with mass m rests on a frictionless horizontal guide system, such as a linear air track, so it can move only along the x-axis. The body is attached to a spring of negligible mass that can be either stretched or compressed. The left end of the spring is held fixed and the right end is attached to the body. The spring force is the only horizontal force acting on the body; the vertical normal and gravitational forces always add to zero.

13.1 A system that can have periodic motion.



13.2 Model for periodic motion. When the body is displaced from its equilibrium position at $x = 0$, the spring exerts a restoring force back toward the equilibrium position.



It's simplest to define our coordinate system so that the origin O is at the equilibrium position, where the spring is neither stretched nor compressed. Then x is the x-component of the **displacement** of the body from equilibrium and is also the change in the length of the spring. The x-component of the force that the spring exerts on the body is F_x, and the x-component of acceleration a_x is given by $a_x = F_x/m$.

Figure 13.2 shows the body for three different displacements of the spring. Whenever the body is displaced from its equilibrium position, the spring force tends to restore it to the equilibrium position. We call a force with this character a **restoring force.** Oscillation can occur only when there is a restoring force tending to return the system to equilibrium.

Let's analyze how oscillation occurs in this system. If we displace the body to the right to $x = A$ and then let go, the net force and the acceleration are to the left (Fig. 13.2a). The speed increases as the body approaches the equilibrium position O. When the body is at O, the net force acting on it is zero (Fig. 13.2b), but because of its motion it *overshoots* the equilibrium position. On the other side of the equilibrium position the body is still moving to the left, but the net force and the acceleration are to the right (Fig. 13.2c); hence the speed decreases until the body comes to a stop. We will show later that with an ideal spring, the stopping point is at $x = -A$. The body then accelerates to the right, overshoots equilibrium again, and stops at the starting point $x = A$, ready to repeat the whole process. The body is oscillating! If there is no friction or other force to remove mechanical energy from the system, this motion repeats forever; the restoring force perpetually draws the body back toward the equilibrium position, only to have the body overshoot time after time.

In different situations the force may depend on the displacement x from equilibrium in different ways. But oscillation *always* occurs if the force is a *restoring* force that tends to return the system to equilibrium.

Amplitude, Period, Frequency, and Angular Frequency

Here are some terms that we'll use in discussing periodic motions of all kinds:

The **amplitude** of the motion, denoted by A, is the maximum magnitude of displacement from equilibrium—that is, the maximum value of $|x|$. It is always positive. If the spring in Fig. 13.2 is an ideal one, the total overall range of the motion is $2A$. The SI unit of A is the meter. A complete vibration, or **cycle,** is one complete round trip—say, from A to $-A$ and back to A, or from O to A, back through O to $-A$, and back to O. Note that motion from one side to the other (say, $-A$ to A) is a half-cycle, not a whole cycle.

The **period,** T, is the time for one cycle. It is always positive. The SI unit is the second, but it is sometimes expressed as "seconds per cycle."

The **frequency,** f, is the number of cycles in a unit of time. It is always positive. The SI unit of frequency is the hertz:

$$1 \text{ hertz} = 1 \text{ Hz} = 1 \text{ cycle/s} = 1 \text{ s}^{-1}$$

This unit is named in honor of the German physicist Heinrich Hertz (1857–1894), a pioneer in investigating electromagnetic waves.

The **angular frequency,** ω, is 2π times the frequency:

$$\omega = 2\pi f$$

We'll learn shortly why ω is a useful quantity. It represents the rate of change of an angular quantity (not necessarily related to a rotational motion) that is always measured in radians, so its units are rad/s. Since f is in cycle/s, we may regard the number 2π as having units rad/cycle.

From the definitions of period T and frequency f we see that each is the reciprocal of the other:

$$f = \frac{1}{T} \qquad T = \frac{1}{f} \qquad \text{(relationships between frequency and period)} \qquad (13.1)$$

Also, from the definition of ω,

$$\omega = 2\pi f = \frac{2\pi}{T} \qquad \text{(angular frequency)} \qquad (13.2)$$

Example 13.1 Period, frequency, and angular frequency

An ultrasonic transducer (a kind of loudspeaker) used for medical diagnosis oscillates at a frequency of 6.7 MHz = 6.7×10^6 Hz. How much time does each oscillation take, and what is the angular frequency?

SOLUTION

IDENTIFY: Our target variables are the period T and the angular frequency ω.

SET UP: We are given the frequency f, so we can find these variables using Eqs. (13.1) and (13.2).

EXECUTE: From Eqs. (13.1) and (13.2),

$$T = \frac{1}{f} = \frac{1}{6.7 \times 10^6\ \text{Hz}} = 1.5 \times 10^{-7}\ \text{s} = 0.15\ \mu\text{s}$$

$$\begin{aligned}\omega &= 2\pi f = 2\pi(6.7 \times 10^6\ \text{Hz})\\ &= (2\pi\ \text{rad/cycle})(6.7 \times 10^6\ \text{cycle/s})\\ &= 4.2 \times 10^7\ \text{rad/s}\end{aligned}$$

EVALUATE: This is a very rapid vibration, with large f and ω and small T. A slow vibration has small f and ω and large T.

Test Your Understanding of Section 13.1 A body like that shown in Fig. 13.2 oscillates back and forth. For each of the following values of the body's x-velocity v_x and x-acceleration a_x, state whether its displacement x is positive, negative, or zero. (a) $v_x > 0$ and $a_x > 0$; (b) $v_x > 0$ and $a_x < 0$; (c) $v_x < 0$ and $a_x > 0$; (d) $v_x < 0$ and $a_x < 0$; (e) $v_x = 0$ and $a_x < 0$; (f) $v_x > 0$ and $a_x = 0$.

MP

13.2 Simple Harmonic Motion

The simplest kind of oscillation occurs when the restoring force F_x is *directly proportional* to the displacement from equilibrium x. This happens if the spring in Figs. 13.1 and 13.2 is an ideal one that obeys Hooke's law. The constant of proportionality between F_x and x is the force constant k. (You may want to review Hooke's law and the definition of the force constant in Section 6.3.) On either side of the equilibrium position, F_x and x always have opposite signs. In Section 6.3 we represented the force acting *on* a stretched ideal spring as $F_x = kx$. The x-component of force the spring exerts *on the body* is the negative of this, so the x-component of force F_x on the body is

$$F_x = -kx \qquad \text{(restoring force exerted by an ideal spring)} \qquad (13.3)$$

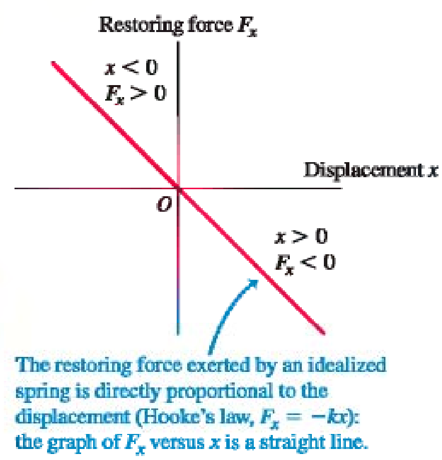
This equation gives the correct magnitude and sign of the force, whether x is positive, negative, or zero (Fig. 13.3). The force constant k is always positive and has units of N/m (a useful alternative set of units is kg/s^2). We are assuming that there is no friction, so Eq. (13.3) gives the *net* force on the body.

When the restoring force is directly proportional to the displacement from equilibrium, as given by Eq. (13.3), the oscillation is called **simple harmonic motion,** *abbreviated* **SHM.** The acceleration $a_x = d^2x/dt^2 = F_x/m$ of a body in SHM is given by

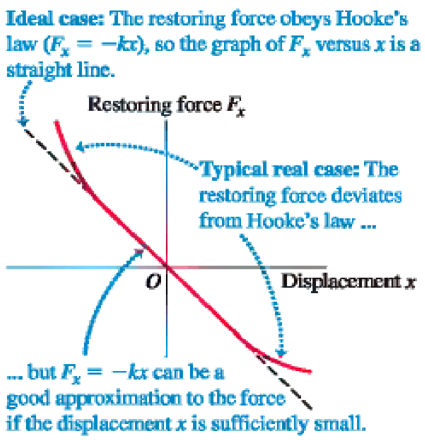
$$a_x = \frac{d^2x}{dt^2} = -\frac{k}{m}x \qquad \text{(simple harmonic motion)} \qquad (13.4)$$

The minus sign means the acceleration and displacement always have opposite signs. This acceleration is *not* constant, so don't even think of using the constant-acceleration equations from Chapter 2. We'll see shortly how to solve this

13.3 An idealized spring exerts a restoring force that obeys Hooke's law, $F_x = -kx$. Oscillation with such a restoring force is called simple harmonic motion.



13.4 In most real oscillations Hooke's law applies provided the body doesn't move too far from equilibrium. In such a case small-amplitude oscillations are approximately simple harmonic.



equation to find the displacement x as a function of time. A body that undergoes simple harmonic motion is called a **harmonic oscillator.**

Why is simple harmonic motion important? Keep in mind that not all periodic motions are simple harmonic; in periodic motion in general, the restoring force depends on displacement in a more complicated way than in Eq. (13.3). But in many systems the restoring force is *approximately* proportional to displacement if the displacement is sufficiently small (Fig. 13.4). That is, if the amplitude is small enough, the oscillations of such systems are approximately simple harmonic and therefore approximately described by Eq. (13.4). Thus we can use SHM as an approximate model for many different periodic motions, such as the vibration of the quartz crystal in a watch, the motion of a tuning fork, the electric current in an alternating-current circuit, and the oscillations of atoms in molecules and solids.

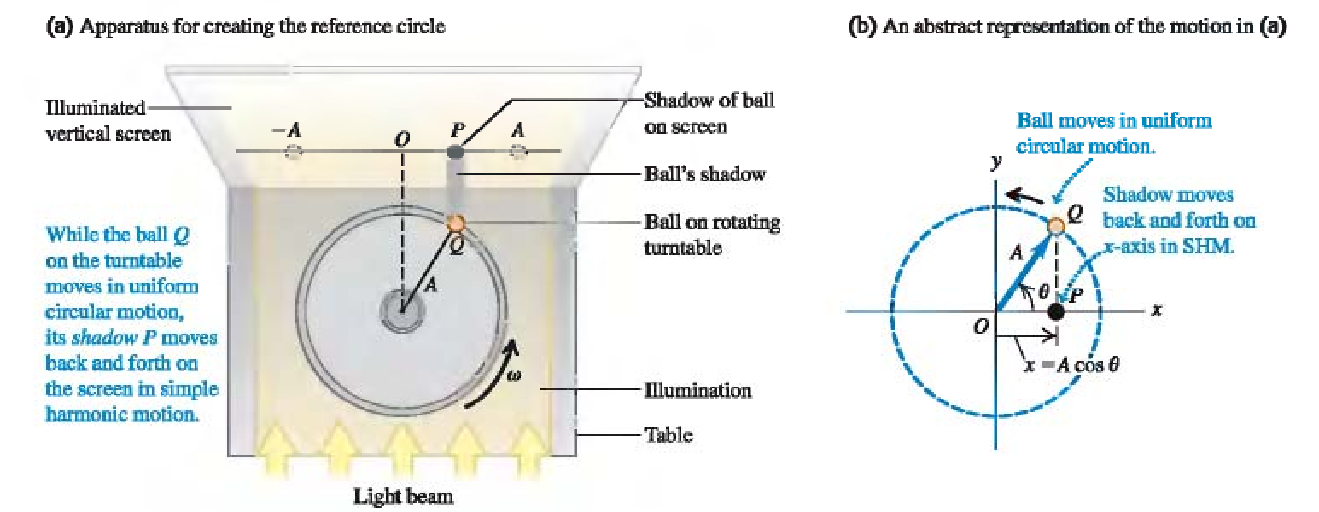
Circular Motion and the Equations of SHM

To explore the properties of simple harmonic motion, we must express the displacement x of the oscillating body as a function of time, $x(t)$. The second derivative of this function, d^2x/dt^2, must be equal to $(-k/m)$ times the function itself, as required by Eq. (13.4). As we mentioned, the formulas for constant acceleration from Section 2.4 are no help because the acceleration changes constantly as the displacement x changes. Instead, we'll find $x(t)$ by noticing a striking similarity between SHM and another form of motion that we've already studied in detail.

Figure 13.5a shows a top view of a horizontal disk of radius A with a ball attached to its rim at point Q. The disk rotates with constant angular speed ω (measured in rad/s), so the ball moves in uniform circular motion. A horizontal light beam shines on the rotating disk and casts a shadow of the ball on a screen. The shadow at point P oscillates back and forth as the ball moves in a circle. We then arrange a body attached to an ideal spring, like the combination shown in Figs. 13.1 and 13.2, so that the body oscillates parallel to the shadow. We will prove that the motion of the body and the motion of the ball's shadow are *identical* if the amplitude of the body's oscillation is equal to the disk radius A, and if the angular frequency $2\pi f$ of the oscillating body is equal to the angular speed ω of the rotating disk. That is, *simple harmonic motion is the projection of uniform circular motion onto a diameter.*

We can verify this remarkable statement by finding the acceleration of the shadow at P and comparing it to the acceleration of a body undergoing SHM, given

13.5 (a) Relating uniform circular motion and simple harmonic motion. (b) The ball's shadow moves exactly like a body oscillating on an ideal spring.



by Eq. (13.4). The circle in which the ball moves so that its projection matches the motion of the oscillating body is called the **reference circle;** we will call the point Q the *reference point.* We take the reference circle to lie in the xy-plane, with the origin O at the center of the circle (Fig. 13.5b). At time t the vector OQ from the origin to the reference point Q makes an angle θ with the positive x-axis. As the point Q moves around the reference circle with constant angular speed ω, the vector OQ rotates with the same angular speed. Such a rotating vector is called a **phasor.** (This term was in use long before the invention of the Star Trek stun gun with a similar name. The phasor method for analyzing oscillations is useful in many areas of physics. We'll use phasors when we study alternating-current circuits in Chapter 31 and the interference of light in Chapters 35 and 36.)

The x-component of the phasor at time t is just the x-coordinate of the point Q:

$$x = A\cos\theta \tag{13.5}$$

This is also the x-coordinate of the shadow P, which is the *projection* of Q onto the x-axis. Hence the x-velocity of the shadow P along the x-axis is equal to the x-component of the velocity vector of the reference point Q (Fig. 13.6a), and the x-acceleration of P is equal to the x-component of the acceleration vector of Q (Fig. 13.6b). Since point Q is in uniform circular motion, its acceleration vector $\vec{a}_Q$ is always directed toward O. Furthermore, the magnitude of $\vec{a}_Q$ is constant and given by the angular speed squared times the radius of the circle (see Section 9.3):

$$a_Q = \omega^2 A \tag{13.6}$$

Figure 13.6b shows that the x-component of $\vec{a}_Q$ is $a_x = -a_Q\cos\theta$. Combining this with Eqs. (13.5) and (13.6), we get that the acceleration of point P is

$$a_x = -a_Q\cos\theta = -\omega^2 A\cos\theta \quad \text{or} \tag{13.7}$$

$$a_x = -\omega^2 x \tag{13.8}$$

The acceleration of the point P is directly proportional to the displacement x and always has the opposite sign. These are precisely the hallmarks of simple harmonic motion.

Equation (13.8) is *exactly* the same as Eq. (13.4) for the acceleration of a harmonic oscillator, provided that the angular speed ω of the reference point Q is related to the force constant k and mass m of the oscillating body by

$$\omega^2 = \frac{k}{m} \quad \text{or} \quad \omega = \sqrt{\frac{k}{m}} \tag{13.9}$$

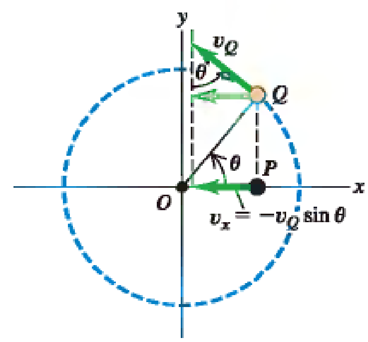
We have been using the same symbol ω for the angular *speed* of the reference point Q and the angular *frequency* of the oscillating point P. The reason is that these quantities are equal! If point Q makes one complete revolution in time T, then point P goes through one complete cycle of oscillation in the same time; hence T is the period of the oscillation. During time T the point Q moves through 2π radians, so its angular speed is $\omega = 2\pi/T$. But this is just the same as Eq. (13.2) for the angular frequency of the point P, which verifies our statement about the two interpretations of ω. This is why we introduced angular frequency in Section 13.1; this quantity makes the connection between oscillation and circular motion. So we reinterpret Eq. (13.9) as an expression for the angular frequency of simple harmonic motion for a body of mass m, acted on by a restoring force with force constant k:

$$\omega = \sqrt{\frac{k}{m}} \quad \text{(simple harmonic motion)} \tag{13.10}$$

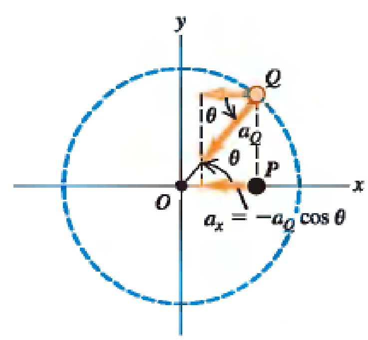
When you start a body oscillating in SHM, the value of ω is not yours to choose; it is predetermined by the values of k and m. The units of k are N/m or kg/s^2, so

13.6 The **(a)** x-velocity and **(b)** x-acceleration of the ball's shadow P (See Fig. 13.5) are the x-components of the velocity and acceleration vectors, respectively, of the ball Q.

(a) Using the reference circle to determine the x-velocity of point P



(b) Using the reference circle to determine the x-acceleration of point P



k/m is in $(\text{kg/s}^2)/\text{kg} = \text{s}^{-2}$. When we take the square root in Eq. (13.10), we get s^{-1}, or more properly rad/s because this is an *angular* frequency (recall that a radian is not a true unit).

According to Eqs. (13.1) and (13.2), the frequency f and period T are

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad \text{(simple harmonic motion)} \qquad (13.11)$$

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \quad \text{(simple harmonic motion)} \qquad (13.12)$$

We see from Eq. (13.12) that a larger mass m, with its greater inertia, will have less acceleration, move more slowly, and take a longer time for a complete cycle (Fig. 13.7). In contrast, a stiffer spring (one with a larger force constant k) exerts a greater force at a given deformation x, causing greater acceleration, higher speeds, and a shorter time T per cycle.

13.7 The greater the mass m in a tuning fork's tines, the lower the frequency of oscillation $f = (1/2\pi)\sqrt{k/m}$ and the lower the pitch of the sound that the tuning fork produces.

Tines with large mass m: low frequency $f = 128$ Hz

Tines with small mass m: high frequency $f = 4096$ Hz

CAUTION **Don't confuse frequency and angular frequency** You can run into trouble if you don't make the distinction between frequency f and angular frequency $\omega = 2\pi f$. Frequency tells you how many cycles of oscillation occur per second, while angular frequency tells you how many radians per second this corresponds to on the reference circle. In solving problems, pay careful attention to whether the goal is to find f or ω.

Period and Amplitude in SHM

Equations (13.11) and (13.12) show that the period and frequency of simple harmonic motion are completely determined by the mass m and the force constant k. *In simple harmonic motion the period and frequency do not depend on the amplitude A.* For given values of m and k, the time of one complete oscillation is the same whether the amplitude is large or small. Equation (13.3) shows why we should expect this. Larger A means that the body reaches larger values of $|x|$ and is subjected to larger restoring forces. This increases the average speed of the body over a complete cycle; this exactly compensates for having to travel a larger distance, so the same total time is involved.

The oscillations of a tuning fork are essentially simple harmonic motion, which means that it always vibrates with the same frequency, independent of amplitude. This is why a tuning fork can be used as a standard for musical pitch. If it were not for this characteristic of simple harmonic motion, it would be impossible to make familiar types of mechanical and electronic clocks run accurately or to play most musical instruments in tune. If you encounter an oscillating body with a period that *does* depend on the amplitude, the oscillation is *not* simple harmonic motion.

Example 13.2 Angular frequency, frequency, and period in SHM

A spring is mounted horizontally, with its left end held stationary. By attaching a spring balance to the free end and pulling toward the right (Fig. 13.8a), we determine that the stretching force is proportional to the displacement and that a force of 6.0 N causes a displacement of 0.030 m. We remove the spring balance and attach a 0.50-kg glider to the end, pull it a distance of 0.020 m along a frictionless air track, release it, and watch it oscillate (Fig. 13.8b). (a) Find the force constant of the spring. (b) Find the angular frequency, frequency, and period of the oscillation.

SOLUTION

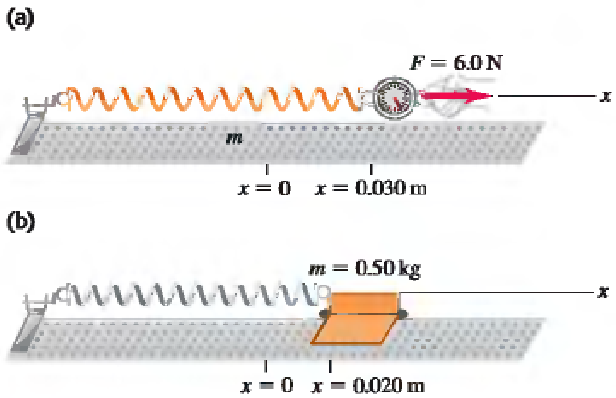
IDENTIFY: Because the spring force (equal in magnitude to the stretching force) is proportional to the displacement, the motion is simple harmonic.

SET UP: We find the value of the force constant k using Hooke's law, Eq. (13.3), and the values of ω, f, and T using Eqs. (13.10), (13.11), and (13.12), respectively.

EXECUTE: (a) When $x = 0.030$ m, the force the spring exerts on the spring balance is $F_x = -6.0$ N. From Eq. (13.3),

$$k = -\frac{F_x}{x} = -\frac{-6.0\text{ N}}{0.030\text{ m}} = 200\text{ N/m} = 200\text{ kg/s}^2$$

13.8 (a) The force exerted *on* the spring (shown by the vector F) has x-component $F_x = +6.0$ N. The force exerted *by* the spring has x-component $F_x = -6.0$ N. (b) A glider is attached to the same spring and allowed to oscillate.



(b) Using $m = 0.50$ kg in Eq. (13.10), we find

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{200\text{ kg/s}^2}{0.50\text{ kg}}} = 20\text{ rad/s}$$

The frequency f is

$$f = \frac{\omega}{2\pi} = \frac{20\text{ rad/s}}{2\pi\text{ rad/cycle}} = 3.2\text{ cycle/s} = 3.2\text{ Hz}$$

The period T is the reciprocal of the frequency f:

$$T = \frac{1}{f} = \frac{1}{3.2\text{ cycle/s}} = 0.31\text{ s}$$

A period is usually stated in "seconds" rather than "seconds per cycle."

EVALUATE: The amplitude of the oscillation is 0.020 m, the distance to the right that we pulled the glider attached to the spring before releasing it. We didn't need to use this information to find the angular frequency, frequency, or period, because in SHM none of these quantities depend on the amplitude.

Displacement, Velocity, and Acceleration in SHM

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9.1 Position Graphs and Equations
9.2 Describing Vibrational Motion
9.5 Ape Drops Tarzan

We still need to find the displacement x as a function of time for a harmonic oscillator. Equation (13.4) for a body in simple harmonic motion along the x-axis is identical to Eq. (13.8) for the x-coordinate of the reference point in uniform circular motion with constant angular speed $\omega = \sqrt{k/m}$. It follows that Eq. (13.5), $x = A\cos\theta$, describes the coordinate x for both of these situations. If at $t = 0$ the phasor OQ makes an angle ϕ (the Greek letter phi) with the positive x-axis, then at any later time t this angle is $\theta = \omega t + \phi$. We substitute this into Eq. (13.5) to obtain

$$x = A\cos(\omega t + \phi) \qquad \text{(displacement in SHM)} \qquad (13.13)$$

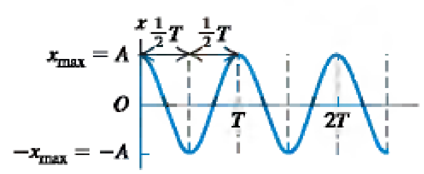
where $\omega = \sqrt{k/m}$. Figure 13.9 shows a graph of Eq. (13.13) for the particular case $\phi = 0$. The displacement x is a periodic function of time, as expected for SHM. We could also have written Eq. (13.13) in terms of a sine function rather than a cosine by using the identity $\cos\alpha = \sin(\alpha + \pi/2)$. *In simple harmonic motion the position is a periodic, sinusoidal function of time.* There are many other periodic functions, but none so smooth and simple as a sine or cosine function.

The value of the cosine function is always between -1 and 1, so in Eq. (13.13), x is always between $-A$ and A. This confirms that A is the amplitude of the motion.

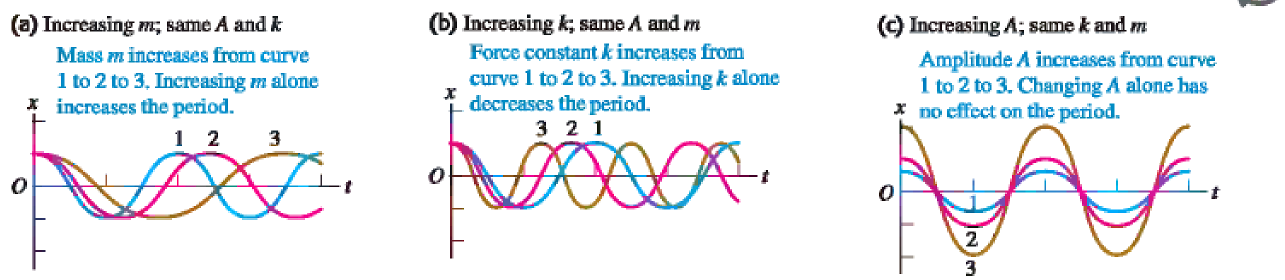
The period T is the time for one complete cycle of oscillation, as Fig. 13.9 shows. The cosine function repeats itself whenever the quantity in parentheses in Eq. (13.13) increases by 2π radians. Thus, if we start at time $t = 0$, the time T to complete one cycle is given by

$$\omega T = \sqrt{\frac{k}{m}}\,T = 2\pi \qquad \text{or} \qquad T = 2\pi\sqrt{\frac{m}{k}}$$

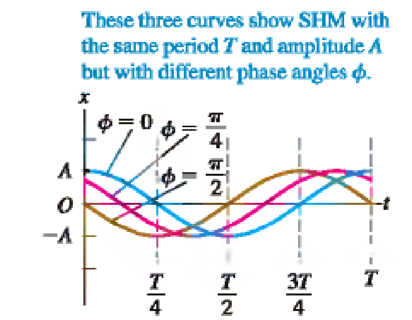
13.9 Graph of x versus t [See Eq. (13.13)] for simple harmonic motion. The case shown has $\phi = 0$.



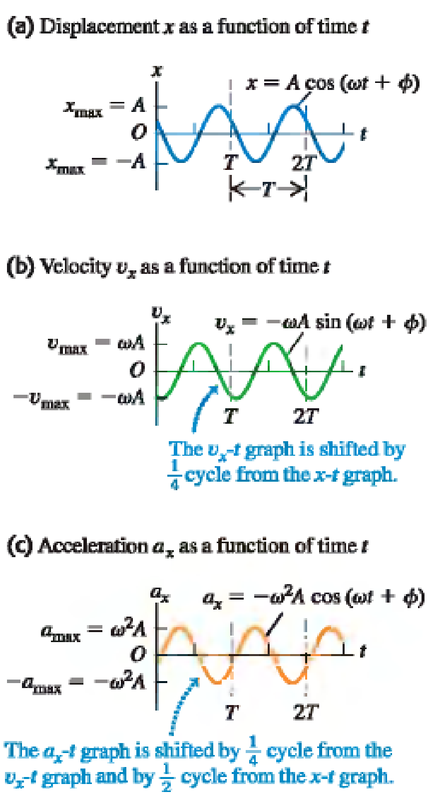
13.10 Variations of simple harmonic motion. All cases shown have $\phi = 0$ [see Eq. (13.13)].



13.11 Variations of SHM: displacement versus time for the same harmonic oscillator with different phase angles ϕ.



13.12 Graphs of (a) x versus t, (b) v_x versus t, and (c) a_x versus t for a body in SHM. For the motion depicted in these graphs, $\phi = \pi/3$.



which is just Eq. (13.12). Changing either m or k changes the period of oscillation, as shown in Figs 13.10a and 13.10b. The period does not depend on the amplitude A (Fig. 13.10c).

The constant ϕ in Eq. (13.13) is called the **phase angle.** It tells us at what point in the cycle the motion was at $t = 0$ (equivalent to where around the circle the point Q was at $t = 0$). We denote the position at $t = 0$ by x_0. Putting $t = 0$ and $x = x_0$ in Eq. (13.13), we get

$$x_0 = A\cos\phi \tag{13.14}$$

If $\phi = 0$, then $x_0 = A\cos 0 = A$, and the body starts at its maximum positive displacement. If $\phi = \pi$, then $x_0 = A\cos\pi = -A$, and the particle starts at its maximum *negative* displacement. If $\phi = \pi/2$, then $x_0 = A\cos(\pi/2) = 0$, and the particle is initially at the origin. Figure 13.11 shows the displacement x versus time for three different phase angles.

We find the velocity v_x and acceleration a_x as functions of time for a harmonic oscillator by taking derivatives of Eq. (13.13) with respect to time:

$$v_x = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi) \qquad \text{(velocity in SHM)} \tag{13.15}$$

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \phi) \qquad \text{(acceleration in SHM)} \tag{13.16}$$

The velocity v_x oscillates between $v_{max} = +\omega A$ and $-v_{max} = -\omega A$, and the acceleration a_x oscillates between $a_{max} = +\omega^2 A$ and $-a_{max} = -\omega^2 A$ (Fig. 13.12). Comparing Eq. (13.16) with Eq. (13.13) and recalling that $\omega^2 = k/m$ from Eq. (13.9), we see that

$$a_x = -\omega^2 x = -\frac{k}{m}x$$

which is just Eq. (13.4) for simple harmonic motion. This confirms that Eq. (13.13) for x as a function of time is correct.

We actually derived Eq. (13.16) earlier in a geometrical way by taking the x-component of the acceleration vector of the reference point Q. This was done in Fig. 13.6b and Eq. (13.7) (recall that $\theta = \omega t + \phi$). In the same way, we could have derived Eq. (13.15) by taking the x-component of the velocity vector of Q, as shown in Fig. 13.6b. We'll leave the details for you to work out (see Problem 13.85).

Note that the sinusoidal graph of displacement versus time (Fig. 13.12a) is shifted by one-quarter period from the graph of velocity versus time (Fig. 13.12b) and by one-half period from the graph of acceleration versus time (Fig. 13.12c). Figure 13.13 shows why this is so. When the body is passing through the equilibrium position so that the displacement is zero, the velocity equals either v_{max} or $-v_{max}$ (depending on which way the body is moving) and the acceleration is zero. When the body is at either its maximum positive displacement, $x = +A$, or its maximum negative displacement, $x = -A$, the velocity is zero and the body is instantaneously at rest. At these points, the restoring force $F_x = -kx$ and the acceleration of the body have their maximum magnitudes. At $x = +A$ the acceleration is negative and equal to $-a_{max}$. At $x = -A$ the acceleration is positive: $a_x = +a_{max}$.

If we are given the initial position x_0 and initial velocity v_{0x} for the oscillating body, we can determine the amplitude A and the phase angle ϕ. Here's how to do it. The initial velocity v_{0x} is the velocity at time $t = 0$; putting $v_x = v_{0x}$ and $t = 0$ in Eq. (13.15), we find

$$v_{0x} = -\omega A \sin\phi \qquad (13.17)$$

To find ϕ, we divide Eq. (13.17) by Eq. (13.14). This eliminates A and gives an equation that we can solve for ϕ:

$$\frac{v_{0x}}{x_0} = \frac{-\omega A \sin\phi}{A\cos\phi} = -\omega\tan\phi$$

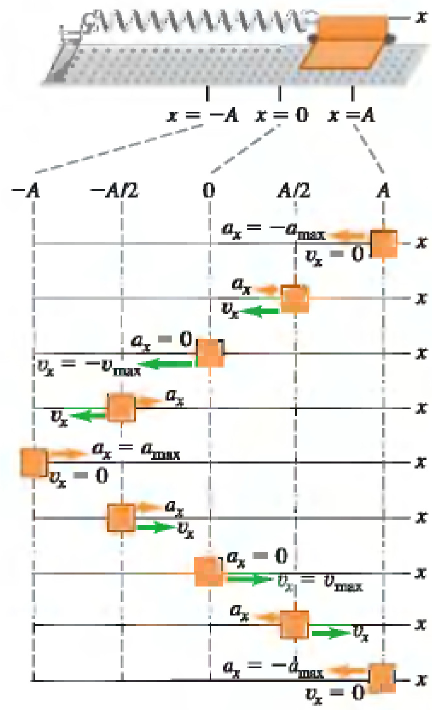
$$\phi = \arctan\left(-\frac{v_{0x}}{\omega x_0}\right) \qquad \text{(phase angle in SHM)} \qquad (13.18)$$

It is also easy to find the amplitude A if we are given x_0 and v_{0x}. We'll sketch the derivation, and you can fill in the details. Square Eq. (13.14); then divide Eq. (13.17) by ω, square it, and add to the square of Eq. (13.14). The right side will be $A^2(\sin^2\phi + \cos^2\phi)$, which is equal to A^2. The final result is

$$A = \sqrt{x_0^2 + \frac{v_{0x}^2}{\omega^2}} \qquad \text{(amplitude in SHM)} \qquad (13.19)$$

Note that when the body has both an initial displacement x_0 and a nonzero initial velocity v_{0x}, the amplitude A is *not* equal to the initial displacement. That's reasonable; if you start the body at a positive x_0 but give it a positive velocity v_{0x}, it will go *farther* than x_0 before it turns and comes back.

13.13 How x-velocity v_x and x-acceleration a_x vary during one cycle of SHM.



Problem-Solving Strategy 13.1 Simple Harmonic Motion I: Describing Motion

MP

IDENTIFY *the relevant concepts:* An oscillating system undergoes simple harmonic motion (SHM) *only* if the restoring force is directly proportional to the displacement. Be certain that this is the case for the problem at hand before attempting to use any of the results of this section. As always, identify the target variables.

SET UP *the problem* using the following steps:
1. Identify the known and unknown quantities, and determine which are the target variables.
2. It's useful to distinguish between two kinds of quantities. *Basic properties* of the system include the mass m and the force constant k as well as quantities derived from m and k, such as the period T, frequency f, and angular frequency ω. *Properties of the motion* describe how the system behaves when it is set into motion in a particular way. They include the amplitude A, maximum velocity v_{max}, and phase angle ϕ as well as the values of x, v_x, and a_x at a particular time.
3. If necessary, define an x-axis as in Fig. 13.13, with the equilibrium position at $x = 0$.

EXECUTE *the solution* as follows:
1. Use the equations given in Sections 13.1 and 13.2 to solve for the target variables.
2. If you need to calculate the phase angle, be certain to express it in radians. The quantity ωt in Eq. (13.13) is naturally in radians, so ϕ must be as well.
3. If you need to find the values of x, v_x, and a_x at various times, use Eqs. (13.13), (13.15), and (13.16), respectively. If the initial position x_0 and initial velocity v_{0x} are both given, you can determine the phase angle and amplitude from Eqs. (13.18) and (13.19). If the body is given an initial positive displacement x_0 but zero initial velocity $(v_{0x} = 0)$, then the amplitude is $A = x_0$ and the phase angle is $\phi = 0$. If it has an initial positive velocity v_{0x} but no initial displacement $(x_0 = 0)$, the amplitude is $A = v_{0x}/\omega$ and the phase angle is $\phi = -\pi/2$.

EVALUATE *your answer:* Check your results to make sure they're consistent. As an example, suppose you've used the initial position and velocity to find general expressions for x and v_x at time t. If you substitute $t = 0$ into these expressions, you should get back the correct values of x_0 and v_{0x}.

Example 13.3 Describing SHM

Let's return to the system of mass and horizontal spring we considered in Example 13.2, with $k = 200\text{ N/m}$ and $m = 0.50\text{ kg}$. This time we give the body an initial displacement of $+0.015\text{ m}$ and an initial velocity of $+0.40\text{ m/s}$. (a) Find the period, amplitude, and phase angle of the motion. (b) Write equations for the displacement, velocity, and acceleration as functions of time.

SOLUTION

IDENTIFY: As in Example 13.2, the oscillations are SHM and we may use the expressions developed in this section.

SET UP: We are given the values of k, m, x_0, and v_{0x}. From them, we calculate the target variables T, A, and ϕ and the expressions for x, v_x, and a_x as functions of time.

EXECUTE: (a) The period is the same as in Example 13.2, $T = 0.31\text{ s}$. In simple harmonic motion the period does not depend on the amplitude, only on the values of k and m. In Example 13.2 we found that $\omega = 20\text{ rad/s}$. So from Eq. (13.19),

$$A = \sqrt{x_0^2 + \frac{{v_{0x}}^2}{\omega^2}} = \sqrt{(0.015\text{ m})^2 + \frac{(0.40\text{ m/s})^2}{(20\text{ rad/s})^2}} = 0.025\text{ m}$$

To find the phase angle ϕ, we use Eq. (13.18):

$$\phi = \arctan\left(-\frac{v_{0x}}{\omega x_0}\right) = \arctan\left(-\frac{0.40\text{ m/s}}{(20\text{ rad/s})(0.015\text{ m})}\right) = -53° = -0.93\text{ rad}$$

(b) The displacement, velocity, and acceleration at any time are given by Eqs. (13.13), (13.15), and (13.16), respectively. Substituting the values, we get

$$x = (0.025\text{ m})\cos[(20\text{ rad/s})t - 0.93\text{ rad}]$$
$$v_x = -(0.50\text{ m/s})\sin[(20\text{ rad/s})t - 0.93\text{ rad}]$$
$$a_x = -(10\text{ m/s}^2)\cos[(20\text{ rad/s})t - 0.93\text{ rad}]$$

The velocity varies sinusoidally between -0.50 m/s and $+0.50\text{ m/s}$, and the acceleration varies sinusoidally between -10 m/s^2 and $+10\text{ m/s}^2$.

EVALUATE: You can check the results for x and v_x as functions of time by substituting $t = 0$ and evaluating the result. You should get $x = x_0 = 0.015\text{ m}$ and $v_x = v_{0x} = 0.40\text{ m/s}$. Do you?

Test Your Understanding of Section 13.2 A glider is attached to a spring as shown in Fig. 13.13. If the glider is moved to $x = 0.10\text{ m}$ and released from rest at time $t = 0$, it will oscillate with amplitude $A = 0.10\text{ m}$ and phase angle $\phi = 0$. (a) Suppose instead that at $t = 0$ the glider is at $x = 0.10\text{ m}$ and is moving to the right in Fig. 13.13. In this situation is the amplitude greater than, less than, or equal to 0.10 m? Is the phase angle greater than, less than, or equal to zero? (b) Suppose instead that at $t = 0$ the glider is at $x = 0.10\text{ m}$ and is moving to the left in Fig. 13.13. In this situation is the amplitude greater than, less than, or equal to 0.10 m? Is the phase angle greater than, less than, or equal to zero?

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9.3 Vibrational Energy
9.4 Two Ways to Weigh Young Tarzan
9.6 Releasing a Vibrating Skier I
9.7 Releasing a Vibrating Skier II
9.8 One- and Two-Spring Vibrating Systems
9.9 Vibro-Ride

13.3 Energy in Simple Harmonic Motion

We can learn even more about simple harmonic motion by using energy considerations. Take another look at the body oscillating on the end of a spring in Figs. 13.2 and 13.13. We've already noted that the spring force is the only horizontal force on the body. The force exerted by an ideal spring is a conservative force, and the vertical forces do no work, so the total mechanical energy of the system is *conserved*. We also assume that the mass of the spring itself is negligible.

The kinetic energy of the body is $K = \frac{1}{2}mv^2$ and the potential energy of the spring is $U = \frac{1}{2}kx^2$, just as in Section 7.2. (You'll find it helpful to review that section.) There are no nonconservative forces that do work, so the total mechanical energy $E = K + U$ is conserved:

$$E = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = \text{constant} \tag{13.20}$$

(Since the motion is one-dimensional, $v^2 = v_x^2$.)

The total mechanical energy E is also directly related to the amplitude A of the motion. When the body reaches the point $x = A$, its maximum displacement from equilibrium, it momentarily stops as it turns back toward the equilibrium position. That is, when $x = A$ (or $-A$), $v_x = 0$. At this point the energy is entirely potential, and $E = \frac{1}{2}kA^2$. Because E is constant, it is equal to $\frac{1}{2}kA^2$ at any other point. Combining this expression with Eq. (13.20), we get

$$E = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \text{constant} \qquad \text{(total mechanical energy in SHM)} \qquad (13.21)$$

We can verify this equation by substituting x and v_x from Eqs. (13.13) and (13.15) and using $\omega^2 = k/m$ from Eq. (13.9):

$$\begin{aligned} E &= \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = \frac{1}{2}m[-\omega A\sin(\omega t + \phi)]^2 + \frac{1}{2}k[A\cos(\omega t + \phi)]^2 \\ &= \frac{1}{2}kA^2\sin^2(\omega t + \phi) + \frac{1}{2}kA^2\cos^2(\omega t + \phi) \\ &= \frac{1}{2}kA^2 \end{aligned}$$

(Recall that $\sin^2\alpha + \cos^2\alpha = 1$.) Hence our expressions for displacement and velocity in SHM are consistent with energy conservation, as they must be.

We can use Eq. (13.21) to solve for the velocity v_x of the body at a given displacement x:

$$v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - x^2} \qquad (13.22)$$

The $\pm$ sign means that at a given value of x the body can be moving in either direction. For example, when $x = \pm A/2$,

$$v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - \left(\pm\frac{A}{2}\right)^2} = \pm\sqrt{\frac{3}{4}}\sqrt{\frac{k}{m}}A$$

Equation (13.22) also shows that the *maximum* speed v_{max} occurs at $x = 0$. Using Eq. (13.10), $\omega = \sqrt{k/m}$, we find that

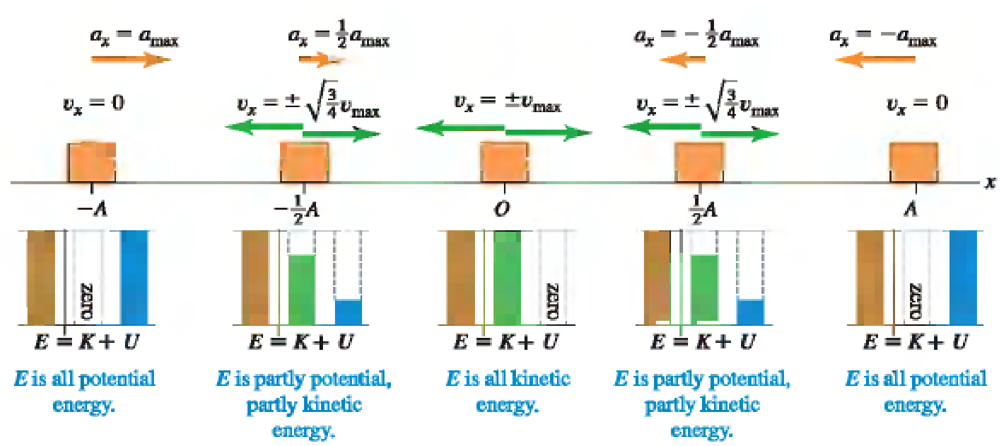
$$v_{max} = \sqrt{\frac{k}{m}}A = \omega A \qquad (13.23)$$

This agrees with Eq. (13.15), which showed that v_x oscillates between $-\omega A$ and $+\omega A$.

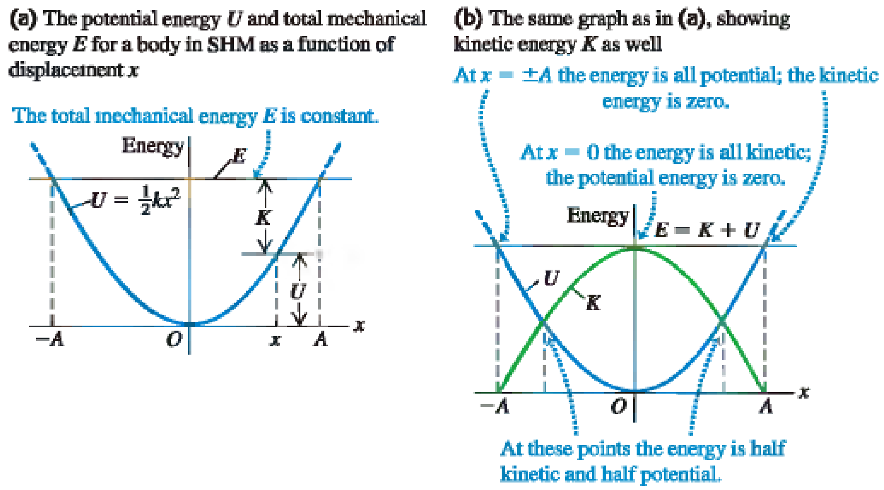
Interpreting E, K, and U in SHM

Figure 13.14 shows the energy quantities E, K, and U at $x = 0$, $x = \pm A/2$, and $x = \pm A$. Figure 13.15 is a graphical display of Eq. (13.21); energy (kinetic, potential, and total) is plotted vertically and the coordinate x is plotted horizontally. The

13.14 Graphs of E, K, and U versus displacement in SHM. The velocity of the body is *not* constant, so these images of the body at equally spaced positions are *not* equally spaced in time.



13.15 Kinetic energy K, potential energy U, and total mechanical energy E as functions of position for SHM. At each value of x the sum of the values of K and U equals the constant value of E. Can you show that the energy is half kinetic and half potential at $x = \pm\sqrt{\frac{1}{2}}A$?



parabolic curve in Fig. 13.15a represents the potential energy $U = \frac{1}{2}kx^2$. The horizontal line represents the total mechanical energy E, which is constant and does not vary with x. At any value of x between $-A$ and A, the vertical distance from the x-axis to the parabola is U; since $E = K + U$, the remaining vertical distance up to the horizontal line is K. Figure 13.15b shows both K and U as functions of x. The horizontal line for E intersects the potential-energy curve at $x = -A$ and $x = A$, so at these points the energy is entirely potential, the kinetic energy is zero, and the body comes momentarily to rest before reversing direction. As the body oscillates between $-A$ and A, the energy is continuously transformed from potential to kinetic and back again.

Figure 13.15a shows the connection between the amplitude A and the corresponding total mechanical energy $E = \frac{1}{2}kA^2$. If we tried to make x greater than A (or less than $-A$), U would be greater than E, and K would have to be negative. But K can never be negative, so x can't be greater than A or less than $-A$.

Problem-Solving Strategy 13.2 Simple Harmonic Motion II: Energy

MP

The energy equation, Eq. (13.21), is a useful alternative relationship between velocity and position, especially when energy quantities are also required. If the problem involves a relationship among position, velocity, and acceleration without reference to time, it is usually easier to use Eq. (13.4) (from Newton's second law) or Eq. (13.21) (from energy conservation) than to use the general expressions for x, v_x, and a_x as functions of time [Eqs. (13.13), (13.15), and (13.16), respectively]. Because the energy equation involves x^2 and v_x^2, it cannot tell you the sign of x or of v_x; you have to infer the sign from the situation. For instance, if the body is moving from the equilibrium position toward the point of greatest positive displacement, then x is positive and v_x is positive.

Example 13.4 Velocity, acceleration, and energy in SHM

In the oscillation described in Example 13.2, $k = 200\ \text{N/m}$, $m = 0.50\ \text{kg}$, and the oscillating mass is released from rest at $x = 0.020\ \text{m}$. (a) Find the maximum and minimum velocities attained by the oscillating body. (b) Compute the maximum acceleration. (c) Determine the velocity and acceleration when the body has moved halfway to the center from its original position. (d) Find the total energy, potential energy, and kinetic energy at this position.

SOLUTION

IDENTIFY: The problem refers to the motion at various *positions* in the motion, not at specified *times*. This is a hint that we can use the energy relationships found in this section to solve for the target variables.

SET UP: Figure 13.13 shows the choice of x-axis. The maximum displacement from equilibrium is $A = 0.020\ \text{m}$. For any position x we use Eqs. (13.22) and (13.4) to find the velocity v_x and acceleration a_x, respectively. Given the velocity and position, we use Eq. (13.21) to find the energy quantities K, U, and E.

EXECUTE: (a) The velocity v_x at any displacement x is given by Eq. (13.22):

$$v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - x^2}$$

The maximum velocity occurs when the body is moving to the right through the equilibrium position, where $x = 0$:

$$v_x = v_{\max} = \sqrt{\frac{k}{m}}A = \sqrt{\frac{200\ \text{N/m}}{0.50\ \text{kg}}}(0.020\ \text{m}) = 0.40\ \text{m/s}$$

The minimum (i.e., most negative) velocity occurs when the body is moving to the left through $x = 0$; its value is $-v_{\max} = -0.40$ m/s.

(b) From Eq. (13.4),

$$a_x = -\frac{k}{m}x$$

The maximum (most positive) acceleration occurs at the most negative value of x, $x = -A$; therefore

$$a_{\max} = -\frac{k}{m}(-A) = -\frac{200\ \text{N/m}}{0.50\ \text{kg}}(-0.020\ \text{m}) = 8.0\ \text{m/s}^2$$

The minimum (most negative) acceleration is $-8.0\ \text{m/s}^2$, occurring at $x = +A = +0.020$ m.

(c) At a point halfway to the center from the initial position, $x = A/2 = 0.010$ m. From Eq. (13.22),

$$v_x = -\sqrt{\frac{200\ \text{N/m}}{0.50\ \text{kg}}}\sqrt{(0.020\ \text{m})^2 - (0.010\ \text{m})^2} = -0.35\ \text{m/s}$$

We choose the negative square root because the body is moving from $x = A$ toward $x = 0$. From Eq. (13.4),

$$a_x = -\frac{200\ \text{N/m}}{0.50\ \text{kg}}(0.010\ \text{m}) = -4.0\ \text{m/s}^2$$

At this point the velocity and the acceleration have the same sign, so the speed is increasing. The conditions at $x = 0$, $\pm A/2$, and $\pm A$ are shown in Fig. 13.14.

(d) The total energy has the same value at all points during the motion:

$$E = \frac{1}{2}kA^2 = \frac{1}{2}(200\ \text{N/m})(0.020\ \text{m})^2 = 0.040\ \text{J}$$

The potential energy is

$$U = \frac{1}{2}kx^2 = \frac{1}{2}(200\ \text{N/m})(0.010\ \text{m})^2 = 0.010\ \text{J}$$

and the kinetic energy is

$$K = \frac{1}{2}mv_x^2 = \frac{1}{2}(0.50\ \text{kg})(-0.35\ \text{m/s})^2 = 0.030\ \text{J}$$

EVALUATE: At the point $x = A/2$, the energy is one-fourth potential energy and three-fourths kinetic energy. You can check this result by inspecting Fig. 13.15b.

Example 13.5 Energy and momentum in SHM

A block with mass M attached to a horizontal spring with force constant k is moving with simple harmonic motion having amplitude A_1. At the instant when the block passes through its equilibrium position, a lump of putty with mass m is dropped vertically onto the block from a very small height and sticks to it. (a) Find the new amplitude and period. (b) Repeat part (a) for the case in which the putty is dropped on the block when it is at one end of its path.

SOLUTION

IDENTIFY: The problem involves the motion at a given position, not a given time, so we can use energy methods. Before the putty lands on the block, the mechanical energy of the oscillating block and spring is constant. When the putty lands on the block, it's a completely inelastic collision (see Section 8.3); the horizontal component of momentum is conserved, but kinetic energy decreases. Once the collision ends, the mechanical energy remains constant at its new value.

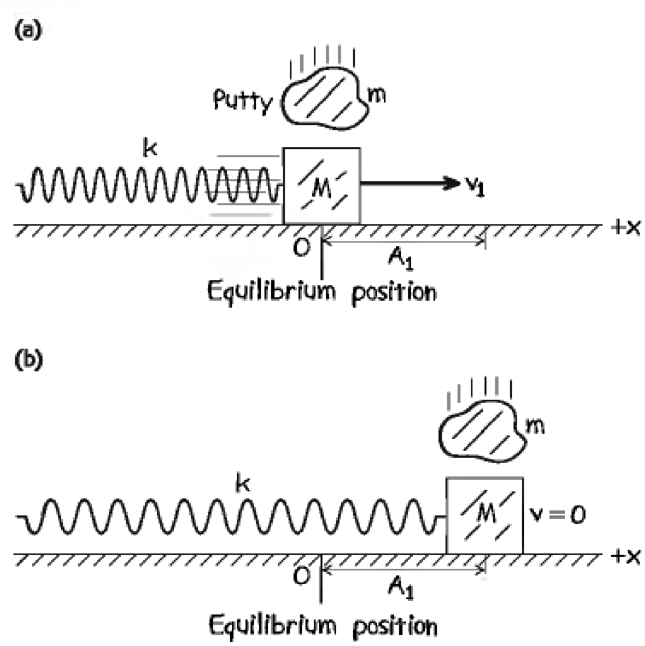
SET UP: Figure 13.16 shows our sketches. In each part we consider what happens before, during, and after the collision. We find the amplitude A_2 after the collision from the final energy of the system, and we find the period T_2 after the collision using the relationship between period and mass.

EXECUTE: (a) Before the collision the total mechanical energy of the block and spring is $E_1 = \frac{1}{2}kA_1^2$. Since the block is at the equilibrium position, $U = 0$, and the energy is purely kinetic (Fig. 13.16a). If we let v_1 be the speed of the block at the equilibrium position, we have

$$E_1 = \frac{1}{2}Mv_1^2 = \frac{1}{2}kA_1^2 \qquad \text{so} \qquad v_1 = \sqrt{\frac{k}{M}}A_1$$

During the collision the x-component of momentum of the system of block and putty is conserved. (Why?) Just before the collision

13.16 Our sketches for this problem.



this component is the sum of Mv_1 (for the block) and zero (for the putty). Just after the collision the block and putty move together with speed v_2, and their combined x-component of momentum is $(M + m)v_2$. From conservation of momentum,

$$Mv_1 + 0 = (M + m)v_2 \qquad \text{so} \qquad v_2 = \frac{M}{M + m}v_1$$

Continued

The collision lasts a very short time, so just after the collision the block and putty are still at the equilibrium position. The energy is still purely kinetic but is *less* than before the collision:

$$E_2 = \frac{1}{2}(M + m)v_2^2 = \frac{1}{2}\frac{M^2}{M + m}v_1^2 = \frac{M}{M + m}\left(\frac{1}{2}Mv_1^2\right)$$
$$= \left(\frac{M}{M + m}\right)E_1$$

Since E_2 equals $\frac{1}{2}kA_2^2$, where A_2 is the amplitude after the collision, we have

$$\frac{1}{2}kA_2^2 = \left(\frac{M}{M + m}\right)\frac{1}{2}kA_1^2$$
$$A_2 = A_1\sqrt{\frac{M}{M + m}}$$

The larger the putty mass m, the smaller the final amplitude.

Finding the period of oscillation after the collision is the easy part. Using Eq. (13.12), we have

$$T_2 = 2\pi\sqrt{\frac{M + m}{k}}$$

(b) When the putty drops, the block is instantaneously at rest (Fig. 13.16b). The x-component of momentum is zero both before and after the collision: The block has zero kinetic energy just before the collision, and the block and putty have zero kinetic energy just after the collision. The energy is all potential energy stored in the spring, so adding the extra mass of the putty has *no effect* on the mechanical energy. That is,

$$E_2 = E_1 = \frac{1}{2}kA_1^2$$

and the amplitude after the collision is unchanged $(A_2 = A_1)$. The period still changes when the putty is added, though; its value doesn't depend on how the mass is added, only on what the total mass is. So T_2 is the same as we found in part (a), $T_2 = 2\pi\sqrt{(M + m)/k}$.

EVALUATE: Why is energy lost in part (a) but not in part (b)? The difference is that in part (a) the putty slides against the moving block during the collision, and energy is dissipated by kinetic friction.

Test Your Understanding of Section 13.3 (a) To double the total energy for a mass-spring system oscillating in SHM, by what factor must the amplitude increase? (i) 4; (ii) 2; (iii) $\sqrt{2} = 1.414$; (iv) $\sqrt[4]{2} = 1.189$. (b) By what factor will the frequency change due to this amplitude increase? (i) 4; (ii) 2; (iii) $\sqrt{2} = 1.414$; (iv) $\sqrt[4]{2} = 1.189$; (v) it does not change.

13.4 Applications of Simple Harmonic Motion

So far, we've looked at a grand total of *one* situation in which simple harmonic motion (SHM) occurs: a body attached to an ideal horizontal spring. But SHM can occur in any system in which there is a restoring force that is directly proportional to the displacement from equilibrium, as given by Eq. (13.3), $F_x = -kx$. The restoring force will originate in different ways in different situations, so the force constant k has to be found for each case by examining the net force on the system. Once this is done, it's straightforward to find the angular frequency ω, frequency f, and period T; we just substitute the value of k into Eqs. (13.10), (13.11), and (13.12), respectively. Let's use these ideas to examine several examples of simple harmonic motion.

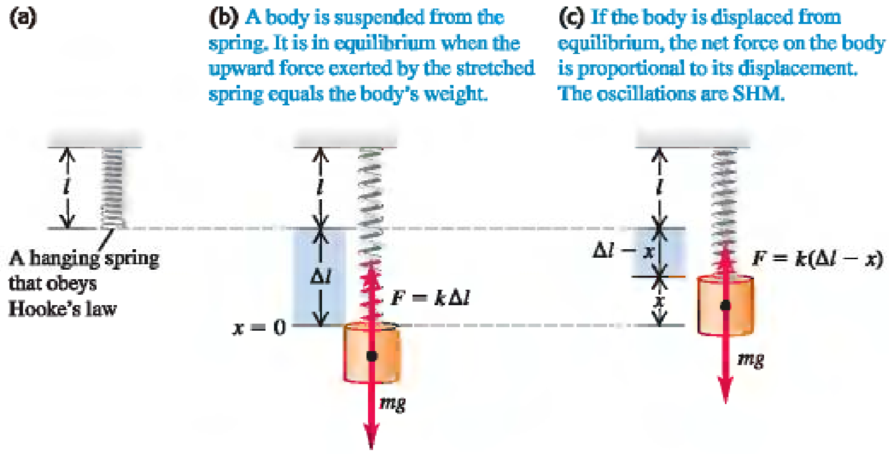
Vertical SHM

Suppose we hang a spring with force constant k (Fig. 13.17a) and suspend from it a body with mass m. Oscillations will now be vertical; will they still be SHM? In Fig. 13.17b the body hangs at rest, in equilibrium. In this position the spring is stretched an amount Δl just great enough that the spring's upward vertical force $k\,\Delta l$ on the body balances its weight mg:

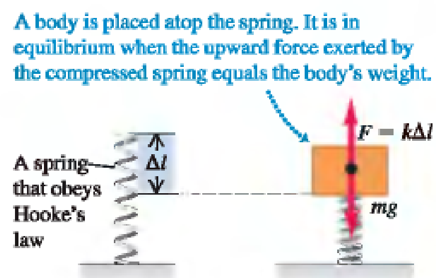
$$k\,\Delta l = mg$$

Take $x = 0$ to be this equilibrium position and take the positive x-direction to be upward. When the body is a distance x *above* its equilibrium position (Fig. 13.17c), the extension of the spring is $\Delta l - x$. The upward force it exerts on the body is then $k(\Delta l - x)$, and the net x-component of force on the body is

$$F_{net} = k(\Delta l - x) + (-mg) = -kx$$



13.17 A body attached to a hanging spring.



13.18 If the weight mg compresses the spring a distance Δl, the force constant is $k = mg/\Delta l$ and the angular frequency for vertical SHM is $\omega = \sqrt{k/m}$—the same as if the body were suspended from the spring (See Fig. 13.17).

that is, a net downward force of magnitude kx. Similarly, when the body is *below* the equilibrium position, there is a net upward force with magnitude kx. In either case there is a restoring force with magnitude kx. If the body is set in vertical motion, it oscillates in SHM with the same angular frequency as though it were horizontal, $\omega = \sqrt{k/m}$. So vertical SHM doesn't differ in any essential way from horizontal SHM. The only real change is that the equilibrium position $x = 0$ no longer corresponds to the point at which the spring is unstretched. The same ideas hold if a body with weight mg is placed atop a compressible spring (Fig. 13.18) and compresses it a distance Δl.

Example 13.6 Vertical SHM in an old car

The shock absorbers in an old car with mass 1000 kg are completely worn out. When a 980-N person climbs slowly into the car to its center of gravity, the car sinks 2.8 cm. When the car, with the person aboard, hits a bump, the car starts oscillating up and down in SHM. Model the car and person as a single body on a single spring, and find the period and frequency of the oscillation.

SOLUTION

IDENTIFY: The situation is like that shown in Fig. 13.18.

SET UP: The compression of the spring when the extra weight is added tells us the force constant, which we can use to find the period and frequency (the target variables).

EXECUTE: When the force increases by 980 N, the spring compresses an additional 0.028 m, and the coordinate x of the car changes by −0.028 m. Hence the effective force constant (including the effect of the entire suspension) is

$$k = -\frac{F_x}{x} = -\frac{980\text{ N}}{-0.028\text{ m}} = 3.5 \times 10^4\text{ kg/s}^2$$

The person's mass is $w/g = (980\text{ N})/(9.8\text{ m/s}^2) = 100\text{ kg}$. The *total* oscillating mass is $m = 1000\text{ kg} + 100\text{ kg} = 1100\text{ kg}$. The period T is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{1100\text{ kg}}{3.5 \times 10^4\text{ kg/s}^2}} = 1.11\text{ s}$$

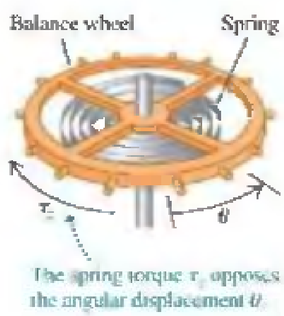
and the frequency is

$$f = \frac{1}{T} = \frac{1}{1.11\text{ s}} = 0.90\text{ Hz}$$

EVALUATE: A persistent oscillation with a period of about 1 second makes for a very unpleasant ride. The purpose of shock absorbers is to make such oscillations die out (see Section 13.7).

Angular SHM

A mechanical watch keeps time based on the oscillations of a balance wheel (Fig. 13.19). The wheel has a moment of inertia I about its axis. A coil spring exerts a restoring torque τ_z that is proportional to the angular displacement θ from the equilibrium position. We write $\tau_z = -\kappa\theta$, where κ (the Greek letter kappa) is a constant called the *torsion constant*. Using the rotational analog of



13.19 The balance wheel of a mechanical watch. The spring exerts a restoring torque that is proportional to the angular displacement θ, so the motion is angular SHM.

Newton's second law for a rigid body, $\Sigma\tau_z = I\alpha_z = I\,d^2\theta/dt^2$, we can find the equation of motion:

$$-\kappa\theta = I\alpha \qquad \text{or} \qquad \frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta$$

The form of this equation is exactly the same as Eq. (13.4) for the acceleration in simple harmonic motion, with x replaced by θ and k/m replaced by κ/I. So we are dealing with a form of *angular* simple harmonic motion. The angular frequency ω and frequency f are given by Eqs. (13.10) and (13.11), respectively, with the same replacement:

$$\omega = \sqrt{\frac{\kappa}{I}} \qquad \text{and} \qquad f = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I}} \qquad \text{(angular SHM)} \tag{13.24}$$

The motion is described by the function

$$\theta = \Theta\cos(\omega t + \phi)$$

where Θ (the Greek letter theta) plays the role of an angular amplitude.

It's a good thing that the motion of a balance wheel *is* simple harmonic. If it weren't, the frequency might depend on the amplitude, and the watch would run too fast or too slow as the spring ran down.

*Vibrations of Molecules

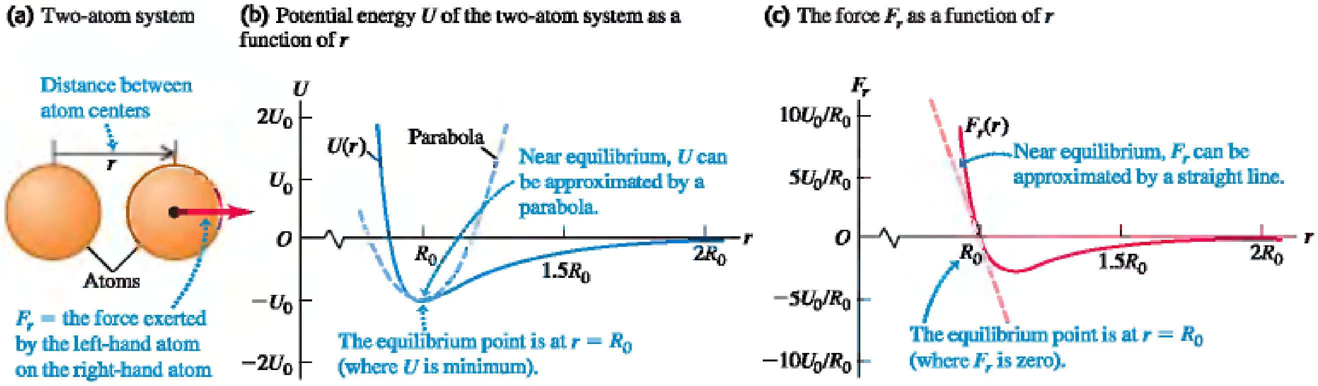
The following discussion of the vibrations of molecules uses the binomial theorem. If you aren't familiar with this theorem, you should read about it in the appropriate section of a math textbook.

When two atoms are separated from each other by a few atomic diameters, they can exert attractive forces on each other. But if the atoms are so close to each other that their electron shells overlap, the forces between the atoms are repulsive. Between these limits, there can be an equilibrium separation distance at which two atoms form a *molecule.* If these atoms are displaced slightly from equilibrium, they will oscillate.

As an example, we'll consider one type of interaction between atoms called the *van der Waals interaction.* Our immediate task here is to study oscillations, so we won't go into the details of how this interaction arises. Let the center of one atom be at the origin and let the center of the other atom be a distance r away (Fig. 13.20a); the equilibrium distance between centers is $r = R_0$. Experiment shows that the van der Waals interaction can be described by the potential-energy function

$$U = U_0\left[\left(\frac{R_0}{r}\right)^{12} - 2\left(\frac{R_0}{r}\right)^{6}\right] \tag{13.25}$$

13.20 (a) Two atoms with centers separated by r. (b) Potential energy U in the van der Waals interaction as a function of r. (c) Force F_r on the right-hand atom as a function of r.



where U_0 is a positive constant with units of joules. When the two atoms are very far apart, $U = 0$; when they are separated by the equilibrium distance $r = R_0$, $U = -U_0$. The force on the second atom is the negative derivative of Eq. (13.25):

$$F_r = -\frac{dU}{dr} = U_0\left[\frac{12R_0^{12}}{r^{13}} - 2\frac{6R_0^6}{r^7}\right] = 12\frac{U_0}{R_0}\left[\left(\frac{R_0}{r}\right)^{13} - \left(\frac{R_0}{r}\right)^7\right] \tag{13.26}$$

The potential energy and force are plotted in Figs. 13.20b and 13.20c, respectively. The force is positive for $r < R_0$ and negative for $r > R_0$, so it is a *restoring* force.

Let's examine the restoring force F_r in Eq. (13.26). We let x represent the displacement from equilibrium:

$$x = r - R_0 \qquad \text{so} \qquad r = R_0 + x$$

In terms of x, the force F_r in Eq. (13.26) becomes

$$\begin{aligned} F_r &= 12\frac{U_0}{R_0}\left[\left(\frac{R_0}{R_0+x}\right)^{13} - \left(\frac{R_0}{R_0+x}\right)^7\right] \\ &= 12\frac{U_0}{R_0}\left[\frac{1}{(1+x/R_0)^{13}} - \frac{1}{(1+x/R_0)^7}\right] \end{aligned} \tag{13.27}$$

This looks nothing like Hooke's law, $F_x = -kx$, so we might be tempted to conclude that molecular oscillations cannot be SHM. But let us restrict ourselves to *small-amplitude* oscillations so that the absolute value of the displacement x is small in comparison to R_0 and the absolute value of the ratio x/R_0 is much less than 1. We can then simplify Eq. (13.27) by using the *binomial theorem:*

$$(1+u)^n = 1 + nu + \frac{n(n-1)}{2!}u^2 + \frac{n(n-1)(n-2)}{3!}u^3 + \cdots \tag{13.28}$$

If $|u|$ is much less than 1, each successive term in Eq. (13.28) is much smaller than the one it follows, and we can safely approximate $(1+u)^n$ by just the first two terms. In Eq. (13.27), u is replaced by x/R_0 and n equals -13 or -7, so

$$\frac{1}{(1+x/R_0)^{13}} = (1+x/R_0)^{-13} \approx 1 + (-13)\frac{x}{R_0}$$

$$\frac{1}{(1+x/R_0)^{7}} = (1+x/R_0)^{-7} \approx 1 + (-7)\frac{x}{R_0}$$

$$F_r \approx 12\frac{U_0}{R_0}\left[\left(1 + (-13)\frac{x}{R_0}\right) - \left(1 + (-7)\frac{x}{R_0}\right)\right] = -\left(\frac{72U_0}{R_0^2}\right)x \tag{13.29}$$

This is just Hooke's law, with force constant $k = 72U_0/R_0^2$. (Note that k has the correct units, J/m^2 or N/m.) So oscillations of molecules bound by the van der Waals interaction can be simple harmonic motion, provided that the amplitude is small in comparison to R_0 so that the approximation $|x/R_0| \ll 1$ used in the derivation of Eq. (13.29) is valid.

You can also show that the potential energy U in Eq. (13.25) can be written as $U \approx \frac{1}{2}kx^2 + C$, where $C = -U_0$ and k is again equal to $72U_0/R_0^2$. Adding a constant to the potential energy has no effect on the physics, so the system of two atoms is fundamentally no different from a mass attached to a horizontal spring for which $U = \frac{1}{2}kx^2$. The proof is left to you (see Exercise 13.39).

Example 13.7 Molecular vibration

Two argon atoms can form a weakly bound molecule, Ar_2, held together by a van der Waals interaction with $U_0 = 1.68 \times 10^{-21}$ J and $R_0 = 3.82 \times 10^{-10}$ m. Find the frequency for small oscillations of one of the atoms about its equilibrium position.

SOLUTION

IDENTIFY: This is just the situation shown in Fig. 13.20.

Continued

SET UP: Because the oscillations are small, we can use Eq. (13.11) to obtain the frequency of simple harmonic motion. The force constant is given by Eq. (13.29).

EXECUTE: The force constant is

$$k = \frac{72U_0}{R_0^2} = \frac{72(1.68 \times 10^{-21}\ \text{J})}{(3.82 \times 10^{-10}\ \text{m})^2} = 0.829\ \text{J/m}^2 = 0.829\ \text{N/m}$$

This is comparable to the force constant of a loose, floppy toy spring like a Slinky™.

From the periodic table of the elements (see Appendix D), the average atomic mass of argon is

$$(39.948\ \text{u})(1.66 \times 10^{-27}\ \text{kg/1 u}) = 6.63 \times 10^{-26}\ \text{kg}.$$

If one of the argon atoms is fixed and the other atom oscillates, the frequency of oscillation is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{0.829\ \text{N/m}}{6.63 \times 10^{-26}\ \text{kg}}} = 5.63 \times 10^{11}\ \text{Hz}$$

The oscillating mass is very small, so even a floppy spring causes very rapid oscillations.

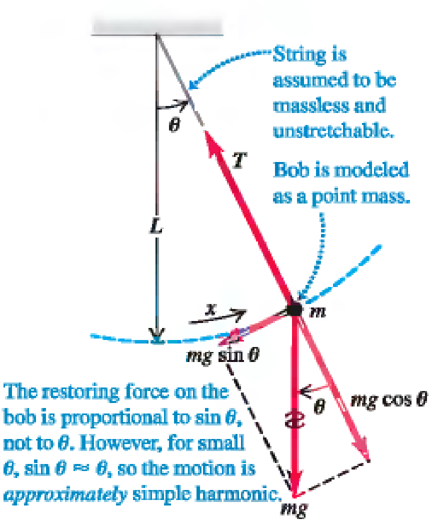
EVALUATE: Our answer for f isn't quite right. If there is no net external force acting on the molecule, the center of mass of the molecule (located halfway between the two atoms) doesn't accelerate. To ensure this, both atoms must oscillate with the same amplitude in opposite directions. It turns out that we can account for this by replacing m with $m/2$ in the expression for f. (See Problem 13.86.) This makes f larger by a factor of $\sqrt{2}$, so $f = \sqrt{2}(5.63 \times 10^{11}\ \text{Hz}) = 7.96 \times 10^{11}\ \text{Hz}$. An additional complication is that on the atomic scale we must use quantum mechanics, not Newtonian mechanics, to describe motion; happily, the frequency has the same value in quantum mechanics.

Test Your Understanding of Section 13.4 A block attached to a hanging ideal spring oscillates up and down with a period of 10 s on earth. If you take the block and spring to Mars, where the acceleration due to gravity is only about 40% as large as on earth, what will be the new period of oscillation? (i) 10 s; (ii) more than 10 s; (iii) less than 10 s.

13.21 The dynamics of a simple pendulum.

(a) A real pendulum

(b) An idealized simple pendulum



13.5 The Simple Pendulum

A **simple pendulum** is an idealized model consisting of a point mass suspended by a massless, unstretchable string. When the point mass is pulled to one side of its straight-down equilibrium position and released, it oscillates about the equilibrium position. Familiar situations such as a wrecking ball on a crane's cable or a person on a swing (Fig. 13.21a) can be modeled as simple pendulums.

The path of the point mass (sometimes called a pendulum bob) is not a straight line but the arc of a circle with radius L equal to the length of the string (Fig. 13.21b). We use as our coordinate the distance x measured along the arc. If the motion is simple harmonic, the restoring force must be directly proportional to x or (because $x = L\theta$) to θ. Is it?

In Fig. 13.21b we represent the forces on the mass in terms of tangential and radial components. The restoring force F_θ is the tangential component of the net force:

$$F_\theta = -mg\sin\theta \tag{13.30}$$

The restoring force is provided by gravity; the tension T merely acts to make the point mass move in an arc. The restoring force is proportional *not* to θ but to $\sin\theta$, so the motion is *not* simple harmonic. However, if the angle θ is *small*, $\sin\theta$ is very nearly equal to θ in radians (Fig. 13.22). For example, when $\theta = 0.1$ rad (about 6°), $\sin\theta = 0.0998$, a difference of only 0.2%. With this approximation, Eq. (13.30) becomes

$$F_\theta = -mg\theta = -mg\frac{x}{L} \quad \text{or}$$

$$F_\theta = -\frac{mg}{L}x \tag{13.31}$$

The restoring force is then proportional to the coordinate for small displacements, and the force constant is $k = mg/L$. From Eq. (13.10) the angular frequency ω of a simple pendulum with small amplitude is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg/L}{m}} = \sqrt{\frac{g}{L}} \quad \text{(simple pendulum, small amplitude)} \tag{13.32}$$

The corresponding frequency and period relationships are

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \quad \text{(simple pendulum, small amplitude)} \tag{13.33}$$

$$T = \frac{2\pi}{\omega} = \frac{1}{f} = 2\pi\sqrt{\frac{L}{g}} \quad \text{(simple pendulum, small amplitude)} \tag{13.34}$$

? Note that these expressions do not involve the *mass* of the particle. This is because the restoring force, a component of the particle's weight, is proportional to m. Thus the mass appears on *both* sides of $\Sigma\vec{F} = m\vec{a}$ and cancels out. (This is the same physics that explains why bodies of different masses fall with the same acceleration in a vacuum.) For small oscillations, the period of a pendulum for a given value of g is determined entirely by its length.

The dependence on L and g in Eqs. (13.32) through (13.34) is just what we should expect. A long pendulum has a longer period than a shorter one. Increasing g increases the restoring force, causing the frequency to increase and the period to decrease.

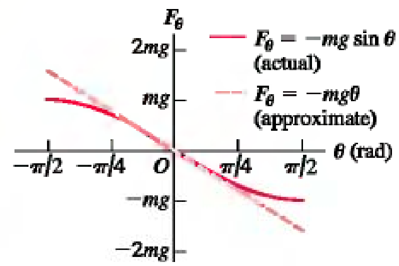
We emphasize again that the motion of a pendulum is only *approximately* simple harmonic. When the amplitude is not small, the departures from simple harmonic motion can be substantial. But how small is "small"? The period can be expressed by an infinite series; when the maximum angular displacement is Θ, the period T is given by

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1^2}{2^2}\sin^2\frac{\Theta}{2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}\sin^4\frac{\Theta}{2} + \cdots\right) \tag{13.35}$$

We can compute the period to any desired degree of precision by taking enough terms in the series. We invite you to check that when $\Theta = 15°$ (on either side of the central position), the true period is longer than that given by the approximate Eq. (13.34) by less than 0.5%.

The usefulness of the pendulum as a timekeeper depends on the period being *very nearly* independent of amplitude, provided that the amplitude is small. Thus, as a pendulum clock runs down and the amplitude of the swings decreases a little, the clock still keeps very nearly correct time.

13.22 For small angular displacements θ, the restoring force $F_\theta = -mg\sin\theta$ on a simple pendulum is approximately equal to $-mg\theta$; that is, it is approximately proportional to the displacement θ. Hence for small angles the oscillations are simple harmonic.



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9.10 Pendulum Frequency
9.11 Risky Pendulum Walk
9.12 Physical Pendulum

Example 13.8 A simple pendulum

Find the period and frequency of a simple pendulum 1.000 m long at a location where $g = 9.800\ \text{m/s}^2$.

SOLUTION

IDENTIFY: Since this is a simple pendulum, we can use the ideas of this section.

SET UP: We use Eq. (13.34) to determine the period T of the pendulum from its length, and Eq. (13.1) to find the frequency f from T.

EXECUTE: From Eqs. (13.34) and (13.1),

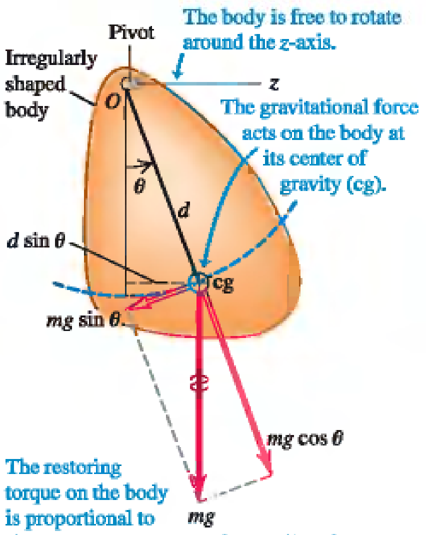
$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{1.000\ \text{m}}{9.800\ \text{m/s}^2}} = 2.007\ \text{s}$$

$$f = \frac{1}{T} = \frac{1}{2.007\text{s}} = 0.4983\ \text{Hz}$$

EVALUATE: The period is almost exactly 2 s. In fact, when the metric system was first established, the second was defined as half the period of a 1-meter pendulum. This wasn't a very good standard for time, however, because the value of g varies from place to place. We discussed more modern time standards in Section 1.3.

Test Your Understanding of Section 13.5 When a body oscillating on a horizontal spring passes through its equilibrium position, its acceleration is zero (see Fig. 13.2b). When the bob of an oscillating simple pendulum passes through its equilibrium position, is its acceleration zero?

13.23 Dynamics of a physical pendulum.



13.6 The Physical Pendulum

A **physical pendulum** is any *real* pendulum that uses an extended body, as contrasted to the idealized model of the *simple* pendulum with all the mass concentrated at a single point. For small oscillations, analyzing the motion of a real, physical pendulum is almost as easy as for a simple pendulum. Figure 13.23 shows a body of irregular shape pivoted so that it can turn without friction about an axis through point O. In the equilibrium position the center of gravity is directly below the pivot; in the position shown in the figure, the body is displaced from equilibrium by an angle θ, which we use as a coordinate for the system. The distance from O to the center of gravity is d, the moment of inertia of the body about the axis of rotation through O is I, and the total mass is m. When the body is displaced as shown, the weight mg causes a restoring torque

$$\tau_z = -(mg)(d\sin\theta) \tag{13.36}$$

The negative sign shows that the restoring torque is clockwise when the displacement is counterclockwise, and vice versa.

When the body is released, it oscillates about its equilibrium position. The motion is not simple harmonic because the torque τ_z is proportional to $\sin\theta$ rather than to θ itself. However, if θ is small, we can approximate $\sin\theta$ by θ in radians, just as we did in analyzing the simple pendulum. Then the motion is *approximately* simple harmonic. With this approximation,

$$\tau_z = -(mgd)\theta$$

The equation of motion is $\sum\tau_z = I\alpha_z$, so

$$-(mgd)\theta = I\alpha_z = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\theta \tag{13.37}$$

Comparing this with Eq. (13.4), we see that the role of (k/m) for the spring-mass system is played here by the quantity (mgd/I). Thus the angular frequency is

$$\omega = \sqrt{\frac{mgd}{I}} \quad \text{(physical pendulum, small amplitude)} \tag{13.38}$$

The frequency f is $1/2\pi$ times this, and the period T is

$$T = 2\pi\sqrt{\frac{I}{mgd}} \quad \text{(physical pendulum, small amplitude)} \tag{13.39}$$

Equation (13.39) is the basis of a common method for experimentally determining the moment of inertia of a body with a complicated shape. First locate the center of gravity of the body by balancing. Then suspend the body so that it is free to oscillate about an axis, and measure the period T of small-amplitude oscillations. Finally, use Eq. (13.39) to calculate the moment of inertia I of the body about this axis from T, the body's mass m, and the distance d from the axis to the center of gravity (see Exercise 13.49). Biomechanics researchers use this method to find the moments of inertia of an animal's limbs. This information is important for analyzing how an animal walks, as we'll see in the second of the two following examples.

Example 13.9 Physical pendulum versus simple pendulum

Suppose the body in Fig. 13.23 is a uniform rod with length L, pivoted at one end. Find the period of its motion.

SOLUTION

IDENTIFY: Our target variable is the oscillation period of a rod, which acts as a physical pendulum. We need to know the rod's moment of inertia to do this.

SET UP: We use Table 9.2 (Section 9.4) to find the moment of inertia of the rod, and then substitute this value into Eq. (13.39) to determine the period of oscillation.

EXECUTE: From Table 9.2, the moment of inertia of a uniform rod about an axis through one end is $I = \frac{1}{3}ML^2$. The distance from the pivot to the center of gravity is $d = L/2$. From Eq. (13.39),

$$T = 2\pi\sqrt{\frac{\frac{1}{3}ML^2}{MgL/2}} = 2\pi\sqrt{\frac{2L}{3g}}$$

EVALUATE: If the rod is a meter stick ($L = 1.00\text{ m}$) and $g = 9.80\text{ m/s}^2$, then

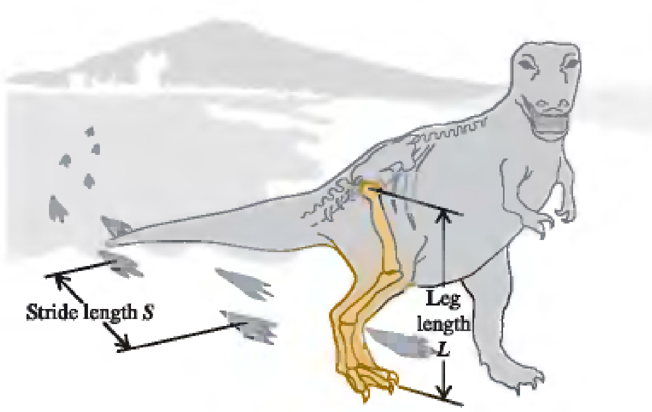
$$T = 2\pi\sqrt{\frac{2(1.00\text{ m})}{3(9.80\text{ m/s}^2)}} = 1.64\text{ s}$$

The period is smaller by a factor of $\sqrt{2/3} = 0.816$ than the period of a simple pendulum with the same length, calculated in Example 13.8. The cg of the rod is half as far from the pivot as the cg of the simple pendulum, which means the torque is half as great. By itself that would give the rod a period $\sqrt{2}$ times greater than the simple pendulum. But the rod's moment of inertia around one end, $I = \frac{1}{3}ML^2$, is one-third that of the simple pendulum, which by itself would make the rod's period $\sqrt{1/3}$ that of the simple pendulum. The moment of inertia factor is more important in this case, which is why the rod has a shorter period than the simple pendulum.

Example 13.10 *Tyrannosaurus rex* and the physical pendulum

All walking animals, including humans, have a natural walking pace—that is, a number of steps per minute that is more comfortable than a faster or slower pace. Suppose this natural pace corresponds to the oscillation of the leg as a physical pendulum. (a) How does the natural walking pace depend on the length L of the leg, measured from hip to foot? Treat the leg as a uniform rod pivoted at the hip joint. (b) Fossil evidence shows that *Tyrannosaurus rex,* a two-legged dinosaur that lived about 65 million years ago at the end of the Cretaceous period, had a leg length $L = 3.1$ m and a stride length $S = 4.0$ m (the distance from one footprint to the next print of the same foot; Fig. 13.24). Estimate the walking speed of *T. rex.*

13.24 The walking speed of *Tyrannosaurus rex* can be estimated from leg length L and stride length S.



SOLUTION

IDENTIFY: Our target variables are (a) the relationship between the walking pace and the leg length and (b) the walking speed of *T. rex.*

SET UP: We treat the leg as a physical pendulum, with a period of oscillation as found in Example 13.9. The shorter the period, the faster the walking pace. We can find the walking speed from the period and the stride length.

EXECUTE: (a) From Example 13.9 the period of oscillation of the leg is $T = 2\pi\sqrt{2L/3g}$, which is proportional to $\sqrt{L}$. Each period (a complete back-and-forth swing of the leg) corresponds to *two* steps, so the walking pace in steps per unit time is just twice the oscillation frequency $f = 1/T$. Hence the walking pace is proportional to $1/\sqrt{L}$. Animals with short legs (small values of L), such as mice or Chihuahuas, have rapid walking paces; humans, giraffes, and other animals with long legs (large values of L) walk at slower paces.

(b) According to our model for the natural walking pace, the elapsed time for one stride of a walking *Tyrannosaurus rex* is

$$T = 2\pi\sqrt{\frac{2L}{3g}} = 2\pi\sqrt{\frac{2(3.1\text{ m})}{3(9.8\text{ m/s}^2)}} = 2.9\text{ s}$$

The distance moved during this time is the stride length S, so the walking speed is

$$v = \frac{S}{T} = \frac{4.0\text{ m}}{2.9\text{ s}} = 1.4\text{ m/s} = 5.0\text{ km/h} = 3.1\text{ mi/h}$$

This is about the same as a typical human walking speed!

Continued

EVALUATE: Our estimate must be somewhat in error because a uniform rod isn't a very good model for a leg. The legs of many animals, including *T. rex* as well as humans, are tapered; there is a lot more mass between the knee and the hip than between the knee and the foot. Thus the center of mass is less than $L/2$ from the hip; a reasonable guess would be about $L/4$. The moment of inertia is therefore *considerably* less than $ML^2/3$, probably somewhere around $ML^2/15$. Try these numbers out with the analysis of Example 13.9; you'll get a shorter oscillation period and an even faster walking speed for *T. rex*.

Test Your Understanding of Section 13.6 The center of gravity of a simple pendulum of mass m and length L is located at the position of the pendulum bob, a distance L from the pivot point. The center of gravity of a uniform rod of the same mass m and length $2L$ pivoted at one end is also a distance L from the pivot point. How does the period of this uniform rod compare to the period of the simple pendulum? (i) The rod has a longer period; (ii) the rod has a shorter period; (iii) the rod has the same period.

13.7 Damped Oscillations

The idealized oscillating systems we have discussed so far are frictionless. There are no nonconservative forces, the total mechanical energy is constant, and a system set into motion continues oscillating forever with no decrease in amplitude.

Real-world systems always have some dissipative forces, however, and oscillations die out with time unless we replace the dissipated mechanical energy (Fig. 13.25). A mechanical pendulum clock continues to run because potential energy stored in the spring or a hanging weight system replaces the mechanical energy lost due to friction in the pivot and the gears. But eventually the spring runs down or the weights reach the bottom of their travel. Then no more energy is available, and the pendulum swings decrease in amplitude and stop.

13.25 A swinging bell left to itself will eventually stop oscillating due to damping forces (air resistance and friction at the point of suspension).

The decrease in amplitude caused by dissipative forces is called **damping,** and the corresponding motion is called **damped oscillation.** The simplest case to analyze in detail is a simple harmonic oscillator with a frictional damping force that is directly proportional to the *velocity* of the oscillating body. This behavior occurs in friction involving viscous fluid flow, such as in shock absorbers or sliding between oil-lubricated surfaces. We then have an additional force on the body due to friction, $F_x = -bv_x$, where $v_x = dx/dt$ is the velocity and b is a constant that describes the strength of the damping force. The negative sign shows that the force is always opposite in direction to the velocity. The *net* force on the body is then

$$\sum F_x = -kx - bv_x \tag{13.40}$$

and Newton's second law for the system is

$$-kx - bv_x = ma_x \qquad \text{or} \qquad -kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2} \tag{13.41}$$

Equation (13.41) is a differential equation for x; it would be the same as Eq. (13.4), the equation for the acceleration in SHM, except for the added term $-b\,dx/dt$. Solving this equation is a straightforward problem in differential equations, but we won't go into the details here. If the damping force is relatively small, the motion is described by

$$x = Ae^{-(b/2m)t}\cos(\omega' t + \phi) \quad \text{(oscillator with little damping)} \tag{13.42}$$

The angular frequency of oscillation ω' is given by

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \quad \text{(oscillator with little damping)} \tag{13.43}$$

You can verify that Eq. (13.42) is a solution of Eq. (13.41) by calculating the first and second derivatives of x, substituting them into Eq. (13.41), and checking whether the left and right sides are equal. This is a straightforward but slightly tedious procedure.

The motion described by Eq. (13.42) differs from the undamped case in two ways. First, the amplitude $Ae^{-(b/2m)t}$ is not constant but decreases with time because of the decreasing exponential factor $e^{-(b/2m)t}$. Figure 13.26 is a graph of Eq. (13.42) for the case $\phi = 0$; it shows that the larger the value of b, the more quickly the amplitude decreases.

Second, the angular frequency ω', given by Eq. (13.43), is no longer equal to $\omega = \sqrt{k/m}$ but is somewhat smaller. It becomes zero when b becomes so large that

$$\frac{k}{m} - \frac{b^2}{4m^2} = 0 \qquad \text{or} \qquad b = 2\sqrt{km} \tag{13.44}$$

When Eq. (13.44) is satisfied, the condition is called **critical damping.** The system no longer oscillates but returns to its equilibrium position without oscillation when it is displaced and released.

If b is greater than $2\sqrt{km}$, the condition is called **overdamping.** Again there is no oscillation, but the system returns to equilibrium more slowly than with critical damping. For the overdamped case the solutions of Eq. (13.41) have the form

$$x = C_1 e^{-a_1 t} + C_2 e^{-a_2 t}$$

where C_1 and C_2 are constants that depend on the initial conditions and a_1 and a_2 are constants determined by m, k, and b.

When b is less than the critical value, as in Eq. (13.42), the condition is called **underdamping.** The system oscillates with steadily decreasing amplitude.

In a vibrating tuning fork or guitar string, it is usually desirable to have as little damping as possible. By contrast, damping plays a beneficial role in the oscillations of an automobile's suspension system. The shock absorbers provide a velocity-dependent damping force so that when the car goes over a bump, it doesn't continue bouncing forever (Fig. 13.27). For optimal passenger comfort, the system should be critically damped or slightly underdamped. Too much damping would be counterproductive; if the suspension is overdamped and the car hits a second bump just after the first one, the springs in the suspension will still be compressed somewhat from the first bump and will not be able to fully absorb the impact.

Energy in Damped Oscillations

In damped oscillations the damping force is nonconservative; the mechanical energy of the system is not constant but decreases continuously, approaching zero after a long time. To derive an expression for the rate of change of energy, we first write an expression for the total mechanical energy E at any instant:

$$E = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2$$

To find the rate of change of this quantity, we take its time derivative:

$$\frac{dE}{dt} = mv_x\frac{dv_x}{dt} + kx\frac{dx}{dt}$$

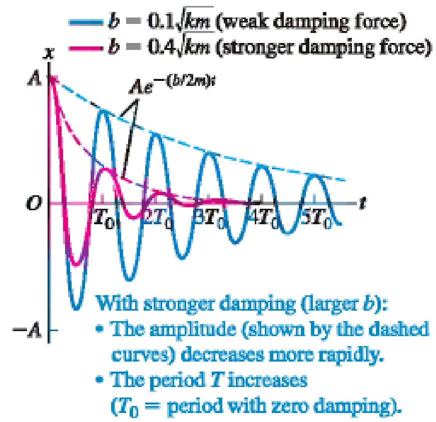
But $dv_x/dt = a_x$ and $dx/dt = v_x$, so

$$\frac{dE}{dt} = v_x(ma_x + kx)$$

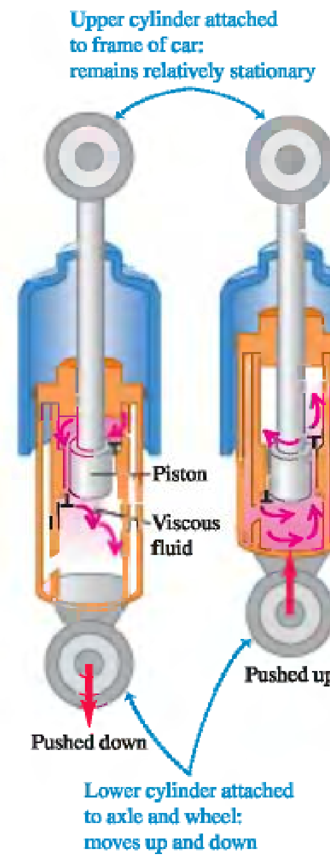
From Eq. (13.41), $ma_x + kx = -b\,dx/dt = -bv_x$, so

$$\frac{dE}{dt} = v_x(-bv_x) = -bv_x^2 \qquad \text{(damped oscillations)} \tag{13.45}$$

13.26 Graph of displacement versus time for an oscillator with little damping [see Eq. (13.42)] and with phase angle $\phi = 0$. The curves are for two values of the damping constant b.



13.27 An automobile shock absorber. The viscous fluid causes a damping force that depends on the relative velocity of the two ends of the unit.



The right side of Eq. (13.45) is **negative** whenever the oscillating body is in motion, whether the x-velocity v_x is positive or negative. This shows that as the body moves, the energy decreases, though not at a uniform rate. The term $-bv_x^2 = (-bv_x)v_x$ (force times velocity) is the rate at which the damping force does (negative) work on the system (that is, the damping *power*). This equals the rate of change of the total mechanical energy of the system.

Similar behavior occurs in electric circuits containing inductance, capacitance, and resistance. There is a natural frequency of oscillation, and the resistance plays the role of the damping constant b. We will study these circuits in detail in Chapters 30 and 31.

Test Your Understanding of Section 13.7 An airplane is flying in a straight line at a constant altitude. If a wind gust strikes and raises the nose of the airplane, the nose will bob up and down until the airplane eventually returns to its original attitude. Are these oscillations (i) undamped, (ii) underdamped, (iii) critically damped, or (iv) overdamped?

13.8 Forced Oscillations and Resonance

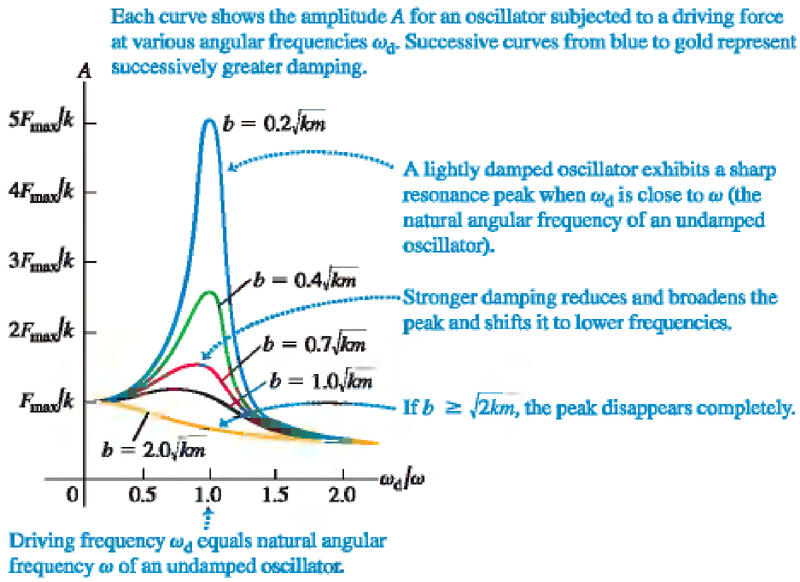
A damped oscillator left to itself will eventually stop moving altogether. But we can maintain a constant-amplitude oscillation by applying a force that varies with time in a periodic or cyclic way, with a definite period and frequency. As an example, consider your cousin Throckmorton on a playground swing. You can keep him swinging with constant amplitude by giving him a little push once each cycle. We call this additional force a **driving force.**

Damped Oscillation with a Periodic Driving Force

If we apply a periodically varying driving force with angular frequency ω_d to a damped harmonic oscillator, the motion that results is called a **forced oscillation** or a *driven oscillation.* It is different from the motion that occurs when the system is simply displaced from equilibrium and then left alone, in which case the system oscillates with a **natural angular frequency** ω' determined by m, k, and b, as in Eq. (13.43). In a forced oscillation, however, the angular frequency with which the mass oscillates is equal to the driving angular frequency ω_d. This does *not* have to be equal to the angular frequency ω' with which the system would oscillate without a driving force. If you grab the ropes of Throckmorton's swing, you can force the swing to oscillate with any frequency you like.

Suppose we force the oscillator to vibrate with an angular frequency ω_d that is nearly *equal* to the angular frequency ω' it would have with no driving force. What happens? The oscillator is naturally disposed to oscillate at $\omega = \omega'$, so we expect the amplitude of the resulting oscillation to be larger than when the two frequencies are very different. Detailed analysis and experiment shows that this is just what happens. The easiest case to analyze is a *sinusoidally* varying force—say, $F(t) = F_{max} \cos \omega_d t$. If we vary the frequency ω_d of the driving force, the amplitude of the resulting forced oscillation varies in an interesting way (Fig. 13.28). When there is very little damping (small b), the amplitude goes through a sharp peak as the driving angular frequency ω_d nears the natural oscillation angular frequency ω'. When the damping is increased (larger b), the peak becomes broader and smaller in height and shifts toward lower frequencies.

We could work out an expression that shows how the amplitude A of the forced oscillation depends on the frequency of a sinusoidal driving force, with



13.28 Graph of the amplitude A of forced oscillation as a function of the angular frequency ω_d of the driving force. The horizontal axis shows the ratio of ω_d to the angular frequency $\omega = \sqrt{k/m}$ of an undamped oscillator. Each curve has a different value of the damping constant b.

maximum value F_{max}. That would involve more differential equations than we're ready for, but here is the result:

$$A = \frac{F_{max}}{\sqrt{(k - m\omega_d^2)^2 + b^2\omega_d^2}} \quad \text{(amplitude of a driven oscillator)} \quad (13.46)$$

When $k - m\omega_d^2 = 0$, the first term under the radical is zero, so A has a maximum near $\omega_d = \sqrt{k/m}$. The height of the curve at this point is proportional to $1/b$; the less damping, the higher the peak. At the low-frequency extreme, when $\omega_d = 0$, we get $A = F_{max}/k$. This corresponds to a *constant* force F_{max} and a constant displacement $A = F_{max}/k$ from equilibrium, as we might expect.

Resonance and Its Consequences

The fact that there is an amplitude peak at driving frequencies close to the natural frequency of the system is called **resonance.** Physics is full of examples of resonance; building up the oscillations of a child on a swing by pushing with a frequency equal to the swing's natural frequency is one. A vibrating rattle in a car that occurs only at a certain engine speed or wheel-rotation speed is au all-too-familiar example. Inexpensive loudspeakers often have an annoying boom or buzz when a musical note happens to coincide with the resonant frequency of the speaker cone or the speaker housing. In Chapter 16 we will study other examples of resonance that involve sound. Resonance also occurs in electric circuits, as we will see in Chapter 31; a tuned circuit in a radio or television receiver responds strongly to waves having frequencies near its resonant frequency, and this fact is used to select a particular station and reject the others.

Resonance in mechanical systems can be destructive. A company of soldiers once destroyed a bridge by marching across it in step; the frequency of their steps was close to a natural vibration frequency of the bridge, and the resulting oscillation had large enough amplitude to tear the bridge apart. Ever since, marching soldiers have been ordered to break step before crossing a bridge. Some years ago, vibrations of the engines of a particular airplane had just the right frequency to resonate with the natural frequencies of its wings. Large oscillations built up, and occasionally the wings fell off.

Nearly everyone has seen the film of the collapse of the Tacoma Narrows suspension bridge in 1940 (Fig. 13.29). This is usually cited as an example of resonance driven by the wind, but there's some doubt whether it should be called that.

13.29 The Tacoma Narrows Bridge collapsed four months and six days after it was opened for traffic. The main span was 2800 ft long and 39 ft wide, with 8-ft-high steel stiffening girders on both sides. The maximum amplitude of the torsional vibrations was 35°; the frequency was about 0.2 Hz.

The wind didn't have to vary *periodically* with a frequency close to a natural frequency of the bridge. The airflow past the bridge was turbulent, and vortices were formed in the air with a regular frequency that depended on the flow speed. It is conceivable that this frequency may have coincided with a natural frequency of the bridge. But the cause may well have been something more subtle called a ***self-excited oscillation,*** in which the aerodynamic forces caused by a *steady* wind blowing on the bridge tended to displace it farther from equilibrium at times when it was already moving away from equilibrium. It is as though we had a damping force such as the $-bv_x$ term in Eq. (13.40) but with the sign reversed. Instead of draining mechanical energy away from the system, this anti-damping force pumps energy into the system, building up the oscillations to destructive amplitudes. The approximate differential equation is Eq. (13.41) with the sign of the b term reversed, and the oscillating solution is Eq. (13.42) with a *positive* sign in the exponent. You can see that we're headed for trouble. Engineers have learned how to stabilize suspension bridges, both structurally and aerodynamically, to prevent such disasters.

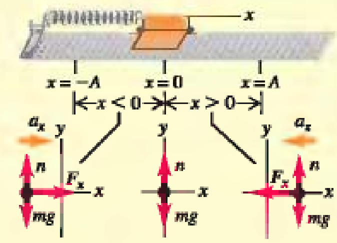
Test Your Understanding of Section 13.8 When driven at a frequency near its natural frequency, an oscillator with very little damping has a much greater response than the same oscillator with more damping. When driven at a frequency that is much higher or lower than the natural frequency, which oscillator will have the greater response: (i) the one with very little damping or (ii) the one with more damping?

CHAPTER 13 SUMMARY

Periodic motion: Periodic motion is motion that repeats itself in a definite cycle. It occurs whenever a body has a stable equilibrium position and a restoring force that acts when it is displaced from equilibrium. Period T is the time for one cycle. Frequency f is the number of cycles per unit time. Angular frequency ω is 2π times the frequency. (See Example 13.1.)

$$f = \frac{1}{T} \qquad T = \frac{1}{f} \tag{13.1}$$

$$\omega = 2\pi f = \frac{2\pi}{T} \tag{13.2}$$



Simple harmonic motion: If the restoring force F_x in periodic motion is directly proportional to the displacement x, the motion is called simple harmonic motion (SHM). In many cases this condition is satisfied if the displacement from equilibrium is small. The angular frequency, frequency, and period in SHM do not depend on the amplitude, but only on the mass m and force constant k. The displacement, velocity, and acceleration in SHM are sinusoidal functions of time; the amplitude A and phase angle ϕ of the oscillation are determined by the initial position and velocity of the body. (See Examples 13.2, 13.3, 13.6, and 13.7.)

$$F_x = -kx \tag{13.3}$$

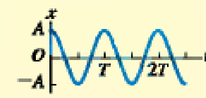
$$a_x = \frac{F_x}{m} = -\frac{k}{m}x \tag{13.4}$$

$$\omega = \sqrt{\frac{k}{m}} \tag{13.10}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{13.11}$$

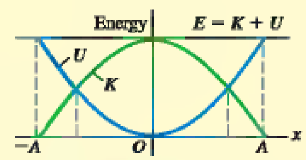
$$T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}} \tag{13.12}$$

$$x = A\cos(\omega t + \phi) \tag{13.13}$$



Energy in simple harmonic motion: Energy is conserved in SHM. The total energy can be expressed in terms of the force constant k and amplitude A. (See Examples 13.4 and 13.5.)

$$E = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \text{constant} \tag{13.21}$$



Angular simple harmonic motion: In angular SHM, the frequency and angular frequency are related to the moment of inertia I and the torsion constant κ.

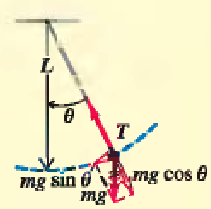
$$\omega = \sqrt{\frac{\kappa}{I}} \quad \text{and} \quad f = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I}} \tag{13.24}$$

Simple pendulum: A simple pendulum consists of a point mass m at the end of a massless string of length L. Its motion is approximately simple harmonic for sufficiently small amplitude; the angular frequency, frequency, and period then depend only on g and L, not on the mass or amplitude. (See Example 13.8.)

$$\omega = \sqrt{\frac{g}{L}} \tag{13.32}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \tag{13.33}$$

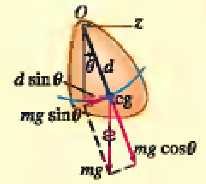
$$T = \frac{2\pi}{\omega} = \frac{1}{f} = 2\pi\sqrt{\frac{L}{g}} \tag{13.34}$$



Physical pendulum: A physical pendulum is any body suspended from an axis of rotation. The angular frequency and period for small-amplitude oscillations are independent of amplitude, but depend on the mass m, distance d from the axis of rotation to the center of gravity, and moment of inertia I about the axis. (See Examples 13.9 and 13.10.)

$$\omega = \sqrt{\frac{mgd}{I}} \quad (13.38)$$

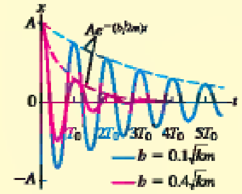
$$T = 2\pi\sqrt{\frac{I}{mgd}} \quad (13.39)$$



Damped oscillations: When a force $F_x = -bv_x$ proportional to velocity is added to a simple harmonic oscillator, the motion is called a damped oscillation. If $b < 2\sqrt{km}$ (called underdamping), the system oscillates with a decaying amplitude and an angular frequency ω' that is lower than it would be without damping. If $b = 2\sqrt{km}$ (called critical damping) or $b > 2\sqrt{km}$ (called overdamping), when the system is displaced it returns to equilibrium without oscillating.

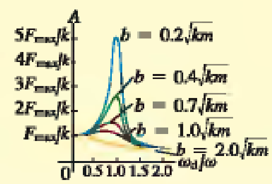
$$x = Ae^{-(b/2m)t}\cos\omega' t \quad (13.42)$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \quad (13.43)$$



Driven oscillations and resonance: When a sinusoidally varying driving force is added to a damped harmonic oscillator, the resulting motion is called a forced oscillation. The amplitude is a function of the driving frequency ω_d and reaches a peak at a driving frequency close to the natural frequency of the system. This behavior is called resonance.

$$A = \frac{F_{max}}{\sqrt{(k - m\omega_d^2)^2 + b^2\omega_d^2}} \quad (13.46)$$



Key Terms

periodic motion (oscillation), *419*
displacement, *420*
restoring force, *420*
amplitude, *420*
cycle, *420*
period, *420*
frequency, *420*
angular frequency, *420*
simple harmonic motion (SHM), *421*
harmonic oscillator, *422*
reference circle, *423*
phasor, *423*
phase angle, *426*
simple pendulum, *436*
physical pendulum, *438*
damping, *440*
damped oscillation, *440*
critical damping, *441*
overdamping, *441*
underdamping, *441*
driving force, *442*
forced oscillation, *442*
natural angular frequency, *442*
resonance, *443*

Answer to Chapter Opening Question ?

Neither—the clock would still keep time correctly. If its rod has negligible mass, then the pendulum is a simple pendulum and its period is independent of the mass [see Eq. (13.34)]. If the rod's mass is included, the pendulum is a physical pendulum. Doubling its mass m also doubles its moment of inertia I, so the ratio I/m is unchanged and the period $T = 2\pi\sqrt{I/mgd}$ [Eq. (13.39)] remains the same.

Answers to Test Your Understanding Questions

13.1 Answers: (a) $x < 0$, (b) $x > 0$, (c) $x < 0$, (d) $x > 0$, (e) $x = 0$, (f) $x > 0$ Figure 13.2 shows that the net x-component of force F_x and the x-acceleration a_x are both positive when $x < 0$ (so the body is displaced to the left and the spring is compressed), while F_x and a_x are both negative when $x > 0$ (so the body is displaced to the right and the spring is stretched). Hence x and a_x always have *opposite* signs. This is true whether the object is moving to the right $(v_x > 0)$, to the left $(v_x < 0)$, or not at all $(v_x = 0)$, since the force exerted by the spring depends only on whether it is compressed or stretched and by what distance. This explains the answers to (a) through (e). If the acceleration is zero as in (f), the net force must also be zero and so the spring must be relaxed; hence $x = 0$.

13.2 Answers: (a) $A > 0.10$ m, $\phi < 0$; (b) $A > 0.10$ m, $\phi > 0$ In both situations the initial $(t = 0)$ x-velocity v_{0x} is nonzero, so from Eq. (13.19) the amplitude $A = \sqrt{x_0^2 + (v_{0x}^2/\omega^2)}$ is greater than the initial x-coordinate $x_0 = 0.10$ m. From Eq. (13.18) the phase angle is $\phi = \arctan(-v_{0x}/\omega x_0)$, which is positive if the quantity $-v_{0x}/\omega x_0$ (the argument of the arctangent function) is positive and negative if $-v_{0x}/\omega x_0$ is negative. In part (a) x_0 and v_{0x} are both positive, so $-v_{0x}/\omega x_0 < 0$ and $\phi < 0$. In part (b) x_0 is positive and v_{0x} is negative, so $-v_{0x}/\omega x_0 > 0$ and $\phi > 0$.

13.3 Answers: (a) (iii), (b) (v) To increase the total energy $E = \frac{1}{2}kA^2$ by a factor of 2, the amplitude A must increase by a factor of $\sqrt{2}$. Because the motion is SHM, changing the amplitude has no effect on the frequency.

13.4 Answer: (i) The oscillation period of a body of mass m attached to a hanging spring of force constant k is given by $T = 2\pi\sqrt{m/k}$, the same expression as for a body attached to a horizontal spring. Neither m nor k changes when the apparatus is taken to Mars, so the period is unchanged. The only difference is that in equilibrium, the spring will stretch a shorter distance on Mars than on earth due to the weaker gravity.

13.5 Answer: no Just as for an object oscillating on a spring, at the equilibrium position the *speed* of the pendulum bob is instantaneously not changing (this is where the speed is maximum, so its derivative at this time is zero). But the *direction* of motion is changing because the pendulum bob follows a circular path. Hence the bob must have a component of acceleration perpendicular to the path and toward the center of the circle (see Section 3.4). To cause this acceleration at the equilibrium position when the string is vertical, the upward tension force at this position must be greater than the weight of the bob. This causes a net upward force on the bob and an upward acceleration toward the center of the circular path.

13.6 Answer: (i) The period of a physical pendulum is given by Eq. (13.39), $T = 2\pi\sqrt{I/mgd}$. The distance $d = L$ from the pivot to the center of gravity is the same for both the rod and the simple pendulum, as is the mass m. This means that for any displacement angle θ the same restoring torque acts on both the rod and the simple pendulum. However, the rod has a greater moment of inertia: $I_{rod} = \frac{1}{3}m(2L)^2 = \frac{4}{3}mL^2$ and $I_{simple} = mL^2$ (all the mass of the pendulum is a distance L from the pivot). Hence the rod has a longer period.

13.7 Answer: (ii) The oscillations are underdamped with a decreasing amplitude on each cycle of oscillation, like those graphed in Fig. 13.26. If the oscillations were undamped, they would continue indefinitely with the same amplitude. If they were critically damped or overdamped, the nose would not bob up and down but would return smoothly to the original equilibrium attitude without overshooting.

13.8 Answer: (i) Figure 13.28 shows that the curve of amplitude versus driving frequency moves upward at *all* frequencies as the value of the damping constant b is decreased. Hence for fixed values of k and m, the oscillator with the least damping (smallest value of b) will have the greatest response at any driving frequency.

PROBLEMS

For instructor-assigned homework, go to www.masteringphysics.com

Discussion Questions

Q13.1. An object is moving with SHM of amplitude A on the end of a spring. If the amplitude is doubled, what happens to the total distance the object travels in one period? What happens to the period? What happens to the maximum speed of the object? Discuss how these answers are related.

Q13.2. Think of several examples in everyday life of motions that are, at least approximately, simple harmonic. In what respects does each differ from SHM?

Q13.3. Does a tuning fork or similar tuning instrument undergo SHM? Why is this a crucial question for musicians?

Q13.4. A box containing a pebble is attached to an ideal horizontal spring and is oscillating on a friction-free air table. When the box has reached its maximum distance from the equilibrium point, the pebble is suddenly lifted out vertically without disturbing the box. Will the following characteristics of the motion increase, decrease, or remain the same in the subsequent motion of the box? Justify each answer. (a) frequency; (b) period; (c) amplitude; (d) the maximum kinetic energy of the box; (e) the maximum speed of the box.

Q13.5. If a uniform spring is cut in half, what is the force constant of each half? Justify your answer. How would the frequency of SHM using a half-spring differ from the frequency using the same mass and the entire spring?

Q13.6. The analysis of SHM in this chapter ignored the mass of the spring. How does the spring's mass change the characteristics of the motion?

Q13.7. Two identical gliders on an air track are connected by an ideal spring. Could such a system undergo SHM? Explain. How would the period compare with that of a single glider attached to a spring whose other end is rigidly attached to a stationary object? Explain.

Q13.8. You are captured by Martians, taken into their ship, and put to sleep. You awake some time later and find yourself locked in a small room with no windows. All the martians have left you with is your digital watch, your school ring, and your long silver-chain necklace. Explain how you can determine whether you are still on earth or have been transported to Mars.

Q13.9. The system shown in Fig. 13.17 is mounted in an elevator. What happens to the period of the motion (does it increase, decrease, or remain the same) if the elevator (a) accelerates upward at 5.0 m/s^2; (b) moves upward at a steady 5.0 m/s; (c) accelerates downward at 5.0 m/s^2? Justify your answers.

Q13.10. If a pendulum has a period of 2.5 s on earth, what would be its period in a space station orbiting the earth? If a mass hung from a vertical spring has a period of 5.0 s on earth, what would its period be in the space station? Justify each of your answers.

Q13.11. A simple pendulum is mounted in an elevator. What happens to the period of the pendulum (does it increase, decrease, or remain the same) if the elevator (a) accelerates upward at 5.0 m/s^2, (b) moves upward at a steady 5.0 m/s, (c) accelerates downward at 5.0 m/s^2, (d) accelerates downward at 9.8 m/s^2? Justify your answers.

Q13.12. What should you do to the length of the string of a simple pendulum to (a) double its frequency; (b) double its period; (c) double its angular frequency?

Q13.13. If a pendulum clock is taken to a mountaintop, does it gain or lose time, assuming it is correct at a lower elevation? Explain your answer.

Q13.14. When the amplitude of a simple pendulum increases, should its period increase or decrease? Give a qualitative argument; do not rely on Eq. (13.35). Is your argument also valid for a physical pendulum?

Q13.15. Why do short dogs (like Chihuahuas) walk with quicker strides than do tall dogs (like Great Danes)?

Q13.16. At what point in the motion of a simple pendulum is the string tension greatest? Least? In each case give the reasoning behind your answer.

Q13.17. Could a standard of time be based on the period of a certain standard pendulum? What advantages and disadvantages would such a standard have compared to the actual present-day standard discussed in Section 1.3?

Q13.18. For a simple pendulum, clearly distinguish between ω (the angular velocity) and ω (the angular frequency). Which is constant and which is variable?

Q13.19. A glider is attached to a fixed ideal spring and oscillates on a horizontal, friction-free air track. A coin is atop the glider and oscillating with it. At what points in the motion is the friction force on the coin greatest? At what points is it least? Justify your answers.

Q13.20. In designing structures in an earthquake-prone region, how should the natural frequencies of oscillation of a structure relate to typical earthquake frequencies? Why? Should the structure have a large or small amount of damping?

Exercises

Section 13.1 Describing Oscillation

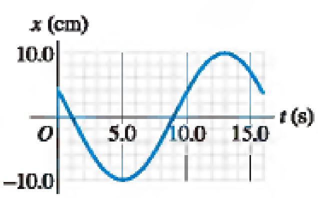
13.1. A piano string sounds a middle A by vibrating primarily at 220 Hz. (a) Calculate the string's period and angular frequency. (b) Calculate the period and angular frequency for a soprano singing an A one octave higher, which is twice the frequency of the piano string.

13.2. If an object on a horizontal, frictionless surface is attached to a spring, displaced, and then released, it will oscillate. If it is displaced 0.120 m from its equilibrium position and released with zero initial speed, then after 0.800 s its displacement is found to be 0.120 m on the opposite side, and it has passed the equilibrium position once during this interval. Find (a) the amplitude; (b) the period; (c) the frequency.

13.3. The tip of a tuning fork goes through 440 complete vibrations in 0.500 s. Find the angular frequency and the period of the motion.

13.4. The displacement of an oscillating object as a function of time is shown in Fig. 13.30. What are (a) the frequency; (b) the amplitude; (c) the period; (d) the angular frequency of this motion?

Figure 13.30 Exercise 13.4.



Section 13.2 Simple Harmonic Motion

13.5. A machine part is undergoing SHM with a frequency of 5.00 Hz and amplitude 1.80 cm. How long does it take the part to go from $x = 0$ to $x = -1.80$ cm?

13.6. In a physics lab, you attach a 0.200-kg air-track glider to the end of an ideal spring of negligible mass and start it oscillating. The elapsed time from when the glider first moves through the equilibrium point to the second time it moves through that point is 2.60 s. Find the spring's force constant.

13.7. When a body of unknown mass is attached to an ideal spring with force constant 120 N/m, it is found to vibrate with a frequency of 6.00 Hz. Find (a) the period of the motion; (b) the angular frequency; (c) the mass of the body.

13.8. When a 0.750-kg mass oscillates on an ideal spring, the frequency is 1.33 Hz. What will the frequency be if 0.220 kg are added to the original mass, and (b) subtracted from the original mass? Try to solve this problem *without* finding the force constant of the spring.

13.9. A harmonic oscillator consists of a 0.500-kg mass attached to an ideal spring with force constant 140 N/m. Find (a) the period; (b) the frequency; (c) the angular frequency of the oscillations.

13.10. Jerk. A guitar string vibrates at a frequency of 440 Hz. A point at its center moves in SHM with an amplitude of 3.0 mm and a phase angle of zero. (a) Write an equation for the position of the center of the string as a function of time. (b) What are the maximum values of the magnitudes of the velocity and acceleration of the center of the string? (c) The derivative of the acceleration with respect to time is a quantity called the *jerk*. Write an equation for the jerk of the center of the string as a function of time, and find the maximum value of the magnitude of the jerk.

13.11. A 2.00-kg, frictionless block is attached to an ideal spring with force constant 300 N/m. At $t = 0$ the spring is neither stretched nor compressed and the block is moving in the negative direction at 12.0 m/s. Find (a) the amplitude and (b) the phase angle. (c) Write an equation for the position as a function of time.

13.12. Repeat Exercise 13.11, but assume that at $t = 0$ the block has velocity -4.00 m/s and displacement $+0.200$ m.

13.13. The point of the needle of a sewing machine moves in SHM along the x-axis with a frequency of 2.5 Hz. At $t = 0$ its position and velocity components are $+1.1$ cm and -15 cm/s, respectively (a) Find the acceleration component of the needle at $t = 0$. (b) Write equations giving the position, velocity, and acceleration components of the point as a function of time.

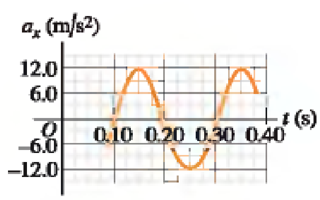
13.14. An object is undergoing SHM with period 1.200 s and amplitude 0.600 m. At $t = 0$ the object is at $x = 0$. How far is the object from the equilibrium position when $t = 0.480$ s?

13.15. Weighing Astronauts. This procedure has actually been used to "weigh" astronauts in space. A 42.5-kg chair is attached to a spring and allowed to oscillate. When it is empty, the chair takes 1.30 s to make one complete vibration. But with an astronaut sitting in it, with her feet off the floor, the chair takes 2.54 s for one cycle. What is the mass of the astronaut?

13.16. A 0.400-kg object undergoing SHM has $a_x = -2.70$ m/s^2 when $x = 0.300$ m. What is the time for one oscillation?

13.17. On a frictionless, horizontal air track, a glider oscillates at the end of an ideal spring of force constant 2.50 N/cm. The graph in Fig. 13.31 shows the acceleration of the glider as a function of time. Find (a) the mass of the glider; (b) the maximum displacement of the glider from the equilibrium point; (c) the maximum force the spring exerts on the glider.

Figure 13.31 Exercise 13.17.



13.18. A 0.500-kg mass on a spring has velocity as a function of time given by $v_x(t) = (3.60 \text{ cm/s}) \sin[(4.71 \text{ s}^{-1})t - \pi/2]$. What are (a) the period; (b) the amplitude; (c) the maximum acceleration of the mass; (d) the force constant of the spring?

13.19. A 1.50-kg mass on a spring has displacement as a function of time given by the equation

$$x(t) = (7.40 \text{ cm}) \cos[(4.16 \text{ s}^{-1})t - 2.42].$$

Find (a) the time for one complete vibration; (b) the force constant of the spring; (c) the maximum speed of the mass; (d) the maximum force on the mass; (e) the position, speed, and acceleration of the mass at $t = 1.00$ s; (f) the force on the mass at that time.

13.20. An object is undergoing SHM with period 0.300 s and amplitude 6.00 cm. At $t = 0$ the object is instantaneously at rest at $x = 6.00$ cm. Calculate the time it takes the object to go from $x = 6.00$ cm to $x = -1.50$ cm.

Section 13.3 Energy in Simple Harmonic Motion

13.21. A tuning fork labeled 392 Hz has the tip of each of its two prongs vibrating with an amplitude of 0.600 mm. (a) What is the maximum speed of the tip of a prong? (b) A housefly (*Musca domestica*) with mass 0.0270 g is holding on to the tip of one of the prongs. As the prong vibrates, what is the fly's maximum kinetic energy? Assume that the fly's mass has a negligible effect on the frequency of oscillation.

13.22. A harmonic oscillator has angular frequency ω and amplitude A. (a) What are the magnitudes of the displacement and velocity when the elastic potential energy is equal to the kinetic energy? (Assume that $U = 0$ at equilibrium.) (b) How often does this occur in each cycle? What is the time between occurrences? (c) At an instant when the displacement is equal to $A/2$, what fraction of the total energy of the system is kinetic and what fraction is potential?

13.23. A 0.500-kg glider, attached to the end of an ideal spring with force constant $k = 450$ N/m, undergoes SHM with an amplitude of 0.040 m. Compute (a) the maximum speed of the glider; (b) the speed of the glider when it is at $x = -0.015$ m; (c) the magnitude of the maximum acceleration of the glider; (d) the acceleration of the glider at $x = -0.015$ m; (e) the total mechanical energy of the glider at any point in its motion.

13.24. A cheerleader waves her pom-pom in SHM with an amplitude of 18.0 cm and a frequency of 0.850 Hz. Find (a) the maximum magnitude of the acceleration and of the velocity; (b) the acceleration and speed when the pom-pom's coordinate is $x = +9.0$ cm; (c) the time required to move from the equilibrium position directly to a point 12.0 cm away. (d) Which of the quantities asked for in parts (a), (b), and (c) can be found using the energy approach used in Section 13.3, and which cannot? Explain.

13.25. For the situation described in part (a) of Example 13.5, what should be the value of the putty mass m so that the amplitude after the collision is one-half the original amplitude? For this value of m, what fraction of the original mechanical energy is converted into heat?

13.26. A 0.150-kg toy is undergoing SHM on the end of a horizontal spring with force constant $k = 300$ N/m. When the object is 0.0120 m from its equilibrium position, it is observed to have a speed of 0.300 m/s. What are (a) the total energy of the object at any point of its motion; (b) the amplitude of the motion; (c) the maximum speed attained by the object during its motion?

13.27. You are watching an object that is moving in SHM. When the object is displaced 0.600 m to the right of its equilibrium position, it has a velocity of 2.20 m/s to the right and an acceleration of 8.40 m/s^2 to the left. How much farther from this point will the object move before it stops momentarily and then starts to move back to the left?

13.28. On a horizontal, frictionless table, an open-topped 5.20-kg box is attached to an ideal horizontal spring having force constant 375 N/m. Inside the box is a 3.44-kg stone. The system is oscillating with an amplitude of 7.50 cm. When the box has reached its maximum speed, the stone is suddenly plucked vertically out of the box without touching the box. Find (a) the period and (b) the amplitude of the resulting motion of the box. (c) Without doing any calculations, is the new period greater or smaller than the original period? How do you know?

13.29. Inside a NASA test vehicle, a 3.50-kg ball is pulled along by a horizontal ideal spring fixed to a friction-free table. The force constant of the spring is 225 N/m. The vehicle has a steady acceleration of 5.00 m/s^2, and the ball is not oscillating. Suddenly, when the vehicle's speed has reached 45.0 m/s, its engines turn off, thus eliminating its acceleration but not its velocity. Find (a) the amplitude and (b) the frequency of the resulting oscillations of the ball. (c) What will be the ball's maximum speed relative to the vehicle?

Section 13.4 Applications of Simple Harmonic Motion

13.30. A proud deep-sea fisherman hangs a 65.0-kg fish from an ideal spring having negligible mass. The fish stretches the spring 0.120 m. (a) Find the force constant of the spring. The fish is now pulled down 5.00 cm and released. (b) What is the period of oscillation of the fish? (c) What is the maximum speed it will reach?

13.31. A 175-g glider on a horizontal, frictionless air track is attached to a fixed ideal spring with force constant 155 N/m. At the instant you make measurements on the glider, it is moving at 0.815 m/s and is 3.00 cm from its equilibrium point. Use *energy conservation* to find (a) the amplitude of the motion and (b) the maximum speed of the glider. (c) What is the angular frequency of the oscillations?

13.32. A thrill-seeking cat with mass 4.00 kg is attached by a harness to an ideal spring of negligible mass and oscillates vertically in SHM. The amplitude is 0.050 m, and at the highest point of the motion the spring has its natural unstretched length. Calculate the elastic potential energy of the spring (take it to be zero for the unstretched spring), the kinetic energy of the cat, the gravitational potential energy of the system relative to the lowest point of the motion, and the sum of these three energies when the cat is (a) at its highest point; (b) at its lowest point; (c) at its equilibrium position.

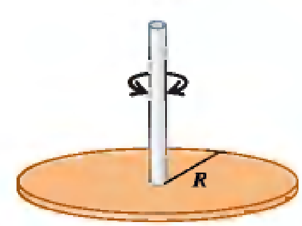
13.33. A 1.50-kg ball and a 2.00-kg ball are glued together with the lighter one below the heavier one. The upper ball is attached to a vertical ideal spring of force constant 165 N/m, and the system is vibrating vertically with amplitude 15.0 cm. The glue connecting the balls is old and weak, and it suddenly comes loose when the balls are at the lowest position in their motion. (a) Why is the glue more likely to fail at the *lowest* point than at any other point in the motion? (b) Find the amplitude and frequency of the vibrations after the lower ball has come loose.

13.34. A uniform, solid metal disk of mass 6.50 kg and diameter 24.0 cm hangs in a horizontal plane, supported at its center by a vertical metal wire. You find that it requires a horizontal force of 4.23 N tangent to the rim of the disk to turn it by 3.34°, thus twisting the wire. You now remove this force and release the disk from rest. (a) What is the torsion constant for the metal wire? (b) What are the frequency and period of the torsional oscillations of the disk? (c) Write the equation of motion for $\theta(t)$ for the disk.

13.35. A certain alarm clock ticks four times each second, with each tick representing half a period. The balance wheel consists of a thin rim with radius 0.55 cm, connected to the balance staff by thin spokes of negligible mass. The total mass of the balance wheel is 0.90 g. (a) What is the moment of inertia of the balance wheel about its shaft? (b) What is the torsion constant of the hairspring?

13.36. A thin metal disk with mass 2.00×10^{-3} kg and radius 2.20 cm is attached at its center to a long fiber (Fig. 13.32). The disk, when twisted and released, oscillates with a period of 1.00 s. Find the torsion constant of the fiber.

Figure 13.32 Exercise 13.36.



13.37. You want to find the moment of inertia of a complicated machine part about an axis through its center of mass. You suspend it from a wire along this axis. The wire has a torsion constant of $0.450\ \text{N}\cdot\text{m/rad}$ You twist the part a small amount about this axis and let it go, timing 125 oscillations in 265 s. What is the moment of inertia you want to find?

13.36. The balance wheel of a watch vibrates with an angular amplitude Θ, angular frequency ω, and phase angle $\phi = 0$. (a) Find expressions for the angular velocity $d\theta/dt$ and angular acceleration $d^2\theta/dt^2$ as functions of time. (b) Find the balance wheel's angular velocity and angular acceleration when its angular displacement is Θ, and when its angular displacement is $\Theta/2$ and θ is decreasing. (*Hint:* Sketch a graph of θ versus t.)

***13.39.** For the van der Waals interaction with potential-energy function given by Eq. (13.25), show that when the magnitude of the displacement x from equilibrium $(r = R_0)$ is small, the potential energy can be written approximately as $U \approx \frac{1}{2}kx^2 - U_0$. [*Hint:* In Eq. (13.25), let $r = R_0 + x$ and $u = x/R_0$. Then approximate $(1 + u)^n$ by the first *three* terms in Eq. (13.28).] How does k in this equation compare with the force constant in Eq. (13.29) for the force?

***13.40.** When displaced from equilibrium, the two hydrogen atoms in an H_2 molecule are acted on by a restoring force $F_x = -kx$ with $k = 580\ \text{N/m}$. Calculate the oscillation frequency of the H_2 molecule. (*Hint:* The mass of a hydrogen atom is 1.008 atomic mass units, or 1 u; see Appendix E. As in Example 13.7 (Section 13.4), use $m/2$ instead of m in the expression for f.)

Section 13.5 The Simple Pendulum

13.41. You pull a simple pendulum 0.240 m long to the side through an angle of 3.50° and release it. (a) How much time does it take the pendulum bob to reach its highest speed? (b) How much time does it take if the pendulum is released at an angle of 1.75° instead of 3.50°?

13.42. An 85.0-kg mountain climber plans to swing down, starting from rest, from a ledge using a light rope 6.50 m long. He holds one end of the rope, and the other end is tied higher up on a rock face. Since the ledge is not very far from the rock face, the rope makes a small angle with the vertical. At the lowest point of his swing, he plans to let go and drop a short distance to the ground. (a) How long after he begins his swing will the climber first reach his lowest point? (b) If he missed the first chance to drop off, how long after first beginning his swing will the climber reach his lowest point for the second time?

13.43. A building in San Francisco has light fixtures consisting of small 2.35-kg bulbs with shades hanging from the ceiling at the end of light thin cords 1.50 m long. If a minor earthquake occurs, how many swings per second will these fixtures make?

13.44. A Pendulum on Mars. A certain simple pendulum has a period on the earth of 1.60 s. What is its period on the surface of Mars, where $g = 3.71\ \text{m/s}^2$?

13.45. An apple weighs 1.00 N. When you hang it from the end of a long spring of force constant 1.50 N/m and negligible mass, it bounces up and down in SHM. If you stop the bouncing and let the apple swing from side to side through a small angle, the frequency of this simple pendulum is half the bounce frequency. (Because the angle is small, the back-and-forth swings do not cause any appreciable change in the length of the spring.) What is the unstretched length of the spring (with the apple removed)?

13.48. A small sphere with mass m is attached to a massless rod of length L that is pivoted at the top, forming a simple pendulum. The pendulum is pulled to one side so that the rod is at an angle Θ from the vertical, and released from rest. (a) In a diagram, show the pendulum just after it is released. Draw vectors representing the *forces* acting on the small sphere and the *acceleration* of the sphere. Accuracy counts! At this point, what is the linear acceleration of the sphere? (b) Repeat part (a) for the instant when the pendulum rod is at an angle $\Theta/2$ from the vertical. (c) Repeat part (a) for the instant when the pendulum rod is vertical. At this point, what is the linear speed of the sphere?

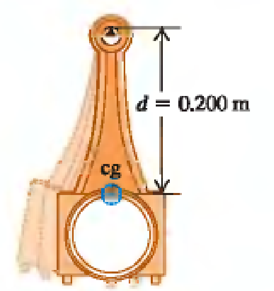
13.47. After landing on an unfamiliar planet, a space explorer constructs a simple pendulum of length 50.0 cm. She finds that the pendulum makes 100 complete swings in 136 s. What is the value of g on this planet?

13.48. A simple pendulum 2.00 m long swings through a maximum angle of 30.0° with the vertical. Calculate its period (a) assuming a small amplitude, and (b) using the first three terms of Eq. (13.35). (c) Which of the answers in parts (a) and (b) is more accurate? For the one that is less accurate, by what percent is it in error from the more accurate answer?

Section 13.6 The Physical Pendulum

13.48. A 1.80-kg connecting rod from a car engine is pivoted about a horizontal knife edge as shown in Fig. 13.33. The center of gravity of the rod was located by balancing and is 0.200 m from the pivot. When the rod is set into small-amplitude oscillation, it makes 100 complete swings in 120 s. Calculate the moment of inertia of the rod about the rotation axis through the pivot.

Figure 13.33 Exercise 13.49.



13.50. We want to hang a thin hoop on a horizontal nail and have the hoop make one complete small-angle oscillation each 2.0 s. What must the hoop's radius be?

13.51. Show that the expression for the period of a physical pendulum reduces to that of a simple pendulum if the physical pendulum consists of a particle with mass m on the end of a massless string of length L.

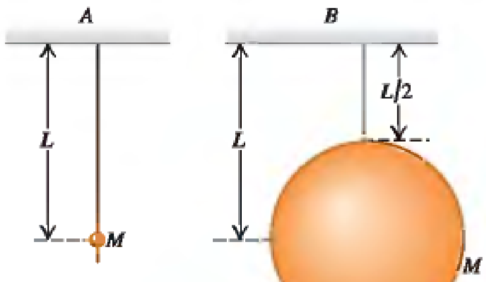
13.52. A 1.80-kg monkey wrench is pivoted 0.250 m from its center of mass and allowed to swing as a physical pendulum. The period for small-angle oscillations is 0.940 s. (a) What is the moment of inertia of the wrench about an axis through the pivot? (b) If the wrench is initially displaced 0.400 rad from its equilibrium position, what is the angular speed of the wrench as it passes through the equilibrium position?

13.53. Two pendulums have the same dimensions (length L) and total mass (m). Pendulum A is a very small ball swinging at the end of a uniform massless bar. In pendulum B, half the mass is in the ball and half is in the uniform bar. Find the period of each pendulum for small oscillations. Which one takes longer for a swing?

13.54. A holiday ornament in the shape of a hollow sphere with mass $M = 0.015$ kg and radius $R = 0.050$ m is hung from a tree limb by a small loop of wire attached to the surface of the sphere. If the ornament is displaced a small distance and released, it swings back and forth as a physical pendulum with negligible friction. Calculate its period. (*Hint:* Use the parallel-axis theorem to find the moment of inertia of the sphere about the pivot at the tree limb.)

13.55. The two pendulums shown in Fig. 13.34 each consist of a uniform solid ball of mass M supported by a massless string, but the ball for pendulum A is very tiny while the ball for pendulum B is much larger. Find the period of each pendulum for small displacements. Which ball takes longer to complete a swing?

Figure 13.34 Exercise 13.55.



Section 13.7 Damped Oscillations

13.56. A 2.20-kg mass oscillates on a spring of force constant 250.0 N/m with a period of 0.615 s. (a) Is this system damped or not? How do you know? If it is damped, find the damping constant b. (b) Is the system undamped, underdamped, critically damped, or overdamped? How do you know?

13.57. An unhappy 0.300-kg rodent, moving on the end of a spring with force constant $k = 2.50$ N/m, is acted on by a damping force $F_x = -bv_x$. (a) If the constant b has the value 0.900 kg/s, what is the frequency of oscillation of the rodent? (b) For what value of the constant b will the motion be critically damped?

13.56. A 50.0-g hard-boiled egg moves on the end of a spring with force constant $k = 25.0$ N/m. Its initial displacement is 0.300 m. A damping force $F_x = -bv_x$ acts on the egg, and the amplitude of the motion decreases to 0.100 m in 5.00 s. Calculate the magnitude of the damping constant b.

13.59. The motion of an underdamped oscillator is described by Eq. (13.42). Let the phase angle ϕ be zero. (a) According to this equation, what is the value of x at $t = 0$? (b) What are the magnitude and direction of the velocity at $t = 0$? What does the result tell you about the slope of the graph of x versus t near $t = 0$? (c) Obtain an expression for the acceleration a_x at $t = 0$. For what value or range of values of the damping constant b (in terms of k and m) is the acceleration at $t = 0$ negative, zero, and positive? Discuss each case in terms of the shape of the graph of x versus t near $t = 0$.

Section 13.8 Forced Oscillations and Resonance

13.60. A sinusoidally varying driving force is applied to a damped harmonic oscillator of force constant k and mass m. If the damping constant has a value b_1, the amplitude is A_1 when the driving angular frequency equals $\sqrt{k/m}$. In terms of A_1, what is the amplitude for the same driving frequency and the same driving force amplitude F_{max}, if the damping constant is (a) $3b_1$ and (b) $b_1/2$?

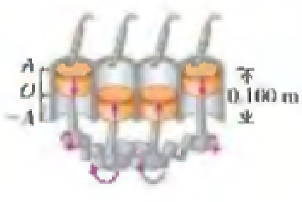
13.61. A sinusoidally varying driving force is applied to a damped harmonic oscillator. (a) What are the units of the damping constant b? (b) Show that the quantity $\sqrt{km}$ has the same units as b. (c) In terms of F_{max} and k, what is the amplitude for $\omega_d = \sqrt{k/m}$ when (i) $b = 0.2\sqrt{km}$ and (ii) $b = 0.4\sqrt{km}$? Compare your results to Fig. 13.28.

13.62. An experimental package and its support structure, which are to be placed on board the International Space Station, act as an underdamped spring–mass system with a force constant of 2.1×10^6 N/m and mass 108 kg. A NASA requirement is that resonance for forced oscillations not occur for any frequency below 35 Hz. Does this package meet the requirement?

Problems

13.63. SHM in a Car Engine. The motion of the piston of an automobile engine (Fig. 13.35) is approximately simple harmonic. (a) If the stroke of an engine (twice the amplitude) is 0.100 m and the engine runs at 3500 rev/min, compute the acceleration of the piston at the endpoint of its stroke. (b) If the piston has mass 0.450 kg, what net force must be exerted on it at this point? (c) What are the speed and kinetic energy of the piston at the midpoint of its stroke? (d) What average power is required to accelerate the piston from rest to the speed found in part (c)? (e) If the engine runs at 7000 rev/min, what are the answers to parts (b), (c), and (d)?

Figure 13.35 Problem 13.63.



13.64. Four passengers with combined mass 250 kg compress the springs of a car with worn-out shock absorbers by 4.00 cm when they get in. Model the car and passengers as a single body on a single ideal spring. If the loaded car has a period of vibration of 1.08 s, what is the period of vibration of the empty car?

13.65. A glider is oscillating in SHM on an air track with an amplitude A_1. You slow it so that its amplitude is halved. What happens to its (a) period, frequency, and angular frequency; (b) total mechanical energy; (c) maximum speed; (d) speed at $x = \pm A_1/4$; (e) potential and kinetic energies at $x = \pm A_1/4$?

13.66. A child with poor table manners is sliding his 250-g dinner plate back and forth in SHM with an amplitude of 0.100 m on a horizontal surface. At a point 0.060 m away from equilibrium, the speed of the plate is 0.300 m/s. (a) What is the period? (b) What is the displacement when the speed is 0.160 m/s? (c) In the center of the dinner plate is a 10.0-g carrot slice. If the carrot slice is just on the verge of slipping at the endpoint of the path, what is the coefficient of static friction between the carrot slice and the plate?

13.67. A 1.50-kg, horizontal, uniform tray is attached to a vertical ideal spring of force constant 185 N/m and a 275-g metal ball is in the tray. The spring is below the tray, so it can oscillate up-and-down. The tray is then pushed down 15.0 cm below its equilibrium point (call this point A) and released from rest. (a) How high above point A will the tray be when the metal ball leaves the tray? (*Hint:* This does *not* occur when the ball and tray reach their maximum speeds.) (b) How much time elapses between releasing the system at point A and the ball leaving the tray? (c) How fast is the ball moving just as it leaves the tray?

13.60. A block with mass M rests on a frictionless surface and is connected to a horizontal spring of force constant k. The other end of the spring is attached to a wall (Fig. 13.36). A second block with mass m rests on top of the first block. The coefficient of static

Figure 13.36 Problem 13.68.

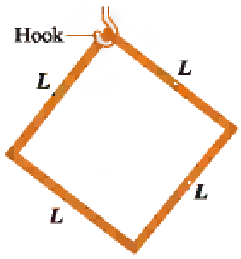
friction between the blocks is μ_s. Find the *maximum* amplitude of oscillation such that the top block will not slip on the bottom block.

13.69. A 10.0-kg mass is traveling to the right with a speed of 2.00 m/s on a smooth horizontal surface when it collides with and sticks to a second 10.0-kg mass that is initially at rest but is attached to a light spring with force constant 80.0 N/m. (a) Find the frequency, amplitude, and period of the subsequent oscillations. (b) How long does it take the system to return the first time to the position it had immediately after the collision?

13.70. A rocket is accelerating upward at 4.00 m/s^2 from the launchpad on the earth. Inside a small 1.50-kg ball hangs from the ceiling by a light 1.10-m wire. If the ball is displaced 8.50° from the vertical and released, find the amplitude and period of the resulting swings of this pendulum.

13.71. A square object of mass m is constructed of four identical uniform thin sticks, each of length L, attached together. This object is hung on a hook at its upper corner (Fig. 13.37). If it is rotated slightly to the left and then released, at what frequency will it swing back and forth?

Figure 13.37 Problem 13.71.



13.72. An object with mass 0.200 kg is acted on by an elastic restoring force with force constant 10.0 N/m. (a) Graph elastic potential energy U as a function of displacement x over a range of x from -0.300 m to $+0.300$ m. On your graph, let 1 cm = 0.05 J vertically and 1 cm = 0.05 m horizontally. The object is set into oscillation with an initial potential energy of 0.140 J and an initial kinetic energy of 0.060 J. Answer the following questions by referring to the graph. (b) What is the amplitude of oscillation? (c) What is the potential energy when the displacement is one-half the amplitude? (d) At what displacement are the kinetic and potential energies equal? (e) What is the value of the phase angle ϕ if the initial velocity is positive and the initial displacement is negative?

13.73. A 2.00-kg bucket containing 10.0 kg of water is hanging from a vertical ideal spring of force constant 125 N/m and oscillating up and down with an amplitude of 3.00 cm. Suddenly the bucket springs a leak in the bottom such that water drops out at a steady rate of 2.00 g/s. When the bucket is half full, find (a) the period of oscillation and (b) the rate at which the period is changing with respect to time. Is the period getting longer or shorter? (c) What is the shortest period this system can have?

13.74. A hanging wire is 1.80 m long. When a 60.0-kg steel ball is suspended from the wire, the wire stretches by 2.00 mm. If the ball is pulled down a small additional distance and released, at what frequency will it vibrate? Assume that the stress on the wire is less than the proportional limit (see Section 11.5).

13.75. A 5.00-kg partridge is suspended from a pear tree by an ideal spring of negligible mass. When the partridge is pulled down 0.100 m below its equilibrium position and released, it vibrates with a period of 4.20 s. (a) What is its speed as it passes through the equilibrium position? (b) What is its acceleration when it is 0.050 m above the equilibrium position? (c) When it is moving upward, how much time is required for it to move from a point 0.050 m below its equilibrium position to a point 0.050 m above it? (d) The motion of the partridge is stopped, and then it is removed from the spring. How much does the spring shorten?

13.76. A 0.0200-kg bolt moves with SHM that has an amplitude of 0.240 m and a period of 1.500 s. The displacement of the bolt is $+0.240$ m when $t = 0$. Compute (a) the displacement of the bolt when $t = 0.500$ s; (b) the magnitude and direction of the force acting on the bolt when $t = 0.500$ s; (c) the minimum time required for the bolt to move from its initial position to the point where $x = -0.180$ m; (d) the speed of the bolt when $x = -0.180$ m.

13.77. SHM of a Butcher's Scale. A spring of negligible mass and force constant $k = 400$ N/m is hung vertically, and a 0.200-kg pan is suspended from its lower end. A butcher drops a 2.2-kg steak onto the pan from a height of 0.40 m. The steak makes a totally inelastic collision with the pan and sets the system into vertical SHM. What are (a) the speed of the pan and steak immediately after the collision; (b) the amplitude of the subsequent motion; (c) the period of that motion?

13.78. A uniform beam is suspended horizontally by two identical vertical springs that are attached between the ceiling and each end of the beam. The beam has mass 225 kg, and a 175-kg sack of gravel sits on the middle of it. The beam is oscillating in SHM, with an amplitude of 40.0 cm and a frequency of 0.600 cycles/s. (a) The sack of gravel falls off the beam when the beam has its maximum upward displacement. What are the frequency and amplitude of the subsequent SHM of the beam? (b) If the gravel instead falls off when the beam has its maximum speed, what are the frequency and amplitude of the subsequent SHM of the beam?

13.79. On the planet Newtonia, a simple pendulum having a bob with mass 1.25 kg and a length of 185.0 cm takes 1.42 s, when released from rest, to swing through an angle of 12.5°, where it again has zero speed. The circumference of Newtonia is measured to be 51,400 km. What is the mass of the planet Newtonia?

13.80. A 40.0-N force stretches a vertical spring 0.250 m. (a) What mass must be suspended from the spring so that the system will oscillate with a period of 1.00 s? (b) If the amplitude of the motion is 0.050 m and the period is that specified in part (a), where is the object and in what direction is it moving 0.35 s after it has passed the equilibrium position, moving downward? (c) What force (magnitude and direction) does the spring exert on the object when it is 0.030 m below the equilibrium position, moving upward?

13.81. Don't Miss the Boat. While on a visit to Minnesota ("Land of 10,000 Lakes"), you sign up to take an excursion around one of the larger lakes. When you go to the dock where the 1500-kg boat is tied, you find that the boat is bobbing up and down in the waves, executing simple harmonic motion with amplitude 20 cm. The boat takes 3.5 s to make one complete up-and-down cycle. When the boat is at its highest point, its deck is at the same height as the stationary dock. As you watch the boat bob up and down, you (mass 60 kg) begin to feel a bit woozy, due in part to the previous night's dinner of lutefisk. As a result, you refuse to board the boat unless the level of the boat's deck is within 10 cm of the dock level. How much time do you have to board the boat comfortably during each cycle of up-and-down motion?

13.82. An interesting, though highly impractical example of oscillation is the motion of an object dropped down a hole that extends from one side of the earth, through its center, to the other side. With the assumption (not realistic) that the earth is a sphere of uniform density, prove that the motion is simple harmonic and find the period. [*Note:* The gravitational force on the object as a function of the object's distance r from the center of the earth was derived in Example 12.10 (Section 12.6). The motion is simple harmonic if the acceleration a_x and the displacement from equilibrium x are related by Eq. (13.8), and the period is then $T = 2\pi/\omega$.]

13.83. Two point masses m are held in place a distance d apart. Another point mass M is midway between them. M is then displaced a small distance x perpendicular to the line connecting the two fixed masses and released. (a) Show that the magnitude of the net gravitational force on M due to the fixed masses is given approximately by $F_{\text{net}} = \dfrac{16\,GmMx}{d^3}$ if $x \ll d$. What is the direction of this force? Is it a restoring force? (b) Show that the mass M will oscillate with an angular frequency of $(4/d)\sqrt{Gm/d}$ and period $\pi d/2\sqrt{d/Gm}$. (c) What would the period be if $m = 100$ kg and $d = 25.0$ cm? Does it seem that you could easily measure this period? What things prevent this experiment from easily being performed in an ordinary physics lab? (d) Will M oscillate if it is displaced from the center a small distance x toward either of the fixed masses? Why?

13.84. For a certain oscillator the net force on the body with mass m is given by $F_x = -cx^3$. (a) What is the potential energy function for this oscillator if we take $U = 0$ at $x = 0$? (b) One-quarter of a period is the time for the body to move from $x = 0$ to $x = A$. Calculate this time and hence the period. [*Hint:* Begin with Eq. (13.20), modified to include the potential-energy function you found in part (a), and solve for the velocity v_x as a function of x. Then replace v_x with dx/dt. Separate the variable by writing all fators containing x on one side and all factors containing t on the other side so that each side can be integrated. In the x-integral make the change of variable $u = x/A$. The resulting integral can be evaluated by numerical methods on a computer and has the value $\int_0^1 du/\sqrt{1-u^4} = 1.31$.] (c) According to the result you obtained in part (b), does the period depend on the amplitude A of the motion? Are the oscillations simple harmonic?

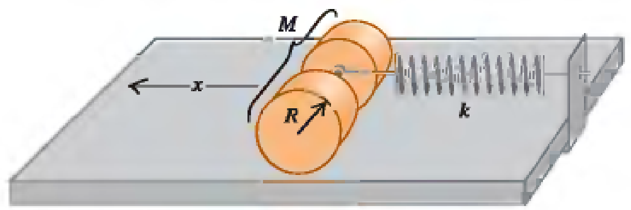
13.85. Consider the circle of reference shown in Fig. 13.6. The x-component of the velocity of Q is the velocity of P. Compute this component, and show that the velocity of P is as given by Eq. (13.15).

***13.86. Diatomic Molecule.** Two identical atoms in a diatomic molecule vibrate as harmonic oscillators. However, their center of mass, midway between them, remains at rest. (a) Show that at any instant, the momenta of the atoms relative to the center of mass are $\vec{p}$ and $-\vec{p}$. (b) Show that the total kinetic energy K of the two atoms at any instant is the same as that of a single object with mass $m/2$ with a momentum of magnitude p. (Use $K = p^2/2m$.) This result shows why $m/2$ should be used in the expression for f in Example 13.7 (Section 13.4). (c) If the atoms are not identical but have masses m_1 and m_2, show that the result of part (a) still holds and the single object's mass in part (b) is $m_1m_2/(m_1 + m_2)$. The quantity $m_1m_2/(m_1 + m_2)$ is called the *reduced mass* of the system.

***13.87.** An approximation for the potential energy of a KCl molecule is $U = A[(R_0^7/8r^8) - 1/r]$, where $R_0 = 2.67 \times 10^{-10}$ m and $A = 2.31 \times 10^{-28}$ J·m. Using this approximation: (a) Show that the radial component of the force on each atom is $F_r = A[(R_0^7/r^9) - 1/r^2]$. (b) Show that R_0 is the equilibrium separation. (c) Find the minimum potential energy. (d) Use $r = R_0 + x$ and the first two terms of the binomial theorem (Eq. 13.28) to show that $F_r \approx -(7A/R_0^3)x$, so that the molecule's force constant is $k = 7A/R_0^3$. (e) With both the K and Cl atoms vibrating in opposite directions on opposite sides of the molecule's center of mass, $m_1m_2/(m_1 + m_2) = 3.06 \times 10^{-26}$ kg is the mass to use in calculating the frequency (see Problem 13.86). Calculate the frequency of small-amplitude vibrations.

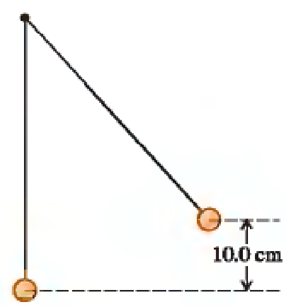
13.88. Two solid cylinders connected along their common axis by a short, light rod have radius R and total mass M and rest on a horizontal tabletop. A spring with force constant k has one end attached to a clamp and the other end attached to a frictionless ring at the center of mass of the cylinders (Fig. 13.38). The cylinders are pulled to the left a distance x, which stretches the spring, and released. There is sufficient friction between the tabletop and the cylinders for the cylinders to roll without slipping as they move back and forth on the end of the spring. Show that the motion of the center of mass of the cylinders is simple harmonic, and calculate its period in terms of M and k. [*Hint:* The motion is simple harmonic if a_x and x are related by Eq. (13.8), and the period then is $T = 2\pi/\omega$. Apply $\sum\tau_z = I_{\text{cm}}\alpha_z$ and $\sum F_x = Ma_{\text{cm-}x}$ to the cylinders in order to relate $a_{\text{cm-}x}$ and the displacement x of the cylinders from their equilibrium position.]

Figure 13.38 Problem 13.88.



13.89. In Fig. 13.39 the upper ball is released from rest, collides with the stationary lower ball, and sticks to it. The strings are both 50.0 cm long. The upper ball has mass 2.00 kg, and it is initially 10.0 cm higher than the lower ball, which has mass 3.00 kg. Find the frequency and maximum angular displacement of the motion after the collision.

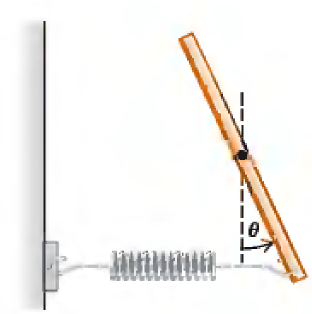
Figure 13.39 Problem 13.89.



13.90. T. rex. Model the leg of the *T. rex* in Example 13.10 (Section 13.6) as two uniform rods, each 1.55 m long, joined rigidly end to end. Let the lower rod have mass M and the upper rod mass $2M$. The composite object is pivoted about the top of the upper rod. Compute the oscillation period of this object for small-amplitude oscillations. Compare your result to that of Example 13.10.

13.91. A slender, uniform, metal rod with mass M is pivoted without friction about an axis through its midpoint and perpendicular to the rod. A horizontal spring with force constant k is attached to the lower end of the rod, with the other end of the spring attached to a rigid support. If the rod is displaced by a small angle Θ from the vertical (Fig. 13.40) and released, show that it moves in angular SHM and calculate the period. (*Hint:* Assume that the angle Θ is small enough for the approximations $\sin\Theta \approx \Theta$ and $\cos\Theta \approx 1$ to be valid. The motion is simple harmonic if $d^2\theta/dt^2 = -\omega^2\theta$, and the period is then $T = 2\pi/\omega$.)

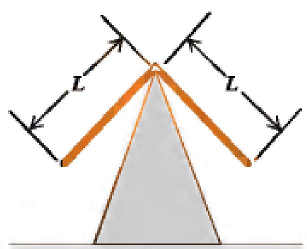
Figure 13.40 Problem 13.91.



13.92. The Silently Ringing Bell Problem. A large bell is hung from a wooden beam so it can swing back and forth with negligible friction. The center of mass of the bell is 0.60 m below the pivot, the bell has mass 34.0 kg, and the moment of inertia of the bell about an axis at the pivot is 18.0 kg·m². The clapper is a small, 1.8-kg mass attached to one end of a slender rod that has length L and negligible mass. The other end of the rod is attached to the inside of the bell so it can swing freely about the same axis as the bell. What should be the length L of the clapper rod for the bell to ring silently—that is, for the period of oscillation for the bell to equal that for the clapper?

13.93. Two identical thin rods, each with mass m and length L, are joined at right angles to form an L-shaped object. This object is balanced on top of a sharp edge (Fig. 13.41). If the L-shaped object is deflected slightly, it oscillates. Find the frequency of oscillation.

Figure 13.41 Problem 13.93.



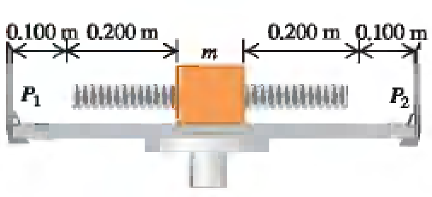
13.94. You want to construct a pendulum with a period of 4.00 s at a location where $g = 9.80\ \text{m/s}^2$. (a) What is the length of a *simple* pendulum having this period? (b) Suppose the pendulum must be mounted in a case that is not more than 0.50 m high. Can you devise a pendulum having a period of 4.00 s that will satisfy this requirement?

13.95. A uniform rod of length L oscillates through small angles about a point a distance x from its center. (a) Prove that its angular frequency is $\sqrt{gx/[(L^2/12) + x^2]}$. (b) Show that its maximum angular frequency occurs when $x = L/\sqrt{12}$. (c) What is the length of the rod if the maximum angular frequency is 2π rad/s?

Challenge Problems

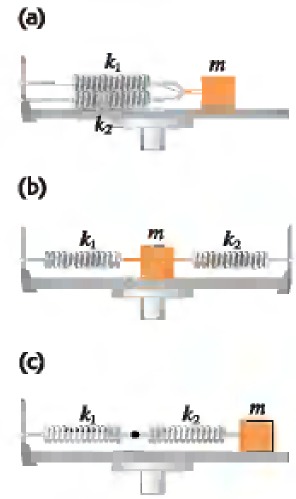
13.96. Two springs, each with unstretched length 0.200 m, but with different force constants k_1 and k_2, are attached to opposite ends of a block with mass m on a level, frictionless surface. The outer ends of the springs are now attached to two pins P_1 and P_2, 0.100 m from the original positions of the ends of the springs (Fig. 13.42). Let $k_1 = 2.00$ N/m, $k_2 = 6.00$ N/m, and $m = 0.100$ kg. (a) Find the length of each spring when the block is in its new equilibrium position after the springs have been attached to the pins. (b) Find the period of vibration of the block if it is slightly displaced from its new equilibrium position and released.

Figure 13.42 Challenge Problem 13.96.



13.97. The Effective Force Constant of Two Springs. Two springs with the same unstretched length, but different force constants k_1 and k_2 are attached to a block with mass m on a level, frictionless surface. Calculate the effective force constant k_{eff} in each of the three cases (a), (b), and (c) depicted in Fig. 13.43. (The effective force constant is defined by $\sum F_x = -k_{\text{eff}}x$.) (d) An object with mass m, suspended from a uniform spring with a force constant k, vibrates with a frequency f_1. When the spring is cut in half and the same object is suspended from one of the halves, the frequency is f_2. What is the ratio f_2/f_1?

Figure 13.43 Challenge Problem 13.97.



13.98. (a) What is the change ΔT in the period of a simple pendulum when the acceleration of gravity g changes by Δg? (*Hint:* The new period $T + \Delta T$ is obtained by substituting $g + \Delta g$ for g:

$$T + \Delta T = 2\pi\sqrt{\frac{L}{g + \Delta g}}$$

To obtain an approximate expression, expand the factor $(g + \Delta g)^{-1/2}$ using the binomial theorem (Appendix B) and keep only the first two terms:

$$(g + \Delta g)^{-1/2} = g^{-1/2} - \frac{1}{2}g^{-3/2}\Delta g + \cdots$$

The other terms contain higher powers of Δg and are very small if Δg is small.) Express your result as the *fractional* change in period $\Delta T/T$ in terms of the fractional change $\Delta g/g$. (b) A pendulum clock keeps correct time at a point where $g = 9.8000\ \text{m/s}^2$, but is found to lose 4.0 s each day at a higher elevation. Use the result of part (a) to find the approximate value of g at this new location.

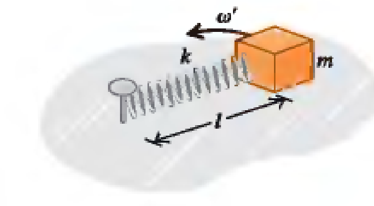
13.99. A Spring with Mass. The preceding problems in this chapter have assumed that the springs had negligible mass. But of course no spring is completely massless. To find the effect of the spring's mass, consider a spring with mass M, equilibrium length L_0, and spring constant k. When stretched or compressed to a length L, the potential energy is $\frac{1}{2}kx^2$, where $x = L - L_0$. (a) Consider a spring, as described above, that has one end fixed and the other end moving with speed v. Assume that the speed of points along the length of the spring varies linearly with distance l from the fixed end. Assume also that the mass M of the spring is distributed uniformly along the length of the spring. Calculate the kinetic energy of the spring in terms of M and v. (*Hint:* Divide the spring into pieces of length dl; find the speed of each piece in terms of l, v, and L; find the mass of each piece in terms of dl, M, and L; and integrate from 0 to L. The result is *not* $\frac{1}{2}Mv^2$, since not all of the spring moves with the same speed.) (b) Take the time derivative of the conservation of energy equation, Eq. (13.21), for a mass m moving on the end of a *massless* spring. By comparing your results to Eq. (13.8), which defines ω, show that the angular frequency of oscillation is $\omega = \sqrt{k/m}$. (c) Apply the procedure of part (b) to obtain the angular frequency of oscillation ω of the spring considered in part (a). If the *effective mass* M' of the spring is defined by $\omega = \sqrt{k/M'}$, what is M' in terms of M?

13.100. A uniform, 1.00-m stick hangs from a horizontal axis at one end and oscillates as a physical pendulum. An object of small dimensions and with mass equal to that of the meter stick can be clamped to the stick at a distance y below the axis. Let T represent the period of the system with the body attached and T_0 the period of the meter stick alone. (a) Find the ratio T/T_0. Evaluate your expression for y ranging from 0 to 1.0 m in steps of 0.1 m, and graph T/T_0 versus y. (b) Is there any value of y, other than $y = 0$, for which $T = T_0$? If so, find it and explain why the period is unchanged when y has this value.

13.101. You measure the period of a physical pendulum about one pivot point to be T. Then you find another pivot point on the opposite side of the center of mass that gives the same period. The two points are separated by a distance L. Use the parallel-axis theorem to show that $g = L(2\pi/T)^2$. (This result shows a way that you can measure g without knowing the mass or any moments of inertia of the physical pendulum.)

13.102. Resonance in a Mechanical System. A mass m is attached to one end of a massless spring with a force constant k and an unstretched length l_0. The other end of the spring is free to turn about a nail driven into a frictionless, horizontal surface (Fig. 13.44). The mass is made to revolve in a circle with an angular frequency of revolution ω'. (a) Calculate the length l of the spring as a function of ω'. (b) What happens to the result in part (a) when ω' approaches the natural frequency $\omega = \sqrt{k/m}$ of the mass-spring system? (If your result bothers you, remember that massless springs and frictionless surfaces don't exist as such, but are only approximate descriptions of real springs and surfaces. Also, Hooke's law is only an approximation of the way real springs behave; the greater the elongation of the spring, the greater the deviation from Hooke's law.)

Figure 13.44 Challenge Problem 13.102.



***13.103. Vibration of a Covalently Bonded Molecule.** Many diatomic (two-atom) molecules are bound together by *covalent bonds* that are much stronger than the van der Waals interaction. Examples include H_2, O_2, and N_2. Experiment shows that for many such molecules, the interaction can be described by a force of the form

$$F_r = A\left[e^{-2b(r-R_0)} - e^{-b(r-R_0)}\right]$$

where A and b are positive constants, r is the center-to-center separation of the atoms, and R_0 is the equilibrium separation. For the hydrogen molecule (H_2), $A = 2.97 \times 10^{-8}$ N, $b = 1.95 \times 10^{10}\ \text{m}^{-1}$, and $R_0 = 7.4 \times 10^{-11}$ m. Find the force constant for small oscillations around equilibrium. (*Hint:* Use the expansion for e^x given in Appendix B.) Compare your result to the value given in Exercise 13.40.

14 FLUID MECHANICS

? This shark must swim constantly to keep from sinking to the bottom of the ocean, yet the orange tropical fish can remain at the same level in the water with little effort. Why is there a difference?

LEARNING GOALS

By studying this chapter, you will learn:

- The meaning of the density of a material and the average density of a body.
- What is meant by the pressure in a fluid, and how it is measured.
- How to calculate the buoyant force that a fluid exerts on a body immersed in it.
- The significance of laminar versus turbulent fluid flow, and how the speed of flow in a tube depends on the tube size.
- How to use Bernoulli's equation to relate pressure and flow speed at different points in certain types of flow.

Fluids play a vital role in many aspects of everyday life. We drink them, breathe them, swim in them. They circulate through our bodies and control our weather. Airplanes fly through them; ships float in them. A fluid is any substance that can flow; we use the term for both liquids and gases. We usually think of a gas as easily compressed and a liquid as nearly incompressible, although there are exceptional cases.

We begin our study with **fluid statics,** the study of fluids at rest in equilibrium situations. Like other equilibrium situations, it is based on Newton's first and third laws. We will explore the key concepts of density, pressure, and buoyancy. **Fluid dynamics,** the study of fluids in motion, is much more complex; indeed, it is one of the most complex branches of mechanics. Fortunately, we can analyze many important situations using simple idealized models and familiar principles such as Newton's laws and conservation of energy. Even so, we will barely scratch the surface of this broad and interesting topic.

14.1 Density

An important property of any material is its **density,** defined as its mass per unit volume. A homogeneous material such as ice or iron has the same density throughout. We use ρ (the Greek letter rho) for density. If a mass m of homogeneous material has volume V, the density ρ is

$$\rho = \frac{m}{V} \qquad \text{(definition of density)} \qquad (14.1)$$

Two objects made of the same material have the same density even though they may have different masses and different volumes. That's because the *ratio* of mass to volume is the same for both objects (Fig. 14.1).

14.1 Two objects with different masses and different volumes but the same density.

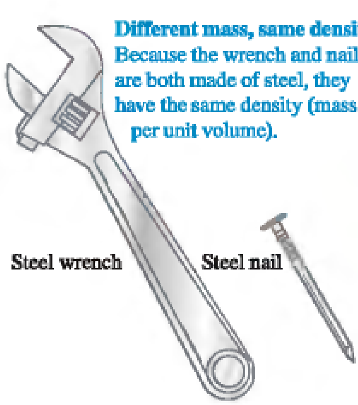


Table 14.1 Densities of Some Common Substances

Material	Density (kg/m^3)*	Material	Density (kg/m^3)*
Air (1 atm, 20°C)	1.20	Iron, steel	7.8×10^3
Ethanol	0.81×10^3	Brass	8.6×10^3
Benzene	0.90×10^3	Copper	8.9×10^3
Ice	0.92×10^3	Silver	10.5×10^3
Water	1.00×10^3	Lead	11.3×10^3
Seawater	1.03×10^3	Mercury	13.6×10^3
Blood	1.06×10^3	Gold	19.3×10^3
Glycerine	1.26×10^3	Platinum	21.4×10^3
Concrete	2×10^3	White dwarf star	10^{10}
Aluminum	2.7×10^3	Neutron star	10^{18}

*To obtain the densities in grams per cubic centimeter, simply divide by 10^3.

The SI unit of density is the kilogram per cubic meter $(1\ kg/m^3)$. The cgs unit, the gram per cubic centimeter $(1\ g/cm^3)$, is also widely used:

$$1\ g/cm^3 = 1000\ kg/m^3$$

The densities of several common substances at ordinary temperatures are given in Table 14.1. Note the wide range of magnitudes (Fig. 14.2). The densest material found on earth is the metal osmium $(\rho = 22{,}500\ kg/m^3)$, but its density pales by comparison to the densities of exotic astronomical objects such as white dwarf stars and neutron stars.

14.2 The price of gold is quoted by weight (say, in dollars per ounce). Because gold is one of the densest of the metals, a fortune in gold can be stored in a small volume.

The **specific gravity** of a material is the ratio of its density to the density of water at 4.0°C, $1000\ kg/m^3$; it is a pure number without units. For example, the specific gravity of aluminum is 2.7. "Specific gravity" is a poor term, since it has nothing to do with gravity; "relative density" would have been better.

The density of some materials varies from point to point within the material. One example is the material of the human body, which includes low-density fat (about $940\ kg/m^3$) and high-density bone (from 1700 to $2500\ kg/m^3$). Two others are the earth's atmosphere (which is less dense at high altitudes) and oceans (which are denser at greater depths). For these materials, Eq. (14.1) describes the **average density.** In general, the density of a material depends on environmental factors such as temperature and pressure.

Measuring density is an important analytical technique. For example, we can determine the charge condition of a storage battery by measuring the density of its electrolyte, a sulfuric acid solution. As the battery discharges, the sulfuric acid (H_2SO_4) combines with lead in the battery plates to form insoluble lead sulfate $(PbSO_4)$, decreasing the concentration of the solution. The density decreases from about $1.30 \times 10^3\ kg/m^3$ for a fully charged battery to $1.15 \times 10^3\ kg/m^3$ for a discharged battery.

Another automotive example is permanent-type antifreeze, which is usually a solution of ethylene glycol $(\rho = 1.12 \times 10^3\ kg/m^3)$ and water. The freezing point of the solution depends on the glycol concentration, which can be determined by measuring the specific gravity. Such measurements can be performed by using a device called a hydrometer, which we'll discuss in Section 14.3.

Example 14.1 The weight of a roomful of air

Find the mass and weight of the air in a living room at 20°C with a 4.0 m × 5.0 m floor and a ceiling 3.0 m high. What are the mass and weight of an equal volume of water?

SOLUTION

IDENTIFY: We assume that air is homogeneous, so that the density is the same throughout the room. (It is true that air is less

Continued

dense at high elevations than near sea level. The variation in density over the 3.0-m height of the room, however, is negligibly small; see Section 14.2.)

SET UP: We use Eq. (14.1) to relate the mass (the target variable) to the volume (which we calculate from the dimensions of the room) and the density (from Table 14.1).

EXECUTE: The volume of the room is $V = (3.0\ \text{m})(4.0\ \text{m}) \times (5.0\ \text{m}) = 60\ \text{m}^3$. The mass m_{air} of air is given by Eq. (14.1):

$$m_{\text{air}} = \rho_{\text{air}} V = (1.20\ \text{kg/m}^3)(60\ \text{m}^3) = 72\ \text{kg}$$

The weight of the air is

$$w_{\text{air}} = m_{\text{air}} g = (72\ \text{kg})(9.8\ \text{m/s}^2) = 700\ \text{N} = 160\ \text{lb}$$

The mass of an equal volume of water is

$$m_{\text{water}} = \rho_{\text{water}} V = (1000\ \text{kg/m}^3)(60\ \text{m}^3) = 6.0 \times 10^4\ \text{kg}$$

The weight is

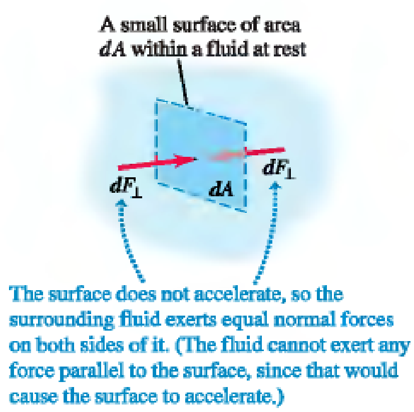
$$\begin{aligned} w_{\text{water}} = m_{\text{water}} g &= (6.0 \times 10^4\ \text{kg})(9.8\ \text{m/s}^2) \\ &= 5.9 \times 10^5\ \text{N} = 1.3 \times 10^5\ \text{lb} = 66\ \text{tons} \end{aligned}$$

EVALUATE: A roomful of air weighs about the same as an average adult! Water is nearly a thousand times denser than air, and its mass and weight are larger by the same factor. The weight of a roomful of water would collapse the floor of an ordinary house.

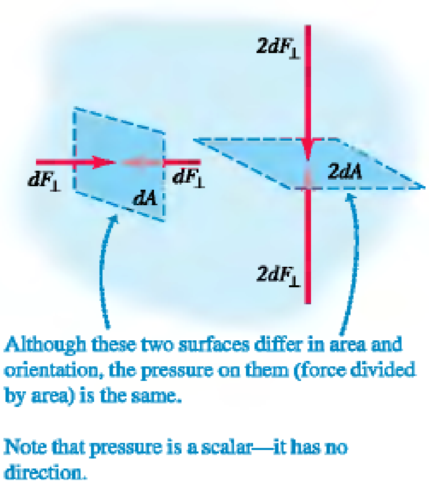
Test Your Understanding of Section 14.1 Rank the following objects in order from highest to lowest average density: (i) mass 4.00 kg, volume $1.60 \times 10^{-3}\ \text{m}^3$; (ii) mass 8.00 kg, volume $1.60 \times 10^{-3}\ \text{m}^3$; (iii) mass 8.00 kg, volume $3.20 \times 10^{-3}\ \text{m}^3$; (iv) mass 2560 kg, volume $0.640\ \text{m}^3$; (v) mass 2560 kg, volume $1.28\ \text{m}^3$.

14.2 Pressure in a Fluid

14.3 Forces acting on a small surface within a fluid at rest.



14.4 The pressure on either side of a surface is force divided by area. Pressure is a scalar with units of newtons per square meter. By contrast, force is a vector with units of newtons.



When a fluid (either liquid or gas) is at rest, it exerts a force perpendicular to any surface in contact with it, such as a container wall or a body immersed in the fluid. This is the force that you feel pressing on your legs when you dangle them in a swimming pool. While the fluid as a whole is at rest, the molecules that make up the fluid are in motion; the force exerted by the fluid is due to molecules colliding with their surroundings.

If we think of an imaginary surface *within* the fluid, the fluid on the two sides of the surface exerts equal and opposite forces on the surface. (Otherwise, the surface would accelerate and the fluid would not remain at rest.) Consider a small surface of area dA centered on a point in the fluid; the normal force exerted by the fluid on each side is $dF_\perp$ (Fig. 14.3). We define the **pressure** p at that point as the normal force per unit area—that is, the ratio of $dF_\perp$ to dA (Fig. 14.4):

$$p = \frac{dF_\perp}{dA} \qquad \text{(definition of pressure)} \tag{14.2}$$

If the pressure is the same at all points of a finite plane surface with area A, then

$$p = \frac{F_\perp}{A} \tag{14.3}$$

where $F_\perp$ is the net normal force on one side of the surface. The SI unit of pressure is the **pascal,** where

$$1\ \text{pascal} = 1\ \text{Pa} = \text{N/m}^2$$

We introduced the pascal in Chapter 11. Two related units, used principally in meteorology, are the *bar,* equal to 10^5 Pa, and the *millibar,* equal to 100 Pa.

Atmospheric pressure p_a is the pressure of the earth's atmosphere, the pressure at the bottom of this sea of air in which we live. This pressure varies with weather changes and with elevation. Normal atmospheric pressure at sea level (an average value) is 1 *atmosphere* (atm), defined to be exactly 101,325 Pa. To four significant figures,

$$\begin{aligned} (p_a)_{\text{av}} &= 1\ \text{atm} = 1.013 \times 10^5\ \text{Pa} \\ &= 1.013\ \text{bar} = 1013\ \text{millibar} = 14.70\ \text{lb/in.}^2 \end{aligned}$$

CAUTION **Don't confuse pressure and force** In everyday language the words "pressure" and "force" mean pretty much the same thing. In fluid mechanics, however, these words describe distinct quantities with different characteristics. Fluid pressure acts perpendicular to any surface in the fluid, no matter how that surface is oriented (Fig. 14.4). Hence pressure has no intrinsic direction of its own; it's a scalar. By contrast, force is a vector with a definite direction. Remember, too, that pressure is force per unit area. As Fig. 14.4 shows, a surface with twice the area has twice as much force exerted on it by the fluid, so the pressure is the same. ❙

Example 14.2 The force of air

In the room described in Example 14.1, what is the total downward force on the surface of the floor due to air pressure of 1.00 atm?

SOLUTION

IDENTIFY: This example uses the relationship among the pressure of a fluid (in this case, air), the normal force exerted by the fluid, and the area over which that force acts. In this situation the surface of the floor is horizontal, so the force exerted by the air is vertical (downward).

SET UP: The pressure is uniform, so we use Eq. (14.3) to determine the force $F_\perp$ from the pressure and area.

EXECUTE: The floor area is $A = (4.0\text{ m})(5.0\text{ m}) = 20\text{ m}^2$, so from Eq. (14.3) the total downward force is

$$\begin{aligned} F_\perp &= pA = (1.013 \times 10^5\text{ N/m}^2)(20\text{ m}^2) \\ &= 2.0 \times 10^6\text{ N} = 4.6 \times 10^5\text{ lb} = 230\text{ tons} \end{aligned}$$

EVALUATE: As in Example 14.1, this should be more than enough force to collapse the floor. Yet it doesn't collapse, because there is an *upward* force of equal magnitude on the underside of the floor. If the house has a basement, this upward force is exerted by the air underneath the floor. In this case, if we neglect the thickness of the floor, the *net* force due to air pressure is zero.

Pressure, Depth, and Pascal's Law

If the weight of the fluid can be neglected, the pressure in a fluid is the same throughout its volume. We used that approximation in our discussion of bulk stress and strain in Section 11.4. But often the fluid's weight is *not* negligible. Atmospheric pressure is less at high altitude than at sea level, which is why an airplane cabin has to be pressurized when flying at 35,000 feet. When you dive into deep water, your ears tell you that the pressure increases rapidly with increasing depth below the surface.

We can derive a general relationship between the pressure p at any point in a fluid at rest and the elevation y of the point. We'll assume that the density ρ has the same value throughout the fluid (that is, the density is *uniform*), as does the acceleration due to gravity g. If the fluid is in equilibrium, every volume element is in equilibrium. Consider a thin element of fluid with thickness dy (Fig. 14.5a). The bottom and top surfaces each have area A, and they are at elevations y and $y + dy$ above some reference level where $y = 0$. The volume of the fluid element is $dV = A\,dy$, its mass is $dm = \rho\,dV = \rho A\,dy$, and its weight is $dw = dm\,g = \rho g A\,dy$.

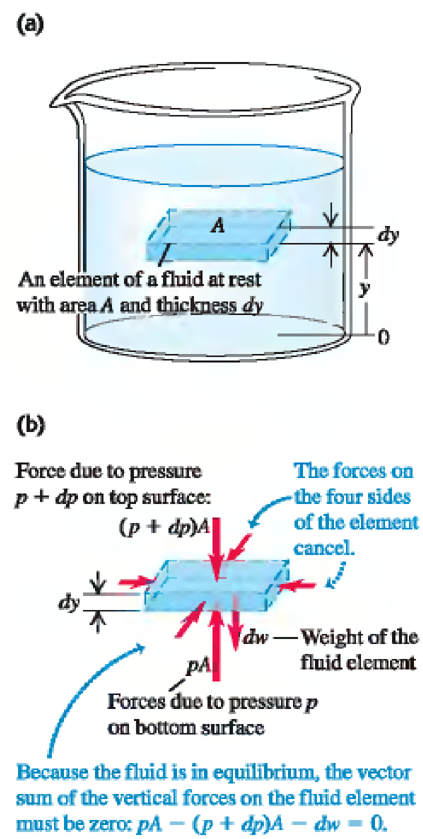
What are the other forces on this fluid element (Fig 14.5b)? Call the pressure at the bottom surface p; the total y-component of upward force on this surface is pA. The pressure at the top surface is $p + dp$, and the total y-component of (downward) force on the top surface is $-(p + dp)A$. The fluid element is in equilibrium, so the total y-component of force, including the weight and the forces at the bottom and top surfaces, must be zero:

$$\sum F_y = 0 \qquad \text{so} \qquad pA - (p + dp)A - \rho g A\,dy = 0$$

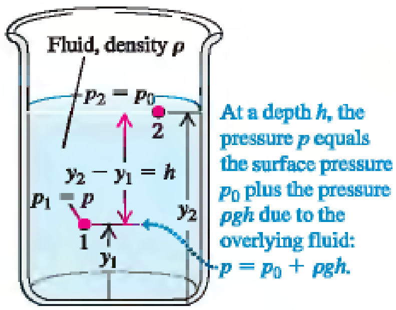
When we divide out the area A and rearrange, we get

$$\frac{dp}{dy} = -\rho g \tag{14.4}$$

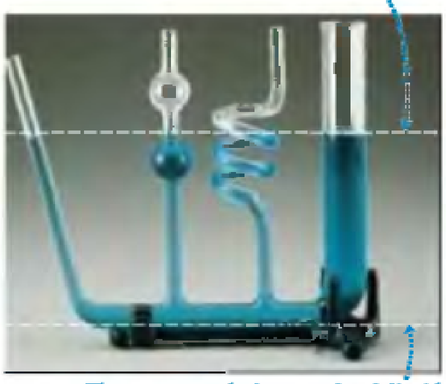
14.5 The forces on an element of fluid in equilibrium.



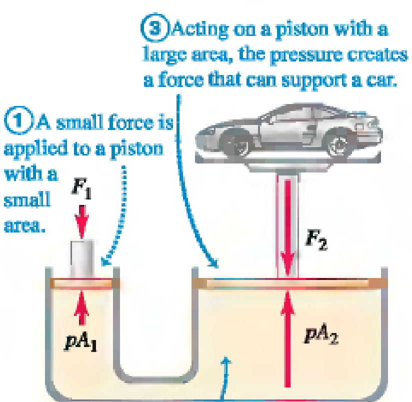
14.6 How pressure varies with depth in a fluid with uniform density.



14.7 Each fluid column has the same height, no matter what its shape.



14.8 The hydraulic lift is an application of Pascal's law. The size of the fluid-filled container is exaggerated for clarity.



This equation shows that when y increases, p decreases; that is, as we move upward in the fluid, pressure decreases, as we expect. If p_1 and p_2 are the pressures at elevations y_1 and y_2, respectively, and if ρ and g are constant, then

$$p_2 - p_1 = -\rho g(y_2 - y_1) \qquad \text{(pressure in a fluid of uniform density)} \qquad (14.5)$$

It's often convenient to express Eq. (14.5) in terms of the *depth* below the surface of a fluid (Fig. 14.6). Take point 1 at any level in the fluid and let p represent the pressure at this point. Take point 2 at the *surface* of the fluid, where the pressure is p_0 (subscript zero for zero depth). The depth of point 1 below the surface is $h = y_2 - y_1$, and Eq. (14.5) becomes

$$p_0 - p = -\rho g(y_2 - y_1) = -\rho gh \qquad \text{or}$$

$$p = p_0 + \rho gh \qquad \text{(pressure in a fluid of uniform density)} \qquad (14.6)$$

The pressure p at a depth h is greater than the pressure p_0 at the surface by an amount ρgh. Note that the pressure is the same at any two points at the same level in the fluid. The *shape* of the container does not matter (Fig. 14.7).

Equation (14.6) shows that if we increase the pressure p_0 at the top surface, possibly by using a piston that fits tightly inside the container to push down on the fluid surface, the pressure p at any depth increases by exactly the same amount. This fact was recognized in 1653 by the French scientist Blaise Pascal (1623–1662) and is called *Pascal's law.*

Pascal's law: Pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid and the walls of the containing vessel.

The hydraulic lift shown schematically in Fig. 14.8 illustrates Pascal's law. A piston with small cross-sectional area A_1 exerts a force F_1 on the surface of a liquid such as oil. The applied pressure $p = F_1/A_1$ is transmitted through the connecting pipe to a larger piston of area A_2. The applied pressure is the same in both cylinders, so

$$p = \frac{F_1}{A_1} = \frac{F_2}{A_2} \qquad \text{and} \qquad F_2 = \frac{A_2}{A_1}F_1 \qquad (14.7)$$

The hydraulic lift is a force-multiplying device with a multiplication factor equal to the ratio of the areas of the two pistons. Dentist's chairs, car lifts and jacks, many elevators, and hydraulic brakes all use this principle.

For gases the assumption that the density ρ is uniform is realistic only over short vertical distances. In a room with a ceiling height of 3.0 m filled with air of uniform density 1.2 kg/m^3, the difference in pressure between floor and ceiling, given by Eq. (14.6), is

$$\rho gh = (1.2\ \text{kg/m}^3)(9.8\ \text{m/s}^2)(3.0\ \text{m}) = 35\ \text{Pa}$$

or about 0.00035 atm, a very small difference. But between sea level and the summit of Mount Everest (8882 m) the density of air changes by nearly a factor of 3, and in this case we cannot use Eq. (14.6). Liquids, by contrast, are nearly incompressible, and it is usually a very good approximation to regard their density as independent of pressure. A pressure of several hundred atmospheres will cause only a few percent increase in the density of most liquids.

Absolute Pressure and Gauge Pressure

If the pressure inside a car tire is equal to atmospheric pressure, the tire is flat. The pressure has to be *greater* than atmospheric to support the car, so the significant quantity is the *difference* between the inside and outside pressures. When we say that the pressure in a car tire is "32 pounds" (actually 32 lb/in.^{2}, equal to 220 kPa or 2.2×10^5 Pa), we mean that it is *greater* than atmospheric pressure

$(14.7\ \text{lb/in.}^2$ or $1.01 \times 10^5\ \text{Pa})$ by this amount. The *total* pressure in the tire is then 47 lb/in.^2 or 320 kPa. The excess pressure above atmospheric pressure is usually called **gauge pressure,** and the total pressure is called **absolute pressure.** Engineers use the abbreviations psig and psia for "pounds per square inch gauge" and "pounds per square inch absolute," respectively. If the pressure is *less* than atmospheric, as in a partial vacuum, the gauge pressure is negative.

Example 14.3 Finding absolute and gauge pressures

A storage tank 12.0 m deep is filled with water. The top of the tank is open to the air. What is the absolute pressure at the bottom of the tank? The gauge pressure?

SOLUTION

IDENTIFY: Water is nearly incompressible. (Imagine trying to use a piston to compress a cylinder full of water—you couldn't do it!) Hence we can treat it as a fluid of uniform density.

SET UP: The level of the top of the tank corresponds to point 2 in Fig. 14.6, and the level of the bottom of the tank corresponds to point 1. Hence our target variable is p in Eq. (14.6). We are told that $h = 12.0$ m, and since the top of the tank is open to the atmosphere, p_0 equals 1 atm $= 1.01 \times 10^{-5}$ Pa.

EXECUTE: From Eq. (14.6), the absolute pressure is

$$\begin{aligned} p &= p_0 + \rho g h \\ &= (1.01 \times 10^5\ \text{Pa}) + (1000\ \text{kg/m}^3)(9.80\ \text{m/s}^2)(12.0\ \text{m}) \\ &= 2.19 \times 10^5\ \text{Pa} = 2.16\ \text{atm} = 31.8\ \text{lb/in.}^2 \end{aligned}$$

The gauge pressure is

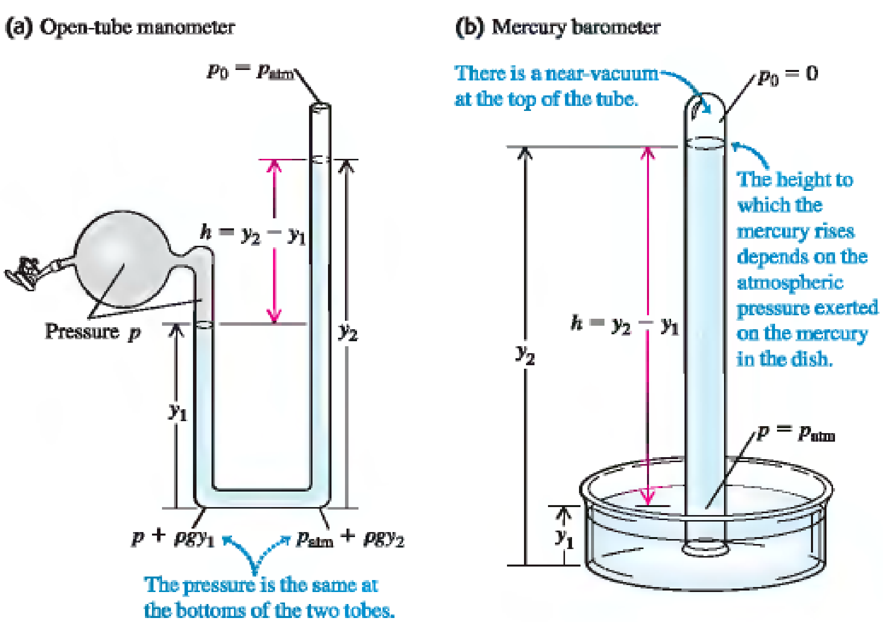
$$\begin{aligned} p - p_0 &= (2.19 - 1.01) \times 10^5\ \text{Pa} \\ &= 1.18 \times 10^5\ \text{Pa} = 1.16\ \text{atm} = 17.1\ \text{lb/in.}^2 \end{aligned}$$

EVALUATE: If such a tank has a pressure gauge, it is usually calibrated to read gauge pressure rather than absolute pressure. As we have mentioned, the variation in *atmospheric* pressure over a height of a few meters is negligibly small.

Pressure Gauges

The simplest pressure gauge is the open-tube *manometer* (Fig. 14.9a). The U-shaped tube contains a liquid of density ρ, often mercury or water. The left end of the tube is connected to the container where the pressure p is to be measured, and the right end is open to the atmosphere at pressure $p_0 = p_{\text{atm}}$. The pressure at the bottom of the tube due to the fluid in the left column is $p + \rho g y_1$, and the pressure at the bottom due to the fluid in the right column is $p_{\text{atm}} + \rho g y_2$. These pressures are measured at the same level, so they must be equal:

$$\begin{aligned} p + \rho g y_1 &= p_{\text{atm}} + \rho g y_2 \\ p - p_{\text{atm}} &= \rho g (y_2 - y_1) = \rho g h \end{aligned} \tag{14.8}$$



14.9 Two types of pressure gauge.

In Eq. (14.8), p is the *absolute pressure,* and the difference $p - p_{atm}$ between absolute and atmospheric pressure is the gauge pressure. Thus the gauge pressure is proportional to the difference in height $h = y_2 - y_1$ of the liquid columns.

Another common pressure gauge is the **mercury barometer.** It consists of a long glass tube, closed at one end, that has been filled with mercury and then inverted in a dish of mercury (Fig. 14.9b). The space above the mercury column contains only mercury vapor; its pressure is negligibly small, so the pressure p_0 at the top of the mercury column is practically zero. From Eq. (14.6),

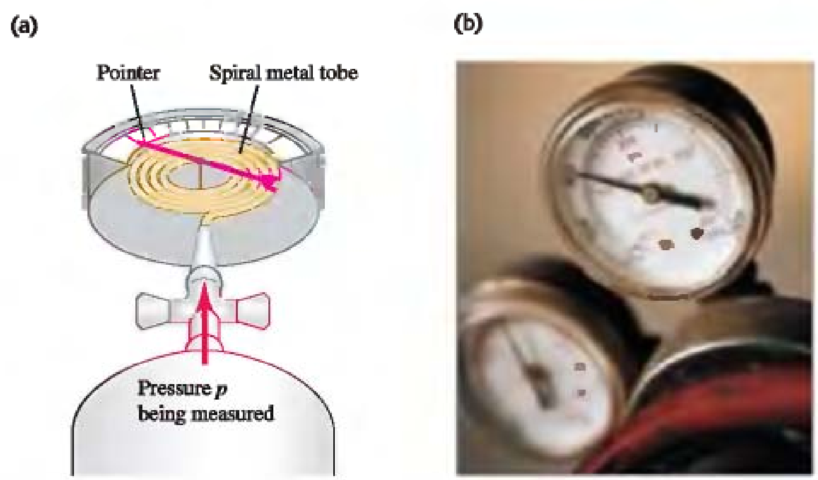
$$p_a = p = 0 + \rho g(y_2 - y_1) = \rho g h \qquad (14.9)$$

Thus the mercury barometer reads the atmospheric pressure p_{atm} directly from the height of the mercury column.

Pressures are often described in terms of the height of the corresponding mercury column, as so many "inches of mercury" or "millimeters of mercury" (abbreviated mm Hg). A pressure of 1 mm Hg is called *1 torr,* after Evangelista Torricelli, inventor of the mercury barometer. But these units depend on the density of mercury, which varies with temperature, and on the value of g, which varies with location, so the pascal is the preferred unit of pressure.

One common type of blood-pressure gauge, called a *sphygmomanometer*, uses a mercury-filled manometer. Blood-pressure readings, such as 130/80, give the maximum and minimum gange pressures in the arteries, measured in mm Hg or torr. Blood pressure varies with height; the standard reference point is the upper arm, level with the heart.

Many types of pressure gauges use a flexible sealed vessel (Fig. 14.10). A change in the pressure either inside or outside the vessel causes a change in its dimensions. This change is detected optically, electrically, or mechanically.



14.10 **(a)** A Bourdon pressure gauge. When the pressure inside the spiral metal tube increases, the tube straightens out a little, deflecting the attached pointer. **(b)** A Bourdon-type pressure gauge used on a compressed-gas tank.

Example 14.4 A tale of two fluids

A manometer tube is partially filled with water. Oil (which does not mix with water and has a lower density than water) is then poured into the left arm of the tube until the oil–water interface is at the midpoint of the tube. Both arms of the tube are open to the air. Find a relationship between the heights h_{oil} and h_{water}.

SOLUTION

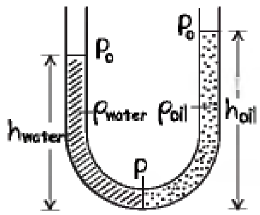
IDENTIFY: The relationship between pressure and depth in a fluid applies only to fluids of uniform density. Hence we can't write a single equation for the oil and the water together. But we can write a pressure–depth relationship for each fluid separately. Note that both fluid columns have the same pressure at the bottom (where they are in contact and in equilibrium, so the pressures must be the same) and at the top (where both are in contact with and in equilibrium with the air).

SET UP: Figure 14.11 shows our sketch. We let p_0 be atmospheric pressure and let p be the pressure at the bottom of the tube. The densities of the two fluids are ρ_{water} and ρ_{oil} (which is less than ρ_{water}). We use Eq. (14.6) for each fluid.

EXECUTE: For the two fluids, Eq. (14.6) becomes

$$p = p_0 + \rho_{\text{water}} g h_{\text{water}}$$
$$p = p_0 + \rho_{\text{oil}} g h_{\text{oil}}$$

14.11 Our sketch for this problem.



Since the pressure p at the bottom of the tube is the same for both fluids, we set these two expressions equal to each other and solve for h_{oil} in terms of h_{water}. You can show that the result is

$$h_{\text{oil}} = \frac{\rho_{\text{water}}}{\rho_{\text{oil}}} h_{\text{water}}$$

EVALUATE: Since oil is less dense than water, the ratio $\rho_{\text{water}}/\rho_{\text{oil}}$ is greater than unity and h_{oil} is greater than h_{water} (as shown in Fig. 14.11). That is, a greater height of low-density oil is needed to produce the same pressure p at the bottom of the tube.

Test Your Understanding of Section 14.2 Mercury is less dense at high temperatures than at low temperatures. Suppose you move a mercury barometer from the cold interior of a tightly sealed refrigerator to outdoors on a hot summer day. You find that the column of mercury remains at the same height in the tube. Compared to the air pressure inside the refrigerator, is the air pressure outdoors (i) higher, (ii) lower, or (iii) the same? (Ignore the very small change in the dimensions of the glass tube due to the temperature change.)

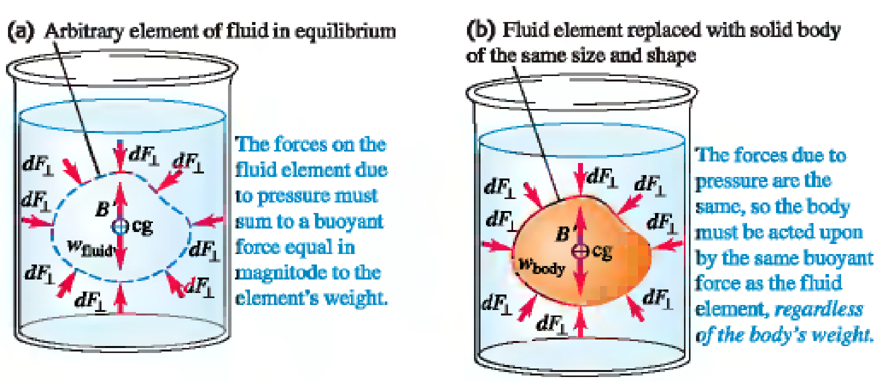
14.3 Buoyancy

Buoyancy is a familiar phenomenon: A body immersed in water seems to weigh less than when it is in air. When the body is less dense than the fluid, it floats. The human body usually floats in water, and a helium-filled balloon floats in air.

Archimedes's principle states: When a body is completely or partially immersed in a fluid, the fluid exerts an upward force on the body equal to the weight of the fluid displaced by the body.

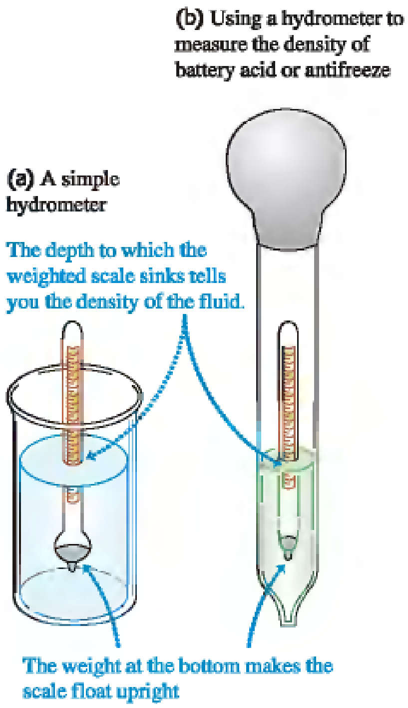
To prove this principle, we consider an arbitrary element of fluid at rest. In Fig. 14.12a the irregular outline is the surface boundary of this element of fluid. The arrows represent the forces exerted on the boundary surface by the surrounding fluid.

The entire fluid is in equilibrium, so the sum of all the y-components of force on this element of fluid is zero. Hence the sum of the y-components of the *surface* forces must be an upward force equal in magnitude to the weight mg of the fluid inside the surface. Also, the sum of the torques on the element of fluid must be zero, so the line of action of the resultant y-component of surface force must pass through the center of gravity of this element of fluid.



14.12 Archimedes's principle.

14.13 Measuring the density of a fluid.



Now we remove the fluid inside the surface and replace it with a solid body having exactly the same shape (Fig. 14.12b). The pressure at every point is exactly the same as before. So the total upward force exerted on the body by the fluid is also the same, again equal in magnitude to the weight mg of the fluid displaced to make way for the body. We call this upward force the **buoyant force** on the solid body. The line of action of the buoyant force again passes through the center of gravity of the displaced fluid (which doesn't necessarily coincide with the center of gravity of the body).

When a balloon floats in equilibrium in air, its weight (including the gas inside it) must be the same as the weight of the air displaced by the balloon. ? A fish's flesh is denser than water, yet a fish can float while submerged because it has a gas-filled cavity within its body. This makes the fish's *average* density the same as water, so its net weight is the same as the weight of the water it displaces. A body whose average density is *less* than that of a liquid can float partially submerged at the free upper surface of the liquid. The greater the density of the liquid, the less of the body is submerged. When you swim in seawater (density 1030 kg/m^3), your body floats higher than in fresh water (1000 kg/m^3).

A practical example of buoyancy is the hydrometer, used to measure the density of liquids (Fig. 14.13a). The calibrated float sinks into the fluid until the weight of the fluid it displaces is exactly equal to its own weight. The hydrometer floats *higher* in denser liquids than in less dense liquids, and a scale in the top stem permits direct density readings. Figure 14.13b shows a type of hydrometer that is commonly used to measure the density of battery acid or antifreeze. The bottom of the large tube is immersed in the liquid; the bulb is squeezed to expel air and is then released, like a giant medicine dropper. The liquid rises into the outer tube, and the hydrometer floats in this sample of the liquid.

Example 14.5 Buoyancy

A 15.0-kg solid gold statue is being raised from a sunken ship (Fig. 14.14a). What is the tension in the hoisting cable when the statue is (a) at rest and completely immersed; and (b) at rest and out of the water?

SOLUTION

IDENTIFY: When the statue is immersed, it experiences an upward buoyant force equal in magnitude to the weight of fluid displaced. To find the tension, we note that the statue is in equilibrium (it is at rest) and consider the three forces acting on it: weight, the buoyant force, and the tension in the cable.

SET UP: Figure 14.14b shows the free-body diagram for the statue in equilibrium. Our target variable is the tension T. We are given the weight mg, and we can calculate the buoyant force B by using Archimedes's principle. We do this for two cases: (a) when the statue is immersed in water and (b) when it is out of the water and immersed in air.

EXECUTE: (a) To find the buoyant force, we first find the volume of the statue, using the density of gold from Table 14.1:

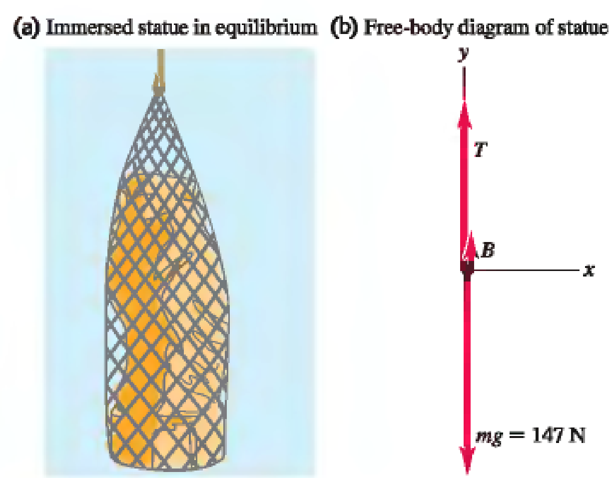
$$V = \frac{m}{\rho_{\text{gold}}} = \frac{15.0 \text{ kg}}{19.3 \times 10^3 \text{ kg/m}^3} = 7.77 \times 10^{-4} \text{ m}^3$$

Using Table 14.1 again, we find the weight of this volume of seawater:

$$\begin{aligned} w_{\text{sw}} &= m_{\text{sw}}g = \rho_{\text{sw}}Vg \\ &= (1.03 \times 10^3 \text{ kg/m}^3)(7.77 \times 10^{-4} \text{ m}^3)(9.80 \text{ m/s}^2) \\ &= 7.84 \text{ N} \end{aligned}$$

This equals the buoyant force B.

14.14 What is the tension in the cable hoisting the statue?



The statue is at rest, so the net external force acting on it is zero. From Fig. 14.14b,

$$\begin{aligned} \sum F_y &= B + T + (-mg) = 0 \\ T &= mg - B = (15.0 \text{ kg})(9.80 \text{ m/s}^2) - 7.84 \text{ N} \\ &= 147 \text{ N} - 7.84 \text{ N} = 139 \text{ N} \end{aligned}$$

If a spring scale is attached to the upper end of the cable, it will indicate 7.84 N less than if the statue were not immersed in seawater. Hence the submerged statue seems to weigh 139 N, about 5% less than its actual weight of 147 N.

(b) The density of air is about 1.2 kg/m^3, so the buoyant force of air on the statue is

$$B = \rho_{air}Vg = (1.2\ \text{kg/m}^3)(7.77 \times 10^{-4}\ \text{m}^3)(9.80\ \text{m/s}^2)$$
$$= 9.1 \times 10^{-3}\ \text{N}$$

This is only 62 parts per million of the statue's actual weight. This effect is not within the precision of our data, and we ignore it. Thus the tension in the cable with the statue in air is equal to the statue's weight, 147 N.

EVALUATE: Note that the buoyant force is proportional to the density of the *fluid*, not the density of the statue. The denser the fluid, the greater the buoyant force and the smaller the cable tension. If the fluid had the same density as the statue, the buoyant force would be equal to the statue's weight and the tension would be zero (the cable would go slack). If the fluid were denser than the statue, the tension would be *negative:* the buoyant force would be greater than the statue's weight, and a downward force would be required to keep the statue from rising upward.

Surface Tension

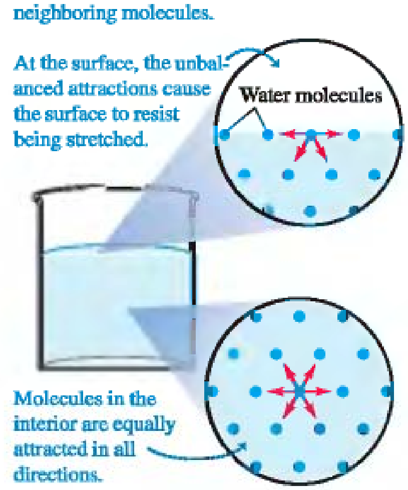
An object less dense than water, such as an air-filled beach ball, floats with part of its volume below the surface. Conversely, a paper clip can rest *atop* a water surface even though its density is several times that of water. This is an example of **surface tension:** The surface of the liquid behaves like a membrane under tension (Fig. 14.15). Surface tension arises because the molecules of the liquid exert attractive forces on each other. There is zero net force on a molecule inside the volume of the liquid, but a surface molecule is drawn into the volume (Fig. 14.16). Thus the liquid tends to minimize its surface area, just as a stretched membrane does.

Surface tension explains why freely falling raindrops are spherical (*not* teardrop-shaped): A sphere has a smaller surface area for its volume than any other shape. It also explains why hot, soapy water is used for washing. To wash clothing thoroughly, water must be forced through the tiny spaces between the fibers (Fig. 14.17). To do so requires increasing the surface area of the water, which is difficult to achieve because of surface tension. The job is made easier by increasing the temperature of the water and adding soap, both of which decrease the surface tension.

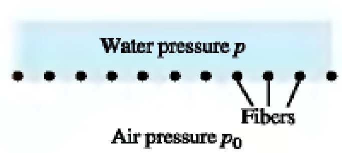
Surface tension is important for a millimeter-sized water drop, which has a relatively large surface area for its volume. (A sphere of radius r has surface area $4\pi r^2$ and volume $(4\pi/3)r^3$. The ratio of surface area to volume is $3/r$, which increases with decreasing radius.) For large quantities of liquid, however, the ratio of surface area to volume is relatively small, and surface tension is negligible compared to pressure forces. For the remainder of this chapter, we will consider only fluids in bulk and hence will ignore the effects of surface tension.

14.15 The surface of the water acts like a membrane under tension, allowing this water strider to literally "walk on water."

14.16 A molecule at the surface of a liquid is attracted into the bulk liquid, which tends to reduce the liquid's surface area.



14.17 Surface tension makes it difficult to force water through small crevices. The required water pressure p can be reduced by using hot, soapy water, which has less surface tension.



Test Your Understanding of Section 14.3 You place a container of seawater on a scale and note the reading on the scale. You now suspend the statue of Example 14.5 in the water (Fig. 14.18). How does the scale reading change? (i) It increases by 7.84 N; (ii) it decreases by 7.84 N; (iii) it remains the same; (iv) none of these.

14.18 How does the scale reading change when the statue is immersed in water?

14.4 Fluid Flow

We are now ready to consider *motion* of a fluid. Fluid flow can be extremely complex, as shown by the currents in river rapids or the swirling flames of a campfire. But some situations can be represented by relatively simple idealized models. An **ideal fluid** is a fluid that is *incompressible* (that is, its density cannot change) and has no internal friction (called **viscosity**). Liquids are approximately incompressible in most situations, and we may also treat a gas as incompressible if the pressure differences from one region to another are not too great. Internal friction in a fluid causes shear stresses when two adjacent layers of fluid move relative to each other, as when fluid flows inside a tube or around an obstacle. In some cases we can neglect these shear forces in comparison with forces arising from gravitation and pressure differences.

The path of an individual particle in a moving fluid is called a **flow line.** If the overall flow pattern does not change with time, the flow is called **steady flow.** In steady flow, every element passing through a given point follows the same flow line. In this case the "map" of the fluid velocities at various points in space remains constant, although the velocity of a particular particle may change in both magnitude and direction during its motion. A **streamline** is a curve whose tangent at any point is in the direction of the fluid velocity at that point. When the flow pattern changes with time, the streamlines do not coincide with the flow lines. We will consider only steady-flow situations, for which flow lines and streamlines are identical.

The flow lines passing through the edge of an imaginary element of area, such as the area A in Fig. 14.19, form a tube called a **flow tube.** From the definition of a flow line, in steady flow no fluid can cross the side walls of a flow tube; the fluids in different flow tubes cannot mix.

14.19 A flow tube bounded by flow lines. In steady flow, fluid cannot cross the walls of a flow tube.

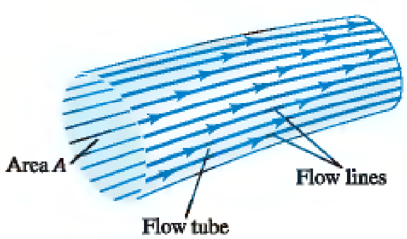
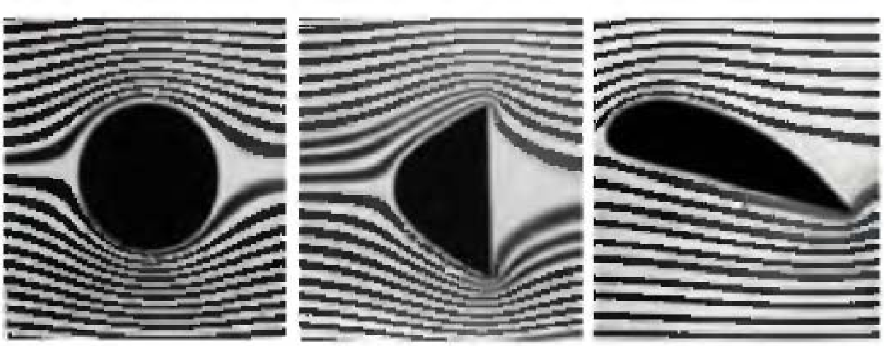


Figure 14.20 shows patterns of fluid flow from left to right around a number of obstacles. The photographs were made by injecting dye into water flowing between two closely spaced glass plates. These patterns are typical of **laminar flow,** in which adjacent layers of fluid slide smoothly past each other and the flow is steady. (A *lamina* is a thin sheet.) At sufficiently high flow rates, or when boundary surfaces cause abrupt changes in velocity, the flow can become irregular and chaotic. This is called **turbulent flow** (Fig. 14.21). In turbulent flow there is no steady-state pattern; the flow pattern changes continuously.

The Continuity Equation

The mass of a moving fluid doesn't change as it flows. This leads to an important quantitative relationship called the **continuity equation.** Consider a portion of a flow tube between two stationary cross sections with areas A_1 and A_2

14.20 Laminar flow around obstacles of different shapes.



14.21 The flow of smoke rising from these incense sticks is laminar up to a certain point, and then becomes turbulent.

(Fig. 14.22). The fluid speeds at these sections are v_1 and v_2, respectively. No fluid flows in or out across the sides of the tube because the fluid velocity is tangent to the wall at every point on the wall. During a small time interval dt, the fluid at A_1 moves a distance $v_1\,dt$, so a cylinder of fluid with height $v_1\,dt$ and volume $dV_1 = A_1v_1\,dt$ flows into the tube across A_1. During this same interval, a cylinder of volume $dV_2 = A_2v_2\,dt$ flows out of the tube across A_2.

Let's first consider the case of an incompressible fluid so that the density ρ has the same value at all points. The mass dm_1 flowing into the tube across A_1 in time dt is $dm_1 = \rho A_1v_1\,dt$. Similarly, the mass dm_2 that flows out across A_2 in the same time is $dm_2 = \rho A_2v_2\,dt$. In steady flow the total mass in the tube is constant, so $dm_1 = dm_2$ and

$$\rho A_1v_1\,dt = \rho A_2v_2\,dt \qquad \text{or}$$

$$A_1v_1 = A_2v_2 \qquad \text{(continuity equation, incompressible fluid)} \qquad (14.10)$$

The product Av is the *volume flow rate* dV/dt, the rate at which volume crosses a section of the tube:

$$\frac{dV}{dt} = Av \qquad \text{(volume flow rate)} \qquad (14.11)$$

The *mass* flow rate is the mass flow per unit time through a cross section. This is equal to the density ρ times the volume flow rate dV/dt.

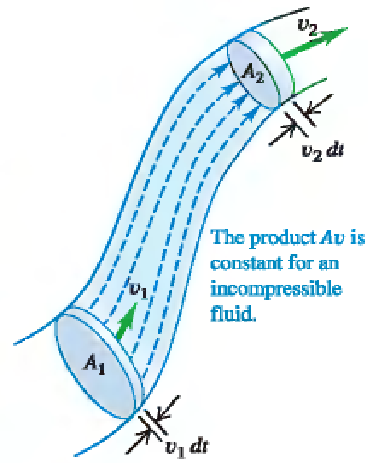
Equation (14.10) shows that the volume flow rate has the same value at all points along any flow tube. When the cross section of a flow tube decreases, the speed increases, and vice versa. The deep part of a river has larger cross section and slower current than the shallow part, but the volume flow rates are the same in both. This is the essence of the familiar maxim, "Still waters run deep." The stream of water from a faucet narrows as it gains speed during its fall, but dV/dt is the same everywhere along the stream. If a water pipe with 2-cm diameter is connected to a pipe with 1-cm diameter, the flow speed is four times as great in the 1-cm part as in the 2-cm part.

We can generalize Eq. (14.10) for the case in which the fluid is *not* incompressible. If ρ_1 and ρ_2 are the densities at sections 1 and 2, then

$$\rho_1A_1v_1 = \rho_2A_2v_2 \qquad \text{(continuity equation, compressible fluid)} \qquad (14.12)$$

If the fluid is denser at point 2 than at point 1 $(\rho_2 > \rho_1)$, the volume flow rate at point 2 will be less than at point 1 $(A_2v_2 < A_1v_1)$. We leave the details to you (see Exercise 14.38). If the fluid is incompressible so that ρ_1 and ρ_2 are always equal, Eq. (14.12) reduces to Eq. (14.10).

14.22 A flow tube with changing cross-sectional area. If the fluid is incompressible, the product Av has the same value at all points along the tube.



Example 14.6 Incompressible fluid flow

As part of a lubricating system for heavy machinery, oil of density 850 kg/m^3 is pumped through a cylindrical pipe of diameter 8.0 cm at a rate of 9.5 liters per second. (a) What is the speed of the oil? What is the mass flow rate? (b) If the pipe diameter is reduced to 4.0 cm, what are the new values of the speed and volume flow rate? Assume that the oil is incompressible.

SOLUTION

IDENTIFY: The key point is that the fluid is incompressible, so we can use the idea of the continuity equation to relate mass flow rate, volume flow rate, flow tube area, and flow speed.

SET UP: We use the definition of volume flow rate, Eq. (14.11), to determine the speed v_1 in the 8.0-cm-diameter section. The mass flow rate is the product of the density and the volume flow rate. The continuity equation for incompressible flow, Eq. (14.10), allows us to find the speed v_2 in the 4.0-cm-diameter section.

EXECUTE: (a) The volume flow rate dV/dt equals the product A_1v_1, where A_1 is the cross-sectional area of the pipe of diameter 8.0 cm and radius 4.0 cm. Hence

$$v_1 = \frac{dV/dt}{A_1} = \frac{(9.5\ \text{L/s})(10^{-3}\ \text{m}^3/\text{L})}{\pi(4.0 \times 10^{-2}\ \text{m})^2} = 1.9\ \text{m/s}$$

Continued

The mass flow rate is $\rho\, dV/dt = (850\ \text{kg/m}^3)(9.5 \times 10^{-3}\ \text{m}^3/\text{s}) = 8.1\ \text{kg/s}$.

(b) Since the oil is incompressible, the volume flow rate has the *same* value (9.5 L/s) in both sections of pipe. From Eq. (14.10),

$$v_2 = \frac{A_1}{A_2}v_1 = \frac{\pi(4.0 \times 10^{-2}\ \text{m})^2}{\pi(2.0 \times 10^{-2}\ \text{m})^2}(1.9\ \text{m/s}) = 7.6\ \text{m/s}$$

EVALUATE: The second section of pipe has one-half the diameter and one-fourth the cross-sectional area of the first section. Hence the speed must be four times greater in the second section, which is just what our result shows ($v_2 = 4v_1$).

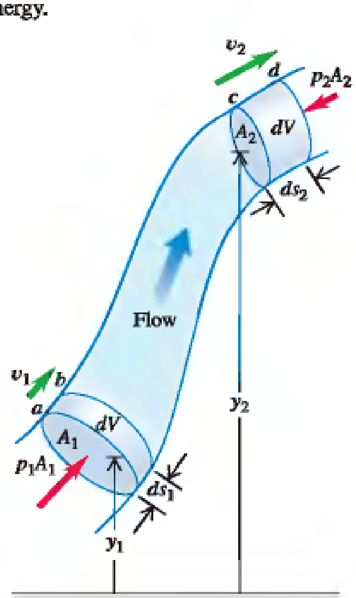
Test Your Understanding of Section 14.4 A maintenance crew is working on a section of a three-lane highway, leaving only one lane open to traffic. The result is much slower traffic flow (a traffic jam). Do cars on a highway behave like (i) the molecules of an incompressible fluid or (ii) the molecules of a compressible fluid?

14.5 Bernoulli's Equation

According to the continuity equation, the speed of fluid flow can vary along the paths of the fluid. The pressure can also vary; it depends on height as in the static situation (see Section 14.2), and it also depends on the speed of flow. We can derive an important relationship called *Bernoulli's equation* that relates the pressure, flow speed, and height for flow of an ideal, incompressible fluid. Bernoulli's equation is an essential tool in analyzing plumbing systems, hydroelectric generating stations, and the flight of airplanes.

The dependence of pressure on speed follows from the continuity equation, Eq. (14.10). When an incompressible fluid flows along a flow tube with varying cross section, its speed *must* change, and so an element of fluid must have an acceleration. If the tube is horizontal, the force that causes this acceleration has to be applied by the surrounding fluid. This means that the pressure *must* be different in regions of different cross section; if it were the same everywhere, the net force on every fluid element would be zero. When a horizontal flow tube narrows and a fluid element speeds up, it must be moving toward a region of lower pressure in order to have a net forward force to accelerate it. If the elevation also changes, this causes an additional pressure difference.

14.23 Deriving Bernoulli's equation. The net work done on a fluid element by the pressure of the surrounding fluid equals the change in the kinetic energy plus the change in the gravitational potential energy.



Deriving Bernoulli's Equation

To derive Bernoulli's equation, we apply the work-energy theorem to the fluid in a section of a flow tube. In Fig. 14.23 we consider the element of fluid that at some initial time lies between the two cross sections a and c. The speeds at the lower and upper ends are v_1 and v_2. In a small time interval dt, the fluid that is initially at a moves to b, a distance $ds_1 = v_1\, dt$, and the fluid that is initially at c moves to d, a distance $ds_2 = v_2\, dt$. The cross-sectional areas at the two ends are A_1 and A_2, as shown. The fluid is incompressible; hence by the continuity equation, Eq. (14.10), the volume of fluid dV passing *any* cross section during time dt is the same. That is, $dV = A_1\, ds_1 = A_2\, ds_2$.

Let's compute the *work* done on this fluid element during dt. We assume that there is negligible internal friction in the fluid (i.e., no viscosity), so the only nongravitational forces that do work on the fluid element are due to the pressure of the surrounding fluid. The pressures at the two ends are p_1 and p_2; the force on the cross section at a is p_1A_1, and the force at c is p_2A_2. The net work dW done on the element by the surrounding fluid during this displacement is therefore

$$dW = p_1A_1\, ds_1 - p_2A_2\, ds_2 = (p_1 - p_2)\, dV \qquad (14.13)$$

The second term has a negative sign because the force at c opposes the displacement of the fluid.

The work dW is due to forces other than the conservative force of gravity, so it equals the change in the total mechanical energy (kinetic energy plus gravitational potential energy) associated with the fluid element. The mechanical energy for the fluid between sections b and c does not change. At the beginning of dt the fluid between a and b has volume $A_1\,ds_1$, mass $\rho A_1\,ds_1$, and kinetic energy $\frac{1}{2}\rho(A_1\,ds_1)v_1^2$. At the end of dt the fluid between c and d has kinetic energy $\frac{1}{2}\rho(A_2\,ds_2)v_2^2$. The net change in kinetic energy dK during time dt is

$$dK = \frac{1}{2}\rho\,dV(v_2^2 - v_1^2) \tag{14.14}$$

What about the change in gravitational potential energy? At the beginning of dt, the potential energy for the mass between a and b is $dm\,gy_1 = \rho\,dV\,gy_1$. At the end of dt, the potential energy for the mass between c and d is $dm\,gy_2 = \rho\,dV\,gy_2$. The net change in potential energy dU during dt is

$$dU = \rho\,dV\,g(y_2 - y_1) \tag{14.15}$$

Combining Eqs. (14.13), (14.14), and (14.15) in the energy equation $dW = dK + dU$, we obtain

$$\begin{aligned}(p_1 - p_2)\,dV &= \frac{1}{2}\rho\,dV(v_2^2 - v_1^2) + \rho\,dV\,g(y_2 - y_1)\\ p_1 - p_2 &= \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(y_2 - y_1)\end{aligned} \tag{14.16}$$

This is **Bernoulli's equation.** It states that the work done on a unit volume of fluid by the surrounding fluid is equal to the sum of the changes in kinetic and potential energies per unit volume that occur during the flow. We may also interpret Eq. (14.16) in terms of pressures. The first term on the right is the pressure difference associated with the change of speed of the fluid. The second term on the right is the additional pressure difference caused by the weight of the fluid and the difference in elevation of the two ends.

We can also express Eq. (14.16) in a more convenient form as

$$p_1 + \rho g y_1 + \frac{1}{2}\rho v_1^2 = p_2 + \rho g y_2 + \frac{1}{2}\rho v_2^2 \quad \text{(Bernoulli's equation)} \tag{14.17}$$

The subscripts 1 and 2 refer to *any* two points along the flow tube, so we can also write

$$p + \rho g y + \frac{1}{2}\rho v^2 = \text{constant} \tag{14.18}$$

Note that when the fluid is *not* moving (so $v_1 = v_2 = 0$), Eq. (14.17) reduces to the pressure relationship we derived for a fluid at rest, Eq. (14.5).

CAUTION **Bernoulli's principle applies only in certain situations** We stress again that Bernoulli's equation is valid for only incompressible, steady flow of a fluid with no internal friction (no viscosity). It's a simple equation that's easy to use; don't let this tempt you to use it in situations in which it doesn't apply!

Problem-Solving Strategy 14.1 Bernoulli's Equation

Bernoulli's equation is derived from the work–energy theorem, so it isn't surprising that much of Problem-Solving Strategy 7.1 (Section 7.1) is applicable here.

IDENTIFY *the relevant concepts:* First ensure that the fluid flow is steady and that the fluid is incompressible and has no internal friction. This case is an idealization, but it holds up surprisingly well for fluids flowing through sufficiently large pipes and for flows within bulk fluids (e.g., air flowing around an airplane or water flowing around a fish).

SET UP *the problem* using the following steps:

1. Always begin by identifying clearly the points 1 and 2 referred to in Bernoulli's equation.
2. Define your coordinate system, particularly the level at which $y = 0$.

Continued

3. Make lists of the unknown and known quantities in Eq. (14.17). The variables are p_1, p_2, v_1, v_2, y_1, and y_2, and the constants are ρ and g. Decide which unknowns are your target variables.

EXECUTE *the solution* as follows: Write Bernoulli's equation and solve for the unknowns. In some problems you will need to use the continuity equation, Eq. (14.10), to get a relationship between the two speeds in terms of cross-sectional areas of pipes or containers. Or perhaps you will know both speeds and need to determine one of the areas. You may also need to use Eq. (14.11) to find the volume flow rate.

EVALUATE *your answer:* As always, verify that the results make physical sense. Double check that you have used consistent units. In SI units, pressure is in pascals, density in kilograms per cubic meter, and speed in meters per second. Also note that the pressures must be either all absolute pressures or all gauge pressures.

Example 14.7 Water pressure in the home

Water enters a house through a pipe with an inside diameter of 2.0 cm at an absolute pressure of 4.0×10^5 Pa (about 4 atm). A 1.0-cm-diameter pipe leads to the second-floor bathroom 5.0 m above (Fig. 14.24). When the flow speed at the inlet pipe is 1.5 m/s, find the flow speed, pressure, and volume flow rate in the bathroom.

SOLUTION

IDENTIFY: We assume that the water flows at a steady rate. The pipe has a relatively large diameter, so it's reasonable to ignore internal friction. Water is rather incompressible, so it's a good approximation to use Bernoulli's equation.

SET UP: Let points 1 and 2 be at the inlet pipe and at the bathroom, respectively. We are given the speed v_1 and pressure p_1 at the inlet pipe, and the pipe diameters at points 1 and 2 (from which we calculate the areas A_1 and A_2). We take $y_1 = 0$ (at the inlet) and $y_2 = 5.0$ m (at the bathroom). Our first two target variables are the speed v_2 and pressure p_2. Since we have more than one unknown, we use both Bernoulli's equation and the continuity equation for an incompressible fluid. Once we find v_2, we can calculate the volume flow rate v_2A_2 at point 2.

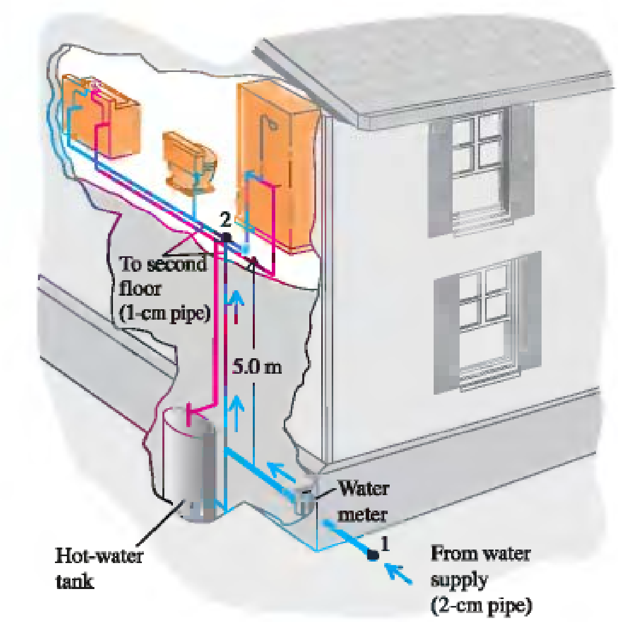
EXECUTE: We find the speed v_2 at the bathroom using the continuity equation, Eq. (14.10):

$$v_2 = \frac{A_1}{A_2}v_1 = \frac{\pi(1.0\text{ cm})^2}{\pi(0.50\text{ cm})^2}(1.5\text{ m/s}) = 6.0\text{ m/s}$$

We are given p_1 and v_1, and we can find p_2 from Bernoulli's equation, Eq. (14.16):

$$\begin{aligned} p_2 &= p_1 - \frac{1}{2}\rho(v_2^2 - v_1^2) - \rho g(y_2 - y_1) = 4.0 \times 10^5\text{ Pa} \\ &\quad - \frac{1}{2}(1.0 \times 10^3\text{ kg/m}^3)(36\text{ m}^2/\text{s}^2 - 2.25\text{ m}^2/\text{s}^2) \\ &\quad - (1.0 \times 10^3\text{ kg/m}^3)(9.8\text{ m/s}^2)(5.0\text{ m}) \\ &= 4.0 \times 10^5\text{ Pa} - 0.17 \times 10^5\text{ Pa} - 0.49 \times 10^5\text{ Pa} \\ &= 3.3 \times 10^5\text{ Pa} = 3.3\text{ atm} = 48\text{ lb/in.}^2 \end{aligned}$$

14.24 What is the water pressure in the second-story bathroom of this house?



The volume flow rate is

$$\begin{aligned} \frac{dV}{dt} &= A_2v_2 = \pi(0.50 \times 10^{-2}\text{ m})^2(6.0\text{ m/s}) \\ &= 4.7 \times 10^{-4}\text{ m}^3/\text{s} = 0.47\text{ L/s} \end{aligned}$$

EVALUATE: This is a reasonable flow rate for a bathroom faucet or shower. Note that after the water is turned off, v_1 and v_2 are both zero, the term $\frac{1}{2}\rho(v_2^2 - v_1^2)$ in the equation for pressure vanishes, and the pressure p_2 rises to 3.5×10^5 Pa.

Example 14.8 Speed of efflux

Figure 14.25 shows a gasoline storage tank with cross-sectional area A_1, filled to a depth h. The space above the gasoline contains air at pressure p_0, and the gasoline flows out through a short pipe with area A_2. Derive expressions for the flow speed in the pipe and the volume flow rate.

SOLUTION

IDENTIFY: We consider the entire volume of moving liquid as a single flow tube of an incompressible fluid with negligible internal friction. Hence, we can use Bernoulli's principle.

SET UP: Points 1 and 2 in Fig. 14.25 are at the surface of the gasoline and at the short exit pipe, respectively. At point 1 the pressure is p_0 and at point 2 it is atmospheric pressure p_{atm}. We take $y = 0$ at the exit pipe, so $y_1 = h$ and $y_2 = 0$. Because A_1 is very much larger than A_2, the upper surface of the gasoline will drop very slowly and we can regard v_1 as essentially equal to zero. We find the target variable v_2 from Eq. (14.17) and the volume flow rate from Eq. (14.11).

EXECUTE: We apply Bernoulli's equation to points 1 and 2

$$p_0 + \frac{1}{2}\rho v_1^2 + \rho g h = p_{atm} + \frac{1}{2}\rho v_2^2 + \rho g(0)$$

$$v_2^2 = v_1^2 + 2\left(\frac{p_0 - p_{atm}}{\rho}\right) + 2gh$$

Using $v_1 = 0$, we find

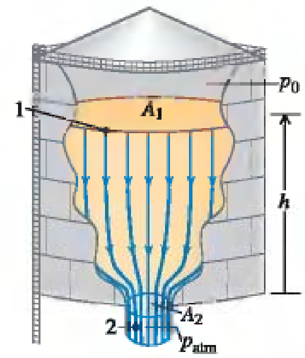
$$v_2^2 = 2\left(\frac{p_0 - p_{atm}}{\rho}\right) + 2gh$$

From Eq. (14.11), the volume flow rate is $dV/dt = v_2 A_2$.

EVALUATE: The speed v_2, sometimes called the *speed of efflux*, depends on both the pressure difference $(p_0 - p_{atm})$ and the height h of the liquid level in the tank. If the top of the tank is vented to the atmosphere, $p_0 = p_{atm}$ and there is zero pressure difference: $p_0 - p_{atm} = 0$. In that case,

$$v_2 = \sqrt{2gh}$$

14.25 Calculating the speed of efflux for gasoline flowing out the bottom of a storage tank.



That is, the speed of efflux from an opening at a distance h below the top surface of the liquid is the *same* as the speed a body would acquire in falling freely through a height h. This result is called *Torricelli's theorem*. It is valid not only for an opening in the bottom of a container, but also for a hole in a side wall at a depth h below the surface. In this case the volume flow rate is

$$\frac{dV}{dt} = A_2\sqrt{2gh}$$

Example 14.9 The Venturi meter

Figure 14.26 shows a *Venturi meter*, used to measure flow speed in a pipe. The narrow part of the pipe is called the *throat*. Derive an expression for the flow speed v_1 in terms of the cross-sectional areas A_1 and A_2 and the difference in height h of the liquid levels in the two vertical tubes.

SOLUTION

IDENTIFY: The flow is steady, and we assume the fluid is incompressible and has negligible internal friction. Hence we can use Bernoulli's equation.

SET UP: We apply Bernoulli's equation to the wide (point 1) and narrow (point 2) parts of the pipe. The difference in height between the two vertical tubes tells us the pressure difference between points 1 and 2.

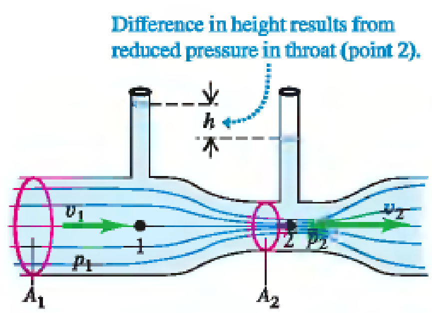
EXECUTE: The two points are at the same vertical coordinate $(y_1 = y_2)$, so Eq. (14.17) says

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

From the continuity equation, $v_2 = (A_1/A_2)v_1$. Substituting this and rearranging, we get

$$p_1 - p_2 = \frac{1}{2}\rho v_1^2\left(\frac{A_1^2}{A_2^2} - 1\right)$$

14.26 The Venturi meter.



From Section 14.2, the pressure difference $p_1 - p_2$ is also equal to $\rho g h$, where h is the difference in the liquid levels in the two tubes. Combining this with the above result and solving for v_1, we get

$$v_1 = \sqrt{\frac{2gh}{(A_1/A_2)^2 - 1}}$$

EVALUATE: Because A_1 is greater than A_2, v_2 is greater than v_1 and the pressure p_2 in the throat is *less* than p_1. A net force to the right accelerates the fluid as it enters the throat, and a net force to the left slows it as it leaves.

Conceptual Example 14.10 Lift on an airplane wing

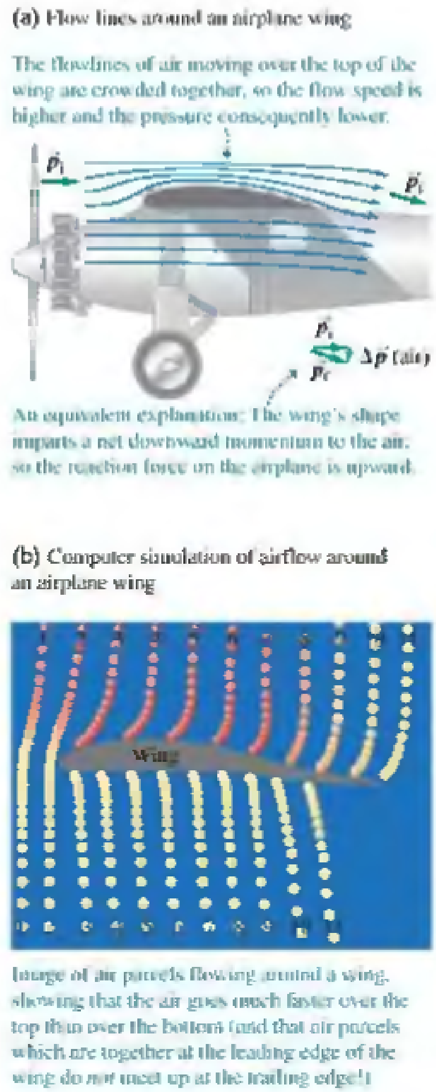
Figure 14.27a shows flow lines around a cross section of an airplane wing. The flow lines crowd together above the wing, corresponding to increased flow speed and reduced pressure in this region, just as in the Venturi throat. The upward force on the underside of the wing is greater than the downward force on the top side; there is a net upward force, or *lift*. Lift is not simply due to the impulse of air striking the underside of the wing; in fact, it turns out that the reduced pressure on the upper wing surface makes the greatest contribution to the lift. (This highly simplified discussion ignores the formation of vortices; a more complete discussion would take these into account.)

We can also understand the lift force on the basis of momentum changes. Figure 14.27a shows that there is a net *downward* change in the vertical component of momentum of the air flowing past the wing, corresponding to the downward force the wing exerts on the air. The reaction force *on* the wing is *upward*, as we concluded above.

A similar flow pattern and lift force are found in the vicinity of any humped object in a wind. In a sufficiently strong wind, the lift force on the top of an open umbrella can collapse the umbrella upward. A lift force also acts on a car driving at high speed due to air moving over the car's curved upper surface. Such lift can reduce traction on the car's tires, which is why many cars are equipped with an aerodynamic "spoiler" at the car's tail. The spoiler is shaped like an upside-down wing and provides a downward force on the rear wheels.

CAUTION A misconception about wings Simplified discussions of wings often claim that air travels faster over the top of a wing because "it has farther to travel." This picture assumes that two adjacent air molecules that part company at the front of the wing, one traveling over the upper surface of the wing and one under the lower surface, must meet again at the wing's trailing edge. Not so! Figure 14.27b shows a computer simulation of parcels of air flowing around an airplane wing. Parcels that are adjacent at the front of the wing do *not* meet at the trailing edge, because the flow over the top of the wing is actually faster than in the simplified (but incorrect) picture. In accordance with Bernoulli's equation, this faster speed means that there is even lower pressure above the wing (and hence greater lift) than the simplified description would suggest.

14.27 (a) Flow lines around an airplane wing. The momentum of a parcel of air (relative to the wing) is $\vec{p}_i$ before encountering the wing and $\vec{p}_f$ afterward. (b) Computer simulation of parcels of air flowing around a wing.



Test Your Understanding of Section 14.5 Which is the most accurate statement of Bernoulli's principle? (i) Fast-moving air causes lower pressure; (ii) lower pressure causes fast-moving air; (iii) both (i) and (ii) are equally accurate.

*14.6 Viscosity and Turbulence

In our discussion of fluid flow we assumed that the fluid had no internal friction and that the flow was laminar. While these assumptions are often quite valid, in many important physical situations the effects of viscosity (internal friction) and turbulence (nonlaminar flow) are extremely important. Let's take a brief look at some of these situations.

Viscosity

Viscosity is internal friction in a fluid. Viscous forces oppose the motion of one portion of a fluid relative to another. Viscosity is the reason it takes effort to paddle a canoe through calm water, but it is also the reason the paddle works. Viscous effects are important in the flow of fluids in pipes, the flow of blood, the lubrication of engine parts, and many other situations.

Fluids that flow readily, such as water or gasoline, have smaller viscosities than do "thick" liquids such as honey or motor oil. Viscosities of all fluids are strongly temperature dependent, increasing for gases and decreasing for liquids as the temperature increases (Fig. 14.28). An important goal in the design of oils for engine lubrication is to *reduce* the temperature variation of viscosity as much as possible.

A viscous fluid always tends to cling to a solid surface in contact with it. There is always a thin *boundary layer* of fluid near the surface, in which the fluid is nearly at rest with respect to the surface. That's why dust particles can cling to a fan blade even when it is rotating rapidly, and why you can't get all the dirt off your car by just squirting a hose at it.

Viscosity has important effects on the flow of liquids through pipes, including the flow of blood in the circulatory system. First think about a fluid with zero viscosity so that we can apply Bernoulli's equation, Eq. (14.17). If the two ends of a long cylindrical pipe are at the same height $(y_1 = y_2)$ and the flow speed is the same at both ends (so $v_1 = v_2$), Bernoulli's equatiou tells us that the pressure is the same at both ends of the pipe. But this result simply isn't true if we take viscosity into account. To see why, consider Fig. 14.29, which shows the flow-speed profile for laminar flow of a viscous fluid in a long cylindrical pipe. Due to viscosity, the speed is *zero* at the pipe walls (to which the fluid clings) and is greatest at the center of the pipe. The motion is like a lot of concentric tubes sliding relative to one another, with the central tube moving fastest and the outermost tube at rest. Viscous forces between the tubes oppose this sliding, so to keep the flow going we must apply a greater pressure at the back of the flow than at the front. That's why you have to keep squeezing a tube of toothpaste or a packet of ketchup (both viscous fluids) to keep the fluid coming out of its container. Your fingers provide a pressure at the back of the flow that is far greater than the atmospheric pressure at the front of the flow.

The pressure difference required to sustain a given volume flow rate through a cylindrical pipe of length L and radius R turns out to be proportional to L/R^4. If we decrease R by one-half, the required pressure increases by $2^4 = 16$; decreasing R by a factor of 0.90 (a 10% reduction) increases the required pressure difference by a factor of $(1/0.90)^4 = 1.52$ (a 52% increase). This simple relationship explains the connection between a high-cholesterol diet (which tends to narrow the arteries) and high blood pressure. Due to the R^4 dependence, even a small narrowing of the arteries can result in substantially elevated blood pressure and added strain on the heart muscle.

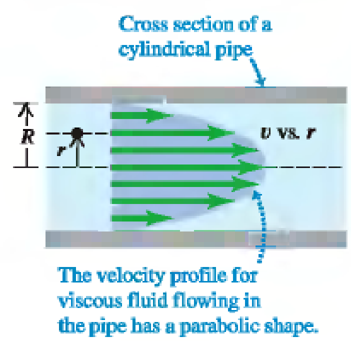
Turbulence

When the speed of a flowing fluid exceeds a certain critical value, the flow is no longer laminar. Instead, the flow pattern becomes extremely irregular and complex, and it changes continuously with time; there is no steady-state pattern. This irregular, chaotic flow is called **turbulence.** Figure 14.21 shows the contrast between laminar and turbulent flow for smoke rising in air. Bernoulli's equatiou is *not* applicable to regions where there is turbulence because the flow is not steady.

Whether a flow is laminar or turbulent depends in part on the fluid's viscosity. The greater the viscosity, the greater the tendency for the fluid to flow in sheets or lamina and the more likely the flow is to be laminar. (When we discussed

14.28 Lava is an example of a viscous fluid. The viscosity decreases with increasing temperature: The hotter the lava, the more easily it can flow.

14.29 Velocity profile for a viscous fluid in a cylindrical pipe.



Bernoulli's equation in Section 14.5, we assumed that the flow was laminar and that the fluid had zero viscosity. In fact, a *little* viscosity is needed to ensure that the flow is laminar.)

For a fluid of a given viscosity, flow speed is a determining factor for the onset of turbulence. A flow pattern that is stable at low speeds suddenly becomes unstable when a critical speed is reached. Irregularities in the flow pattern can be caused by roughness in the pipe wall, variations in the density of the fluid, and many other factors. At low flow speeds, these disturbances damp out; the flow pattern is *stable* and tends to maintain its laminar nature (Fig. 14.30a). When the critical speed is reached, however, the flow pattern becomes unstable. The disturbances no longer damp out but grow until they destroy the entire laminar-flow pattern (Fig. 14.30b).

Normal blood flow in the human aorta is laminar, but a small disturbance such as a heart pathology can cause the flow to become turbulent. Turbulence makes noise, which is why listening to blood flow with a stethoscope is a useful diagnostic technique.

14.30 The flow of water from a faucet is (a) laminar at low speeds but (b) turbulent at sufficiently high speeds.

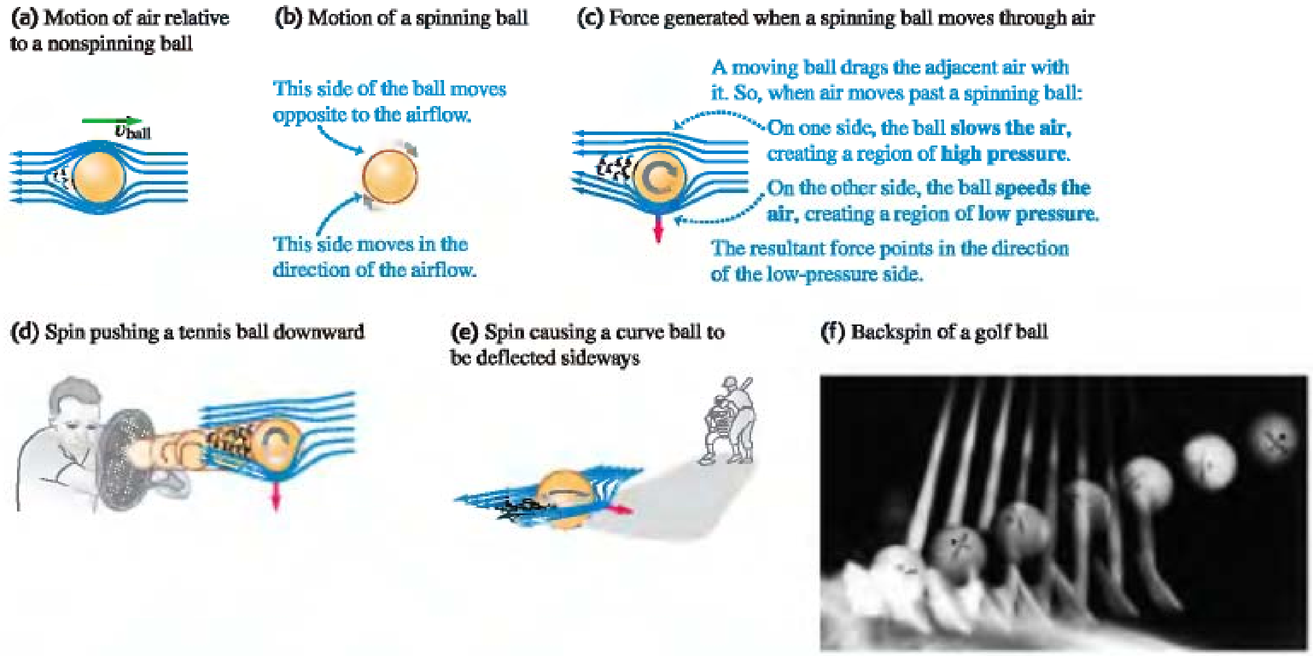
Conceptual Example 14.11 The curve ball

Does a curve ball *really* curve? Yes, it certainly does, and the reason is turbulence. Fig. 14.31a shows a ball moving through the air from left to right. To an observer moving with the center of the ball, the air stream appears to move from right to left, as shown by the flow lines in the figure. Because of the high speeds that are ordinarily involved (near 160 km/h, or 100 mi/h), there is a region of *turbulent* flow behind the ball.

Figure 14.31b shows a *spinning* ball with "top spin." Layers of air near the ball's surface are pulled around in the direction of the spin by friction between the ball and air and by the air's internal friction (viscosity). The speed of air relative to the ball's surface becomes slower at the top of the ball than at the bottom, and turbulence occurs farther forward on the top side than on the bottom. This asymmetry causes a pressure difference; the average pressure at the top of the ball is now greater than that at the bottom. The net force deflects the ball downward, as shown in Fig. 14.31c. This is why "top spin" is used in tennis to keep a very fast serve in the court (Fig. 14.31d). In a baseball curve pitch, the ball spins about a nearly vertical axis, and the actual deflection is sideways. In that case, Fig. 14.31c is a *top* view of the situation. A curve ball thrown by a left-handed pitcher curves *toward* a right-handed batter, making it harder to hit (Fig. 14.31e).

A similar effect occurs with golf balls, which always have "back spin" from impact with the slanted face of the golf club. The resulting pressure difference between the top and bottom of the ball causes a lift force that keeps the ball in the air considerably longer than would be possible without spin. A well-hit drive appears from the tee to "float" or even curve *upward* during the initial portion of its flight. This is a real effect, not an illusion. The dimples on the ball play an essential role; the viscosity of air gives an undimpled ball a much shorter trajectory than a dimpled one with the same initial velocity and spin. Figure 14.31f shows the backspin of a golf ball just after it is struck by a club.

14.31 (a)–(e) Analyzing the motion of a spinning ball through the air. (f) Stroboscopic photograph of a golf ball being struck by a club. The picture was taken at 1000 flashes per second. The ball rotates about once in eight pictures, corresponding to an angular speed of 125 rev/s, or 7500 rpm.



Test Your Understanding of Section 14.6 How much more thumb pressure must a nurse use to administer an injection with a hypodermic needle of inside diameter 0.30 mm compared to one with inside diameter 0.60 mm? Assume that the two needles have the same length and that the volume flow rate is the same in both cases. (i) twice as much; (ii) 4 times as much; (iii) 8 times as much; (iv) 16 times as much; (v) 32 times as much.

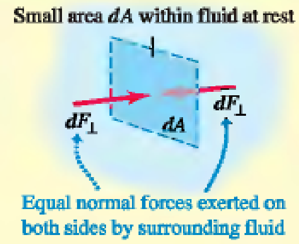
CHAPTER 14 SUMMARY

Density and pressure: Density is mass per unit volume. If a mass m of homogeneous material has volume V, its density ρ is the ratio m/V. Specific gravity is the ratio of the density of a material to the density of water. (See Example 14.1.)

Pressure is normal force per unit area. Pascal's law states that pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid. Absolute pressure is the total pressure in a fluid; gauge pressure is the difference between absolute pressure and atmospheric pressure. The SI unit of pressure is the pascal (Pa): 1 Pa = 1 N/m^2. (See Example 14.2.)

$$\rho = \frac{m}{V} \qquad (14.1)$$

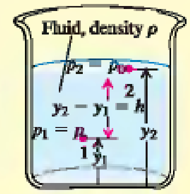
$$p = \frac{dF_\perp}{dA} \qquad (14.2)$$



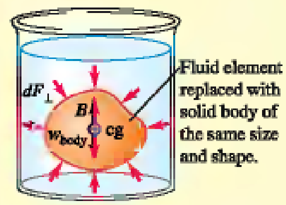
Pressures in a fluid at rest: The pressure difference between points 1 and 2 in a static fluid of uniform density ρ (an incompressible fluid) is proportional to the difference between the elevations y_1 and y_2. If the pressure at the surface of an incompressible liquid at rest is p_0, then the pressure at a depth h is greater by an amount $\rho g h$. (See Examples 14.3 and 14.4.)

$$p_2 - p_1 = -\rho g(y_2 - y_1) \qquad (14.5)$$
(pressure in a fluid of uniform density)

$$p = p_0 + \rho g h \qquad (14.6)$$
(pressure in a fluid of uniform density)



Buoyancy: Archimedes's principle states that when a body is immersed in a fluid, the fluid exerts an upward buoyant force on the body equal to the weight of the fluid that the body displaces. (See Example 14.5.)



Fluid flow: An ideal fluid is incompressible and has no viscosity (no internal friction). A flow line is the path of a fluid particle; a streamline is a curve tangent at each point to the velocity vector at that point. A flow tube is a tube bounded at its sides by flow lines. In laminar flow, layers of fluid slide smoothly past each other. In turbulent flow, there is great disorder and a constantly changing flow pattern.

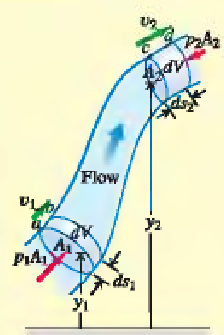
Conservation of mass in an incompressible fluid is expressed by the continuity equation, which relates the flow speeds v_1 and v_2 for two cross sections A_1 and A_2 in a flow tube. The product Av equals the volume flow rate, dV/dt, the rate at which volume crosses a section of the tube. (See Example 14.6.)

Bernoulli's equation relates the pressure p, flow speed v, and elevation y for any two points, assuming steady flow in an ideal fluid. (See Examples 14.7–14.10.)

$$A_1 v_1 = A_2 v_2 \qquad (14.10)$$
(continuity equation, incompressible fluid)

$$\frac{dV}{dt} = Av \qquad (14.11)$$
(volume flow rate)

$$p_1 + \rho g y_1 + \frac{1}{2}\rho v_1^2 = p_2 + \rho g y_2 + \frac{1}{2}\rho v_2^2 \qquad (14.17)$$
(Bernoulli's equation)



Key Terms

Answer to Chapter Opening Question ?

The flesh of both the shark and the tropical fish is denser than seawater, so left to themselves they would sink. However, a tropical fish has a gas-filled body cavity called a swimbladder, so that the *average* density of the fish's body is the same as seawater and the fish neither sinks nor rises. Sharks have no such cavity. Hence they must swim constantly to keep from sinking, using their pectoral fins to provide lift much like the wings of an airplane (see Section 14.5).

Answers to Test Your Understanding Questions

14.1 Answer: (ii), (iv), (i) and (iii) (tie) (v) In each case the average density equals the mass divided by the volume. Hence we have (i) $\rho = (4.00\,\text{kg})/(1.60 \times 10^{-3}\,\text{m}^3) = 2.50 \times 10^3\,\text{kg/m}^3$; (ii) $\rho = (8.00\,\text{kg})/(1.60 \times 10^{-3}\,\text{m}^3) = 5.00 \times 10^3\,\text{kg/m}^3$; (iii) $\rho = (8.00\,\text{kg})/(3.20 \times 10^{-3}\,\text{m}^3) = 2.50 \times 10^3\,\text{kg/m}^3$; (iv) $\rho = (2560\,\text{kg})/(0.640\,\text{m}^3) = 4.00 \times 10^3\,\text{kg/m}^3$; (v) $\rho = (2560\,\text{kg})/(1.28\,\text{m}^3) = 2.00 \times 10^3\,\text{kg/m}^3$. Note that compared to object (i), object (ii) has double the mass but the same volume and so has double the average density. Object (iii) has double the mass and double the volume of object (i), so (i) and (iii) have the same average density. Finally, object (v) has the same mass as object (iv) but double the volume, so (v) has half the average density of (iv).

14.2 Answer: (ii) From Eq. (14.9), the pressure outside the barometer is equal to the product $\rho g h$. When the barometer is taken out of the refrigerator, the density ρ decreases while the height h of the mercury column remains the same. Hence the air pressure must be lower outdoors than inside the refrigerator.

14.3 Answer: (i) Consider the water, the statue, and the container together as a system; the total weight of the system does not depend on whether the statue is immersed. The total supporting force, including the tension T and the upward force F of the scale on the container (equal to the scale reading), is the same in both cases. But we saw in Example 14.5 that T decreases by 7.84 N when the statue is immersed, so the scale reading F must *increase* by 7.84 N. An alternative viewpoint is that the water exerts an upward buoyant force of 7.84 N on the statue, so the statue must exert an equal downward force on the water, making the scale reading 7.84 N greater than the weight of water and container.

14.4 Answer: (ii) A highway that narrows from three lanes to one is like a pipe whose cross-sectional area narrows to one-third of its value. If cars behaved like the molecules of an incompressible fluid, then as the cars encountered the one-lane section, the spacing between cars (the "density") would stay the same but the cars would triple their speed. This would keep the "volume flow rate" (number of cars per second passing a point on the highway) the same. In real life cars behave like the molecules of a *compressible* fluid: They end up packed closer (the "density" increases) and fewer cars per second pass a point on the highway (the "volume flow rate" decreases).

14.5 Answer: (ii) Newton's second law tells us that a body accelerates (its velocity changes) in response to a net force. In fluid flow, a pressure difference between two points means that fluid particles moving between those two points experience a force, and this force causes the fluid particles to accelerate and change speed.

14.6 Answer: (iv) The required pressure is proportional to $1/R^4$, where R is the inside radius of the needle (half the inside diameter). With the smaller-diameter needle, the pressure is greater by a factor of $[(0.60\,\text{mm})/(0.30\,\text{mm})]^4 = 2^4 = 16$.

PROBLEMS

Discussion Questions

Q14.1. A cube of oak wood with very smooth faces normally floats in water. Suppose you submerge it completely and press one face flat against the bottom of a tank so that no water is under that face. Will the block float to the surface? Is there a buoyant force on it? Explain.

Q14.2. A rubber hose is attached to a funnel, and the free end is bent around to point upward. When water is poured into the funnel, it rises in the hose to the same level as in the funnel, even though the funnel has a lot more water in it than the hose does. Why? What supports the extra weight of the water in the funnel?

Q14.3. Comparing Example 14.1 (Section 14.1) and Example 14.2 (Section 14.2), it seems that 700 N of air is exerting a downward force of 2.0×10^6 N on the floor. How is this possible?

Q14.4. Equation (14.7) shows that an area ratio of 100 to 1 can give 100 times more output force than input force. Doesn't this violate conservation of energy? Explain.

Q14.5. You have probably noticed that the lower the tire pressure, the larger the contact area between the tire and the road. Why?

Q14.6. In hot-air ballooning, a large balloon is filled with air heated by a gas burner at the bottom. Why must the air be heated? How does the balloonist control ascent and descent?

Q14.7. In describing the size of a large ship, one uses such expressions as "it displaces 20,000 tons." What does this mean? Can the weight of the ship be obtained from this information?

Q14.8. You drop a solid sphere of aluminum in a bucket of water that sits on the ground. The buoyant force equals the weight of water displaced; this is less than the weight of the sphere, so the sphere sinks to the bottom. If you take the bucket with you on an elevator that accelerates upward, the apparent weight of the water increases and the buoyant force on the sphere increases. Could the acceleration of the elevator be great enough to make the sphere pop up out of the water? Explain.

Q14.9. A rigid, lighter-than-air dirigible filled with helium cannot continue to rise indefinitely. Why? What determines the maximum height it can attain?

Q14.10. Air pressure decreases with increasing altitude. So why is air near the surface not continuously drawn upward toward the lower-pressure regions above?

Q14.11. The purity of gold can be tested by weighing it in air and in water. How? Do you think you could get away with making a fake gold brick by gold-plating some cheaper material?

Q14.12. During the Great Mississippi Flood of 1993, the levees in St. Louis tended to rupture first at the bottom. Why?

Q14.13. A cargo ship travels from the Atlantic Ocean (salt water) to Lake Ontario (freshwater) via the St. Lawrence River. The ship rides several centimeters lower in the water in Lake Ontario than it did in the ocean. Explain why.

Q14.14. You push a piece of wood under the surface of a swimming pool. After it is completely submerged, you keep pushing it deeper and deeper. As you do this, what will happen to the buoyant force on it? Will the force keep increasing, stay the same, or decrease? Why?

Q14.15. An old question is "Which weighs more, a pound of feathers or a pound of lead?" If the weight in pounds is the gravitational force, will a pound of feathers balance a pound of lead on opposite pans of an equal-arm balance? Explain, taking into account buoyant forces.

Q14.16. Suppose the door of a room makes an airtight but frictionless fit in its frame. Do you think you could open the door if the air pressure on one side were standard atmospheric pressure and the air pressure on the other side differed from standard by 1%? Explain.

Q14.17. At a certain depth in an incompressible liquid, the absolute pressure is p. At twice this depth, will the absolute pressure be equal to $2p$, greater than $2p$, or less than $2p$? Justify your answer.

Q14.18. A piece of iron is glued to the top of a block of wood. When the block is placed in a bucket of water with the iron on top, the block floats. The block is now turned over so that the iron is submerged beneath the wood. Does the block float or sink? Does the water level in the bucket rise, drop, or stay the same? Explain your answers.

Q14.19. You take an empty glass jar and push it into a tank of water with the open mouth of the jar downward, so that the air inside the jar is trapped and cannot get out. If you push the jar deeper into the water, does the buoyant force on the jar stay the same? If not, does it increase or decrease? Explain your answer.

Q14.20. You are floating in a canoe in the middle of a swimming pool. Your friend is at the edge of the pool, carefully noting the level of the water on the side of the pool. You have a bowling ball with you in the canoe. If you carefully drop the bowling ball over the side of the canoe and it sinks to the bottom of the pool, does the water level in the pool rise or fall?

Q14.21. You are floating in a canoe in the middle of a swimming pool. A large bird flies up and lights on your shoulder. Does the water level in the pool rise or fall?

Q14.22. At a certain depth in the incompressible ocean the gauge pressure is p_g. At three times this depth, will the gauge pressure be greater than $3p_g$, equal to $3p_g$, or less than $3p_g$? Justify your answer.

Q14.23. An ice cube floats in a glass of water. When the ice melts, will the water level in the glass rise, fall, or remain unchanged? Explain.

Q14.24. You are told, "Bernoulli's equation tells us that where there is higher fluid speed, there is lower fluid pressure, and vice versa." Is this statement always true, even for an idealized fluid? Explain.

Q14.25. If the velocity at each point in space in steady-state fluid flow is constant, how can a fluid particle accelerate?

Q14.26. In a store-window vacuum cleaner display, a table-tennis ball is suspended in midair in a jet of air blown from the outlet hose of a tank-type vacuum cleaner. The ball bounces around a little but always moves back toward the center of the jet, even if the jet is tilted from the vertical. How does this behavior illustrate Bernoulli's equation?

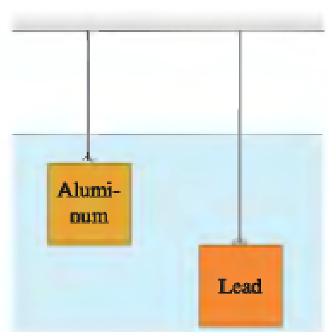
Q14.27. A tornado consists of a rapidly whirling air vortex. Why is the pressure always much lower in the center than at the outside? How does this condition account for the destructive power of a tornado?

Q14.28. Airports at high elevations have longer runways for takeoffs and landings than do airports at sea level. One reason is that aircraft engines develop less power in the thin air well above sea level. What is another reason?

Q14.29. When a smooth-flowing stream of water comes out of a faucet, it narrows as it falls. Explain why this happens.

Q14.30. Identical-size lead and aluminum cubes are suspended at different depths by two wires in a large vat of water (Fig. 14.32). (a) Which cube experiences a greater buoyant force? (b) For which cube is the tension in the wire greater? (c) Which cube experiences a greater force on its lower face? (d) For which cube is the difference in pressure between the upper and lower faces greater?

Figure 14.32 Question Q14.30.



Exercises

Section 14.1 Density

14.1. On a part-time job, you are asked to bring a cylindrical iron rod of length 85.8 cm and diameter 2.85 cm from a storage room to a machinist. Will you need a cart? (To answer, calculate the weight of the rod.)

14.2. Miles per Kilogram. The density of gasoline is 737 kg/m^3. If your new hybrid car gets 45.0 miles per gallon of gasoline, what is its mileage in miles per kilogram of gasoline? (See Appendix E.)

14.3. You purchase a rectangular piece of metal that has dimensions $5.0 \times 15.0 \times 30.0$ mm and mass 0.0158 kg. The seller tells you that the metal is gold. To check this, you compute the average density of the piece. What value do you get? Were you cheated?

14.4. Gold Brick. You win the lottery and decide to impress your friends by exhibiting a million-dollar cube of gold. At the time, gold is selling for \$426.60 per troy ounce, and 1.0000 troy ounce equals 31.1035 g. How tall would your million-dollar cube be?

14.5. A uniform lead sphere and a uniform aluminum sphere have the same mass. What is the ratio of the radius of the aluminum sphere to the radius of the lead sphere?

14.6. (a) What is the average density of the sun? (b) What is the average density of a neutron star that has the same mass as the sun but a radius of only 20.0 km?

14.7. A hollow cylindrical copper pipe is 1.50 m long and has an outside diameter of 3.50 cm and an inside diameter of 2.50 cm. How much does it weigh?

Section 14.2 Pressure in a Fluid

14.8. Black Smokers. Black smokers are hot volcanic vents that emit smoke deep in the ocean floor. Many of them teem with exotic creatures, and some biologists think that life on earth may have begun around such vents. The vents range in depth from about 1500 m to 3200 m below the surface. What is the gauge pressure at a 3200-m deep vent, assuming that the density of water does not vary? Express your answer in pascals and atmospheres.

14.9. Oceans on Mars. Scientists have found evidence that Mars may once have had an ocean 0.500 km deep. The acceleration due to gravity on Mars is 3.71 m/s^2. (a) What would be the gauge pressure at the bottom of such an ocean, assuming it was freshwater? (b) To what depth would you need to go in the earth's ocean to experience the same gauge pressure?

14.10. (a) Calculate the difference in blood pressure between the feet and top of the head for a person who is 1.65 m tall. (b) Consider a cylindrical segment of a blood vessel 2.00 cm long and 1.50 mm in diameter. What *additional* outward force would such a vessel need to withstand in the person's feet compared to a similar vessel in her head?

14.11. In intravenous feeding, a needle is inserted in a vein in the patient's arm and a tube leads from the needle to a reservoir of fluid (density 1050 kg/m^3) located at height h above the arm. The top of the reservoir is open to the air. If the gauge pressure inside the vein is 5980 Pa, what is the minimum value of h that allows fluid to enter the vein? Assume the needle diameter is large enough that you can ignore the viscosity (see Section 14.6) of the fluid.

14.12. A barrel contains a 0.120-m layer of oil floating on water that is 0.250 m deep. The density of the oil is 600 kg/m^3. (a) What is the gauge pressure at the oil–water interface? (b) What is the gauge pressure at the bottom of the barrel?

14.13. A 975-kg car has its tires each inflated to "32.0 pounds." (a) What are the absolute and gauge pressures in these tires in $lb/in.^2$, Pa, and atm? (b) If the tires were perfectly round, could the tire pressure exert any force on the pavement? (Assume that the tire walls are flexible so that the pressure exerted by the tire on the pavement equals the air pressure inside the tire.) (c) If you examine a car's tires, it is obvious that there is some flattening at the bottom. What is the total contact area for all four tires of the flattened part of the tires at the pavement?

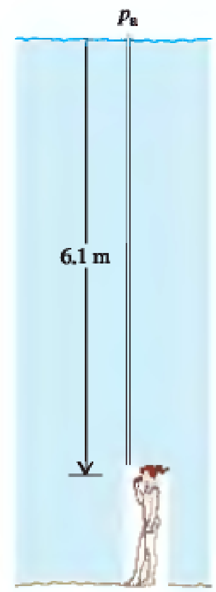
14.14. You are designing a diving bell to withstand the pressure of seawater at a depth of 250 m. (a) What is the gauge pressure at this depth? (You can ignore changes in the density of the water with depth.) (b) At this depth, what is the net force due to the water outside and the air inside the bell on a circular glass window 30.0 cm in diameter if the pressure inside the diving bell equals the pressure at the surface of the water? (You can ignore the small variation of pressure over the surface of the window.)

14.15. What gauge pressure must a pump produce to pump water from the bottom of the Grand Canyon (elevation 730 m) to Indian Gardens (elevation 1370 m)? Express your results in pascals and in atmospheres.

14.16. The liquid in the open-tube manometer in Fig. 14.9a is mercury, $y_1 = 3.00$ cm, and $y_2 = 7.00$ cm. Atmospheric pressure is 980 millibars. (a) What is the absolute pressure at the bottom of the U-shaped tube? (b) What is the absolute pressure in the open tube at a depth of 4.00 cm below the free surface? (c) What is the absolute pressure of the gas in the tank? (d) What is the gauge pressure of the gas in pascals?

14.17. There is a maximum depth at which a diver can breathe through a snorkel tube (Fig. 14.33) because as the depth increases, so does the pressure difference, which tends to collapse the diver's lungs. Since the snorkel connects the air in the lungs to the atmosphere at the surface, the pressure inside the lungs is atmospheric pressure. What is the external–internal pressure difference when the diver's lungs are at a depth of 6.1 m (about 20 ft)? Assume that the diver is in freshwater. (A scuba diver breathing from compressed air tanks can operate at greater depths than can a snorkeler, since the pressure of the air inside the scuba diver's lungs increases to match the external pressure of the water.)

Figure 14.33 Exercise 14.17.



14.18. A tall cylinder with a cross-sectional area 12.0 cm^2 is partially filled with mercury; the surface of the mercury is 5.00 cm above the bottom of the cylinder. Water is slowly poured in on top of the mercury, and the two fluids don't mix. What volume of water must be added to double the gauge pressure at the bottom of the cylinder?

14.19. A lake in the far north of the Yukon is covered with a 1.75-m-thick layer of ice. Find the absolute pressure and the gauge pressure at a depth of 2.50 m in the lake.

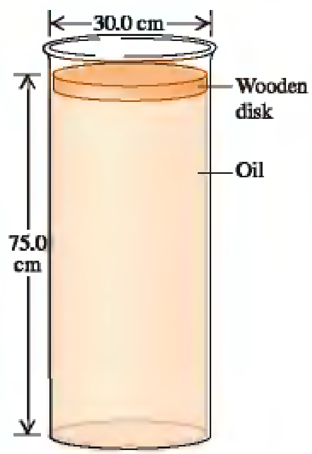
14.20. A closed container is partially filled with water. Initially, the air above the water is at atmospheric pressure $(1.01 \times 10^5 \text{ Pa})$ and the gauge pressure at the bottom of the water is 2500 Pa. Then additional air is pumped in, increasing the pressure of the air above the water by 1500 Pa. (a) What is the gauge pressure at the bottom of the water? (b) By how much must the water level in the container be reduced, by drawing some water out through a valve at the bottom of the container, to return the gauge pressure at the bottom of the water to its original value of 2500 Pa? The pressure of the air above the water is maintained at 1500 Pa above atmospheric pressure.

14.21. An electrical short cuts off all power to a submersible diving vehicle when it is 30 m below the surface of the ocean. The crew must push out a hatch of area 0.75 m^2 and weight 300 N on the bottom to escape. If the pressure inside is 1.0 atm, what downward force must the crew exert on the hatch to open it?

14.22. Exploring Venus. The surface pressure on Venus is 92 atm, and the acceleration due to gravity there is $0.894g$. In a future exploratory mission, an upright cylindrical tank of benzene is sealed at the top but still pressurized at 92 atm just above the benzene. The tank has a diameter of 1.72 m, and the benzene column is 11.50 m tall. Ignore any effects due to the very high temperature on Venus. (a) What total force is exerted on the inside surface of the bottom of the tank? (b) What force does the Venusian atmosphere exert on the outside surface of the bottom of the tank? (c) What total inward force does the atmosphere exert on the vertical walls of the tank?

14.23. A cylindrical disk of wood weighing 45.0 N and having a diameter of 30.0 cm floats on a cylinder of oil of density 0.850 g/cm^3 (Fig. 14.34). The cylinder of oil is 75.0 cm deep and has a diameter the same as that of the wood. (a) What is the gauge pressure at the top of the oil column? (b) Suppose now that someone puts a weight of 83.0 N on top of the wood, but no oil seeps around the edge of the wood. What is the *change* in pressure at (i) the bottom of the oil, and (ii) halfway down in the oil?

Figure 14.34 Exercise 14.23.



14.24. Hydraulic Lift I. For the hydraulic lift shown in Fig. 14.8, what must be the ratio of the diameter of the vessel at the car to the diameter of the vessel where the force F_1 is applied so that a 1520-kg car can be lifted with a force F_1 of just 125 N?

14.25. Hydraulic Lift II. The piston of a hydraulic automobile lift is 0.30 m in diameter. What gauge pressure, in pascals, is required to lift a car with a mass of 1200 kg? Also express this pressure in atmospheres.

Section 14.3 Buoyancy

14.26. A slab of ice floats on a freshwater lake. What minimum volume must the slab have for a 45.0-kg woman to be able to stand on it without getting her feet wet?

14.27. An ore sample weighs 17.50 N in air. When the sample is suspended by a light cord and totally immersed in water, the tension in the cord is 11.20 N. Find the total volume and the density of the sample.

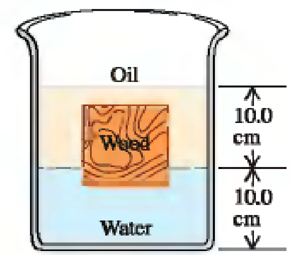
14.26. You are preparing some apparatus for a visit to a newly discovered planet Caasi having oceans of glycerine and a surface acceleration due to gravity of 4.15 m/s^2. If your apparatus floats in the oceans on earth with 25.0% of its volume submerged, what percentage will be submerged in the glycerine oceans of Caasi?

14.29. An object of average density ρ floats at the surface of a fluid of density ρ_{fluid}. (a) How must the two densities be related? (b) In view of the answer to part (a), how can steel ships float in water? (c) In terms of ρ and ρ_{fluid}, what fraction of the object is submerged and what fraction is above the fluid? Check that your answers give the correct limiting behavior as $\rho \rightarrow \rho_{\text{fluid}}$ and as $\rho \rightarrow 0$. (d) While on board your yacht, your cousin Throckmorton cuts a rectangular piece (dimensions $5.0 \times 4.0 \times 3.0$ cm) out of a life preserver and throws it into the ocean. The piece has a mass of 42 g. As it floats in the ocean, what percentage of its volume is above the surface?

14.30. A hollow plastic sphere is held below the surface of a freshwater lake by a cord anchored to the bottom of the lake. The sphere has a volume of 0.650 m^3 and the tension in the cord is 900 N. (a) Calculate the buoyant force exerted by the water on the sphere. (b) What is the mass of the sphere? (c) The cord breaks and the sphere rises to the surface. When the sphere comes to rest, what fraction of its volume will be submerged?

14.31. A cubical block of wood, 10.0 cm on a side, floats at the interface between oil and water with its lower surface 1.50 cm below the interface (Fig. 14.35). The density of the oil is 790 kg/m^3. (a) What is the gauge pressure at the upper face of the block? (b) What is the gauge pressure at the lower face of the block? (c) What are the mass and density of the block?

Figure 14.35 Exercise 14.31.



14.32. A solid aluminum ingot weighs 89 N in air. (a) What is its volume? (b) The ingot is suspended from a rope and totally immersed in water. What is the tension in the rope (the *apparent* weight of the ingot in water)?

14.33. A rock is suspended by a light string. When the rock is in air, the tension in the string is 39.2 N. When the rock is totally immersed in water, the tension is 28.4 N. When the rock is totally immersed in an unknown liquid, the tension is 18.6 N. What is the density of the unknown liquid?

Section 14.4 Fluid Flow

14.34. Water runs into a fountain, filling all the pipes, at a steady rate of 0.750 m/s^3. (a) How fast will it shoot out of a hole 4.50 cm in diameter? (b) At what speed will it shoot out if the diameter of the hole is three times as large?

14.35. A shower head has 20 circular openings, each with radius 1.0 mm. The shower head is connected to a pipe with radius 0.80 cm. If the speed of water in the pipe is 3.0 m/s, what is its speed as it exits the shower-head openings?

14.36. Water is flowing in a pipe with a varying cross-sectional area, and at all points the water completely fills the pipe. At point 1 the cross-sectional area of the pipe is 0.070 m^2, and the magnitude of the fluid velocity is 3.50 m/s. (a) What is the fluid speed at points in the pipe where the cross-sectional area is (a) 0.105 m^2 and (b) 0.047 m^2? (c) Calculate the volume of water discharged from the open end of the pipe in 1.00 hour.

14.37. Water is flowing in a pipe with a circular cross section but with varying cross-sectional area, and at all points the water completely fills the pipe. (a) At one point in the pipe the radius is 0.150 m. What is the speed of the water at this point if water is flowing into this pipe at a steady rate of $1.20 \text{ m}^3\text{/s}$? (b) At a second point in the pipe the water speed is 3.80 m/s. What is the radius of the pipe at this point?

14.36. (a) Derive Eq. (14.12). (b) If the density increases by 1.50% from point 1 to point 2, what happens to the volume flow rate?

Section 14.5 Bernoulli's Equation

14.39. A sealed tank containing seawater to a height of 11.0 m also contains air above the water at a gauge pressure of 3.00 atm. Water flows out from the bottom through a small hole. How fast is this water moving?

14.40. A small circular hole 6.00 mm in diameter is cut in the side of a large water tank, 14.0 m below the water level in the tank. The top of the tank is open to the air. Find (a) the speed of efflux of the water, and (b) the volume discharged per second.

14.41. What gauge pressure is required in the city water mains for a stream from a fire hose connected to the mains to reach a vertical height of 15.0 m? (Assume that the mains have a much larger diameter than the fire hose.)

14.42. At one point in a pipeline the water's speed is 3.00 m/s and the gauge pressure is 5.00×10^4 Pa. Find the gauge pressure at a second point in the line, 11.0 m lower than the first, if the pipe diameter at the second point is twice that at the first.

14.43. Lift on an Airplane. Air streams horizontally past a small airplane's wings such that the speed is 70.0 m/s over the top surface and 60.0 m/s past the bottom surface. If the plane has a wing area of 16.2 m^2 on the top and on the bottom, what is the net vertical force that the air exerts on the airplane? The density of the air is 1.20 kg/m^3.

14.44. A soft drink (mostly water) flows in a pipe at a beverage plant with a mass flow rate that would fill 220 0.355-L cans per minute. At point 2 in the pipe, the gauge pressure is 152 kPa and the cross-sectional area is 8.00 cm^2. At point 1, 1.35 m above point 2, the cross-sectional area is 2.00 cm^2. Find the (a) mass flow rate; (b) volume flow rate; (c) flow speeds at points 1 and 2; (d) gauge pressure at point 1.

14.45. At a certain point in a horizontal pipeline, the water's speed is 2.50 m/s and the gauge pressure is 1.80×10^4 Pa. Find the gauge pressure at a second point in the line if the cross-sectional area at the second point is twice that at the first.

14.46. A golf course sprinkler system discharges water from a horizontal pipe at the rate of 7200 cm^3/s. At one point in the pipe, where the radius is 4.00 cm, the water's absolute pressure is 2.40×10^5 Pa. At a second point in the pipe, the water passes through a constriction where the radius is 2.00 cm. What is the water's absolute pressure as it flows through this constriction?

Problems

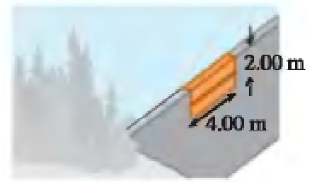
14.47. In a lecture demonstration, a professor pulls apart two hemispherical steel shells (diameter D) with ease using their attached handles. She then places them together, pumps out the air to an absolute pressure of p, and hands them to a bodybuilder in the back row to pull apart. (a) If atmospheric pressure is p_0, how much force must the bodybuilder exert on each shell? (b) Evaluate your answer for the case $p = 0.025$ atm, $D = 10.0$ cm.

14.40. The deepest point known in any of the earth's oceans is in the Marianas Trench, 10.92 km deep. (a) Assuming water is incompressible, what is the pressure at this depth? Use the density of seawater. (b) The actual pressure is 1.16×10^8 Pa; your calculated value will be less because the density actually varies with depth. Using the compressibility of water and the actual pressure, find the density of the water at the bottom of the Marianas Trench. What is the percent change in the density of the water?

14.49. A swimming pool is 5.0 m long, 4.0 m wide, and 3.0 m deep. Compute the force exerted by the water against (a) the bottom; and (b) either end. (*Hint:* Calculate the force on a thin, horizontal strip at a depth h, and integrate this over the end of the pool.) Do not include the force due to air pressure.

14.50. The upper edge of a gate in a dam runs along the water surface. The gate is 2.00 m high and 4.00 m wide and is hinged along a horizontal line through its center (Fig. 14.36). Calculate the torque about the hinge arising from the force due to the water. (*Hint:* Use a procedure similar to that used in Problem 14.49; calculate the torque on a thin, horizontal strip at a depth h and integrate this over the gate.)

Figure 14.36 Problem 14.50.



14.51. Force and Torque on a Dam. A dam has the shape of a rectangular solid. The side facing the lake has area A and height H. The surface of the freshwater lake behind the dam is at the top of the dam. (a) Show that the net horizontal force exerted by the water on the dam equals $\frac{1}{2}\rho gHA$—that is, the average gauge pressure across the face of the dam times the area (see Problem 14.49). (b) Show that the torque exerted by the water about an axis along the bottom of the dam is $\rho gH^2A/6$. (c) How do the force and torque depend on the size of the lake?

14.52. Submarines on Europa. Some scientists are eager to send a remote-controlled submarine to Jupiter's moon Europa to search for life in its oceans below an icy crust. Europa's mass has been measured to be 4.78×10^{22} kg, its diameter is 3130 km, and it has no appreciable atmosphere. Assume that the layer of ice at the surface is not thick enough to exert substantial force on the water. If the windows of the submarine you are designing are 25.0 cm square and can stand a maximum inward force of 9750 N per window, what is the greatest depth to which this submarine can safely dive?

14.53. An astronaut is standing at the north pole of a newly discovered, spherically symmetric planet of radius R. In his hands he holds a container full of a liquid with mass m and volume V. At the surface of the liquid, the pressure is p_0; at a depth d below the surface, the pressure has a greater value p. From this information, determine the mass of the planet.

14.54. Ballooning on Mars. It has been proposed that we could explore Mars using inflated balloons to hover just above the surface. The buoyancy of the atmosphere would keep the balloon aloft. The density of the Martian atmosphere is 0.0154 kg/m^3 (although this varies with temperature). Suppose we construct these balloons of a thin but tough plastic having a density such that each square meter has a mass of 5.00 g. We inflate them with a very light gas whose mass we can neglect. (a) What should be the radius and mass of these balloons so they just hover above the surface of Mars? (b) If we released one of the balloons from part (a) on earth, where the atmospheric density is 1.20 kg/m^3, what would be its initial acceleration assuming it was the same size as on Mars? Would it go up or down? (c) If on Mars these balloons have five times the radius found in part (a), how heavy an instrument package could they carry?

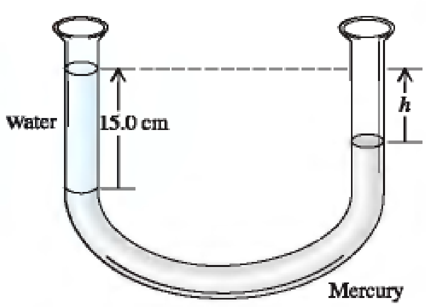
14.55. The earth does not have a uniform density; it is most dense at its center and least dense at its surface. An approximation of its density is $\rho(r) = A - Br$, where $A = 12{,}700\ kg/m^3$ and $B = 1.50 \times 10^{-3}\ kg/m^4$. Use $R = 6.37 \times 10^6$ m for the radius of the earth approximated as a sphere. (a) Geological evidence indicates that the densities are 13,100 kg/m^3 and 2,400 kg/m^3 at the earth's center and surface, respectively. What values does the linear approximation model give for the densities at these two locations? (b) Imagine dividing the earth into concentric, spherical shells. Each shell has radius r, thickness dr, volume $dV = 4\pi r^2\,dr$, and mass $dm = \rho(r)dV$. By integrating from $r = 0$ to $r = R$, show that the mass of the earth in this model is $M = \frac{4}{3}\pi R^3\left(A - \frac{3}{4}BR\right)$. (c) Show that the given values of A and B give the correct mass of the earth to within 0.4%. (d) We saw in Section 12.6 that a uniform spherical shell gives no contribution to g inside it. Show that

$g(r) = \frac{4}{3}\pi G r\left(A - \frac{3}{4}Br\right)$ inside the earth in this model. (e) Verify that the expression of part (d) gives $g = 0$ at the center of the earth and $g = 9.85\ \mathrm{m/s^2}$ at the surface. (f) Show that in this model g does *not* decrease uniformly with depth but rather has a maximum of $4\pi GA^2/9B = 10.01\ \mathrm{m/s^2}$ at $r = 2A/3B = 5640$ km.

14.56. In Example 12.10 (Section 12.6) we saw that inside a planet of uniform density (not a realistic assumption for the earth) the acceleration due to gravity increases uniformly with distance from the center of the planet. That is, $g(r) = g_s r/R$, where g_s is the acceleration due to gravity at the surface, r is the distance from the center of the planet, and R is the radius of the planet. The interior of the planet can be treated approximately as an incompressible fluid of density ρ. (a) Replace the height y in Eq. (14.4) with the radial coordinate r and integrate to find the pressure inside a uniform planet as a function of r. Let the pressure at the surface be zero. (This means ignoring the pressure of the planet's atmosphere.) (b) Using this model, calculate the pressure at the center of the earth. (Use a value of ρ equal to the average density of the earth, calculated from the mass and radius given in Appendix F.) (c) Geologists estimate the pressure at the center of the earth to be approximately 4×10^{11} Pa. Does this agree with your calculation for the pressure at $r = 0$? What might account for any differences?

14.57. A U-shaped tube open to the air at both ends contains some mercury. A quantity of water is carefully poured into the left arm of the U-shaped tube until the vertical height of the water column is 15.0 cm (Fig. 14.37). (a) What is the gauge pressure at the water–mercury interface? (b) Calculate the vertical distance h from the top of the mercury in the right-hand arm of the tube to the top of the water in the left-hand arm.

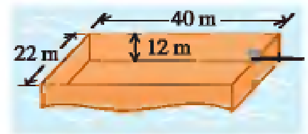
Figure 14.37 Problem 14.57.



14.58. The Great Molasses Flood. On the afternoon of January 15, 1919, an unusually warm day in Boston, a 27.4-m-high, 27.4-m-diameter cylindrical metal tank used for storing molasses ruptured. Molasses flooded into the streets in a 9-m-deep stream, killing pedestrians and horses, and knocking down buildings. The molasses had a density of $1600\ \mathrm{kg/m^3}$. If the tank was full before the accident, what was the total outward force the molasses exerted on its sides? (*Hint:* Consider the outward force on a circular ring of the tank wall of width dy and at a depth y below the surface. Integrate to find the total outward force. Assume that before the tank ruptured, the pressure at the surface of the molasses was equal to the air pressure outside the tank.)

14.58. An open barge has the dimensions shown in Fig. 14.38. If the barge is made out of 4.0-cm-thick steel plate on each of its four sides and its bottom, what mass of coal can the barge carry in freshwater without sinking? Is there enough room in the barge to hold this amount of coal? (The density of coal is about $1500\ \mathrm{kg/m^3}$.)

Figure 14.38 Problem 14.59.



14.60. A hot-air balloon has a volume of $2200\ \mathrm{m^3}$. The balloon fabric (the envelope) weighs 900 N. The basket with gear and full propane tanks weighs 1700 N. If the balloon can barely lift an additional 3200 N of passengers, breakfast, and champagne when the outside air density is $1.23\ \mathrm{kg/m^3}$, what is the average density of the heated gases in the envelope?

14.61. Advertisements for a certain small car claim that it floats in water. (a) If the car's mass is 900 kg and its interior volume is $3.0\ \mathrm{m^3}$, what fraction of the car is immersed when it floats? You can ignore the volume of steel and other materials. (b) Water gradually leaks in and displaces the air in the car. What fraction of the interior volume is filled with water when the car sinks?

14.62. A single ice cube with mass 9.70 g floats in a glass completely full of $420\ \mathrm{cm^3}$ of water. You can ignore the water's surface tension and its variation in density with temperature (as long as it remains a liquid). (a) What volume of water does the ice cube displace? (b) When the ice cube has completely melted, has any water overflowed? If so, how much? If not, explain why this is so. (c) Suppose the water in the glass had been very salty water of density $1050\ \mathrm{kg/m^3}$. What volume of salt water would the 9.70-g ice cube displace? (d) Redo part (b) for the freshwater ice cube in the salty water.

14.63. A piece of wood is 0.600 m long, 0.250 m wide, and 0.080 m thick. Its density is $600\ \mathrm{kg/m^3}$. What volume of lead must be fastened underneath it to sink the wood in calm water so that its top is just even with the water level? What is the mass of this volume of lead?

14.64. A hydrometer consists of a spherical bulb and a cylindrical stem with a cross-sectional area of $0.400\ \mathrm{cm^2}$ (see Fig. 14.13a). The total volume of bulb and stem is $13.2\ \mathrm{cm^3}$. When immersed in water, the hydrometer floats with 8.00 cm of the stem above the water surface. When the hydrometer is immersed in an organic fluid, 3.20 cm of the stem is above the surface. Find the density of the organic fluid. (*Note:* This illustrates the precision of such a hydrometer. Relatively small density differences give rise to relatively large differences in hydrometer readings.)

14.65. The densities of air, helium, and hydrogen (at $p = 1.0$ atm and $T = 20°\mathrm{C}$) are $1.20\ \mathrm{kg/m^3}$, $0.166\ \mathrm{kg/m^3}$, and $0.0899\ \mathrm{kg/m^3}$, respectively. (a) What is the volume in cubic meters displaced by a hydrogen-filled airship that has a total "lift" of 120 kN? (The "lift" is the amount by which the buoyant force exceeds the weight of the gas that fills the airship.) (b) What would be the "lift" if helium were used instead of hydrogen? In view of your answer, why is helium used in modern airships like advertising blimps?

14.66. SHM of a Floating Object. An object with height h, mass M, and a uniform cross-sectional area A floats upright in a liquid with density ρ. (a) Calculate the vertical distance from the surface of the liquid to the bottom of the floating object at equilibrium. (b) A downward force with magnitude F is applied to the top of the object. At the new equilibrium position, how much farther below the surface of the liquid is the bottom of the object than it was in part (a)? (Assume that some of the object remains above the surface of the liquid.) (c) Your result in part (b) shows that if the force is suddenly removed, the object will oscillate up and down in SHM. Calculate the period of this motion in terms of the density ρ of the liquid, the mass M, and cross-sectional area A of the object. You can ignore the damping due to fluid friction (see Section 13.7).

14.67. A 950-kg cylindrical can buoy floats vertically in salt water. The diameter of the buoy is 0.900 m. (a) Calculate the additional distance the buoy will sink when a 70.0-kg man stands on top. (Use the expression derived in part (b) of Problem 14.66.) (b) Calculate the period of the resulting vertical SHM when the man dives off. (Use the expression derived in part (c) of Problem 14.66, and as in that problem, you can ignore the damping due to fluid friction.)

14.68. A firehose must be able to shoot water to the top of a building 35.0 m tall when aimed straight up. Water enters this hose at a steady rate of $0.500\ \mathrm{m^3/s}$ and shoots out of a round nozzle. (a) What is the maximum diameter this nozzle can have? (b) If the only nozzle available has a diameter twice as great, what is the highest point the water can reach?

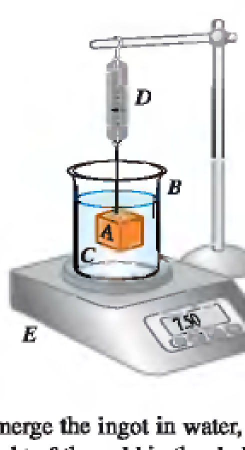
14.69. You drill a small hole in the side of a vertical cylindrical water tank that is standing on the ground with its top open to the air. (a) If the water level has a height H, at what height above the base should you drill the hole for the water to reach its greatest distance from the base of the cylinder when it hits the ground? (b) What is the greatest distance the water will reach?

14.70. A vertical cylindrical tank of cross-sectional area A_1 is open to the air at the top and contains water to a depth h_0. A worker accidentally pokes a hole of area A_2 in the bottom of the tank. (a) Derive an equation for the depth h of the water as a function of time t after the hole is poked. (b) How long after the hole is made does it take for the tank to empty out?

14.71. A block of balsa wood placed in one scale pan of an equal-arm balance is exactly balanced by a 0.0950-kg brass mass in the other scale pan. Find the true mass of the balsa wood if its density is $150\ \mathrm{kg/m^3}$. Explain why it is accurate to ignore the buoyancy in air of the brass but *not* the buoyancy in air of the balsa wood.

14.72. Block A in Fig. 14.39 hangs by a cord from spring balance D and is submerged in a liquid C contained in beaker B. The mass of the beaker is 1.00 kg; the mass of the liquid is 1.80 kg. Balance D reads 3.50 kg, and balance E reads 7.50 kg. The volume of block A is $3.80 \times 10^{-3}\ \mathrm{m^3}$. (a) What is the density of the liquid? (b) What will each balance read if block A is pulled up out of the liquid?

Figure 14.39 Problem 14.72.



14.73. A hunk of aluminum is completely covered with a gold shell to form an ingot of weight 45.0 N. When you suspend the ingot from a spring balance and submerge the ingot in water, the balance reads 39.0 N. What is the weight of the gold in the shell?

14.74. A plastic ball has radius 12.0 cm and floats in water with 16.0% of its volume submerged. (a) What force must you apply to the ball to hold it at rest totally below the surface of the water? (b) If you let go of the ball, what is its acceleration the instant you release it?

14.75. The weight of a king's solid crown is w. When the crown is suspended by a light rope and completely immersed in water, the tension in the rope (the crown's apparent weight) is fw. (a) Prove that the crown's relative density (specific gravity) is $1/(1-f)$. Discuss the meaning of the limits as f approaches 0 and 1. (b) If the crown is solid gold and weighs 12.9 N in air, what is its apparent weight when completely immersed in water? (c) Repeat part (b) if the crown is solid lead with a very thin gold plating, but still has a weight in air of 12.9 N.

14.76. A piece of steel has a weight w, an apparent weight (see Problem 14.75) w_{water} when completely immersed in water, and an apparent weight w_{fluid} when completely immersed in an unknown fluid. (a) Prove that the fluid's density relative to water (specific gravity) is $(w - w_{\mathrm{fluid}})/(w - w_{\mathrm{water}})$. (b) Is this result reasonable for the three cases of w_{fluid} greater than, equal to, or less than w_{water}? (c) The apparent weight of the piece of steel in water of density $1000\ \mathrm{kg/m^3}$ is 87.2% of its weight. What percentage of its weight will its apparent weight be in formic acid (density $1220\ \mathrm{kg/m^3}$)?

14.77. You cast some metal of density ρ_m in a mold, but you are worried that there might be cavities within the casting. You measure the weight of the casting to be w, and the buoyant force when it is completely surrounded by water to be B. (a) Show that $V_0 = B/(\rho_{\mathrm{water}} g) - w/(\rho_m g)$ is the total volume of any enclosed cavities. (b) If your metal is copper, the casting's weight is 156 N, and the buoyant force is 20 N, what is the total volume of any enclosed cavities in your casting? What fraction is this of the total volume of the casting?

14.78. A cubical block of wood 0.100 m on a side and with a density of $550\ \mathrm{kg/m^3}$ floats in a jar of water. Oil with a density of $750\ \mathrm{kg/m^3}$ is poured on the water until the top of the oil layer is 0.035 m below the top of the block. (a) How deep is the oil layer? (b) What is the gauge pressure at the block's lower face?

14.79. Dropping Anchor. An iron anchor with mass 35.0 kg and density $7860\ \mathrm{kg/m^3}$ lies on the deck of a small barge that has vertical sides and floats in a freshwater river. The area of the bottom of the barge is $8.00\ \mathrm{m^2}$. The anchor is thrown overboard but is suspended above the bottom of the river by a rope; the mass and volume of the rope are small enough to ignore. After the anchor is overboard and the barge has finally stopped bobbing up and down, has the barge risen or sunk down in the water? By what vertical distance?

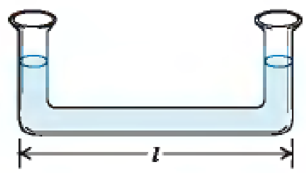
14.80. Assume that crude oil from a supertanker has density $750\ \mathrm{kg/m^3}$. The tanker runs aground on a sandbar. To refloat the tanker, its oil cargo is pumped out into steel barrels, each of which has a mass of 15.0 kg when empty and holds $0.120\ \mathrm{m^3}$ of oil. You can ignore the volume occupied by the steel from which the barrel is made. (a) If a salvage worker accidentally drops a filled, sealed barrel overboard, will it float or sink in the seawater? (b) If the barrel floats, what fraction of its volume will be above the water surface? If it sinks, what minimum tension would have to be exerted by a rope to haul the barrel up from the ocean floor? (c) Repeat parts (a) and (b) if the density of the oil is $910\ \mathrm{kg/m^3}$ and the mass of each empty barrel is 32.0 kg.

14.81. A cubical block of density ρ_B and with sides of length L floats in a liquid of greater density ρ_L. (a) What fraction of the block's volume is above the surface of the liquid? (b) The liquid is denser than water (density ρ_W) and does not mix with it. If water is poured on the surface of the liquid, how deep must the water layer be so that the water surface just rises to the top of the block? Express your answer in terms of L, ρ_B, ρ_L, and ρ_W. (c) Find the depth of the water layer in part (b) if the liquid is mercury, the block is made of iron, and the side length is 10.0 cm.

14.82. A barge is in a rectangular lock on a freshwater river. The lock is 60.0 m long and 20.0 m wide, and the steel doors on each end are closed. With the barge floating in the lock, a 2.50×10^6 N load of scrap metal is put onto the barge. The metal has density 9000 kg/m^3. (a) When the load of scrap metal, initially on the bank, is placed onto the barge, what vertical distance does the water in the lock rise? (b) The scrap metal is now pushed overboard into the water. Does the water level in the lock rise, fall, or remain the same? If it rises or falls, by what vertical distance does it change?

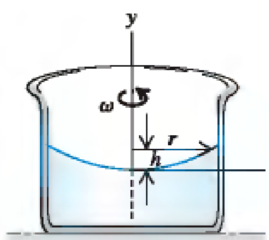
14.83. A U-shaped tube with a horizontal portion of length l (Fig. 14.40) contains a liquid. What is the difference in height between the liquid columns in the vertical arms (a) if the tube has an acceleration a toward the right? and (b) if the tube is mounted on a horizontal turntable rotating with an angular speed ω with one of the vertical arms on the axis of rotation? (c) Explain why the difference in height does not depend on the density of the liquid or on the cross-sectional area of the tube. Would it be the same if the vertical tubes did not have equal cross-sectional areas? Would it be the same if the horizontal portion were tapered from one end to the other? Explain.

Figure 14.40 Problem 14.83.



14.84. A cylindrical container of an incompressible liquid with density ρ rotates with constant angular speed ω about its axis of symmetry, which we take to be the y-axis (Fig. 14.41). (a) Show that the pressure at a given height within the fluid increases in the radial direction (outward from the axis of rotation) according to $\partial p/\partial r = \rho\omega^2 r$. (b) Integrate this partial differential equation to find the pressure as a function of distance from the axis of rotation along a horizontal line at $y = 0$. (c) Combine the result of part (b) with Eq. (14.5) to show that the surface of the rotating liquid has a *parabolic* shape, that is, the height of the liquid is given by $h(r) = \omega^2 r^2/2g$. (This technique is nsed for making parabolic telescope mirrors; liquid glass is rotated and allowed to solidify while rotating.)

Figure 14.41 Problem 14.84.

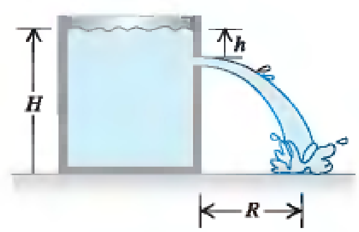


14.85. An incompressible fluid with density ρ is in a horizontal test tube of inner cross-sectional area A. The test tube spins in a horizontal circle in an ultracentrifuge at an angular speed ω. Gravitational forces are negligible. Consider a volume element of the fluid of area A and thickness dr' a distance r' from the rotation axis. The pressure on its inner surface is p and on its outer surface is $p + dp$. (a) Apply Newton's second law to the volume element to show that $dp = \rho\omega^2 r' dr'$. (b) If the surface of the fluid is at a radius r_0 where the pressure is p_0, show that the pressure p at a distance $r \geq r_0$ is $p = p_0 + \rho\omega^2(r^2 - r_0^2)/2$. (c) An object of volume V and density ρ_{ob} has its center of mass at a distance R_{cmob} from the axis. Show that the net horizontal force on the object is $\rho V\omega^2 R_{cm}$, where R_{cm} is the distance from the axis to the center of mass of the displaced fluid. (d) Explain why the object will move inward if $\rho R_{cm} > \rho_{ob} R_{cmob}$ and outward if $\rho R_{cm} < \rho_{ob} R_{cmob}$. (e) For small objects of uniform density, $R_{cm} = R_{cmob}$. What happens to a mixture of small objects of this kind with different densities in an ultracentrifuge?

14.86. Untethered helium balloons, floating in a car that has all the windows rolled up and outside air vents closed, move in the direction of the car's acceleration, but loose balloons filled with air move in the opposite direction. To show why, consider only the horizontal forces acting on the balloons. Let a be the magnitude of the car's forward acceleration. Consider a horizontal tube of air with a cross-sectional area A that extends from the windshield, where $x = 0$ and $p = p_0$, back along the x-axis. Now consider a volume element of thickness dx in this tube. The pressure on its front surface is p and the pressure on its rear surface is $p + dp$. Assume the air has a constant density ρ. (a) Apply Newton's second law to the volume element to show that $dp = \rho a\, dx$. (b) Integrate the result of part (a) to find the pressure at the front surface in terms of a and x. (c) To show that considering ρ constant is reasonable, calculate the pressure difference in atm for a distance as long as 2.5 m and a large acceleration of 5.0 m/s^2. (d) Show that the net horizontal force on a balloon of volume V is ρVa. (e) For negligible friction forces, show that the acceleration of the balloon (average density ρ_{bal}) is $(\rho/\rho_{bal})a$, so that the acceleration relative to the car is $a_{rel} = [(\rho/\rho_{bal}) - 1]a$. (f) Use the expression for a_{rel} in part (e) to explain the movement of the balloons.

14.87. Water stands at a depth H in a large, open tank whose side walls are vertical (Fig. 14.42). A hole is made in one of the walls at a depth h below the water surface. (a) At what distance R from the foot of the wall does the emerging stream strike the floor? (b) How far above the bottom of the tank could a second hole be cut so that the stream emerging from it could have the same range as for the first hole?

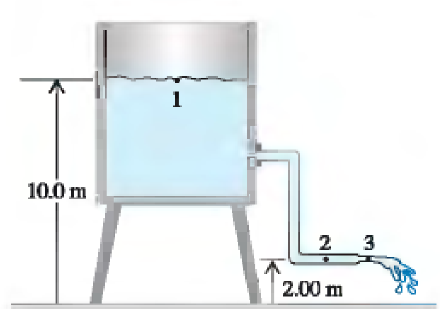
Figure 14.42 Problem 14.87.



14.88. A cylindrical bucket, open at the top, is 25.0 cm high and 10.0 cm in diameter. A circular hole with a cross-sectional area 1.50 cm^2 is cut in the center of the bottom of the bucket. Water flows into the bucket from a tube above it at the rate of 2.40×10^{-4} m^3/s. How high will the water in the bucket rise?

14.88. Water flows steadily from an open tank as in Fig. 14.43. The elevation of point 1 is 10.0 m, and the elevation of points 2 and 3 is 2.00 m. The cross-sectional area at point 2 is 0.0480 m^2; at point 3 it is 0.0160 m^2. The area of the tank is very large compared with the cross-sectional area of the pipe. Assuming that Bernoulli's equation applies, compute (a) the discharge rate in cubic meters per second; and (b) the gauge pressure at point 2.

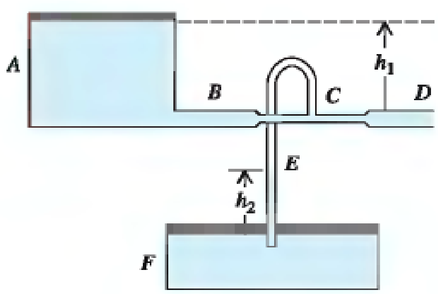
Figure 14.43 Problem 14.89.



14.90. In 1993 the radius of Hurricane Emily was about 350 km. The wind speed near the center ("eye") of the hurricane, whose radius was about 30 km, reached about 200 km/h. As air swirled in from the rim of the hurricane toward the eye, its angular momentum remained roughly constant. (a) Estimate the wind speed at the rim of the hurricane. (b) Estimate the pressure difference at the earth's surface between the eye and the rim. (*Hint:* See Table 14.1.). Where is the pressure greater? (c) If the kinetic energy of the swirling air in the eye could be converted completely to gravitational potential energy, how high would the air go? (d) In fact, the air in the eye is lifted to heights of several kilometers. How can you reconcile this with your answer to part (c)?

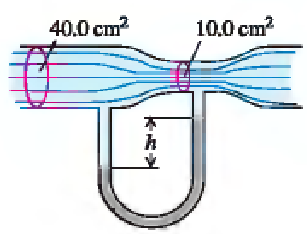
14.91. Two very large open tanks A and F (Fig. 14.44) contain the same liquid. A horizontal pipe BCD, having a constriction at C and open to the air at D, leads out of the bottom of tank A, and a vertical pipe E opens into the constriction at C and dips into the liquid in tank F. Assume streamline flow and no viscosity. If the cross-sectional area at C is one-half the area at D and if D is a distance h_1 below the level of the liquid in A, to what height h_2 will liquid rise in pipe E? Express your answer in terms of h_1.

Figure 14.44 Problem 14.91.



14.92. The horizontal pipe shown in Fig. 14.45 has a cross-sectional area of $40.0\ \text{cm}^2$ at the wider portions and $10.0\ \text{cm}^2$ at the constriction. Water is flowing in the pipe, and the discharge from the pipe is $6.00 \times 10^{-3}\ \text{m}^3/\text{s}\,(6.00\ \text{L/s})$. Find (a) the flow speeds at the wide and the narrow portions; (b) the pressure difference between these portions; (c) the difference in height between the mercury columns in the U-shaped tube.

Figure 14.45 Problem 14.92.



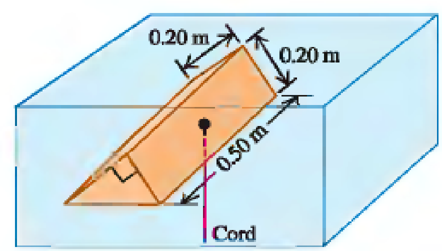
14.93. A liquid flowing from a vertical pipe has a definite shape as it flows from the pipe. To get the equation for this shape, assume that the liquid is in free fall once it leaves the pipe. Just as it leaves the pipe, the liquid has speed v_0 and the radius of the stream of liquid is r_0. (a) Find an equation for the speed of the liquid as a function of the distance y it has fallen. Combining this with the equation of continuity, find an expression for the radius of the stream as a function of y. (b) If water flows out of a vertical pipe at a speed of 1.20 m/s, how far below the outlet will the radius be one-half the original radius of the stream?

Challenge Problems

14.94. A rock with mass $m = 3.00\ \text{kg}$ is suspended from the roof of an elevator by a light cord. The rock is totally immersed in a bucket of water that sits on the floor of the elevator, but the rock doesn't touch the bottom or sides of the bucket. (a) When the elevator is at rest, the tension in the cord is 21.0 N. Calculate the volume of the rock. (b) Derive an expression for the tension in the cord when the elevator is accelerating *upward* with an acceleration of magnitude a. Calculate the tension when $a = 2.50\ \text{m/s}^2$ upward. (c) Derive an expression for the tension in the cord when the elevator is accelerating *downward* with an acceleration of magnitude a. Calculate the tension when $a = 2.50\ \text{m/s}^2$ downward. (d) What is the tension when the elevator is in free fall with a downward acceleration equal to g?

14.95. Suppose a piece of styrofoam, $\rho = 180\ \text{kg/m}^3$, is held completely submerged in water (Fig. 14.46). (a) What is the tension in the cord? Find this using Archimedes's principle. (b) Use $p = p_0 + \rho g h$ to calculate directly the force exerted by the water on the two sloped sides and the bottom of the styrofoam; then show that the vector sum of these forces is the buoyant force.

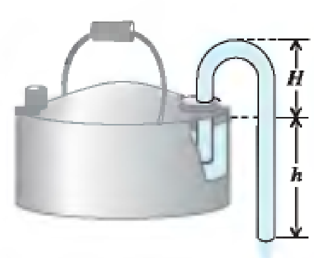
Figure 14.46 Challenge Problem 14.95.



14.96. A large tank with diameter D, open to the air, contains water to a height H. A small hole with diameter d $(d \ll D)$ is made at the base of the tank. Ignoring any effects of viscosity, calculate the time it takes for the tank to drain completely.

14.97. A *siphon*, as shown in Fig. 14.47, is a convenient device for removing liquids from containers. To establish the flow, the tube must be initially filled with fluid. Let the fluid have density ρ, and let the atmospheric pressure be p_a. Assume that the cross-sectional area of the tube is the same at all points along it. (a) If the lower end of the siphon is at a distance h below the surface of the liquid in the container, what is the speed of the fluid as it flows out the lower end of the siphon? (Assume that the container has a very large diameter, and ignore any effects of viscosity.) (b) A curious

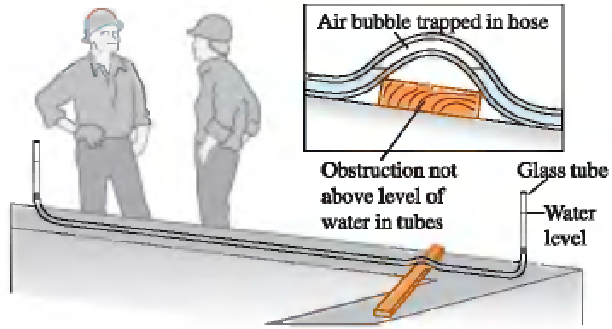
Figure 14.47 Challenge Problem 14.97.



feature of a siphon is that the fluid initially flows "uphill." What is the greatest height H that the high point of the tube can have if flow is still to occur?

14.98. The following passage is quoted from a letter. *It is the practice of carpenters hereabouts, when laying out and leveling up the foundations of relatively long buildings, to use a garden hose filled with water, with glass tubes 10 to 12 inches long thrust into the ends of the hose. The theory is that water, seeking a common level, will be the same height in both the tubes and thus effect a level. Now the question rises as to what happens if a bubble of air is left in the hose. Our greybeards contend the air will not affect the reading from one end to the other. Others say that it will cause important inaccuracies.* Can you give a relatively simple solution to this problem, together with an explanation? Figure 14.48 gives a rough sketch of the situation that caused the dispute.

Figure 14.48 Challenge Problem 14.98.



17 TEMPERATURE AND HEAT

LEARNING GOALS

By studying this chapter, you will learn:

- The meaning of thermal equilibrium, and what thermometers really measure.
- How different types of thermometers function.
- The physics behind the absolute, or Kelvin, temperature scale.
- How the dimensions of an object change as a result of a temperature change.
- The meaning of heat, and how it differs from temperature.
- How to do calculations that involve heat flow, temperature changes, and changes of phase.
- How heat is transferred by conduction, convection, and radiation.

? At a steelworks, molten iron is heated to 1500° Celsius to remove impurities. Is it accurate to say that the molten iron contains heat?

Whether it's a sweltering summer day or a frozen midwinter night, your body needs to be kept at a nearly constant temperature. It has effective temperature-control mechanisms, but sometimes it needs help. On a hot day you wear less clothing to improve heat transfer from your body to the air and for better cooling by evaporation of perspiration. You drink cold beverages and may sit near a fan or in an air-conditioned room. On a cold day you wear more clothes or stay indoors where it's warm. When you're outside, you keep active and drink hot liquids to stay warm. The concepts in this chapter will help you understand the basic physics of keeping warm or cool.

The terms "temperature" and "heat" are often used interchangeably in everyday language. In physics, however, these two terms have very different meanings. In this chapter we'll define temperature in terms of how it's measured and see how temperature changes affect the dimensions of objects. We'll see that heat refers to energy transfer caused by temperature differences and learn how to calculate and control such energy transfers.

Our emphasis in this chapter is on the concepts of temperature and heat as they relate to *macroscopic* objects such as cylinders of gas, ice cubes, and the human body. In Chapter 18 we'll look at these same concepts from a *microscopic* viewpoint in terms of the behavior of individual atoms and molecules. These two chapters lay the groundwork for the subject of **thermodynamics,** the study of energy transformations involving heat, mechanical work, and other aspects of energy and how these transformations relate to the properties of matter. Thermodynamics forms an indispensable part of the foundation of physics, chemistry, and the life sciences, and its applications turn up in such places as car engines, refrigerators, biochemical processes, and the structure of stars. We'll explore the key ideas of thermodynamics in Chapters 19 and 20.

17.1 Temperature and Thermal Equilibrium

The concept of **temperature** is rooted in qualitative ideas of "hot" and "cold" based on our sense of touch. A body that feels hot usually has a higher temperature than a similar body that feels cold. That's pretty vague, and the senses can be deceived. But many properties of matter that we can *measure* depend on temperature. The length of a metal rod, steam pressure in a boiler, the ability of a wire to conduct an electric current, and the color of a very hot glowing object—all these depend on temperature.

Temperature is also related to the kinetic energies of the molecules of a material. In general this relationship is fairly complex, so it's not a good place to start in *defining* temperature. In Chapter 18 we will look at the relationship between temperature and the energy of molecular motion for an ideal gas. It is important to understand, however, that temperature and heat can be defined independently of any detailed molecular picture. In this section we'll develop a *macroscopic* definition of temperature.

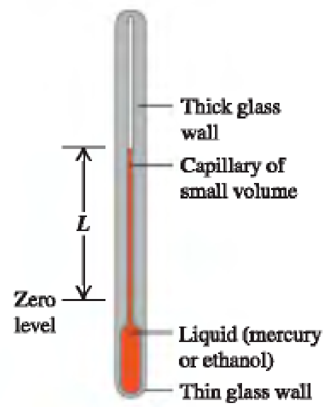
To use temperature as a measure of hotness or coldness, we need to construct a temperature scale. To do this, we can use any measurable property of a system that varies with its "hotness" or "coldness." Figure 17.1a shows a familiar system that is used to measure temperature. When the system becomes hotter, the colored liquid (usually mercury or ethanol) expands and rises in the tube, and the value of L increases. Another simple system is a quantity of gas in a constant-volume container (Fig. 17.1b). The pressure p, measured by the gauge, increases or decreases as the gas becomes hotter or colder. A third example is the electrical resistance R of a conducting wire, which also varies when the wire becomes hotter or colder. Each of these properties gives us a number (L, p, or R) that varies with hotness and coldness, so each property can be used to make a **thermometer.**

To measure the temperature of a body, you place the thermometer in contact with the body. If you want to know the temperature of a cup of hot coffee, you stick the thermometer in the coffee; as the two interact, the thermometer becomes hotter and the coffee cools off a little. After the thermometer settles down to a steady value, you read the temperature. The system has reached an *equilibrium* condition, in which the interaction between the thermometer and the coffee causes no further change in the system. We call this a state of **thermal equilibrium.**

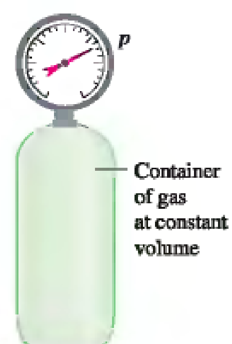
If two systems are separated by an insulating material or **insulator** such as wood, plastic foam, or fiberglass, they influence each other more slowly. Camping coolers are made with insulating materials to delay the ice and cold food inside from warming up and attaining thermal equilibrium with the hot summer air outside. An *ideal insulator* is a material that permits no interaction at all between the two systems. It prevents the systems from attaining thermal equilibrium if they aren't in thermal equilibrium at the start. An ideal insulator is just that, an idealization; real insulators, like those in camping coolers, aren't ideal, so the contents of the cooler will warm up eventually.

17.1 Two devices for measuring temperature.

(a) Changes in temperature cause the liquid's volume to change.



(b) Changes in temperature cause the pressure of the gas to change.

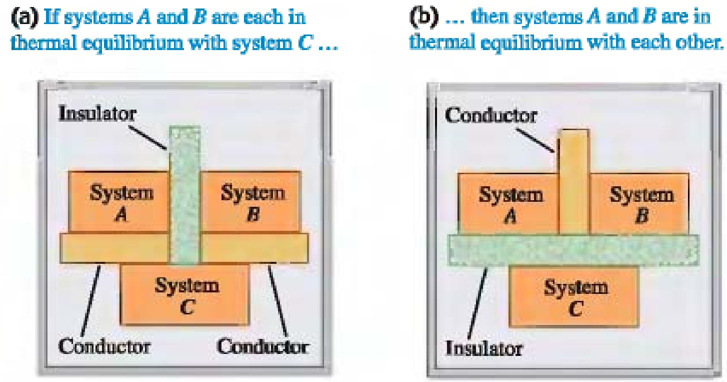


The Zeroth Law of Thermodynamics

We can discover an important property of thermal equilibrium by considering three systems, A, B, and C, that initially are not in thermal equilibrium (Fig. 17.2). We surround them with an ideal insulating box so that they cannot interact with anything except each other. We separate systems A and B with an ideal insulating wall (the green slab in Fig. 17.2a), but we let system C interact with both systems A and B. This interaction is shown in the figure by a yellow slab representing a thermal **conductor,** a material that *permits* thermal interactions through it. We wait until thermal equilibrium is attained; then A and B are each in thermal equilibrium with C. But are they in thermal equilibrium *with each other?*

To find out, we separate system C from systems A and B with an ideal insulating wall (Fig. 17.2b), and then we replace the insulating wall between A and B with a

17.2 The zeroth law of thermodynamics.



conducting wall that lets A and B interact. What happens? Experiment shows that *nothing* happens; there are no additional changes to A or B. We conclude

If C is initially in thermal equilibrium with both A and B, then A and B are also in thermal equilibrium with each other. This result is called the zeroth law of thermodynamics.

(The importance of this law was recognized only after the first, second, and third laws of thermodynamics had been named. Since it is fundamental to all of them, the name "zeroth" seemed appropriate.)

Now suppose system C is a thermometer, such as the tube-and-liquid system of Fig. 17.1a. In Fig. 17.2a the thermometer C is in contact with both A and B. In thermal equilibrium, when the thermometer reading reaches a stable value, the thermometer measures the temperature of both A and B; hence A and B both have the *same* temperature. Experiment shows that thermal equilibrium isn't affected by adding or removing insulators, so the reading of thermometer C wouldn't change if it were in contact only with A or only with B. We conclude

Two systems are in thermal equilibrium if and only if they have the same temperature.

This is what makes a thermometer useful; a thermometer actually measures *its own* temperature, but when a thermometer is in thermal equilibrium with another body, the temperatures must be equal. When the temperatures of two systems are different, they *cannot* be in thermal equilibrium.

Test Your Understanding of Section 17.1 You put a thermometer in a pot of hot water and record the reading. What temperature have you recorded? (i) the temperature of the water; (ii) the temperature of the thermometer; (iii) an equal average of the temperatures of the water and thermometer; (iv) a weighted average of the temperatures of the water and thermometer, with more emphasis on the temperature of the water; (v) a weighted average of the water and thermometer, with more emphasis on the temperature of the thermometer.

17.2 Thermometers and Temperature Scales

To make the liquid-in-tube device shown in Fig. 17.1a into a useful thermometer, we need to mark a scale on the tube wall with numbers on it. These numbers are arbitrary, and historically many different schemes have been used. Suppose we label the thermometer's liquid level at the freezing temperature of pure water "zero" and the level at the boiling temperature "100," and divide the distance between these two points into 100 equal intervals called *degrees*. The result is the **Celsius temperature scale** (formerly called the *centigrade* scale in English-speaking countries). The Celsius temperature for a state colder than freezing water is a negative number. The Celsius scale is used, both in everyday life and in science and industry, almost everywhere in the world.

Another common type of thermometer uses a *bimetallic strip,* made by bonding strips of two different metals together (Fig. 17.3a). When the temperature of the composite strip increases, one metal expands more than the other and the strip bends (Fig. 17.3b). This strip is usually formed into a spiral, with the outer end anchored to the thermometer case and the inner end attached to a pointer (Fig. 17.3c). The pointer rotates in response to temperature changes.

In a *resistance thermometer* the changing electrical resistance of a coil of fine wire, a carbon cylinder, or a germanium crystal is measured. Because resistance can be measured very precisely, resistance thermometers are usually more precise than most other types.

Some thermometers work by detecting the amount of infrared radiation emitted by an object. (We'll see in Section 17.7 that *all* objects emit electromagnetic radiation, including infrared, as a consequence of their temperature.) A modern example is a *temporal artery thermometer* (Fig. 17.4). A nurse runs this over a patient's forehead in the vicinity of the temporal artery, and an infrared sensor in the thermometer measures the radiation from the skin. Tests show that this device gives more accurate values of body temperature than do oral or ear thermometers.

In the **Fahrenheit temperature scale,** still used in everyday life in the United States, the freezing temperature of water is 32°F (thirty-two degrees Fahrenheit) and the boiling temperature is 212°F, both at standard atmospheric pressure. There are 180 degrees between freezing and boiling, compared to 100 on the Celsius scale, so one Fahrenheit degree represents only $\frac{100}{180}$, or $\frac{5}{9}$, as great a temperature change as one Celsius degree.

To convert temperatures from Celsius to Fahrenheit, note that a Celsius temperature T_C is the number of Celsius degrees above freezing; the number of Fahrenheit degrees above freezing is $\frac{9}{5}$ of this. But freezing on the Fahrenheit scale is at 32°F, so to obtain the actual Fahrenheit temperature T_F, multiply the Celsius value by $\frac{9}{5}$ and then add 32°. Symbolically,

$$T_F = \frac{9}{5}T_C + 32° \qquad (17.1)$$

To convert Fahrenheit to Celsius, solve this equation for T_C:

$$T_C = \frac{5}{9}(T_F - 32°) \qquad (17.2)$$

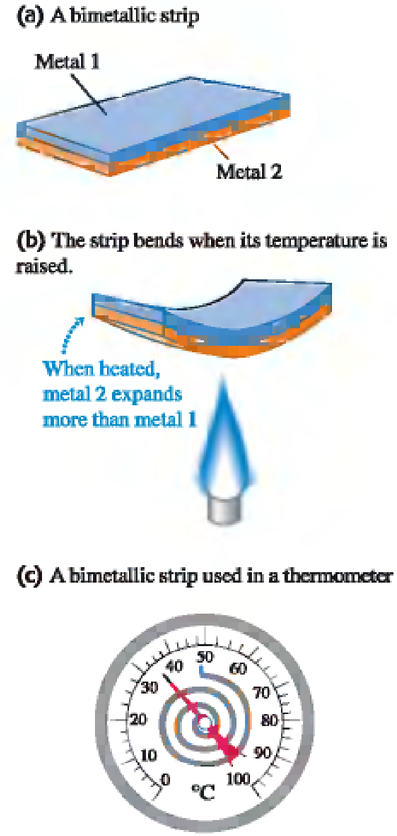
In words, subtract 32° to get the number of Fahrenheit degrees above freezing, and then multiply by $\frac{5}{9}$ to obtain the number of Celsius degrees above freezing—that is, the Celsius temperature.

We don't recommend memorizing Eqs. (17.1) and (17.2). Instead, try to understand the reasoning that led to them so that you can derive them on the spot when you need them, checking your reasoning with the relationship 100°C = 212°F.

It is useful to distinguish between an actual temperature and a temperature *interval* (a difference or change in temperature). An actual temperature of 20° is stated as 20°C (twenty degrees Celsius), and a temperature *interval* of 10° is 10 C° (ten Celsius degrees). A beaker of water heated from 20°C to 30°C undergoes a temperature change of 10 C°.

Test Your Understanding of Section 17.2 Which of the following types of thermometers have to be in thermal equilibrium with the object being measured in order to give accurate readings? (i) a bimetallic strip; (ii) a resistance thermometer; (iii) a temporal artery thermometer; (iv) both (i) and (ii); (v) all of (i), (ii), and (iii).

17.3 Use of a bimetallic strip as a thermometer.



17.4 A temporal artery thermometer measures infrared radiation from the skin that overlies one of the important arteries in the head. Although the thermometer cover touches the skin, the infrared detector inside the cover does not.

17.3 Gas Thermometers and the Kelvin Scale

When we calibrate two thermometers, such as a liquid-in-tube system and a resistance thermometer, so that they agree at 0°C and 100°C, they may not agree exactly at intermediate temperatures. Any temperature scale defined in this way always depends somewhat on the specific properties of the material used. Ideally, we would like to define a temperature scale that *doesn't* depend on the properties of a particular material. To establish a truly material-independent scale, we first need to develop some principles of thermodynamics. We'll return to this fundamental problem in Chapter 20. Here we'll discuss a thermometer that comes close to the ideal, the *gas thermometer.*

The principle of a gas thermometer is that the pressure of a gas at constant volume increases with temperature. A quantity of gas is placed in a constant-volume container (Fig. 17.5a), and its pressure is measured by one of the devices described in Section 14.2. To calibrate a constant-volume gas thermometer, we measure the pressure at two temperatures, say 0°C and 100°C, plot these points on a graph, and draw a straight line between them. Then we can read from the graph the temperature corresponding to any other pressure. Figure 17.5b shows the results of three such experiments, each using a different type and quantity of gas.

By extrapolating this graph, we see that there is a hypothetical temperature, −273.15°C, at which the absolute pressure of the gas would become zero. We might expect that this temperature would be different for different gases, but it turns out to be the *same* for many different gases (at least in the limit of very low gas density). We can't actually observe this zero-pressure condition. Gases liquefy and solidify at very low temperatures, and the proportionality of pressure to temperature no longer holds.

We use this extrapolated zero-pressure temperature as the basis for a temperature scale with its zero at this temperature. This is the **Kelvin temperature scale,** named for the British physicist Lord Kelvin (1824–1907). The units are the same size as those on the Celsius scale, but the zero is shifted so that 0 K = −273.15°C and 273.15 K = 0°C; that is,

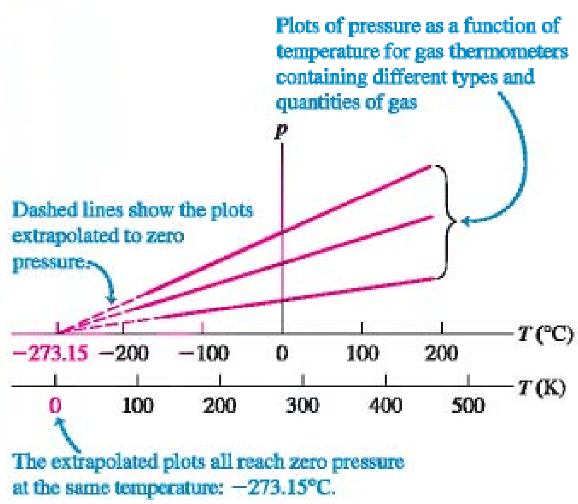
$$T_K = T_C + 273.15 \tag{17.3}$$

This scale is shown in Fig. 17.5b. A common room temperature, 20°C (= 68°F), is 20 + 273.15, or about 293 K.

17.5 (a) Using a constant-volume gas thermometer to measure temperature.
(b) The greater the amount of gas in the thermometer, the higher the graph of pressure p versus temperature T.

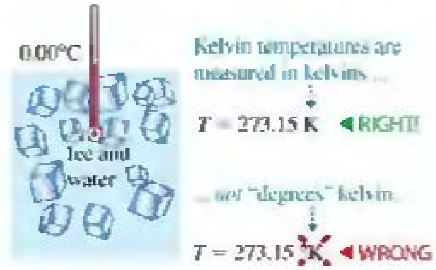
(a) A constant-volume gas thermometer

(b) Graphs of pressure versus temperature at constant volume for three different types and quantities of gas



CAUTION **Never say "degrees kelvin"** In SI nomenclature, "degree" is not used with the Kelvin scale; the temperature mentioned above is read "293 kelvins," not "degrees kelvin" (Fig. 17.6). We capitalize Kelvin when it refers to the temperature scale; however, the *unit* of temperature is the *kelvin,* which is not capitalized (but is nonetheless abbreviated as a capital K).

17.6 Correct and incorrect uses of the Kelvin scale.



Example 17.1 Body temperature

You place a small piece of melting ice in your mouth. Eventually, the water all converts from ice at $T_1 = 32.00°\text{F}$ to body temperature, $T_2 = 98.60°\text{F}$. Express these temperatures as °C and K, and find $\Delta T = T_2 - T_1$ in both cases.

SOLUTION

IDENTIFY: Our target variables are temperatures T_1 and T_2 expressed in Celsius degrees and in kelvins, as well as the difference between these two temperatures.

SET UP: We convert Fahrenheit to Celsius temperatures using Eq. (17.2), and Celsius to Kelvin temperatures using Eq. (17.3).

EXECUTE: First we find the Celsius temperatures. We know that $T_1 = 32.00°\text{F} = 0.00°\text{C}$, and 98.60°F is $98.60 - 32.00 =$ 66.60 F° above freezing; we multiply this by $(5\text{ C°}/9\text{ F°})$ to find 37.00 C° above freezing, or $T_2 = 37.00°\text{C}$.

To get the Kelvin temperatures, we just add 273.15 to each Celsius temperature: $T_1 = 273.15\text{ K}$ and $T_2 = 310.15\text{ K}$. "Normal" body temperature is 37.0°C, but if your doctor says that your temperature is 310 K, don't be alarmed.

The temperature *difference* $\Delta T = T_2 - T_1$ is $37.00\text{ C°} = 37.00\text{ K}$.

EVALUATE: The Celsius and Kelvin scales have different zero points but the same size degrees. Therefore any temperature difference is the *same* on the Celsius and Kelvin scales but not the same on the Fahrenheit scale.

The Kelvin Scale and Absolute Temperature

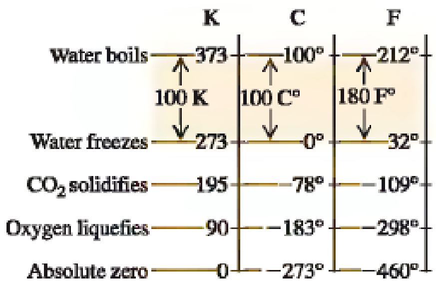
The Celsius scale has two fixed points, the normal freezing and boiling temperatures of water. But we can define the Kelvin scale using a gas thermometer with only a single reference temperature. We define the ratio of any two temperatures T_1 and T_2 on the Kelvin scale as the ratio of the corresponding gas-thermometer pressures p_1 and p_2:

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} \quad \text{(constant-volume gas thermometer, } T \text{ in kelvins)} \qquad (17.4)$$

The pressure p is directly proportional to the Kelvin temperature, as shown in Fig. 17.5b. To complete the definition of T, we need only specify the Kelvin temperature of a single specific state. For reasons of precision and reproducibility, the state chosen is the *triple point* of water. This is the unique combination of temperature and pressure at which solid water (ice), liquid water, and water vapor can all coexist. It occurs at a temperature of 0.01°C and a water-vapor pressure of 610 Pa (about 0.006 atm). (This is the pressure of the *water;* it has nothing to do directly with the gas pressure in the *thermometer.*) The triple-point temperature T_{triple} of water is *defined* to have the value $T_{triple} = 273.16\text{ K}$, corresponding to 0.01°C. From Eq. (17.4), if p_{triple} is the pressure in a gas thermometer at temperature T_{triple} and p is the pressure at some other temperature T, then T is given on the Kelvin scale by

$$T = T_{triple}\frac{p}{p_{triple}} = (273.16\text{ K})\frac{p}{p_{triple}} \qquad (17.5)$$

17.7 Relationships among Kelvin (K), Celsius (C), and Fahrenheit (F) temperature scales. Temperatures have been rounded off to the nearest degree.



Low-pressure gas thermometers using various gases are found to agree very closely, but they are large, bulky, and very slow to come to thermal equilibrium. They are used principally to establish high-precision standards and to calibrate other thermometers.

Figure 17.7 shows the relationships among the three temperature scales we have discussed. The Kelvin scale is called an **absolute temperature scale,** and its zero point ($T = 0\text{ K} = -273.15°\text{C}$, the temperature at which $p = 0$ in Eq. (17.5)) is called **absolute zero.** At absolute zero a system of molecules (such as a quantity of a gas, a liquid, or a solid) has its *minimum* possible total energy (kinetic plus potential); because of quantum effects, however, it is *not* correct to say that all molecular motion ceases at absolute zero. To define more completely what we mean by absolute zero, we need to use the thermodynamic principles developed in the next several chapters. We will return to this concept in Chapter 20.

Test Your Understanding of Section 17.3 Rank the following temperatures from highest to lowest: (i) 0.00°C; (ii) 0.00°F (iii) 260.00 K; (iv) 77.00 K; (v) −180.00°C.

17.4 Thermal Expansion

Most materials expand when their temperatures increase. Rising temperatures make the liquid expand in a liquid-in-tube thermometer (Fig. 17.1a) and bend bimetallic strips (Fig. 17.3b). The decks of bridges need special joints and supports to allow for expansion. A completely filled and tightly capped bottle of water cracks when it is heated, but you can loosen a metal jar lid by running hot water over it. These are all examples of *thermal expansion.*

Linear Expansion

Suppose a rod of material has a length L_0 at some initial temperature T_0. When the temperature changes by ΔT, the length changes by ΔL. Experiments show that if ΔT is not too large (say, less than 100 C° or so), ΔL is *directly proportional* to ΔT (Fig. 17.8a). If two rods made of the same material have the same temperature change, but one is twice as long as the other, then the *change* in its length is also twice as great. Therefore ΔL must also be proportional to L_0 (Fig. 17.8b). Introducing a proportionality constant α (which is different for different materials), we may express these relationships in an equation:

$$\Delta L = \alpha L_0 \Delta T \qquad \text{(linear thermal expansion)} \tag{17.6}$$

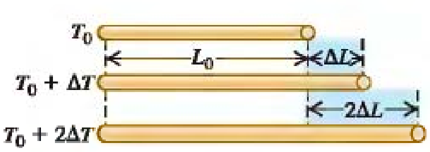
If a body has length L_0 at temperature T_0, then its length L at a temperature $T = T_0 + \Delta T$ is

$$L = L_0 + \Delta L = L_0 + \alpha L_0 \Delta T = L_0(1 + \alpha \Delta T) \tag{17.7}$$

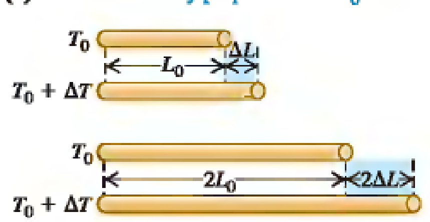
The constant α, which describes the thermal expansion properties of a particular material, is called the **coefficient of linear expansion.** The units of α are K^{-1} or $(\text{C}°)^{-1}$. (Remember that a temperature *interval* is the same in the Kelvin and Celsius scales.) For many materials, every linear dimension changes according to Eq. (17.6) or (17.7). Thus L could be the thickness of a rod, the side length of a square sheet, or the diameter of a hole. Some materials, such as wood or single crystals, expand differently in different directions. We won't consider this complication.

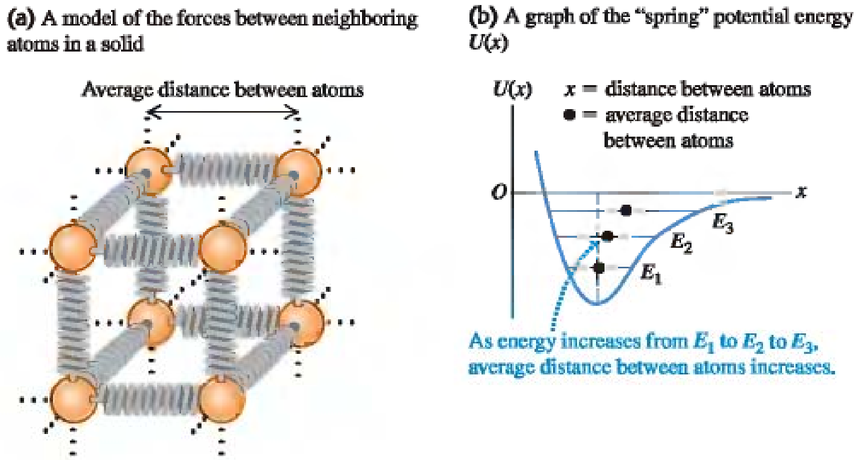
17.8 How the length of a rod changes with a change in temperature. (Length changes are exaggerated for clarity.)

(a) For moderate temperature changes, ΔL is directly proportional to ΔT.



(b) ΔL is also directly proportional to L_0.



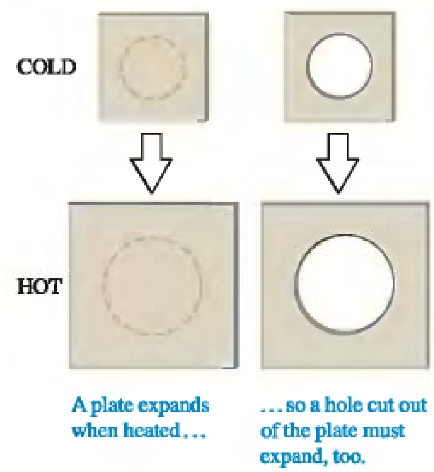


17.9 (a) We can model atoms in a solid as being held together by "springs" that are easier to stretch than to compress. (b) A graph of the "spring" potential energy $U(x)$ versus distance x between neighboring atoms is *not* symmetrical (compare Fig. 13.20b). As the energy increases and the atoms oscillate with greater amplitude, the *average* distance increases.

We can understand thermal expansion qualitatively on a molecular basis. Picture the interatomic forces in a solid as springs, as in Fig. 17.9. (We explored the analogy between spring forces and interatomic forces in Section 13.4.) Each atom vibrates about its equilibrium position. When the temperature increases, the energy and amplitude of the vibration also increase. The interatomic spring forces are not symmetrical about the equilibrium position; they usually behave like a spring that is easier to stretch than to compress. As a result, when the amplitude of vibration increases, the *average* distance between atoms also increases. As the atoms get farther apart, every dimension increases.

CAUTION **Heating an object with a hole** If a solid object has a hole in it, what happens to the size of the hole when the temperature of the object increases? A common misconception is that if the object expands, the hole will shrink because material expands into the hole. But the truth of the matter is that if the object expands, the hole will expand too (Fig. 17.10); as we stated above, *every* linear dimension of an object changes in the same way when the temperature changes. If you're not convinced, think of the atoms in Fig. 17.9a as outlining a cubical hole. When the object expands, the atoms move apart and the hole increases in size. The only situation in which a "hole" will fill in due to thermal expansion is when two separate objects expand and close the gap between them (Fig. 17.11).

17.10 When an object undergoes thermal expansion, any holes in the object expand as well. (The expansion is exaggerated.)



The direct proportionality expressed by Eq. (17.6) is not exact; it is *approximately* correct only for sufficiently small temperature changes. For a given material, α varies somewhat with the initial temperature T_0 and the size of the temperature interval. We'll ignore this complication here, however. Average values of α for several materials are listed in Table 17.1 on page 578. Within the precision of these values we don't need to worry whether T_0 is 0°C or 20°C or some other temperature. Note that typical values of α are very small; even for a temperature change of 100 C°, the fractional length change $\Delta L/L_0$ is only of the order of $1/1000$ for the metals in the table.

17.11 When this SR-71 aircraft is sitting on the ground, its wing panels fit together so loosely that fuel leaks out of the wings onto the ground. But once it is in flight at over three times the speed of sound, air friction heats the panels so much that they expand to make a perfect fit. (In-flight refueling makes up for the lost fuel.)

Volume Expansion

Increasing temperature usually causes increases in *volume* for both solid and liquid materials. Just as with linear expansion, experiments show that if the temperature change ΔT is not too great (less than 100 C° or so), the increase in volume ΔV is approximately proportional to both the temperature change ΔT and the initial volume V_0:

$$\Delta V = \beta V_0 \Delta T \qquad \text{(volume thermal expansion)} \tag{17.8}$$

Table 17.1 Coefficients of Linear Expansion

Material	α [K^{-1} or $(C°)^{-1}$]
Aluminum	2.4×10^{-5}
Brass	2.0×10^{-5}
Copper	1.7×10^{-5}
Glass	$0.4–0.9 \times 10^{-5}$
Invar (nickel–iron alloy)	0.09×10^{-5}
Quartz (fused)	0.04×10^{-5}
Steel	1.2×10^{-5}

The constant β characterizes the volume expansion properties of a particular material; it is called the **coefficient of volume expansion.** The units of β are K^{-1} or $(C°)^{-1}$. As with linear expansion, β varies somewhat with temperature, and Eq. (17.8) is an approximate relationship that is valid only for small temperature changes. For many substances, β decreases at low temperatures. Several values of β in the neighborhood of room temperature are listed in Table 17.2. Note that the values for liquids are generally much larger than those for solids.

For solid materials there is a simple relationship between the volume expansion coefficient β and the linear expansion coefficient α. To derive this relationship, we consider a cube of material with side length L and volume $V = L^3$. At the initial temperature the values are L_0 and V_0. When the temperature increases by dT, the side length increases by dL and the volume increases by an amount dV given by

$$dV = \frac{dV}{dL}\,dL = 3L^2\,dL$$

Now we replace L and V by the initial values L_0 and V_0. From Eq. (17.6), dL is

$$dL = \alpha L_0\,dT$$

Since $V_0 = L_0^3$, this means that dV can also be expressed as

$$dV = 3L_0^2\alpha L_0\,dT = 3\alpha V_0\,dT$$

This is consistent with the infinitesimal form of Eq. (17.8), $dV = \beta V_0\,dT$, only if

$$\beta = 3\alpha \tag{17.9}$$

You should check this relationship for some of the materials listed in Tables 17.1 and 17.2.

Table 17.2 Coefficients of Volume Expansion

Solids	β [K^{-1} or $(C°)^{-1}$]	Liquids	β [K^{-1} or $(C°)^{-1}$]
Aluminum	7.2×10^{-5}	Ethanol	75×10^{-5}
Brass	6.0×10^{-5}	Carbon disulfide	115×10^{-5}
Copper	5.1×10^{-5}	Glycerin	49×10^{-5}
Glass	$1.2–2.7 \times 10^{-5}$	Mercury	18×10^{-5}
Invar	0.27×10^{-5}		
Quartz (fused)	0.12×10^{-5}		
Steel	3.6×10^{-5}		

Problem-Solving Strategy 17.1 Thermal Expansion

MP

IDENTIFY *the relevant concepts:* Decide whether the problem involves changes in length (linear thermal expansion) or in volume (volume thermal expansion).

SET UP *the problem* using the following steps:

1. Choose Eq. (17.6) for linear expansion and Eq. (17.8) for volume expansion.
2. Identify which quantities in Eq. (17.6) or (17.8) are known and which are the unknown target variables.

EXECUTE *the solution* as follows:

1. Solve for the target variables. Often you will be given two temperatures and asked to compute ΔT. Or you may be given an initial temperature T_0 and asked to find a final temperature corresponding to a given length or volume change. In this case, plan to find ΔT first; then the final temperature is $T_0 + \Delta T$.
2. Unit consistency is crucial, as always. L_0 and ΔL (or V_0 and ΔV) must have the same units, and if you use a value of α or β in K^{-1} or $(C°)^{-1}$, then ΔT must be in kelvins or Celsius degrees $(C°)$. But you can use K and C° interchangeably.

EVALUATE *your answer:* Check whether your results make sense. Remember that the sizes of holes in a material expand with temperature just the same way as any other linear dimension, and the volume of a hole (such as the volume of a container) expands the same way as the corresponding solid shape.

Example 17.2 Length change due to temperature change I

A surveyor uses a steel measuring tape that is exactly 50.000 m long at a temperature of 20°C. What is its length on a hot summer day when the temperature is 35°C?

SOLUTION

IDENTIFY: This problem concerns linear expansion. We are given the initial length and initial temperature of the tape, and our target variable is the tape's length at the final temperature.

SET UP: We use Eq. (17.6) to find the change ΔL in the tape's length. We are given $L = 50.000\text{ m}$, $T_0 = 20°\text{C}$, and $T = 35°\text{C}$, and the value of α is found from Table 17.1. The target variable is the new length $L = L_0 + \Delta L$.

EXECUTE: The temperature change is $\Delta T = T - T_0 = 15\text{ C}°$, so from Eq. (17.6) the change in length ΔL and the final length $L = L_0 + \Delta L$ are

$$\begin{aligned}\Delta L &= \alpha L_0 \Delta T = (1.2 \times 10^{-5}\text{ K}^{-1})(50\text{ m})(15\text{ K}) \\ &= 9.0 \times 10^{-3}\text{ m} = 9.0\text{ mm} \\ L &= L_0 + \Delta L = 50.000\text{ m} + 0.009\text{ m} = 50.009\text{ m}\end{aligned}$$

Thus the length at 35°C is 50.009 m.

EVALUATE: Note that L_0 is given to five significant figures but that we need only two of them to compute ΔL. Note also that ΔL is proportional to the initial length L_0: A 5.0-m tape would expand by 0.90 mm, and a 0.50-m (50-cm) tape would expand by a mere 0.090 mm.

This example shows that metals expand very little under moderate temperature changes. Even a metal baking pan in a 200°C (392°F) oven is only slightly larger than it is at room temperature.

Example 17.3 Length change due to temperature change II

In Example 17.2 the surveyor uses the measuring tape to measure a distance when the temperature is 35°C; the value that she reads off the tape is 35.794 m. What is the actual distance? Assume that the tape is calibrated for use at 20°C.

SOLUTION

IDENTIFY: As we saw in Example 17.2, at 35°C the tape has expanded slightly. The distance between two successive meter marks is slightly more than 1 meter, so the scale underestimates the actual distance.

SET UP: The actual distance (our target variable) is *larger* than the distance read off the tape by a factor equal to the ratio of the tape's length L at 35°C to its length L_0 at 20°C.

EXECUTE: The ratio L/L_0 is $(50.009\text{ m})/(50.000\text{ m})$, so the true distance is

$$\frac{50.009\text{ m}}{50.000\text{ m}}(35.794\text{ m}) = 35.800\text{ m}$$

EVALUATE: Although the difference of $0.008\text{ m} = 8\text{ mm}$ between the scale reading and the actual distance seems small, it can be important in precision work.

Example 17.4 Volume change due to temperature change

A glass flask with volume 200 cm^3 is filled to the brim with mercury at 20°C. How much mercury overflows when the temperature of the system is raised to 100°C? The coefficient of linear expansion of the glass is $0.40 \times 10^{-5}\text{ K}^{-1}$.

SOLUTION

IDENTIFY: This problem involves the volume expansion of the glass and of the mercury. The amount of overflow depends on the *difference* between the volume changes for these two materials.

SET UP: The amount of overflow is equal to the difference between the values of ΔV for mercury and for glass, both given by Eq. (17.8). For the mercury to overflow, its coefficient of volume expansion β must be larger than that for glass. The value for mercury is $\beta_{\text{mercury}} = 18 \times 10^{-5}\text{ K}^{-1}$ from Table 17.2, and we find the value of β for this type of glass from Eq. (17.9), $\beta = 3\alpha$.

EXECUTE: The coefficient of volume expansion for the glass is

$$\beta_{\text{glass}} = 3\alpha_{\text{glass}} = 3(0.40 \times 10^{-5}\text{ K}^{-1}) = 1.2 \times 10^{-5}\text{ K}^{-1}$$

Continued

The increase in volume of the glass flask is

$$\begin{aligned}\Delta V_{\text{glass}} &= \beta_{\text{glass}} V_0 \Delta T \\ &= (1.2 \times 10^{-5}\ \text{K}^{-1})(200\ \text{cm}^3)(100°\text{C} - 20°\text{C}) \\ &= 0.19\ \text{cm}^3\end{aligned}$$

The increase in volume of the mercury is

$$\begin{aligned}\Delta V_{\text{mercury}} &= \beta_{\text{mercury}} V_0 \Delta T \\ &= (18 \times 10^{-5}\ \text{K}^{-1})(200\ \text{cm}^3)(100°\text{C} - 20°\text{C}) \\ &= 2.9\ \text{cm}^3\end{aligned}$$

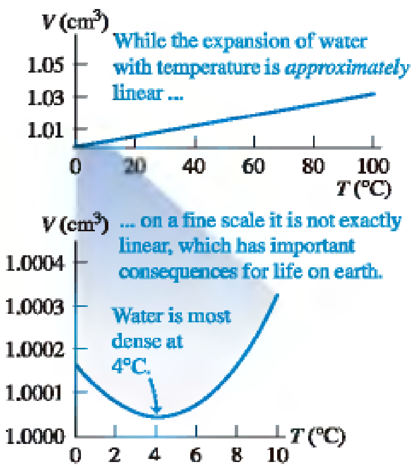
The volume of mercury that overflows is

$$\Delta V_{\text{mercury}} - \Delta V_{\text{glass}} = 2.9\ \text{cm}^3 - 0.19\ \text{cm}^3 = 2.7\ \text{cm}^3$$

EVALUATE: This is basically how a mercury-in-glass thermometer works, except that instead of letting the mercury overflow and run all over the place, the thermometer has it rise inside a sealed tube as T increases.

As Tables 17.1 and 17.2 show, glass has smaller coefficients of expansion α and β than do most metals. This is why you can use hot water to loosen a metal lid on a glass jar; the metal expands more than the glass does.

17.12 The volume of 1 gram of water in the temperature range from 0°C to 10°C. By 100°C the volume has increased to 1.034 cm³. If the coefficient of volume expansion were constant, the curve would be a straight line.



17.13 The interlocking teeth of an expansion joint on a bridge. These joints are needed to accommodate changes in length that result from thermal expansion.

Thermal Expansion of Water

Water, in the temperature range from 0°C to 4°C, *decreases* in volume with increasing temperature. In this range its coefficient of volume expansion is *negative.* Above 4°C, water expands when heated (Fig. 17.12). Hence water has its greatest density at 4°C. Water also expands when it freezes, which is why ice humps up in the middle of the compartments in an ice cube tray. By contrast, most materials contract when they freeze.

This anomalous behavior of water has an important effect on plant and animal life in lakes. A lake cools from the surface down; above 4°C, the cooled water at the surface flows to the bottom because of its greater density. But when the surface temperature drops below 4°C, the water near the surface is less dense than the warmer water below. Hence the downward flow ceases, and the water near the surface remains colder than that at the bottom. As the surface freezes, the ice floats because it is less dense than water. The water at the bottom remains at 4°C until nearly the entire lake is frozen. If water behaved like most substances, contracting continuously on cooling and freezing, lakes would freeze from the bottom up. Circulation due to density differences would continuously carry warmer water to the surface for efficient cooling, and lakes would freeze solid much more easily. This would destroy all plant and animal life that cannot withstand freezing. If water did not have this special property, the evolution of life would have taken a very different course.

Thermal Stress

If we clamp the ends of a rod rigidly to prevent expansion or contraction and then change the temperature, tensile or compressive stresses called **thermal stresses** develop. The rod would like to expand or contract, but the clamps won't let it. The resulting stresses may become large enough to strain the rod irreversibly or even break it. (You may want to review the discussion of stress and strain in Section 11.4).

Engineers must account for thermal stress when designing structures. Concrete highways and bridge decks usually have gaps between sections, filled with a flexible material or bridged by interlocking teeth (Fig. 17.13), to permit expansion and contraction of the concrete. Long steam pipes have expansion joints or U-shaped sections to prevent buckling or stretching with temperature changes. If one end of a steel bridge is rigidly fastened to its abutment, the other end usually rests on rollers.

To calculate the thermal stress in a clamped rod, we compute the amount the rod *would* expand (or contract) if not held and then find the stress needed to compress (or stretch) it back to its original length. Suppose that a rod with length L_0 and cross-sectional area A is held at constant length while the temperature is reduced (negative ΔT), causing a tensile stress. The fractional change in length if the rod were free to contract would be

$$\left(\frac{\Delta L}{L_0}\right)_{\text{thermal}} = \alpha \Delta T \tag{17.10}$$

Both ΔL and ΔT are negative. The tension must increase by an amount F that is just enough to produce an equal and opposite fractional change in length $(\Delta L/L_0)_{\text{tension}}$. From the definition of Young's modulus, Eq. (11.10),

$$Y = \frac{F/A}{\Delta L/L_0} \qquad \text{so} \qquad \left(\frac{\Delta L}{L_0}\right)_{\text{tension}} = \frac{F}{AY} \tag{17.11}$$

If the length is to be constant, the *total* fractional change in length must be zero. From Eqs. (17.10) and (17.11), this means that

$$\left(\frac{\Delta L}{L_0}\right)_{\text{thermal}} + \left(\frac{\Delta L}{L_0}\right)_{\text{tension}} = \alpha \Delta T + \frac{F}{AY} = 0$$

Solving for the tensile stress F/A required to keep the rod's length constant, we find

$$\frac{F}{A} = -Y\alpha\,\Delta T \qquad \text{(thermal stress)} \tag{17.12}$$

For a decrease in temperature, ΔT is negative, so F and F/A are positive; this means that a *tensile* force and stress are needed to maintain the length. If ΔT is positive, F and F/A are negative, and the required force and stress are *compressive*.

If there are temperature differences within a body, nonuniform expansion or contraction will result and thermal stresses can be induced. You can break a glass bowl by pouring very hot water into it; the thermal stress between the hot and cold parts of the bowl exceeds the breaking stress of the glass, causing cracks. The same phenomenon makes ice cubes crack when dropped into warm water. Heat-resistant glasses such as Pyrex™ have exceptionally low expansion coefficients and high strength.

Example 17.5 Thermal stress

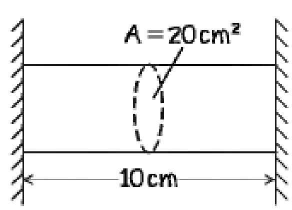
An aluminum cylinder 10 cm long, with a cross-sectional area of $20\ \text{cm}^2$, is to be used as a spacer between two steel walls. At 17.2°C it just slips in between the walls. When it warms to 22.3°C, calculate the stress in the cylinder and the total force it exerts on each wall, assuming that the walls are perfectly rigid and a constant distance apart.

SOLUTION

IDENTIFY: Our target variables are the thermal stress in the cylinder and the associated force it exerts on each of the walls that holds it in place.

SET UP: Figure 17.14 shows our sketch of the situation. We use Eq. (17.12) to relate the stress to the temperature change. The relevant values of Young's modulus Y and the coefficient of linear expansion α are those for aluminum, the material of which the cylinder is made; we find these values from Tables 11.1 and 17.1, respectively.

17.14 Our sketch for this problem.



Continued

EXECUTE: For aluminum, $Y = 7.0 \times 10^{10}\,\text{Pa}$ and $\alpha = 2.4 \times 10^{-5}\,\text{K}^{-1}$. The temperature change is $\Delta T = 22.3°\text{C} - 17.2°\text{C} = 5.1\,\text{C}° = 5.1\,\text{K}$. The stress is F/A; from Eq. (17.12),

$$\frac{F}{A} = -Y\alpha\Delta T = -(0.70 \times 10^{11}\,\text{Pa})(2.4 \times 10^{-5}\,\text{K}^{-1})(5.1\,\text{K})$$
$$= -8.6 \times 10^{6}\,\text{Pa (or } -1200\,\text{lb/in.}^2)$$

The negative sign indicates that compressive rather than tensile stress is needed to keep the cylinder's length constant. This stress is independent of the length and cross-sectional area of the cylinder. The total force F is the cross-sectional area times the stress:

$$F = A\left(\frac{F}{A}\right) = (20 \times 10^{-4}\,\text{m}^2)(-8.6 \times 10^{6}\,\text{Pa})$$
$$= -1.7 \times 10^{4}\,\text{N}$$

or nearly 2 tons. The negative sign indicates compression.

EVALUATE: The stress on the cylinder and the force it exerts on each wall are immense.This points out the importance of accounting for such thermal stresses in engineering.

Test Your Understanding of Section 17.4 In the bimetallic strip shown in Fig. 17.3a, metal 1 is copper. Which of the following materials could be used for metal 2? (There may be more than one correct answer). (i) steel; (ii) brass; (iii) aluminum.

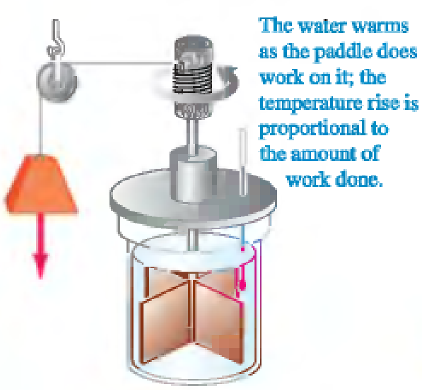
17.5 Quantity of Heat

When you put a cold spoon into a cup of hot coffee, the spoon warms up and the coffee cools down as they approach thermal equilibrium. The interaction that causes these temperature changes is fundamentally a transfer of *energy* from one substance to another. Energy transfer that takes place solely because of a temperature difference is called *heat flow* or *heat transfer,* and energy transferred in this way is called **heat.**

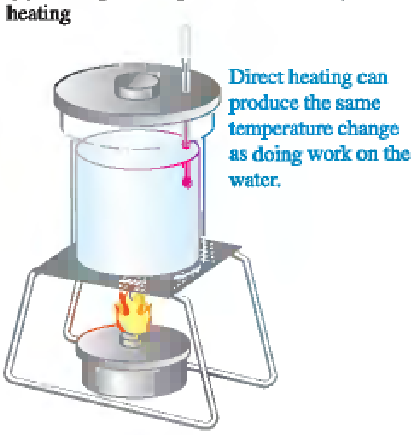
An understanding of the relationship between heat and other forms of energy emerged gradually during the 18th and 19th centuries. Sir James Joule (1818–1889) studied how water can be warmed by vigorous stirring with a paddle wheel (Fig. 17.15a). The paddle wheel adds energy to the water by doing *work* on it, and Joule found that *the temperature rise is directly proportional to the amount of work done.* The same temperature change can also be caused by putting the water in contact with some hotter body (Fig. 17.15b); hence this interaction must also involve an energy exchange. We will explore the relationship between heat and mechanical energy in greater detail in Chapters 19 and 20.

17.15 The same temperature change of the same system may be accomplished by (a) doing work on it or (b) adding heat to it.

(a) Raising the temperature of water by doing work on it



(b) Raising the temperature of water by direct heating



CAUTION **Temperature vs. heat** It is absolutely essential for you to keep clearly in mind the distinction between *temperature* and *heat.* Temperature depends on the physical state of a material and is a quantitative description of its hotness or coldness. In physics the term "heat" always refers to energy in transit from one body or system to another because of a temperature difference, never to the amount of energy contained within a particular system. We can change the temperature of a body by adding heat to it or taking heat away, or by adding or subtracting energy in other ways, such as mechanical work (Fig. 17.15a). If we cut a body in half, each half has the same temperature as the whole; but to raise the temperature of each half by a given interval, we add *half* as much heat as for the whole.

We can define a *unit* of quantity of heat based on temperature changes of some specific material. The **calorie** (abbreviated cal) is defined as *the amount of heat required to raise the temperature of 1 gram of water from 14.5°C to 15.5°C.* The kilocalorie (kcal), equal to 1000 cal, is also used; a food-value calorie is actually a kilocalorie (Fig. 17.16 on the next page). A corresponding unit of heat using Fahrenheit degrees and British units is the **British thermal unit,** or Btu. One Btu is the quantity of heat required to raise the temperature of 1 pound (weight) of water 1 F° from 63°F to 64°F.

Because heat is energy in transit, there must be a definite relationship between these units and the familiar mechanical energy units such as the joule. Experiments similar in concept to Joule's have shown that

$$1 \text{ cal} = 4.186 \text{ J}$$
$$1 \text{ kcal} = 1000 \text{ cal} = 4186 \text{ J}$$
$$1 \text{ Btu} = 778 \text{ ft} \cdot \text{lb} = 252 \text{ cal} = 1055 \text{ J}$$

The calorie is not a fundamental SI unit. The International Committee on Weights and Measures recommends using the joule as the basic unit of energy in all forms, including heat. We will follow that recommendation in this book.

17.16 The motto "Komm in Schwung mit Zucker" on these German sugar packets can be translated as "Sugar gives you momentum." In fact, sugar gives you *energy:* According to the label, each packet has an energy content of 22 kilocalories (22 food-value calories) or 92 kilojoules. (We discussed the difference between energy and momentum in Section 8.1.)

Specific Heat

We use the symbol Q for quantity of heat. When it is associated with an infinitesimal temperature change dT, we call it dQ. The quantity of heat Q required to increase the temperature of a mass m of a certain material from T_1 to T_2 is found to be approximately proportional to the temperature change $\Delta T = T_2 - T_1$. It is also proportional to the mass m of material. When you're heating water to make tea, you need twice as much heat for two cups as for one if the temperature change is the same. The quantity of heat needed also depends on the nature of the material; raising the temperature of 1 kilogram of water by 1 C° requires 4190 J of heat, but only 910 J is needed to raise the temperature of 1 kilogram of aluminum by 1 C°.

Putting all these relationships together, we have

$$Q = mc\,\Delta T \qquad \text{(heat required for temperature change } \Delta T \text{ of mass } m) \tag{17.13}$$

where c is a quantity, different for different materials, called the **specific heat** of the material. For an infinitesimal temperature change dT and corresponding quantity of heat dQ,

$$dQ = mc\,dT \tag{17.14}$$

$$c = \frac{1}{m}\frac{dQ}{dT} \qquad \text{(specific heat)} \tag{17.15}$$

In Eqs. (17.13), (17.14), and (17.15), Q (or dQ) and ΔT (or dT) can be either positive or negative. When they are positive, heat enters the body and its temperature increases; when they are negative, heat leaves the body and its temperature decreases.

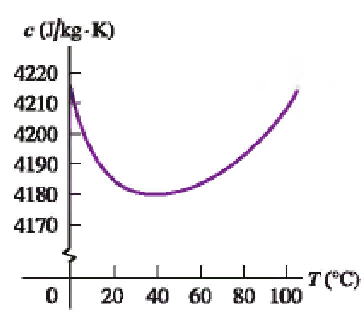
CAUTION **The definition of heat** Remember that dQ does not represent a change in the amount of heat *contained* in a body; this is a meaningless concept. Heat is always energy *in transit* as a result of a temperature difference. There is no such thing as "the amount of heat in a body."

The specific heat of water is approximately

$$4190 \text{ J/kg} \cdot \text{K} \qquad 1 \text{ cal/g} \cdot \text{C}° \qquad \text{or} \qquad 1 \text{ Btu/lb} \cdot \text{F}°$$

The specific heat of a material always depends somewhat on the initial temperature and the temperature interval. Figure 17.17 shows this dependence for water. In the problems and examples in this chapter we will usually ignore this small variation.

17.17 Specific heat of water as a function of temperature. The value of c varies by less than 1% between 0°C and 100°C.



Example 17.6 Feed a cold, starve a fever

During a bout with the flu an 80-kg man ran a fever of 39.0°C (102.2°F) instead of the normal body temperature of 37.0°C (98.6°F). Assuming that the human body is mostly water, how much heat is required to raise his temperature by that amount?

SOLUTION

IDENTIFY: This problem uses the relationship among heat (the target variable), mass, specific heat, and temperature change.

SET UP: We are given the values of $m = 80\text{ kg}$, $c = 4190\text{ J/kg}\cdot\text{K}$ (for water), and $\Delta T = 39.0°\text{C} - 37.0°\text{C} = 2.0\text{ C}° = 2.0\text{ K}$. We use Eq. (17.13) to determine the required heat.

EXECUTE: From Eq. (17.13),

$$Q = mc\,\Delta T = (80\text{ kg})(4190\text{ J/kg}\cdot\text{K})(2.0\text{ K}) = 6.7 \times 10^5\text{ J}$$

EVALUATE: This corresponds to 160 kcal, or 160 food-value calories. In fact, the specific heat of the human body is more nearly equal to $3480\text{ J/kg}\cdot\text{K}$, about 83% that of water. The difference is due to the presence of protein, fat, and minerals, which have lower specific heats. With this value of c, the required heat is $5.6 \times 10^5\text{ J} = 133\text{ kcal}$. Either result shows us that were it not for the body's temperature-regulating systems, taking in energy in the form of food would produce measurable changes in body temperature. (In the case of a person with the flu, the elevated temperature results from the body's extra activity in fighting infection.)

Example 17.7 Overheating electronics

You are designing an electronic circuit element made of 23 mg of silicon. The electric current through it adds energy at the rate of $7.4\text{ mW} = 7.4 \times 10^{-3}\text{ J/s}$. If your design doesn't allow any heat transfer out of the element, at what rate does its temperature increase? The specific heat of silicon is $705\text{ J/kg}\cdot\text{K}$.

SOLUTION

IDENTIFY: The energy added to the circuit element gives rise to a temperature increase, just as if heat were flowing into the element at a rate of $7.4 \times 10^{-3}\text{ J/s}$. Our target variable is the *rate* of temperature change.

SET UP: From Eq. (17.13), the temperature change ΔT in kelvins is proportional to the heat transferred in joules, so the rate of temperature change in K/s is proportional to the rate of heat transfer in J/s.

EXECUTE: In 1 second, $Q = (7.4 \times 10^{-3}\text{ J/s})(1\text{ s}) = 7.4 \times 10^{-3}\text{ J}$. From Eq. (17.13), $Q = mc\,\Delta T$, the temperature change in 1 second is

$$\Delta T = \frac{Q}{mc} = \frac{7.4 \times 10^{-3}\text{ J}}{(23 \times 10^{-6}\text{ kg})(705\text{ J/kg}\cdot\text{K})} = 0.46\text{ K}$$

Alternatively, we can divide both sides of Eq. (17.14) by dt and rearrange:

$$\frac{dT}{dt} = \frac{dQ/dt}{mc}$$
$$= \frac{7.4 \times 10^{-3}\text{ J/s}}{(23 \times 10^{-6}\text{ kg})(705\text{ J/kg}\cdot\text{K})} = 0.46\text{ K/s}$$

EVALUATE: At this rate of temperature rise (27 K every minute), the circuit element would soon self-destruct. Heat transfer is an important design consideration in electronic circuit elements.

Molar Heat Capacity

Sometimes it's more convenient to describe a quantity of substance in terms of the number of *moles* n rather than the *mass* m of material. Recall from your study of chemistry that a mole of any pure substance always contains the same number of molecules. (We will discuss this point in more detail in Chapter 18.) The *molar mass* of any substance, denoted by M, is the mass per mole. (The quantity M is sometimes called *molecular weight,* but *molar mass* is preferable; the quantity depends on the mass of a molecule, not its weight.) For example, the molar mass of water is $18.0\text{ g/mol} = 18.0 \times 10^{-3}\text{ kg/mol}$; 1 mole of water has a mass of $18.0\text{ g} = 0.0180\text{ kg}$. The total mass m of material is equal to the mass per mole M times the number of moles n:

$$m = nM \tag{17.16}$$

Replacing the mass m in Eq. (17.13) by the product nM, we find

$$Q = nMc\,\Delta T \tag{17.17}$$

The product Mc is called the **molar heat capacity** (or *molar specific heat*) and is denoted by C (capitalized). With this notation we rewrite Eq. (17.17) as

$$Q = nC\,\Delta T \qquad \text{(heat required for temperature change of } n \text{ moles)} \tag{17.18}$$

Comparing to Eq. (17.15), we can express the molar heat capacity C (heat per mole per temperature change) in terms of the specific heat c (heat per mass per temperature change) and the molar mass M (mass per mole):

$$C = \frac{1}{n}\frac{dQ}{dT} = Mc \qquad \text{(molar heat capacity)} \qquad (17.19)$$

For example, the molar heat capacity of water is

$$C = Mc = (0.0180\ \text{kg/mol})(4190\ \text{J/kg}\cdot\text{K}) = 75.4\ \text{J/mol}\cdot\text{K}$$

Values of specific heat and molar heat capacity for several substances are given in Table 17.3. Note the remarkably large specific heat for water (Fig. 17.18).

CAUTION **The meaning of "heat capacity"** The term "heat capacity" is unfortunate because it gives the erroneous impression that a body *contains* a certain amount of heat. Remember, heat is energy in transit to or from a body, not the energy residing in the body.

Precise measurements of specific heats and molar heat capacities require great experimental skill. Usually, a measured quantity of energy is supplied by an electric current in a heater wire wound around the specimen. The temperature change ΔT is measured with a resistance thermometer or thermocouple embedded in the specimen. This sounds simple, but great care is needed to avoid or compensate for unwanted heat transfer between the sample and its surroundings. Measurements for solid materials are usually made at constant atmospheric pressure; the corresponding values are called the *specific heat* and *molar heat capacity at constant pressure*, denoted by c_p and C_p. For a gas it is usually easier to keep the substance in a container with constant *volume;* the corresponding values are called the *specific heat* and *molar heat capacity at constant volume*, denoted by c_V and C_V. For a given substance, C_V and C_p are different. If the system can expand while heat is added, there is additional energy exchange through the performance of *work* by the system on its surroundings. If the volume is constant, the system does no work. For gases the difference between C_p and C_V is substantial. We will study heat capacities of gases in detail in Section 19.7.

The last column of Table 17.3 shows something interesting. The molar heat capacities for most elemental solids are about the same: about 25 J/mol · K. This correlation, named the *rule of Dulong and Petit* (for its discoverers), forms the basis for a very important idea. The number of atoms in 1 mole is the same for all

17.18 Water has a much higher specific heat than the glass or metals used to make cookware. This helps explain why it takes several minutes to boil water on a stove, even though the pot or kettle reaches a high temperature very quickly.

Table 17.3 Approximate Specific Heats and Molar Heat Capacities (Constant Pressure)

Substance	Specific Heat, c (J/kg · K)	Molar Mass, M (kg/mol)	Molar Heat Capacity, C (J/mol · K)
Aluminum	910	0.0270	24.6
Beryllium	1970	0.00901	17.7
Copper	390	0.0635	24.8
Ethanol	2428	0.0461	111.9
Ethylene glycol	2386	0.0620	148.0
Ice (near 0°C)	2100	0.0180	37.8
Iron	470	0.0559	26.3
Lead	130	0.207	26.9
Marble ($CaCO_3$)	879	0.100	87.9
Mercury	138	0.201	27.7
Salt (NaCl)	879	0.0585	51.4
Silver	234	0.108	25.3
Water (liquid)	4190	0.0180	75.4

elemental substances. This means that on a *per atom* basis, about the same amount of heat is required to raise the temperature of each of these elements by a given amount, even though the *masses* of the atoms are very different. The heat required for a given temperature increase depends only on *how many* atoms the sample contains, not on the mass of an individual atom. We will see the reason the rule of Dulong and Petit works so well when we study the molecular basis of heat capacities in greater detail in Chapter 18.

Test Your Understanding of Section 17.5 You wish to raise the temperature of each the following samples from 20°C to 21°C. Rank these in order of the amount of heat needed to do this, from highest to lowest. (i) 1 kilogram of mercury; (ii) 1 kilogram of ethanol; (iii) 1 mole of mercury; (iv) 1 mole of ethanol.

MP

17.6 Calorimetry and Phase Changes

Calorimetry means "measuring heat." We have discussed the energy transfer (heat) involved in temperature changes. Heat is also involved in *phase changes*, such as the melting of ice or boiling of water. Once we understand these additional heat relationships, we can analyze a variety of problems involving quantity of heat.

Phase Changes

We use the term **phase** to describe a specific state of matter, such as a solid, liquid, or gas. The compound H_2O exists in the *solid phase* as ice, in the *liquid phase* as water, and in the *gaseous phase* as steam. (These are also referred to as **states of matter:** the solid state, the liquid state, and the gaseous state.) A transition from one phase to another is called a **phase change** or *phase transition*. For any given pressure a phase change takes place at a definite temperature, usually accompanied by absorption or emission of heat and a change of volume and density.

17.19 The surrounding air is at room temperature, but this ice–water mixture remains at 0°C until all of the ice has melted and the phase change is complete.

A familiar example of a phase change is the melting of ice. When we add heat to ice at 0°C and normal atmospheric pressure, the temperature of the ice *does not* increase. Instead, some of it melts to form liquid water. If we add the heat slowly, to maintain the system very close to thermal equilibrium, the temperature remains at 0°C until all the ice is melted (Fig. 17.19). The effect of adding heat to this system is not to raise its temperature but to change its *phase* from solid to liquid.

To change 1 kg of ice at 0°C to 1 kg of liquid water at 0°C and normal atmospheric pressure requires 3.34×10^5 J of heat. The heat required per unit mass is called the **heat of fusion** (or sometimes *latent heat of fusion*), denoted by L_f. For water at normal atmospheric pressure the heat of fusion is

$$L_f = 3.34 \times 10^5\ \text{J/kg} = 79.6\ \text{cal/g} = 143\ \text{Btu/lb}$$

More generally, to melt a mass m of material that has a heat of fusion L_f requires a quantity of heat Q given by

$$Q = mL_f$$

This process is *reversible*. To freeze liquid water to ice at 0°C, we have to *remove* heat; the magnitude is the same, but in this case, Q is negative because heat is removed rather than added. To cover both possibilities and to include other kinds of phase changes, we write

$$Q = \pm mL \qquad \text{(heat transfer in a phase change)} \qquad (17.20)$$

The plus sign (heat entering) is used when the material melts; the minus sign (heat leaving) is used when it freezes. The heat of fusion is different for different materials, and it also varies somewhat with pressure.

For any given material at any given pressure, the freezing temperature is the same as the melting temperature. At this unique temperature the liquid and solid phases (liquid water and ice, for example) can coexist in a condition called **phase equilibrium.**

We can go through this whole story again for *boiling* or *evaporation,* a phase transition between liquid and gaseous phases. The corresponding heat (per unit mass) is called the **heat of vaporization** L_v. At normal atmospheric pressure the heat of vaporization L_v for water is

$$L_v = 2.256 \times 10^6 \text{ J/kg} = 539 \text{ cal/g} = 970 \text{ Btu/lb}$$

That is, it takes 2.256×10^6 J to change 1 kg of liquid water at 100°C to 1 kg of water vapor at 100°C. By comparison, to raise the temperature of 1 kg of water from 0°C to 100°C requires $Q = mc\,\Delta T = (1.00\text{ kg})(4190\text{ J/kg}\cdot\text{C}°) \times (100\text{ C}°) = 4.19 \times 10^5$ J, less than one-fifth as much heat as is required for vaporization at 100°C. This agrees with everyday kitchen experience; a pot of water may reach boiling temperature in a few minutes, but it takes a much longer time to completely evaporate all the water away.

Like melting, boiling is a reversible transition. When heat is removed from a gas at the boiling temperature, the gas returns to the liquid phase, or *condenses,* giving up to its surroundings the same quantity of heat (heat of vaporization) that was needed to vaporize it. At a given pressure the boiling and condensation temperatures are always the same; at this temperature the liquid and gaseous phases can coexist in phase equilibrium.

Both L_v and the boiling temperature of a material depend on pressure. Water boils at a lower temperature (about 95°C) in Denver than in Pittsburgh because Denver is at higher elevation and the average atmospheric pressure is lower. The heat of vaporization is somewhat greater at this lower pressure, about 2.27×10^6 J/kg.

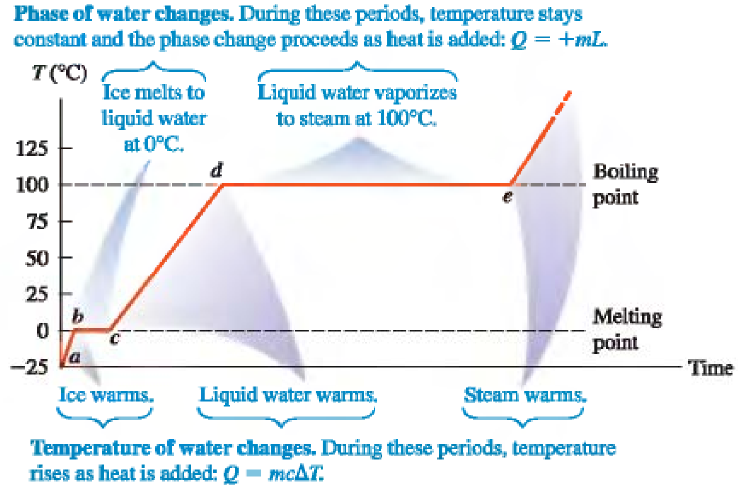
Table 17.4 lists heats of fusion and vaporization for several materials and their melting and boiling temperatures at normal atmospheric pressure. Very few *elements* have melting temperatures in the vicinity of ordinary room temperatures; one of the few is the metal gallium, shown in Fig. 17.20.

17.20 The metal gallium, shown here melting in a person's hand, is one of the few elements that melt in the vicinity of room temperature. Its melting temperature is 29.8°C, and its heat of fusion is 8.04×10^4 J/kg.

Table 17.4 Heats of Fusion and Vaporization

	Normal Melting Point		Heat of Fusion, L_f	Normal Boiling Point		Heat of Vaporization, L_v
Substance	**K**	**°C**	**(J/kg)**	**K**	**°C**	**(J/kg)**
Helium	*	*	*	4.216	−268.93	20.9×10^3
Hydrogen	13.84	−259.31	58.6×10^3	20.26	−252.89	452×10^3
Nitrogen	63.18	−209.97	25.5×10^3	77.34	−195.8	201×10^3
Oxygen	54.36	−218.79	13.8×10^3	90.18	−183.0	213×10^3
Ethanol	159	−114	104.2×10^3	351	78	854×10^3
Mercury	234	−39	11.8×10^3	630	357	272×10^3
Water	273.15	0.00	334×10^3	373.15	100.00	2256×10^3
Sulfur	392	119	38.1×10^3	717.75	444.60	326×10^3
Lead	600.5	327.3	24.5×10^3	2023	1750	871×10^3
Antimony	903.65	630.50	165×10^3	1713	1440	561×10^3
Silver	1233.95	960.80	88.3×10^3	2466	2193	2336×10^3
Gold	1336.15	1063.00	64.5×10^3	2933	2660	1578×10^3
Copper	1356	1083	134×10^3	1460	1187	5069×10^3

*A pressure in excess of 25 atmospheres is required to make helium solidify. At 1 atmosphere pressure, helium remains a liquid down to absolute zero.



17.21 Graph of temperature versus time for a specimen of water initially in the solid phase (ice). Heat is added to the specimen at a constant rate. The temperature remains constant during each change of phase, provided that the pressure remains constant.

Figure 17.21 shows how the temperature varies when we add heat continuously to a specimen of ice with an initial temperature below 0°C (point a). The temperature rises until we reach the melting point (point b). As more heat is added, the temperature remains constant until all the ice has melted (point c). Then the temperature rises again until the boiling temperature is reached (point d). At that point the temperature again is constant until all the water is transformed into the vapor phase (point e). If the rate of heat input is constant, the line for the solid phase (ice) has a steeper slope than does the line for the liquid phase (water). Do you see why? (See Table 17.3.)

A substance can sometimes change directly from the solid to the gaseous phase. This process is called *sublimation,* and the solid is said to *sublime.* The corresponding heat is called the *heat of sublimation,* L_s. Liquid carbon dioxide cannot exist at a pressure lower than about 5×10^5 Pa (about 5 atm), and "dry ice" (solid carbon dioxide) sublimes at atmospheric pressure. Sublimation of water from frozen food causes freezer burn. The reverse process, a phase change from gas to solid, occurs when frost forms on cold bodies such as refrigerator cooling coils.

Very pure water can be cooled several degrees below the freezing temperature without freezing; the resulting unstable state is described as *supercooled.* When a small ice crystal is dropped in or the water is agitated, it crystallizes within a second or less. Supercooled water *vapor* condenses quickly into fog droplets when a disturbance, such as dust particles or ionizing radiation, is introduced. This principle is used in "seeding" clouds, which often contain supercooled water vapor, to cause condensation and rain.

A liquid can sometimes be *superheated* above its normal boiling temperature. Any small disturbance such as agitation causes local boiling with bubble formation.

Steam heating systems for buildings use a boiling–condensing process to transfer heat from the furnace to the radiators. Each kilogram of water that is turned to steam in the boiler absorbs over 2×10^6 J (the heat of vaporization L_v of water) from the boiler and gives it up when it condenses in the radiators. Boiling–condensing processes are also used in refrigerators, air conditioners, and heat pumps. We will discuss these systems in Chapter 20.

The temperature-control mechanisms of many warm-blooded animals make use of heat of vaporization, removing heat from the body by using it to evaporate water from the tongue (panting) or from the skin (sweating). Evaporative cooling enables humans to maintain normal body temperature in hot, dry desert climates where the air temperature may reach 55°C (about 130°F). The skin temperature may be as much as 30°C cooler than the surrounding air. Under these conditions a normal person may perspire several liters per day, and this lost water must be replaced. Old-time desert rats (such as one of the authors) state that in the desert, any canteen that holds less than a gallon should be viewed as a toy! Evaporative cooling also explains why you feel cold when you first step out of a swimming pool (Fig. 17.22).

Evaporative cooling is also used to cool buildings in hot, dry climates and to condense and recirculate "used" steam in coal-fired or nuclear-powered electric-generating plants. That's what goes on in the large, tapered concrete towers that you see at such plants.

Chemical reactions such as combustion are analogous to phase changes in that they involve definite quantities of heat. Complete combustion of 1 gram of gasoline produces about 46,000 J or about 11,000 cal, so the **heat of combustion L_c** of gasoline is

$$L_c = 46{,}000\ \text{J/g} = 4.6 \times 10^7\ \text{J/kg}$$

Energy values of foods are defined similarly. When we say that a gram of peanut butter "contains 6 calories," we mean that 6 kcal of heat (6,000 cal or 25,000 J) is released when the carbon and hydrogen atoms in the peanut butter react with oxygen (with the help of enzymes) and are completely converted to CO_2 and H_2O. Not all of this energy is directly useful for mechanical work. We will study the *efficiency* of energy utilization in Chapter 20.

17.22 The water may be warm and it may be a hot day, but these children will feel cold when they first step out of the swimming pool. That's because as water evaporates from their skin, it removes the heat of vaporization from their bodies. To stay warm, they will need to dry off immediately.

Heat Calculations

Let's look at some examples of calorimetry calculations (calculations with heat). The basic principle is very simple: When heat flow occurs between two bodies that are isolated from their surroundings, the amount of heat lost by one body must equal the amount gained by the other. Heat is energy in transit, so this principle is really just conservation of energy. Calorimetry, dealing entirely with one conserved quantity, is in many ways the simplest of all physical theories!

Problem-Solving Strategy 17.2 Calorimetry Problems

MP

IDENTIFY *the relevant concepts:* When heat flow occurs between two bodies that are isolated from their surroundings, the amount of heat lost by one body must equal the amount gained by the other body.

SET UP *the problem* using the following steps:

1. Identify which objects exchange heat. To avoid confusion with algebraic signs, take each quantity of heat *added* to a body as *positive* and each quantity *leaving* a body as *negative*. When two or more bodies interact, the *algebraic sum* of the quantities of heat transferred to all the bodies must be zero.
2. Each object will undergo a temperature change with no phase change, a phase change at constant temperature, or both. Use Eq. (17.13) to describe temperature changes and Eq. (17.20) to describe phase changes.
3. Consult Table 17.3 for values of the specific heat or molar heat capacity and Table 17.4 for heats of fusion or vaporization.
4. Be certain to identify which quantities are known and which are the unknown target variables.

EXECUTE *the solution* as follows:

1. Solve Eq. (17.13) and/or Eq. (17.20) for the target variables. Often you will need to find an unknown temperature. Represent it by an algebraic symbol such as T. Then if a body has an initial temperature of 20°C and an unknown final temperature T, the temperature change for the body is $\Delta T = T_{\text{final}} - T_{\text{initial}} = T - 20°\text{C}$ (*not* $20°\text{C} - T$).
2. In problems where a phase change takes place, as when ice melts, you may not know in advance whether *all* the material undergoes a phase change or only part of it. You can always assume one or the other, and if the resulting calculation gives an absurd result (such as a final temperature higher or lower than *any* of the initial temperatures), you know the initial assumption was wrong. Back up and try again!

EVALUATE *your answer:* A common error is to use the wrong algebraic sign for either a Q or ΔT term. Double check your calculations, and make sure that the final results are physically sensible.

Example 17.8 A temperature change with no phase change

A geologist working in the field drinks her morning coffee out of an aluminum cup. The cup has a mass of 0.120 kg and is initially at 20.0°C when she pours in 0.300 kg of coffee initially at 70.0°C. What is the final temperature after the coffee and the cup attain thermal equilibrium? (Assume that coffee has the same specific heat as water and that there is no heat exchange with the surroundings.)

SOLUTION

IDENTIFY: The two objects we must consider are the cup and the coffee, and the target variable is their common final temperature.

SET UP: No phase changes occur in this situation, so the only equation we need is Eq. (17.13).

Continued

EXECUTE: By using Table 17.3, the (negative) heat gained by the coffee is

$$Q_{coffee} = m_{coffee}c_{water}\,\Delta T_{coffee}$$
$$= (0.300\ \text{kg})(4190\ \text{J/kg}\cdot\text{K})(T - 70.0°\text{C})$$

The (positive) heat gained by the aluminum cup is

$$Q_{aluminum} = m_{aluminum}\,c_{aluminum}\,\Delta T_{aluminum}$$
$$= (0.120\ \text{kg})(910\ \text{J/kg}\cdot\text{K})(T - 20.0°\text{C})$$

We equate the sum of these two quantities of heat to zero, obtaining an algebraic equation for T:

$$Q_{coffee} + Q_{aluminum} = 0 \qquad \text{or}$$
$$(0.300\ \text{kg})(4190\ \text{J/kg}\cdot\text{K})(T - 70.0°\text{C}) + (0.120\ \text{kg})(910\ \text{J/kg}\cdot\text{K})(T - 20.0°\text{C}) = 0$$

Solution of this equation gives $T = 66.0°\text{C}$.

EVALUATE: The final temperature is much closer to the initial temperature of the coffee than to that of the cup; water has a much higher specific heat than aluminum, and we have more than twice as much mass of water. We can also find the quantities of heat by substituting the value $T = 66.0°\text{C}$ back into the original equations. We find that $Q_{coffee} = -5.0 \times 10^3\ \text{J}$ and $Q_{aluminum} = +5.0 \times 10^3\ \text{J}$; Q_{coffee} is negative, which means that the coffee loses heat.

Example 17.9 Changes in both temperature and phase

A physics student wants to cool 0.25 kg of Diet Omni-Cola (mostly water), initially at 25°C, by adding ice initially at −20°C. How much ice should she add so that the final temperature will be 0°C with all the ice melted if the heat capacity of the container may be neglected?

SOLUTION

IDENTIFY: The ice and the Omni-Cola are the objects that exchange heat. The Omni-Cola undergoes a temperature change only, while the ice undergoes both a temperature change and a phase change from solid to liquid. The target variable is the mass of ice, m_{ice}.

SET UP: We use Eq. (17.13) to find the amount of heat involved in warming the ice to 0°C and cooling the Omni-Cola to 0°C. In addition, we'll need Eq. (17.20) to calculate the heat required to melt the ice at 0°C.

EXECUTE: The Omni-Cola loses heat, so the heat added to it is negative:

$$Q_{Omni} = m_{Omni}c_{water}\,\Delta T_{Omni}$$
$$= (0.25\ \text{kg})(4190\ \text{J/kg}\cdot\text{K})(0°\text{C} - 25°\text{C})$$
$$= -26{,}000\ \text{J}$$

From Table 17.3, the specific heat of ice (not the same as for liquid water) is $2.1 \times 10^3\ \text{J/kg}\cdot\text{K}$. Let the mass of ice be m_{ice}; then the heat Q_1 needed to warm it from −20°C to 0°C is

$$Q_1 = m_{ice}\,c_{ice}\,\Delta T_{ice}$$
$$= m_{ice}(2.1 \times 10^3\ \text{J/kg}\cdot\text{K})[0°\text{C} - (-20°\text{C})]$$
$$= m_{ice}(4.2 \times 10^4\ \text{J/kg})$$

From Eq. (17.20) the additional heat Q_2 needed to melt this mass of ice is the mass times the heat of fusion. Using Table 17.4, we find

$$Q_2 = m_{ice}\,L_f$$
$$= m_{ice}(3.34 \times 10^5\ \text{J/kg})$$

The sum of these three quantities must equal zero:

$$Q_{Omni} + Q_1 + Q_2 = -26{,}000\ \text{J} + m_{ice}(42{,}000\ \text{J/kg}) + m_{ice}(334{,}000\ \text{J/kg}) = 0$$

Solving this for m_{ice}, we get $m_{ice} = 0.069\ \text{kg} = 69\ \text{g}$.

EVALUATE: This mass of ice corresponds to three or four medium-size ice cubes, which seems reasonable for the quantity of Omni-Cola in this problem.

Example 17.10 What's cooking?

A heavy copper pot of mass 2.0 kg (including the copper lid) is at a temperature of 150°C. You pour 0.10 kg of water at 25°C into the pot, then quickly close the lid of the pot so that no steam can escape. Find the final temperature of the pot and its contents, and determine the phase (liquid or gas) of the water. Assume that no heat is lost to the surroundings.

SOLUTION

IDENTIFY: The two objects that exchange heat are the water and the pot. Note that there are three conceivable outcomes in this situation. One, none of the water boils, and the final temperature is less than 100°C; two, a portion of the water boils, giving a mixture of water and steam at 100°C; or three, all the water boils, giving 0.10 kg of steam at a temperature of 100°C or greater.

SET UP: We again use Eq. (17.13) for the heat transferred in a temperature change and Eq. (17.20) for the heat transferred in a phase change.

EXECUTE: The simplest case to calculate is the first possibility. Let the common final temperature of the liquid water and the copper pot be T. Since we are assuming that no phase changes take place, the sum of the quantities of heat added to the two materials is

$$Q_{water} + Q_{copper} = m_{water}c_{water}(T - 25°\text{C}) + m_{copper}c_{copper}(T - 150°\text{C})$$
$$= (0.10\ \text{kg})(4190\ \text{J/kg}\cdot\text{K})(T - 25°\text{C}) + (2.0\ \text{kg})(390\ \text{J/kg}\cdot\text{K})(T - 150°\text{C})$$
$$= 0$$

Solving this for T, we find $T = 106°C$. But this is above the boiling point of water, which contradicts our assumption that none of the water boils! So this assumption can't be correct; at least some of the water undergoes a phase change.

If we try the second possibility, in which the final temperature is 100°C, we have to find the fraction of water that changes to the gaseous phase. Let this fraction be x; the (positive) amount of heat needed to vaporize this water is $(xm_{\text{water}})L_v$. Setting the final temperature T equal to 100°C, we have

$$
\begin{aligned}
Q_{\text{water}} &= m_{\text{water}}c_{\text{water}}(100°C - 25°C) + xm_{\text{water}}L_v \\
&= (0.10\,\text{kg})(4190\,\text{J/kg}\cdot\text{K})(75\,\text{K}) \\
&\quad + x(0.10\,\text{kg})(2.256 \times 10^6\,\text{J/kg}) \\
&= 3.14 \times 10^4\,\text{J} + x(2.256 \times 10^5\,\text{J}) \\
Q_{\text{copper}} &= m_{\text{copper}}c_{\text{copper}}(100°C - 150°C) \\
&= (2.0\,\text{kg})(390\,\text{J/kg}\cdot\text{K})(-50\,\text{K}) = -3.90 \times 10^4\,\text{J}
\end{aligned}
$$

Now require that the sum of all the quantities of heat be zero:

$$
\begin{aligned}
Q_{\text{water}} + Q_{\text{copper}} &= 3.14 \times 10^4\,\text{J} + x(2.256 \times 10^5\,\text{J}) \\
&\quad -3.90 \times 10^4\,\text{J} = 0 \\
x &= \frac{3.90 \times 10^4\,\text{J} - 3.14 \times 10^4\,\text{J}}{2.256 \times 10^3\,\text{J}} = 0.034
\end{aligned}
$$

This makes sense, and we conclude that the final temperature of the water and copper is 100°C. Of the original 0.10 kg of water, $0.034(0.10\,\text{kg}) = 0.0034\,\text{kg} = 3.4\,\text{g}$ has been converted to steam at 100°C.

EVALUATE: Had x turned out to be greater than 1, we would have again had a contradiction (the fraction of water that vaporized can't be greater than 1). In this case the third possibility would have been the correct description, all the water would have vaporized, and the final temperature would have been greater than 100°C. Can you show that this would have been the case if we had originally poured less than 15 g of 25°C water into the pot?

Example 17.11 Combustion, temperature change, and phase change

In a particular gasoline camp stove, 30% of the energy released in burning the fuel actually goes to heating the water in the pot on the stove. If we heat 1.00 L (1.00 kg) of water from 20°C to 100°C and boil 0.25 kg of it away, how much gasoline do we burn in the process?

SOLUTION

IDENTIFY: In this problem all of the water undergoes a temperature change and part of the water also undergoes a phase change from liquid to gas. This requires a certain amount of heat, which we use to determine the amount of gasoline that must be burned (the target variable).

SET UP: We use Eqs. (17.13) and (17.20) as well as the idea of heat of combustion.

EXECUTE: The heat required to raise the temperature of the water from 20°C to 100°C is

$$
\begin{aligned}
Q_1 &= mc\,\Delta T = (1.00\,\text{kg})(4190\,\text{J/kg}\cdot\text{K})(80\,\text{K}) \\
&= 3.35 \times 10^5\,\text{J}
\end{aligned}
$$

To boil 0.25 kg of water at 100°C requires

$$Q_2 = mL_v = (0.25\,\text{kg})(2.256 \times 10^6\,\text{J/kg}) = 5.64 \times 10^5\,\text{J}$$

The total energy needed is the sum of these, or 8.99×10^5 J This is only 0.30 of the total heat of combustion, so that energy is $(8.99 \times 10^5\,\text{J})/0.30 = 3.00 \times 10^6\,\text{J}$. As we mentioned earlier, 1 gram of gasoline releases 46,000 J, so the mass of gasoline required is

$$\frac{3.00 \times 10^6\,\text{J}}{46{,}000\,\text{J/g}} = 65\,\text{g}$$

or a volume of about 0.09 L of gasoline.

EVALUATE: This result is a testament to the tremendous amount of energy that can be released by burning even a small quantity of gasoline. Note that most of the heat delivered was used to boil away 0.25 L of water. Can you show that another 123 g of gasoline would be required to boil away the remaining water?

Test Your Understanding of Section 17.6 You take a block of ice at 0°C and add heat to it at a steady rate. It takes a time t to completely convert the block of ice to steam at 100°C. What do you have at time $t/2$? (i) all ice at 0°C; (ii) a mixture of ice and water at 0°C; (iii) water at a temperature between 0°C and 100°C; (iv) a mixture of water and steam at 100°C.

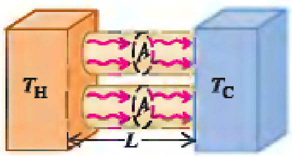
17.7 Mechanisms of Heat Transfer

We have talked about *conductors* and *insulators*, materials that permit or prevent heat transfer between bodies. Now let's look in more detail at *rates* of energy transfer. In the kitchen you use a metal or glass pot for good heat transfer from the stove to whatever you're cooking, but your refrigerator is insulated with a material that *prevents* heat from flowing into the food inside the refrigerator. How do we describe the difference between these two materials?

17.23 Steady-state heat flow due to conduction in a uniform rod.

(a) Heat current H

(b) Doubling the cross-sectional area of the conductor doubles the heat current (H is proportional to A).



(c) Doubling the length of the conductor halves the heat current (H is inversely proportional to L).

Table 17.5 Thermal Conductivities

Substance	k (W/m·K)
Metals	
Aluminum	205.0
Brass	109.0
Copper	385.0
Lead	34.7
Mercury	8.3
Silver	406.0
Steel	50.2
Solids (representative values)	
Brick, insulating	0.15
Brick, red	0.6
Concrete	0.8
Cork	0.04
Felt	0.04
Fiberglass	0.04
Glass	0.8
Ice	1.6
Rock wool	0.04
Styrofoam	0.01
Wood	0.12–0.04
Gases	
Air	0.024
Argon	0.016
Helium	0.14
Hydrogen	0.14
Oxygen	0.023

The three mechanisms of heat transfer are conduction, convection, and radiation. *Conduction* occurs within a body or between two bodies in contact. *Convection* depends on motion of mass from one region of space to another. *Radiation* is heat transfer by electromagnetic radiation, such as sunshine, with no need for matter to be present in the space between bodies.

Conduction

If you hold one end of a copper rod and place the other end in a flame, the end you are holding gets hotter and hotter, even though it is not in direct contact with the flame. Heat reaches the cooler end by **conduction** through the material. On the atomic level, the atoms in the hotter regions have more kinetic energy, on the average, than their cooler neighbors. They jostle their neighbors, giving them some of their energy. The neighbors jostle *their* neighbors, and so on through the material. The atoms themselves do not move from one region of material to another, but their energy does.

Most metals also use another, more effective mechanism to conduct heat. Within the metal, some electrons can leave their parent atoms and wander through the crystal lattice. These "free" electrons can rapidly carry energy from the hotter to the cooler regions of the metal, so metals are generally good conductors of heat. A metal rod at 20°C feels colder than a piece of wood at 20°C because heat can flow more easily from your hand into the metal. The presence of "free" electrons also causes most metals to be good electrical conductors.

Heat transfer occurs only between regions that are at different temperatures, and the direction of heat flow is always from higher to lower temperature. Figure 17.23a shows a rod of conducting material with cross-sectional area A and length L. The left end of the rod is kept at a temperature T_H and the right end at a lower temperature T_C, so heat flows from left to right. The sides of the rod are covered by an ideal insulator, so no heat transfer occurs at the sides.

When a quantity of heat dQ is transferred through the rod in a time dt, the rate of heat flow is dQ/dt. We call this rate the **heat current,** denoted by H. That is, $H = dQ/dt$. Experiments show that the heat current is proportional to the cross-sectional area A of the rod (Fig. 17.23b) and to the temperature difference $(T_H - T_C)$ and is inversely proportional to the rod length L (Fig. 17.23c). Introducing a proportionality constant k called the **thermal conductivity** of the material, we have

$$H = \frac{dQ}{dt} = kA\frac{T_H - T_C}{L} \qquad \text{(heat current in conduction)} \qquad (17.21)$$

The quantity $(T_H - T_C)/L$ is the temperature difference *per unit length;* it is called the magnitude of the **temperature gradient.** The numerical value of k depends on the material of the rod. Materials with large k are good conductors of heat; materials with small k are poor conductors or insulators. Equation (17.21) also gives the heat current through a slab or through *any* homogeneous body with uniform cross section A perpendicular to the direction of flow; L is the length of the heat-flow path.

The units of heat current H are units of energy per time, or power; the SI unit of heat current is the watt $(1\text{ W} = 1\text{ J/s})$. We can find the units of k by solving Eq. (17.21) for k; you can show that the SI units are W/m·K. Some numerical values of k are given in Table 17.5.

The thermal conductivity of "dead" (that is, nonmoving) air is very small. A wool sweater keeps you warm because it traps air between the fibers. In fact, many insulating materials such as Styrofoam and fiberglass are mostly dead air. Figure 17.24 shows a ceramic material with very unusual thermal properties, including very small conductivity.

If the temperature varies in a nonuniform way along the length of the conducting rod, we introduce a coordinate x along the length and generalize the temperature gradient to be dT/dx. The corresponding generalization of Eq. (17.21) is

$$H = \frac{dQ}{dt} = -kA\frac{dT}{dx} \tag{17.22}$$

The negative sign shows that heat always flows in the direction of *decreasing* temperature.

For thermal insulation in buildings, engineers use the concept of **thermal resistance,** denoted by R. The thermal resistance R of a slab of material with area A is defined so that the heat current H through the slab is

$$H = \frac{A(T_H - T_C)}{R} \tag{17.23}$$

where T_H and T_C are the temperatures on the two sides of the slab. Comparing this with Eq. (17.21), we see that R is given by

$$R = \frac{L}{k} \tag{17.24}$$

where L is the thickness of the slab. The SI unit of R is $1\ \text{m}^2 \cdot \text{K/W}$. In the units used for commercial insulating materials in the United States, H is expressed in Btu/h, A is in ft^2, and $T_H - T_C$ in F°. $(1\ \text{Btu/h} = 0.293\ \text{W.})$ The units of R are then $\text{ft}^2 \cdot \text{F}° \cdot \text{h/Btu}$, though values of R are usually quoted without units; a 6-inch-thick layer of fiberglass has an R value of 19 (that is, $R = 19\ \text{ft}^2 \cdot \text{F}° \cdot \text{h/Btu}$), a 2-inch-thick slab of polyurethane foam has an R value of 12, and so on. Doubling the thickness doubles the R value. Common practice in new construction in severe northern climates is to specify R values of around 30 for exterior walls and ceilings. When the insulating material is in layers, such as a plastered wall, fiberglass insulation, and wood exterior siding, the R values are additive. Do you see why? (See Problem 17.110.)

17.24 This protective tile, developed for use in the space shuttle, has extraordinary thermal properties. The extremely small thermal conductivity and small heat capacity of the material make it possible to hold the tile by its edges, even though its temperature is high enough to emit the light for this photograph.

Problem-Solving Strategy 17.3 Heat Conduction

IDENTIFY *the relevant concepts:* The concept of heat conduction comes into play whenever two objects at different temperature are placed in contact.

SET UP *the problem* using the following steps:

1. Identify the direction of heat flow in the problem (from hot to cold). In Eq. (17.21), L is always measured along this direction, and A is always an area perpendicular to this direction. Often when a box or other container has an irregular shape but uniform wall thickness, you can approximate it as a flat slab with the same thickness and total wall area.
2. Identify the target variable.

EXECUTE *the solution* as follows:

1. If heat flows through a single object, use Eq. (17.21) to solve for the target variable.
2. In some problems the heat flows through two different materials in succession. The temperature at the interface between the two materials is then intermediate between T_H and T_C; represent it by a symbol such as T. The temperature differences for the two materials are then $(T_H - T)$ and $(T - T_C)$. In steady-state heat flow, the same heat has to pass through both materials in succession, so the heat current H must be *the same* in both materials.
3. If there are two *parallel* heat-flow paths, so that some heat flows through each, then the total H is the sum of the quantities H_1 and H_2 for the separate paths. An example is heat flow from inside to outside a house, both through the glass in a window and through the surrounding frame. In this case the temperature difference is the same for the two paths, but L, A, and k may be different for the two paths.
4. As always, it is essential to use a consistent set of units. If you use a value of k expressed in $\text{W/m} \cdot \text{K}$, don't use distances in centimeters, heat in calories, or T in degrees Fahrenheit!

EVALUATE *your answer:* As always, ask yourself whether the results are physically reasonable.

Example 17.12 Conduction through a picnic cooler

A Styrofoam box used to keep drinks cold at a picnic (Fig. 17.25a) has total wall area (including the lid) of 0.80 m^2 and wall thickness 2.0 cm. It is filled with ice, water, and cans of Omni-Cola at 0°C. What is the rate of heat flow into the box if the temperature of the outside wall is 30°C? How much ice melts in one day?

SOLUTION

IDENTIFY: The first target variable is the heat current H. The second is the amount of ice melted, which depends on the heat current (heat per unit time), the elapsed time, and the heat of fusion.

SET UP: We use Eq. (17.21) to describe the heat current and Eq. (17.20), $Q = mL_f$, to determine the mass m of ice that melts due to the heat flow.

EXECUTE: We assume that the total heat flow is approximately the same as it would be through a flat slab of area 0.80 m^2 and thickness 2.0 cm = 0.020 m (Fig. 17.25b). We find k from Table 17.5. From Eq. (17.21) the heat current (rate of heat flow) is

$$H = kA\frac{T_H - T_C}{L} = (0.010\ \mathrm{W/m\cdot K})(0.80\ \mathrm{m^2})\frac{30°\mathrm{C} - 0°\mathrm{C}}{0.020\ \mathrm{m}}$$
$$= 12\ \mathrm{W} = 12\ \mathrm{J/s}$$

The total heat flow Q in one day (86,400 s) is

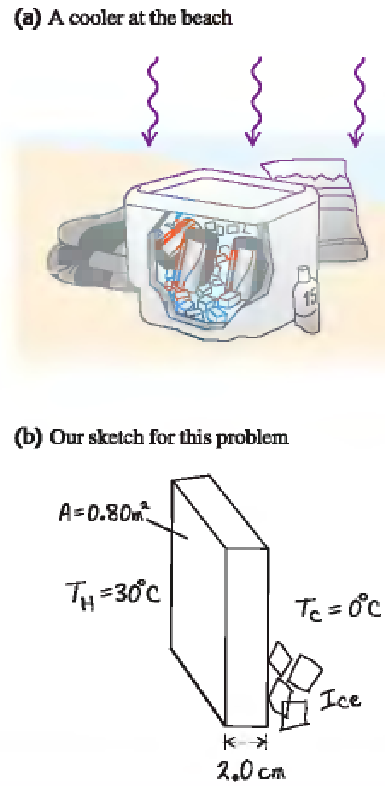
$$Q = Ht = (12\ \mathrm{J/s})(86{,}400\ \mathrm{s}) = 1.04 \times 10^6\ \mathrm{J}$$

The heat of fusion of ice is 3.34×10^5 J/kg, so the quantity of ice melted by this quantity of heat is

$$m = \frac{Q}{L_f} = \frac{1.04 \times 10^6\ \mathrm{J}}{3.34 \times 10^5\ \mathrm{J/kg}} = 3.1\ \mathrm{kg}$$

EVALUATE: The low heat current is a result of the low thermal conductivity of Styrofoam. A substantial amount of heat flows in 24 hours, but a relatively small amount of ice melts because the heat of fusion is high.

17.25 Conduction of heat across the walls of a Styrofoam cooler.



Example 17.13 Conduction through two bars I

A steel bar 10.0 cm long is welded end to end to a copper bar 20.0 cm long. Both bars are insulated perfectly on their sides. Each bar has a square cross section, 2.00 cm on a side. The free end of the steel bar is maintained at 100°C by placing it in contact with steam, and the free end of the copper bar is maintained at 0°C by placing it in contact with ice. Find the temperature at the junction of the two bars and the total rate of heat flow.

SOLUTION

IDENTIFY: In this problem there is heat flow through two bars of different composition. As we discussed in Problem-Solving Strategy 17.3, the heat currents in the two end-to-end bars must be the same.

SET UP: Figure 17.26 shows the situation. We write Eq. (17.21) twice, once for each bar, and set the heat currents H_{steel} and H_{copper} equal to each other. Both expressions for the heat current involve the temperature T at the junction, which is one of our target variables.

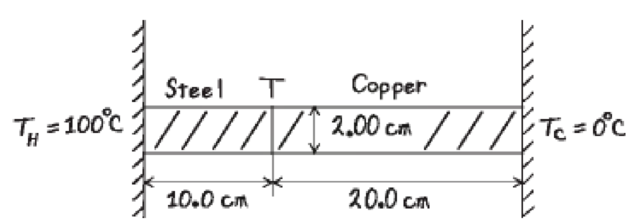
EXECUTE: Setting the two heat currents equal, we have

$$H_{steel} = \frac{k_{steel}A(100°\mathrm{C} - T)}{L_{steel}} = H_{copper} = \frac{k_{copper}A(T - 0°\mathrm{C})}{L_{copper}}$$

The areas A are equal and may be divided out. Substituting $L_{steel} = 0.100$ m, $L_{copper} = 0.200$ m, and numerical values of k from Table 17.5, we find

$$\frac{(50.2\ \mathrm{W/m\cdot K})(100°\mathrm{C} - T)}{0.100\ \mathrm{m}} = \frac{(385\ \mathrm{W/m\cdot K})(T - 0°\mathrm{C})}{0.200\ \mathrm{m}}$$

17.26 Our sketch for this problem.



Rearranging and solving for T, we find

$$T = 20.7°C$$

We can find the total heat current by substituting this value for T back into either of the above expressions:

$$H_{steel} = \frac{(50.2\ W/m \cdot K)(0.0200\ m)^2(100°C - 20.7°C)}{0.100\ m} = 15.9\ W$$

or

$$H_{copper} = \frac{(385\ W/m \cdot K)(0.0200\ m)^2(20.7°C)}{0.200\ m} = 15.9\ W$$

EVALUATE: Even though the steel bar is shorter, the temperature drop across it is much greater than across the copper bar (from 100°C to 20.7°C in the steel versus from 20.7°C to 0°C in the copper). This difference arises because steel is a much poorer conductor than copper.

Example 17.14 Conduction through two bars II

In Example 17.13, suppose the two bars are separated. One end of each bar is maintained at 100°C and the other end of each bar is maintained at 0°C. What is the *total* rate of heat flow in the two bars?

SOLUTION

IDENTIFY: In this case the bars are side by side rather than end to end. The total heat current is now the *sum* of the currents in the two bars.

SET UP: Figure 17.27 shows the situation. For each bar, $T_H - T_C = 100°C - 0°C = 100\ K$.

EXECUTE: We write the heat currents for the two rods individually, and then add them to get the total heat current:

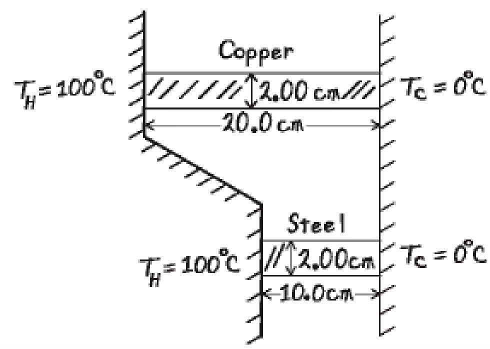
$$H = H_{steel} + H_{copper} = \frac{k_{steel}A(T_H - T_C)}{L_{steel}} + \frac{k_{copper}A(T_H - T_C)}{L_{copper}}$$

$$= \frac{(50.2\ W/m \cdot K)(0.0200\ m)^2(100\ K)}{0.100\ m} + \frac{(385\ W/m \cdot K)(0.0200\ m)^2(100\ K)}{0.200\ m}$$

$$= 20.1\ W + 77.0\ W = 97.1\ W$$

EVALUATE: The heat flow in the copper bar is much greater than that in the steel bar, even though it is longer, because the thermal conductivity of copper is much larger. The total heat flow is much greater than in Example 17.13, partly because the total cross section for heat flow is greater and partly because the full 100-K temperature difference appears across each bar.

17.27 Our sketch for this problem.



Convection

Convection is the transfer of heat by mass motion of a fluid from one region of space to another. Familiar examples include hot-air and hot-water home heating systems, the cooling system of an automobile engine, and the flow of blood in the body. If the fluid is circulated by a blower or pump, the process is called *forced convection;* if the flow is caused by differences in density due to thermal expansion, such as hot air rising, the process is called *natural convection* or *free convection* (Fig. 17.28).

Free convection in the atmosphere plays a dominant role in determining the daily weather, and convection in the oceans is an important global heat-transfer mechanism. On a smaller scale, soaring hawks and glider pilots make use of thermal updrafts from the warm earth. The most important mechanism for heat transfer within the human body (needed to maintain nearly constant temperature in various environments) is *forced* convection of blood, with the heart serving as the pump.

Convective heat transfer is a very complex process, and there is no simple equation to describe it. Here are a few experimental facts:

1. The heat current due to convection is directly proportional to the surface area. This is the reason for the large surface areas of radiators and cooling fins.
2. The viscosity of fluids slows natural convection near a stationary surface, giving a surface film that on a vertical surface typically has about the same insulating value as 1.3 cm of plywood (R value $= 0.7$). Forced convection

17.28 A heating element in the tip of this submerged tube warms the surrounding water, producing a complex pattern of free convection.

decreases the thickness of this film, increasing the rate of heat transfer. This is the reason for the "wind-chill factor"; you get cold faster in a cold wind than in still air with the same temperature.

3. The heat current due to convection is found to be approximately proportional to the $\frac{5}{4}$ power of the temperature difference between the surface and the main body of fluid.

Radiation

Radiation is the transfer of heat by electromagnetic waves such as visible light, infrared, and ultraviolet radiation. Everyone has felt the warmth of the sun's radiation and the intense heat from a charcoal grill or the glowing coals in a fireplace. Most of the heat from these very hot bodies reaches you not by conduction or convection in the intervening air but by *radiation*. This heat transfer would occur even if there were nothing but vacuum between you and the source of heat.

Every body, even at ordinary temperatures, emits energy in the form of electromagnetic radiation. At ordinary temperatures, say 20°C, nearly all the energy is carried by infrared waves with wavelengths much longer than those of visible light (see Figs. 17.4 and 17.29). As the temperature rises, the wavelengths shift to shorter values. At 800°C a body emits enough visible radiation to be self-luminous and appears "red-hot," although even at this temperature most of the energy is carried by infrared waves. At 3000°C, the temperature of an incandescent lamp filament, the radiation contains enough visible light that the body appears "white-hot."

17.29 This false-color infrared photograph reveals radiation emitted by various parts of the man's body. The strongest emission (colored red) comes from the warmest areas, while there is very little emission from the bottle of cold beverage.

The rate of energy radiation from a surface is proportional to the surface area A. The rate increases very rapidly with temperature, depending on the fourth power of the absolute (Kelvin) temperature. The rate also depends on the nature of the surface; this dependence is described by a quantity e called the **emissivity.** A dimensionless number between 0 and 1, it represents the ratio of the rate of radiation from a particular surface to the rate of radiation from an equal area of an ideal radiating surface at the same temperature. Emissivity also depends somewhat on temperature. Thus the heat current $H = dQ/dt$ due to radiation from a surface area A with emissivity e at absolute temperature T can be expressed as

$$H = Ae\sigma T^4 \qquad \text{(heat current in radiation)} \qquad (17.25)$$

where σ is a fundamental physical constant called the **Stefan–Boltzmann constant.** This relationship is called the **Stefan–Boltzmann law** in honor of its late-19th-century discoverers. The current best numerical value of σ is

$$\sigma = 5.670400(40) \times 10^{-8}\ \mathrm{W/m^2 \cdot K^4}$$

We invite you to check unit consistency in Eq. (17.25). Emissivity (e) is often larger for dark surfaces than for light ones. The emissivity of a smooth copper surface is about 0.3, but e for a dull black surface can be close to unity.

Example 17.15 Heat transfer by radiation

A thin square steel plate, 10 cm on a side, is heated in a blacksmith's forge to a temperature of 800°C. If the emissivity is 0.60, what is the total rate of radiation of energy?

SOLUTION

IDENTIFY: The target variable is H, the rate of emission of energy.

SET UP: We use Eq. (17.25) to calculate H from the given values.

EXECUTE: The total surface area, including both sides, is $2(0.10\ \mathrm{m})^2 = 0.020\ \mathrm{m}^2$. We must convert the temperature to the Kelvin scale; 800°C = 1073 K. Then Eq. (17.25) gives

$$\begin{aligned} H &= Ae\sigma T^4 \\ &= (0.020\ \mathrm{m^2})(0.60)(5.67 \times 10^{-8}\ \mathrm{W/m^2 \cdot K^4})(1073\ \mathrm{K})^4 \\ &= 900\ \mathrm{W} \end{aligned}$$

EVALUATE: A blacksmith standing nearby will easily feel heat being radiated from this plate.

Radiation and Absorption

While a body at absolute temperature T is radiating, its surroundings at temperature T_s are also radiating, and the body *absorbs* some of this radiation. If it is in thermal equilibrium with its surroundings, $T = T_s$ and the rates of radiation and absorption must be equal. For this to be true, the rate of absorption must be given in general by $H = Ae\sigma T_s^4$. Then the *net* rate of radiation from a body at temperature T with surroundings at temperature T_s is

$$H_{net} = Ae\sigma T^4 - Ae\sigma T_s^4 = Ae\sigma(T^4 - T_s^4) \qquad (17.26)$$

In this equation a positive value of H means a net heat flow *out of* the body. Equation (17.26) shows that for radiation, as for conduction and convection, the heat current depends on the temperature *difference* between two bodies.

Example 17.16 Radiation from the human body

If the total surface area of the human body is 1.20 m^2 and the surface temperature is $30°C = 303 \text{ K}$, find the total rate of radiation of energy from the body. If the surroundings are at a temperature of 20°C, what is the *net* rate of heat loss from the body by radiation? The emissivity of the body is very close to unity, irrespective of skin pigmentation.

SOLUTION

IDENTIFY: We must take into account both the radiation that the body emits and the radiation that the body absorbs from its surroundings.

SET UP: The rate of radiation of energy from the body is given by Eq. (17.25), and the net rate of heat loss is given by Eq. (17.26).

EXECUTE: Taking $e = 1$ in Eq. (17.25), we find that the body radiates at a rate

$$\begin{aligned} H &= Ae\sigma T^4 \\ &= (1.20 \text{ m}^2)(1)(5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4)(303 \text{ K})^4 \\ &= 574 \text{ W} \end{aligned}$$

This loss is partly offset by absorption of radiation, which depends on the temperature of the surroundings. The *net* rate of radiative energy transfer is given by Eq. (17.26):

$$\begin{aligned} H_{net} &= Ae\sigma(T^4 - T_s^4) \\ &= (1.20 \text{ m}^2)(1)(5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4) \\ &\quad \times [(303 \text{ K})^4 - (293 \text{ K})^4] = 72 \text{ W} \end{aligned}$$

EVALUATE: The value of H_{net} is positive because the body is losing heat to its colder surroundings.

Applications of Radiation

Heat transfer by radiation is important in some surprising places. A premature baby in an incubator can be cooled dangerously by radiation if the walls of the incubator happen to be cold, even when the *air* in the incubator is warm. Some incubators regulate the air temperature by measuring the baby's skin temperature.

A body that is a good absorber must also be a good emitter. An ideal radiator, with an emissivity of unity, is also an ideal absorber, absorbing *all* of the radiation that strikes it. Such an ideal surface is called an ideal black body or simply a **blackbody.** Conversely, an ideal *reflector,* which absorbs *no* radiation at all, is also a very ineffective radiator.

This is the reason for the silver coatings on vacuum ("Thermos") bottles, invented by Sir James Dewar (1842–1923). A vacuum bottle has double glass walls. The air is pumped out of the spaces between the walls; this eliminates nearly all heat transfer by conduction and convection. The silver coating on the walls reflects most of the radiation from the contents back into the container, and the wall itself is a very poor emitter. Thus a vacuum bottle can keep coffee or soup hot for several hours. The Dewar flask, used to store very cold liquefied gases, is exactly the same in principle.

Test Your Understanding of Section 17.7 A room has one wall made of concrete, one wall made of copper, and one wall made of steel. All of the walls are the same size and at the same temperature of 20°C. Which wall feels coldest to the touch? (i) the concrete wall; (ii) the copper wall; (iii) the steel wall; (iv) all three walls feel equally cold to the touch.

CHAPTER 17 SUMMARY

Temperature and temperature scales: A thermometer measures temperature. Two bodies in thermal equilibrium must have the same temperature. A conducting material between two bodies permits them to interact and come to thermal equilibrium; an insulating material impedes this interaction.

The Celsius and Fahrenheit temperature scales are based on the freezing temperature ($0°C = 32°F$) and boiling temperature ($100°C = 212°F$). One Celsius degree equals $\frac{9}{5}$ Fahrenheit degrees. (See Example 17.1.)

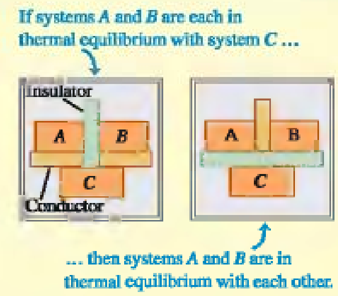
The Kelvin scale has its zero at the extrapolated zero-pressure temperature for a gas thermometer, $-273.15°C = 0\text{ K}$. In the gas-thermometer scale, the ratio of two temperatures T_1 and T_2 is defined to be equal to the ratio of the two corresponding gas-thermometer pressures p_1 and p_2. The triple-point temperature of water ($0.01°C$) is defined to be 273.16 K.

$$T_F = \frac{9}{5}T_C + 32° \quad (17.1)$$

$$T_C = \frac{5}{9}(T_F - 32°) \quad (17.2)$$

$$T_K = T_C + 273.15 \quad (17.3)$$

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} \quad (17.4)$$



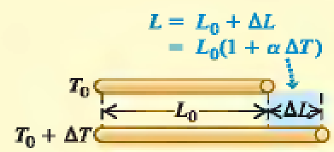
Thermal expansion and thermal stress: A temperature change ΔT causes a change in any linear dimension L_0 of a solid body. The change ΔL is approximately proportional to L_0 and ΔT. Similarly, a temperature change causes a change ΔV in the volume V_0 of any solid or liquid material that is approximately proportional to V_0 and ΔT. The quantities α and β are the coefficients of linear expansion and volume expansion, respectively. For solids, $\beta = 3\alpha$. (See Examples 17.2–17.4.)

When a material is cooled or heated and held so it cannot contract or expand, it is under a tensile stress F/A. (See Example 17.5.)

$$\Delta L = \alpha L_0 \, \Delta T \quad (17.6)$$

$$\Delta V = \beta V_0 \, \Delta T \quad (17.8)$$

$$\frac{F}{A} = -Y\alpha \, \Delta T \quad (17.12)$$



Heat, phase changes, and calorimetry: Heat is energy in transit from one body to another as a result of a temperature difference. The quantity of heat Q required to raise the temperature of a quantity of material by a small amount ΔT is proportional to ΔT. This proportionality can be expressed either in terms of the mass m and specific heat capacity c or in terms of the number of moles n and the molar heat capacity $C = Mc$. Here M is the molar mass and $m = nM$. (See Examples 17.6 and 17.7.)

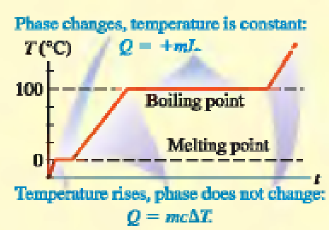
To change a mass m of a material to a different phase at the same temperature (such as liquid to solid or liquid to vapor) requires the addition or subtraction of a quantity of heat. The amount of heat is equal to the product of m and L, the heat of fusion, vaporization, or sublimation.

When heat is added to a body, the corresponding Q is positive; when it is removed, Q is negative. The basic principle of calorimetry comes from conservation of energy. In an isolated system whose parts interact by heat exchange, the algebraic sum of the Q's for all parts of the system must be zero. (See Examples 17.8–17.11.)

$$Q = mc \, \Delta T \quad (17.13)$$

$$Q = nC \, \Delta T \quad (17.18)$$

$$Q = \pm mL \quad (17.20)$$



Conduction, convection, and radiation: Conduction is the transfer of energy of molecular motion within bulk materials without bulk motion of the materials. The heat current H or conduction depends on the area A through which the heat flows, the length L of the heat-flow path, the temperature difference $(T_H - T_C)$, and the thermal conductivity k of the material. (See Examples 17.12–17.14.)

Convection is a complex heat-transfer process that involves mass motion from one region to another. It depends on surface area, orientation, and the temperature difference between a body and its surroundings.

Radiation is energy transfer through electromagnetic radiation. The radiation heat current H depends on the surface area A, the emissivity e of the surface (a pure number between 0 and 1), and the Kelvin temperature T. It involves a fundamental constant σ called the Stefan–Boltzmann constant. When a body at temperature T is surrounded by material at temperature T_s, the *net* heat current H_{net} from the body to its surroundings depends on both T and T_s. (See Examples 17.15 and 17.16.)

$$H = \frac{dQ}{dt} = kA\frac{T_H - T_C}{L} \quad (17.21)$$

$$H = Ae\sigma T^4 \quad (17.25)$$

$$H_{net} = Ae\sigma(T^4 - T_s^4) \quad (17.26)$$

Key Terms

thermodynamics, *570*
temperature, *571*
thermometer, *571*
thermal equilibrium, *571*
insulator, *571*
conductor, *571*
zeroth law of thermodynamics, *572*
Celsius temperature scale, *572*
Fahrenheit temperature scale, *573*
Kelvin temperature scale, *574*
absolute temperature scale, *576*
absolute zero, *576*
coefficient of linear expansion, *576*
coefficient of volume expansion, *578*
thermal stress, *580*
heat, *582*
calorie, *582*
British thermal unit, *583*
specific heat, *583*
molar heat capacity, *584*
phase, *586*
states of matter, *586*
phase change, *586*
heat of fusion, *586*
phase equilibrium, *587*
heat of vaporization, *587*
heat of combustion, *589*
conduction, *592*
heat current, *592*
thermal conductivity, *592*
temperature gradient, *592*
thermal resistance, *593*
convection, *595*
radiation, *596*
emissivity, *596*
Stefan–Boltzmann constant, *596*
Stefan–Boltzmann law, *596*
blackbody, *597*

Answer to Chapter Opening Question ?

No. By "heat" we mean energy that is in transit from one body to another as a result of temperature difference between the bodies. Bodies do not *contain* heat.

Answers to Test Your Understanding Questions

17.1 Answer: (ii) A liquid-in-tube thermometer actually measures its own temperature. If the thermometer stays in the hot water long enough, it will come to thermal equilibrium with the water and its temperature will be the same as that of the water.

17.2 Answer: (iv) Both a bimetallic strip and a resistance thermometer measure their own temperature. For this to be equal to the temperature of the object being measured, the thermometer and object must be in contact and in thermal equilibrium. A temporal artery thermometer detects the infrared radiation from a person's skin, so there is no need for the detector and skin to be at the same temperature.

17.3 Answer: (i), (iii), (ii), (v), (iv) To compare these temperatures, convert them all to the Kelvin scale. For (i), the Kelvin temperature is $T_K = T_C + 273.15 = 0.00 + 273.15 = 273.15$ K; for (ii), $T_C = \frac{5}{9}(T_F - 32°) = \frac{5}{9}(0.00° - 32°) = -17.78$°C and $T_K = T_C + 273.15 = -17.78 + 273.15 = 255.37$ K; for (iii), $T_K = 260.00$ K; for (iv), $T_K = 77.00$ K; and for (v), $T_K = T_C + 273.15 = -180.00 + 273.15 = 93.15$ K.

17.4 Answer: (ii) and (iii) Metal 2 must expand more than metal 1 when heated and so must have a larger coefficient of linear expansion α. From Table 17.1, brass and aluminum have larger values of α than copper, but steel does not.

17.5 Answer: (ii), (i), (iv), (iii) For (i) and (ii), the relevant quantity is the specific heat c of the substance, which is the amount of heat required to raise the temperature of 1 *kilogram* of that substance by 1 K (1 C°). From Table 17.3, these values are (i) 138 J for mercury and (ii) 2428 J for ethanol. For (iii) and (iv) we need the molar heat capacity C, which is the amount of heat required to raise the temperature of 1 *mole* of that substance by 1 C°. Again from Table 17.3, these values are (iii) 27.7 J for mercury and (iv) 111.9 J for ethanol. (The ratio of molar heat capacities is different from the ratio of the specific heats because a mole of mercury and a mole of ethanol have different masses.)

17.6 Answer: (iv) In time t the system goes from point b to point e in Fig. 17.21. According to this figure, at time $t/2$ (halfway along the horizontal axis from b to e), the system is at 100°C and is still boiling; that is, it is a mixture of liquid and gas. This says that most of the heat added goes into boiling the water.

17.7 Answer: (ii) When you touch one of the walls, heat flows from your hand to the lower-temperature wall. The more rapidly heat flows from your hand, the colder you will feel. Equation (17.21) shows that the rate of heat flow is proportional to the thermal conductivity k. From Table 17.5, copper has a much higher thermal conductivity $(385.0\ \mathrm{W/m \cdot K})$ than steel $(50.2\ \mathrm{W/m \cdot K})$ or concrete $(0.8\ \mathrm{W/m \cdot K})$, and so the copper wall feels the coldest to the touch.

PROBLEMS

For instructor-assigned homework, go to www.masteringphysics.com

Discussion Questions

Q17.1. Explain why it would not make sense to use a full-size glass thermometer to measure the temperature of a thimbleful of hot water.

Q17.2. If you heat the air inside a rigid, sealed container until its Kelvin temperature doubles, the air pressure in the container will also double. Is the same thing true if you double the Celsius temperature of the air in the container? Explain.

Q17.3. Many automobile engines have cast-iron cylinders and aluminum pistons. What kinds of problems could occur if the engine gets too hot? (The coefficient of volume expansion of cast iron is approximately the same as that of steel.)

Q17.4. Why do frozen water pipes burst? Would a mercury thermometer break if the temperature went below the freezing temperature of mercury? Why or why not?

Q17.5. Two bodies made of the same material have the same external dimensions and appearance, but one is solid and the other is hollow. When their temperature is increased, is the overall volume expansion the same or different? Why?

Q17.6. The inside of an oven is at a temperature of 200°C (392°F). You can put your hand in the oven without injury as long as you don't touch anything. But since the air inside the oven is also at 200°C, why isn't your hand burned just the same?

Q17.7. A newspaper article about the weather states that "the temperature of a body measures how much heat the body contains." Is this description correct? Why or why not?

Q17.8. To raise the temperature of an object, must you add heat to it? If you add heat to an object, must you raise its temperature? Explain.

Q17.9. A student asserts that a suitable unit for specific heat capacity is $1\ \mathrm{m^2/s^2 \cdot C°}$. Is she correct? Why or why not?

Q17.10. In some household air conditioners used in dry climates, air is cooled by blowing it through a water-soaked filter, evaporating some of the water. How does this cool the air? Would such a system work well in a high-humidity climate? Why or why not?

Q17.11. The units of specific heat capacity c are $\mathrm{J/kg \cdot K}$, but the units of heat of fusion L_f or heat of vaporization L_v are simply J/kg. Why do the units of L_f and L_v not include a factor of $(\mathrm{K})^{-1}$ to account for a temperature change?

Q17.12. Why is a hot, humid day in the tropics generally more uncomfortable for human beings than a hot, dry day in the desert?

Q17.13. A piece of aluminum foil used to wrap a potato for baking in a hot oven can usually be handled safely within a few seconds after the potato is removed from the oven. The same is not true of the potato, however! Give two reasons for this difference.

Q17.14. Desert travelers sometimes keep water in a canvas bag. Some water seeps through the bag and evaporates. How does this cool the water inside the bag?

Q17.15. When you first step out of the shower, you feel cold. But as soon as you are dry you feel warmer, even though the room temperature does not change. Why?

Q17.16. The climate of regions adjacent to large bodies of water (like the Pacific and Atlantic coasts) usually features a narrower range of temperature than the climate of regions far from large bodies of water (like the prairies). Why?

Q17.17. When water is placed in ice-cube trays in a freezer, why doesn't the water freeze all at once when the temperature has reached 0°C? In fact, the water freezes first in a layer adjacent to the sides of the tray. Why?

Q17.16. Before giving you an injection, a physician swabs your arm with isopropyl alcohol at room temperature. Why does this make your arm feel cold? (*Hint:* The reason is *not* the fear of the injection! The boiling point of isopropyl alcohol is 82.4°C.)

Q17.19. A cold block of metal feels colder than a block of wood at the same temperature. Why? A *hot* block of metal feels hotter than a block of wood at the same temperature. Again, why? Is there any temperature at which the two blocks feel equally hot or cold? What temperature is this?

Q17.20. A person pours a cup of hot coffee, intending to drink it five minutes later. To keep the coffee as hot as possible, should she put cream in it now, or wait until just before she drinks it? Explain.

Q17.21. When a freshly baked apple pie has just been removed from the oven, the crust and filling are both at the same temperature. Yet if you sample the pie, the filling will burn your tongue but the crust will not. Why is there a difference? (*Hint:* The filling is moist while the crust is dry.)

Q17.22. Old-time kitchen lore suggests that things cook better (evenly and without burning) in heavy cast-iron pots. What desirable characteristics do such pots have?

Q17.23. In coastal regions in the winter, the temperature over the land is generally colder than the temperature over the nearby ocean; in the summer, the reverse is usually true. Explain. (*Hint:* The specific heat of soil is only 0.2–0.8 times as great as that of water.)

Q17.24. It is well known that a potato bakes faster if a large nail is stuck through it. Why? Does an aluminum nail work better than a steel one? Why or why not? (*Note:* Don't try this in a microwave oven!) There is also a gadget on the market to hasten the roasting of meat; it consists of a hollow metal tube containing a wick and some water. This is claimed to work much better than a solid metal rod. How does it work?

Q17.25. Glider pilots in the Midwest know that thermal updrafts are likely to occur in the vicinity of freshly plowed fields. Why?

Q17.26. Some folks claim that ice cubes freeze faster if the trays are filled with hot water, because hot water cools off faster than cold water. What do you think?

Q17.27. We're lucky that the earth isn't in thermal equilibrium with the sun (which has a surface temperature of 5800 K). But why aren't the two bodies in thermal equilibrium?

Q17.26. When energy shortages occur, magazine articles sometimes urge us to keep our homes at a constant temperature day and night to conserve fuel. They argue that when we turn down the heat at night, the walls, ceilings, and other areas cool off and must be reheated in the morning. So if we keep the temperature constant, these parts of the house will not cool off and will not have to be reheated. Does this argument make sense? Would we really save energy by following this advice?

Exercises

Section 17.2 Thermometers and Temperature Scales

17.1. Convert the following Celsius temperatures to Fahrenheit: (a) $-62.8°C$, the lowest temperature ever recorded in North America (February 3, 1947, Snag, Yukon); (b) 56.7°C, the highest temperature ever recorded in the United States (July 10, 1913, Death Valley, California); (c) 31.1°C, the world's highest average annual temperature (Lugh Ferrandi, Somalia).

17.2. Find the Celsius temperatures corresponding to (a) a winter night in Seattle $(41.0°F)$; (b) a hot summer day in Palm Springs $(107.0°F)$; (c) a cold winter day in northern Manitoba $(-18.0°F)$.

17.3. While vacationing in Italy, you see on local TV one summer morning that the temperature will rise from the current 18°C to a high of 39°C. What is the corresponding increase in the Fahrenheit temperature?

17.4. Two beakers of water, A and B, initially are at the same temperature. The temperature of the water in beaker A is increased 10 F°, and the temperature of the water in beaker B is increased 10 K. After these temperature changes, which beaker of water has the higher temperature? Explain.

17.5. You put a bottle of soft drink in a refrigerator and leave it until its temperature has dropped 10.0 K. What is its temperature change in (a) F° and (b) C°?

17.6. (a) On January 22, 1943, the temperature in Spearfish, South Dakota, rose from $-4.0°F$ to 45.0°F in just 2 minutes. What was the temperature change in Celsius degrees? (b) The temperature in Browning, Montana, was 44.0°F on January 23, 1916. The next day the temperature plummeted to $-56°C$. What was the temperature change in Celsius degrees?

17.7. (a) You feel sick and are told that you have a temperature of 40.2°C. What is your temperature in °F? Should you be concerned? (b) The morning weather report in Sydney gives a current temperature of 12°C. What is this temperature in °F?

Section 17.3 Gas Thermometers and the Kelvin Scale

17.8. (a) Calculate the one temperature at which Fahrenheit and Celsius thermometers agree with each other. (b) Calculate the one temperature at which Fahrenheit and Kelvin thermometers agree with each other.

17.9. Convert the following record-setting temperatures to the Kelvin scale: (a) the lowest temperature recorded in the 48 contiguous states ($-70.0°F$ at Rogers Pass, Montana, on January 20, 1954); (b) Australia's highest temperature (127.0°F at Cloncurry, Queensland, on January 16, 1889); (c) the lowest temperature recorded in the northern hemisphere ($-90.0°F$ at Verkhoyansk, Siberia, in 1892).

17.10. Convert the following Kelvin temperatures to the Celsius and Fahrenheit scales: (a) the midday temperature at the surface of the moon (400 K); (b) the temperature at the tops of the clouds in the atmosphere of Saturn (95 K); (c) the temperature at the center of the sun $(1.55 \times 10^7\ \text{K})$.

17.11. Liquid nitrogen is a relatively inexpensive material that is often used to perform entertaining low-temperature physics demonstrations. Nitrogen gas liquefies at a temperature of $-346°F$. Convert this temperature to (a) °C and (b) K.

17.12. A gas thermometer registers an absolute pressure corresponding to 325 mm of mercury when in contact with water at the triple point. What pressure does it read when in contact with water at the normal boiling point?

17.13. The pressure of a gas at the triple point of water is 1.35 atm. If its volume remains unchanged, what will its pressure be at the temperature at which CO_2 solidifies?

17.14. Like the Kelvin scale, the *Rankine scale* is an absolute temperature scale: Absolute zero is zero degrees Rankine $(0°R)$. However, the units of this scale are the same size as those of the Fahrenheit scale rather than the Celsius scale. What is the numerical value of the triple-point temperature of water on the Rankine scale?

17.15. A Constant-Volume Gas Thermometer. An experimenter using a gas thermometer found the pressure at the triple point of water (0.01°C) to be 4.80×10^4 Pa and the pressure at the normal boiling point $(100°C)$ to be 6.50×10^4 Pa. (a) Assuming that the pressure varies linearly with temperature, use these two data points to find the Celsius temperature at which the gas pressure would be zero (that is, find the Celsius temperature of absolute zero). (b) Does the gas in this thermometer obey Eq. (17.4) precisely? If that equation were precisely obeyed and the pressure at 100°C were 6.50×10^4Pa, what pressure would the experimenter have measured at 0.01°C? (As we will learn in Section 18.1, Eq. (17.4) is accurate only for gases at very low density.)

Section 17.4 Thermal Expansion

17.16. The tallest building in the world, according to some architectural standards, is the Taipei 101 in Taiwan, at a height of 1671 feet. Assume that this height was measured on a cool spring day when the temperature was 15.5°C. You could use the building as a sort of giant thermometer on a hot summer day by carefully measuring its height. Suppose you do this and discover that the Taipei 101 is 0.471 foot taller than its official height. What is the temperature, assuming that the building is in thermal equilibrium with the air and that its entire frame is made of steel?

17.17. The Humber Bridge in England has the world's longest single span, 1410 m. Calculate the change in length of the steel deck of the span when the temperature increases from $-5.0°C$ to 18.0°C.

17.18. Ensuring a Tight Fit. Aluminum rivets used in airplane construction are made slightly larger than the rivet holes and cooled by "dry ice" (solid CO_2) before being driven. If the diameter of a hole is 4.500 mm, what should be the diameter of a rivet at 23.0°C, if its diameter is to equal that of the hole when the rivet is cooled to $-78.0°C$, the temperature of dry ice? Assume that the expansion coefficient remains constant at the value given in Table 17.1.

17.19. A U.S. penny has a diameter of 1.9000 cm at 20.0°C. The coin is made of a metal alloy (mostly zinc) for which the coefficient of linear expansion is $2.6 \times 10^{-5}\ \text{K}^{-1}$. What would its diameter be on a hot day in Death Valley $(48.0°C)$? On a cold night in the mountains of Greenland $(-53°C)$?

17.20. A geodesic dome constructed with an aluminum framework is a nearly perfect hemisphere; its diameter measures 55.0 m on a winter day at a temperature of $-15°C$. How much more interior space does the dome have in the summer, when the temperature is 35°C?

17.21. A metal rod is 40.125 cm long at 20.0°C and 40.148 cm long at 45.0°C. Calculate the average coefficient of linear expansion of the rod for this temperature range.

17.22. A copper cylinder is initially at 20.0°C. At what temperature will its volume be 0.150% larger than it is at 20.0°C?

17.23. The density of water is 999.73 kg/m^3 at a temperature of 10°C and 958.38 kg/m^3 at a temperature of 100°C. Calculate the average coefficient of volume expansion for water in that range of temperature.

17.24. A steel tank is completely filled with 2.80 m^3 of ethanol when both the tank and the ethanol are at a temperature of 32.0°C. When the tank and its contents have cooled to 18.0°C, what additional volume of ethanol can be put into the tank?

17.25. A glass flask whose volume is 1000.00 cm^3 at 0.0°C is completely filled with mercury at this temperature. When flask and mercury are warmed to 55.0°C, 8.95 cm^3 of mercury overflow. If the coefficient of volume expansion of mercury is $18.0 \times 10^{-5}\ \text{K}^{-1}$, compute the coefficient of volume expansion of the glass.

17.26. (a) If an area measured on the surface of a solid body is A_0 at some initial temperature and then changes by ΔA when the temperature changes by ΔT, show that

$$\Delta A = (2\alpha)\, A_0 \Delta T$$

where α is the coefficient of linear expansion. (b) A circular sheet of aluminum is 55.0 cm in diameter at 15.0°C. By how much does the area of one side of the sheet change when the temperature increases to 27.5°C?

17.27. A machinist bores a hole of diameter 1.35 cm in a steel plate at a temperature of 25.0°C. What is the cross-sectional area of the hole (a) at 25.0°C and (b) when the temperature of the plate is increased to 175°C? Assume that the coefficient of linear expansion remains constant over this temperature range. (*Hint:* See Exercise 17.26.)

17.28. As a new mechanical engineer for Engines Inc., you have been assigned to design brass pistons to slide inside steel cylinders. The engines in which these pistons will be used will operate between 20.0°C and 150.0°C. Assume that the coefficients of expansion are constant over this temperature range. (a) If the piston just fits inside the chamber at 20.0°C, will the engines be able to run at higher temperatures? Explain. (b) If the cylindrical pistons are 25.000 cm in diameter at 20.0°C, what should be the minimum diameter of the cylinders at that temperature so the pistons will operate at 150.0°C?

17.29. The outer diameter of a glass jar and the inner diameter of its iron lid are both 725 mm at room temperature (20.0°C). What will be the size of the mismatch between the lid and the jar if the lid is briefly held under hot water until its temperature rises to 50.0°C, without changing the temperature of the glass?

17.30. A brass rod is 185 cm long and 1.60 cm in diameter. What force must be applied to each end of the rod to prevent it from contracting when it is cooled from 120.0°C to 10.0°C?

17.31. (a) A wire that is 1.50 m long at 20.0°C is found to increase in length by 1.90 cm when warmed to 420.0°C. Compute its average coefficient of linear expansion for this temperature range. (b) The wire is stretched just taut (zero tension) at 420.0°C. Find the stress in the wire if it is cooled to 20.0°C without being allowed to contract. Young's modulus for the wire is 2.0×10^{11} Pa.

17.32. Steel train rails are laid in 12.0-m-long segments placed end to end. The rails are laid on a winter day when their temperature is −2.0°C. (a) How much space must be left between adjacent rails if they are just to touch on a summer day when their temperature is 33.0°C? (b) If the rails are originally laid in contact, what is the stress in them on a summer day when their temperature is 33.0°C?

Section 17.5 Quantity of Heat

17.33. An aluminum tea kettle with mass 1.50 kg and containing 1.80 kg of water is placed on a stove. If no heat is lost to the surroundings, how much heat must be added to raise the temperature from 20.0°C to 85.0°C?

17.34. In an effort to stay awake for an all-night study session, a student makes a cup of coffee by first placing a 200-W electric immersion heater in 0.320 kg of water. (a) How much heat must be added to the water to raise its temperature from 20.0°C to 80.0°C? (b) How much time is required? Assume that all of the heater's power goes into heating the water.

17.35. You are given a sample of metal and asked to determine its specific heat. You weigh the sample and find that its weight is 28.4 N. You carefully add 1.25×10^4 J of heat energy to the sample and find that its temperature rises 18.0 C°. What is the sample's specific heat?

17.36. Heat Loss During Breathing. In very cold weather a significant mechanism for heat loss by the human body is energy expended in warming the air taken into the lungs with each breath. (a) On a cold winter day when the temperature is −20°C, what amount of heat is needed to warm to body temperature (37°C) the 0.50 L of air exchanged with each breath? Assume that the specific heat of air is 1020 $\text{J/kg} \cdot \text{K}$ and that 1.0 L of air has mass 1.3×10^{-3} kg. (b) How much heat is lost per hour if the respiration rate is 20 breaths per minute?

17.37. While running, a 70-kg student generates thermal energy at a rate of 1200 W. To maintain a constant body temperature of 37°C, this energy must be removed by perspiration or other mechanisms. If these mechanisms failed and the heat could not flow out of the student's body, for what amount of time could a student run before irreversible body damage occurred? (*Note:* Protein structures in the body are irreversibly damaged if body temperature rises to 44°C or higher. The specific heat of a typical human body is 3480 $\text{J/kg} \cdot \text{K}$, slightly less than that of water. The difference is due to the presence of protein, fat, and minerals, which have lower specific heats.)

17.38. While painting the top of an antenna 225 m in height, a worker accidentally lets a 1.00-L water bottle fall from his lunchbox. The bottle lands in some bushes at ground level and does not break. If a quantity of heat equal to the magnitude of the change in mechanical energy of the water goes into the water, what is its increase in temperature?

17.39. A crate of fruit with mass 35.0 kg and specific heat 3650 $\text{J/kg} \cdot \text{K}$ slides down a ramp inclined at 36.9°C below the horizontal. The ramp is 8.00 m long. (a) If the crate was at rest at the top of the incline and has a speed of 2.50 m/s at the bottom, how much work was done on the crate by friction? (b) If an amount of heat equal to the magnitude of the work done by friction goes into the crate of fruit and the fruit reaches a uniform final temperature, what is its temperature change?

17.40. A 25,000-kg subway train initially traveling at 15.5 m/s slows to a stop in a station and then stays there long enough for its brakes to cool. The station's dimensions are 65.0 m long by 20.0 m wide by 12.0 m high. Assuming all the work done by the brakes in stopping the train is transferred as heat uniformly to all the air in the station, by how much does the air temperature in the station rise? Take the density of the air to be 1.20 kg/m^3 and its specific heat to be $1020 \text{ J/kg} \cdot \text{K}$.

17.41. A nail driven into a board increases in temperature. If we assume that 60% of the kinetic energy delivered by a 1.80-kg hammer with a speed of 7.80 m/s is transformed into heat that flows into the nail and does not flow out, what is the temperature increase of an 8.00-g aluminum nail after it is struck ten times?

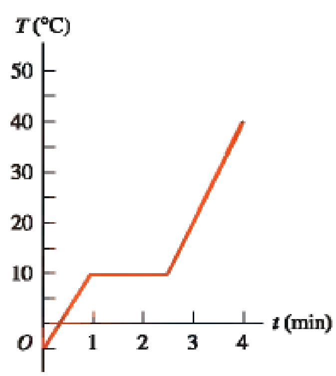
17.42. A technician measures the specific heat of an unidentified liquid by immersing an electrical resistor in it. Electrical energy is converted to heat transferred to the liquid for 120 s at a constant rate of 65.0 W. The mass of the liquid is 0.780 kg, and its temperature increases from 18.55°C to 22.54°C. (a) Find the average specific heat of the liquid in this temperature range. Assume that negligible heat is transferred to the container that holds the liquid and that no heat is lost to the surroundings. (b) Suppose that in this experiment heat transfer from the liquid to the container or surroundings cannot be ignored. Is the result calculated in part (a) an *overestimate* or an *underestimate* of the average specific heat? Explain.

17.43. You add 8950 J of heat to 3.00 mol of iron. (a) What is the temperature increase of the iron? (b) If this same amount of heat is added to 3.00 kg of iron, what is the iron's temperature increase? (c) Explain the difference in your results for parts (a) and (b).

Section 17.6 Calorimetry and Phase Changes

17.44. As a physicist, you put heat into a 500.0-g solid sample at the rate of 10.0 kJ/min, while recording its temperature as a function of time. You plot your data and obtain the graph shown in Fig. 17.30. (a) What is the latent heat of fusion for this solid? (b) What are the specific heats of the liquid and solid states of the material?

Figure 17.30 Exercise 17.44.



17.45. A 500.0-g chunk of an unknown metal, which has been in boiling water for several minutes, is quickly dropped into an insulating Styrofoam beaker containing 1.00 kg of water at room temperature (20.0°C). After waiting and gently stirring for 5.00 minutes, you observe that the water's temperature has reached a constant value of 22.0°C. (a) Assuming that the Styrofoam absorbs a negligibly small amount of heat and that no heat was lost to the surroundings, what is the specific heat of the metal? (b) Which is more useful for storing thermal energy: this metal or an equal weight of water? Explain. (c) What if the heat absorbed by the Styrofoam actually is not negligible. How would the specific heat you calculated in part (a) be in error? Would it be too large, too small, or still correct? Explain.

17.46. Before going in for his annual physical, a 70.0-kg man whose body temperature is 37.0°C consumes an entire 0.355-L can of a soft drink (mostly water) at 12.0°C. (a) What will his body temperature be after equilibrium is attained? Ignore any heating by the man's metabolism. The specific heat of the man's body is $3480 \text{ J/kg} \cdot \text{K}$. (b) Is the change in his body temperature great enough to be measured by a medical thermometer?

17.47. In the situation described in Exercise 17.46, the man's metabolism will eventually return the temperature of his body (and of the soft drink that he consumed) to 37.0°C. If his body releases energy at a rate of 7.00×10^3 kJ/day (the *basal metabolic rate*, or BMR), how long does this take? Assume that all of the released energy goes into raising the temperature.

17.48. An ice-cube tray of negligible mass contains 0.350 kg of water at 18.0°C. How much heat must be removed to cool the water to 0.00°C and freeze it? Express your answer in joules, calories, and Btu.

17.49. How much heat is required to convert 12.0 g of ice at −10.0°C to steam at 100.0°C? Express your answer in joules, calories, and Btu.

17.50. An open container holds 0.550 kg of ice at −15.0°C. The mass of the container can be ignored. Heat is supplied to the container at the constant rate of 800.0 J/min for 500.0 min. (a) After how many minutes does the ice *start* to melt? (b) After how many minutes, from the time when the heating is first started, does the temperature begin to rise above 0.0°C? (c) Plot a curve showing the temperature as a function of the elapsed time.

17.51. The capacity of commercial air conditioners is sometimes expressed in "tons," the number of tons of ice (1 ton = 2000 lb) that can be frozen from water at 0°C in 24 h by the unit. Express the capacity of a 2-ton air conditioner in Btu/h and in watts.

17.52. Steam Burns Versus Water Burns. What is the amount of heat input to your skin when it receives the heat released (a) by 25.0 g of steam initially at 100.0°C, when it is cooled to skin temperature (34.0°C)? (b) By 25.0 g of water initially at 100.0°C, when it is cooled to 34.0°C? (c) What does this tell you about the relative severity of steam and hot water burns?

17.53. What must the initial speed of a lead bullet be at a temperature of 25.0°C so that the heat developed when it is brought to rest will be just sufficient to melt it? Assume that all the initial mechanical energy of the bullet is converted to heat and that no heat flows from the bullet to its surroundings. (Typical rifles have muzzle speeds that exceed the speed of sound in air, which is 347 m/s at 25.0°C.)

17.54. Evaporation of sweat is an important mechanism for temperature control in some warm-blooded animals. (a) What mass of water must evaporate from the skin of a 70.0-kg man to cool his body 1.00 C°? The heat of vaporization of water at body temperature (37°C) is $2.42 \times 10^6 \text{ J/kg} \cdot \text{K}$. The specific heat of a typical human body is $3480 \text{ J/kg} \cdot \text{K}$ (see Exercise 17.37). (b) What volume of water must the man drink to replenish the evaporated water? Compare to the volume of a soft-drink can (355 cm^3).

17.55. "The Ship of the Desert." Camels require very little water because they are able to tolerate relatively large changes in their body temperature. While humans keep their body temperatures constant to within one or two Celsius degrees, a dehydrated camel permits its body temperature to drop to 34.0°C overnight and rise to 40.0°C during the day. To see how effective this mechanism is for saving water, calculate how many liters of water a 400 kg camel would have to drink if it attempted to keep its body temperature at a constant 34.0°C by evaporation of sweat during the day (12 hours) instead of letting it rise to 40.0°C. (*Note:* The specific heat of a camel or other mammal is about the same as that of a typical human, 3480 J/kg · K. The heat of vaporization of water at 34°C is 2.42×10^6 J/kg.)

17.56. An asteroid with a diameter of 10 km and a mass of 2.60×10^{15} kg impacts the earth at a speed of 32.0 km/s, landing in the Pacific Ocean. If 1.00% of the asteroid's kinetic energy goes to boiling the ocean water (assume an initial water temperature of 10.0°C), what mass of water will be boiled away by the collision? (For comparison, the mass of water contained in Lake Superior is about 2×10^{15} kg.)

17.57. A refrigerator door is opened and room-temperature air (20.0°C) fills the 1.50-m³ compartment. A 10.0-kg turkey, also at room temperature, is placed in the refrigerator and the door is closed. The density of air is 1.20 kg/m³ and its specific heat is 1020 J/kg · K. Assume the specific heat of a turkey, like that of a human, is 3480 J/kg · K. How much heat must the refrigerator remove from its compartment to bring the air and the turkey to thermal equilibrium at a temperature of 5.00°C? Assume no heat exchange with the surrounding environment.

17.56. A laboratory technician drops a 0.0850-kg sample of unknown material, at a temperature of 100.0°C, into a calorimeter. The calorimeter can, initially at 19.0°C, is made of 0.150 kg of copper and contains 0.200 kg of water. The final temperature of the calorimeter can and contents is 26.1°C. Compute the specific heat capacity of the sample.

17.56. An insulated beaker with negligible mass contains 0.250 kg of water at a temperature of 75.0°C. How many kilograms of ice at a temperature of −20.0°C must be dropped into the water to make the final temperature of the system 30.0°C?

17.60. A glass vial containing a 16.0-g sample of an enzyme is cooled in an ice bath. The bath contains water and 0.120 kg of ice. The sample has specific heat 2250J/kg · K; the glass vial has mass 6.00 g and specific heat 2800 J/kg · K. How much ice melts in cooling the enzyme sample from room temperature (19.5°C) to the temperature of the ice bath?

17.61. A 4.00-kg silver ingot is taken from a furnace, where its temperature is 750.0°C, and placed on a large block of ice at 0.0°C. Assuming that all the heat given up by the silver is used to melt the ice, how much ice is melted?

17.62. A copper calorimeter can with mass 0.100 kg contains 0.160 kg of water and 0.0180 kg of ice in thermal equilibrium at atmospheric pressure. If 0.750 kg of lead at a temperature of 255°C is dropped into the calorimeter can, what is the final temperature? Assume that no heat is lost to the surroundings.

17.63. A vessel whose walls are thermally insulated contains 2.40 kg of water and 0.450 kg of ice, all at a temperature of 0.0°C. The outlet of a tube leading from a boiler in which water is boiling at atmospheric pressure is inserted into the water. How many grams of steam must condense inside the vessel (also at atmospheric pressure) to raise the temperature of the system to 28.0°C? You can ignore the heat transferred to the container.

Section 17.7 Mechanisms of Heat Transfer

17.64. Use Eq. (17.21) to show that the SI units of thermal conductivity are W/m · K.

17.65. Suppose that the rod in Fig. 17.23a is made of copper, is 45.0 cm long, and has a cross-sectional area of 1.25 cm². Let $T_H = 100.0$°C and $T_C = 0.0$°C. (a) What is the final steady-state temperature gradient along the rod? (b) What is the heat current in the rod in the final steady state? (c) What is the final steady-state temperature at a point in the rod 12.0 cm from its left end?

17.66. One end of an insulated metal rod is maintained at 100.0°C, and the other end is maintained at 0.00°C by an ice–water mixture. The rod is 60.0 cm long and has a cross-sectional area of 1.25 cm². The heat conducted by the rod melts 8.50 g of ice in 10.0 min. Find the thermal conductivity k of the metal.

17.67. A carpenter builds an exterior house wall with a layer of wood 3.0 cm thick on the outside and a layer of Styrofoam insulation 2.2 cm thick on the inside wall surface. The wood has $k = 0.080$ W/m · K, and the Styrofoam has $k = 0.010$ W/m · K. The interior surface temperature is 19.0°C, and the exterior surface temperature is −10.0°C. (a) What is the temperature at the plane where the wood meets the Styrofoam? (b) What is the rate of heat flow per square meter through this wall?

17.66. An electric kitchen range has a total wall area of 1.40 m² and is insulated with a layer of fiberglass 4.00 cm thick. The inside surface of the fiberglass has a temperature of 175°C, and its outside surface is at 35.0°C. The fiberglass has a thermal conductivity of 0.040 W/m · K. (a) What is the heat current through the insulation, assuming it may be treated as a flat slab with an area of 1.40 m²? (b) What electric-power input to the heating element is required to maintain this temperature?

17.69. The ceiling of a room has an area of 125 ft². The ceiling is insulated to an R value of 30 (in units of ft² · F° · h/Btu). The surface in the room is maintained at 69°F, and the surface in the attic has a temperature of 35°F. What is the heat flow through the ceiling into the attic in 5.0 h? Express your answer in Btu and in joules.

17.70. A long rod, insulated to prevent heat loss along its sides, is in perfect thermal contact with boiling water (at atmospheric pressure) at one end and with an ice–water mixture at the other (Fig. 17.31). The rod consists of a 1.00-m section of copper (one end in boiling water) joined end to end to a length L_2 of steel (one end in the ice–water mixture). Both sections of the rod have cross-sectional areas of 4.00 cm². The temperature of the copper–steel junction is 65.0°C after a steady state has been set up. (a) How much heat per second flows from the boiling water to the ice–water mixture? (b) What is the length L_2 of the steel section?

Figure 17.31 Exercise 17.70.

17.71. A pot with a steel bottom 8.50 mm thick rests on a hot stove. The area of the bottom of the pot is 0.150 m². The water inside the pot is at 100.0°C, and 0.390 kg are evaporated every 3.00 min. Find the temperature of the lower surface of the pot, which is in contact with the stove.

17.72. You are asked to design a cylindrical steel rod 50.0 cm long, with a circular cross section, that will conduct 150.0 J/s from a furnace at 400.0°C to a container of boiling water under 1 atmosphere. What must the rod's diameter be?

17.73. A picture window has dimensions of 1.40 m × 2.50 m and is made of glass 5.20 mm thick. On a winter day, the outside temperature is −20.0°C, while the inside temperature is a comfortable 19.5°C. (a) At what rate is heat being lost through the window by conduction? (b) At what rate would heat be lost through the window if you covered it with a 0.750-mm-thick layer of paper (thermal conductivity 0.0500)?

17.74. What is the rate of energy radiation per unit area of a blackbody at a temperature of (a) 273 K and (b) 2730 K?

17.75. What is the net rate of heat loss by radiation in Example 17.16 (Section 17.7) if the temperature of the surroundings is 5.0°C?

17.76. The emissivity of tungsten is 0.350. A tungsten sphere with radius 1.50 cm is suspended within a large evacuated enclosure whose walls are at 290.0 K. What power input is required to maintain the sphere at a temperature of 3000.0 K if heat conduction along the supports is neglected?

17.77. Size of a Light-Bulb Filament. The operating temperature of a tungsten filament in an incandescent light bulb is 2450 K, and its emissivity is 0.350. Find the surface area of the filament of a 150-W bulb if all the electrical energy consumed by the bulb is radiated by the filament as electromagnetic waves. (Only a fraction of the radiation appears as visible light.)

17.78. The Sizes of Stars. The hot glowing surfaces of stars emit energy in the form of electromagnetic radiation. It is a good approximation to assume $e = 1$ for these surfaces. Find the radii of the following stars (assumed to be spherical): (a) Rigel, the bright blue star in the constellation Orion, which radiates energy at a rate of 2.7×10^{32} W and has surface temperature 11,000 K; (b) Procyon B (visible only using a telescope), which radiates energy at a rate of 2.1×10^{23} W and has surface temperature 10,000 K. (c) Compare your answers to the radius of the earth, the radius of the sun, and the distance between the earth and the sun. (Rigel is an example of a *supergiant* star, and Procyon B is an example of a *white dwarf* star.)

Problems

17.78. You propose a new temperature scale with temperatures given in °M. You define 0.0°M to be the normal melting point of mercury and 100.0° to be the normal boiling point of mercury. (a) What is the normal boiling point of water in °M? (b) A temperature change of 10.0 M° corresponds to how many C°?

17.80. Suppose that a steel hoop could be constructed to fit just around the earth's equator at a temperature of 20.0°C. What would be the thickness of space between the hoop and the earth if the temperature of the hoop were increased by 0.500 C°?

17.81. At an absolute temperature T_0, a cube has sides of length L_0 and has density ρ_0. The cube is made of a material with coefficient of volume expansion β. (a) Show that if the temperature increases to $T_0 + \Delta T$, the density of the cube becomes approximately

$$\rho \approx \rho_0(1 - \beta\Delta T)$$

(*Hint:* Use the expression $(1 + x)^n \approx 1 + nx$, valid for $|x| \ll 1$.) Explain why this approximate result is valid only if ΔT is much less than $1/\beta$, and explain why you would expect this to be the case in most situations. (b) A copper cube has sides of length 1.25 cm at 20.0°C. Find the change in its volume and density when its temperature is increased to 70.0°C.

17.82. A 250-kg weight is hanging from the ceiling by a thin copper wire. In its fundamental mode, this wire vibrates at the frequency of concert A (440 Hz). You then increase the temperature of the wire by 40 C°. (a) By how much will the fundamental frequency change? Will it increase or decrease? (b) By what percentage will the speed of a wave on the wire change? (c) By what percentage will the wavelength of the fundamental standing wave change? Will it increase or decrease?

17.83. You are making pesto for your pasta and have a cylindrical measuring cup 10.0 cm high made of ordinary glass $[\beta = 2.7 \times 10^{-5}\,(\text{C}°)^{-1}]$ that is filled with olive oil $[\beta = 6.8 \times 10^{-4}\,(\text{C}°)^{-1}]$ to a height of 1.00 mm below the top of the cup. Initially, the cup and oil are at room temperature (22.0°C). You get a phone call and forget about the olive oil, which you inadvertently leave on the hot stove. The cup and oil heat up slowly and have a common temperature. At what temperature will the olive oil start to spill out of the cup?

17.84. Use Fig. 17.12 to find the approximate coefficient of volume expansion of water at 2.0°C and at 8.0°C.

17.85. A Foucault pendulum consists of a brass sphere with a diameter of 35.0 cm suspended from a steel cable 10.5 m long (both measurements made at 20.0°C). Due to a design oversight, the swinging sphere clears the floor by a distance of only 2.00 mm when the temperature is 20.0°C. At what temperature will the sphere begin to brush the floor?

17.86. You pour 108 cm³ of ethanol, at a temperature of −10.0°C, into a graduated cylinder initially at 20.0°C, filling it to the very top. The cylinder is made of glass with a specific heat of 840 J/kg · K and a coefficient of volume expansion of $1.2 \times 10^{-5}\,\text{K}^{-1}$; its mass is 0.110 kg. The mass of the ethanol is 0.0873 kg. (a) What will be the final temperature of the ethanol, once thermal equilibrium is reached? (b) How much ethanol will overflow the cylinder before thermal equilibrium is reached?

17.87. A metal rod that is 30.0 cm long expands by 0.0650 cm when its temperature is raised from 0.0°C to 100.0°C. A rod of a different metal and of the same length expands by 0.0350 cm for the same rise in temperature. A third rod, also 30.0 cm long, is made up of pieces of each of the above metals placed end to end and expands 0.0580 cm between 0.0°C and 100.0°C. Find the length of each portion of the composite rod.

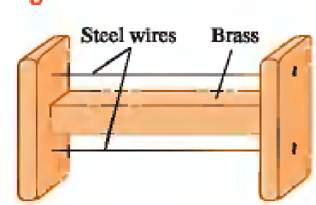
17.86. On a cool (4.0°C) Saturday morning, a pilot fills the fuel tanks of her Pitts S-2C (a two-seat aerobatic airplane) to their full capacity of 106.0 L. Before flying on Sunday morning, when the temperature is again 4.0°C, she checks the fuel level and finds only 103.4 L of gasoline in the tanks. She realizes that it was hot on Saturday afternoon, and that thermal expansion of the gasoline caused the missing fuel to empty out of the tank's vent. (a) What was the maximum temperature (in °C) reached by the fuel and the tank on Saturday afternoon? The coefficient of volume expansion of gasoline is $9.5 \times 10^{-4}\,\text{K}^{-1}$, and the tank is made of aluminum. (b) In order to have the maximum amount of fuel available for flight, when should the pilot have filled the fuel tanks?

17.86. (a) Equation (17.12) gives the stress required to keep the length of a rod constant as its temperature changes. Show that if the length is permitted to change by an amount ΔL when its temperature changes by ΔT, the stress is equal to

$$\frac{F}{A} = Y\left(\frac{\Delta L}{L_0} - \alpha\Delta T\right)$$

where F is the tension on the rod, L_0 is the original length of the rod, A its cross-sectional area, α its coefficient of linear expansion, and Y its Young's modulus. (b) A heavy brass bar has projections at its ends, as in Fig. 17.32. Two fine steel wires, fastened between the projections, are just taut (zero tension) when the whole system is at 20°C. What is the tensile stress in the steel wires when the temperature of the system is raised to 140°C? Make any simplifying assumptions you think are justified, but state what they are.

Figure 17.32 Problem 17.89.



17.90. A steel rod 0.350 m long and an aluminum rod 0.250 m long, both with the same diameter, are placed end to end between rigid supports with no initial stress in the rods. The temperature of the rods is now raised by 60.0 C°. What is the stress in each rod? (*Hint:* The length of the combined rod remains the same, but the lengths of the individual rods do not. See Problem 17.89.)

17.91. A steel ring with a 2.5000-in. inside diameter at 20.0°C is to be warmed and slipped over a brass shaft with a 2.5020-in. outside diameter at 20.0°C. (a) To what temperature should the ring be warmed? (b) If the ring and the shaft together are cooled by some means such as liquid air, at what temperature will the ring just slip off the shaft?

17.92. Bulk Stress Due to a Temperature Increase. (a) Prove that, if an object under pressure has its temperature raised but is not allowed to expand, the increase in pressure is

$$\Delta p = B\beta\Delta T$$

where the bulk modulus B and the average coefficient of volume expansion β are both assumed positive and constant. (b) What pressure is necessary to prevent a steel block from expanding when its temperature is increased from 20.0°C to 35.0°C?

17.93. A liquid is enclosed in a metal cylinder that is provided with a piston of the same metal. The system is originally at a pressure of 1.00 atm $(1.013 \times 10^5\ \text{Pa})$ and at a temperature of 30.0°C. The piston is forced down until the pressure on the liquid is increased by 50.0 atm, and then clamped in this position. Find the new temperature at which the pressure of the liquid is again 1.00 atm. Assume that the cylinder is sufficiently strong so that its volume is not altered by changes in pressure, but only by changes in temperature. Use the result derived in Problem 17.92. (*Hint:* See Section 11.4.)

Compressibility of liquid: $k = 8.50 \times 10^{-10}\ \text{Pa}^{-1}$

Coefficient of volume expansion of liquid: $\beta = 4.80 \times 10^{-4}\ \text{K}^{-1}$

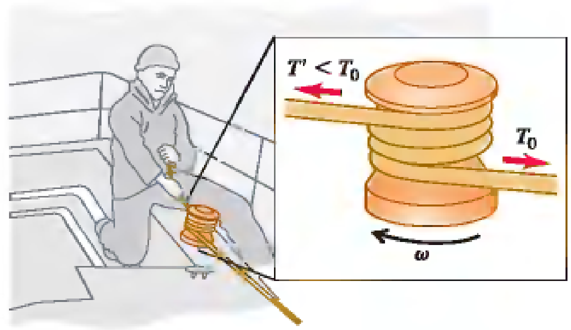
Coefficient of volume expansion of metal: $\beta = 3.90 \times 10^{-5}\ \text{K}^{-1}$

17.94. You cool a 100.0-g slug of red-hot iron (temperature 745°C) by dropping it into an insulated cup of negligible mass containing 75.0 g of water at 20.0°C. Assuming no heat exchange with the surroundings, a) what is the final temperature of the water and b) what is the final mass of the iron and the remaining water?

17.95. Spacecraft Reentry. A spacecraft made of aluminum circles the earth at a speed of 7700 m/s. (a) Find the ratio of its kinetic energy to the energy required to raise its temperature from 0°C to 600°C. (The melting point of aluminum is 660°C. Assume a constant specific heat of 910 J/kg · K.) (b) Discuss the bearing of your answer on the problem of the reentry of a manned space vehicle into the earth's atmosphere.

17.96. A capstan is a rotating drum or cylinder over which a rope or cord slides in order to provide a great amplification of the rope's tension while keeping both ends free (Fig. 17.33). Since the added tension in the rope is due to friction, the capstan generates thermal energy. (a) If the difference in tension between the two ends of the rope is 520.0 N and the capstan has a diameter of 10.0 cm and turns once in 0.900 s, find the rate at which thermal energy is generated. Why does the number of turns not matter? (b) If the capstan is made of iron and has mass 6.00 kg, at what rate does its temperature rise? Assume that the temperature in the capstan is uniform and that all the thermal energy generated flows into it.

Figure 17.33 Problem 17.96.



17.97. Debye's T^3 Law. At very low temperatures the molar heat capacity of rock salt varies with temperature according to Debye's T^3 law:

$$C = k\frac{T^3}{\Theta^3}$$

where $k = 1940$ J/mol · K and $\Theta = 281$ K. (a) How much heat is required to raise the temperature of 1.50 mol of rock salt from 10.0 K to 40.0 K? (*Hint:* Use Eq. (17.18) in the form $dQ = nC\,dT$ and integrate.) (b) What is the average molar heat capacity in this range? (c) What is the true molar heat capacity at 40.0 K?

17.98. A person of mass 70.0 kg is sitting in the bathtub. The bathtub is 190.0 cm by 80.0 cm; before the person got in, the water was 10.0 cm deep. The water is at a temperature of 37.0°C. Suppose that the water were to cool down spontaneously to form ice at 0.0°C, and that all the energy released was used to launch the hapless bather vertically into the air. How high would the bather go? (As you will see in Chapter 20, this event is allowed by energy conservation but is prohibited by the second law of thermodynamics.)

17.98. Hot Air in a Physics Lecture. (a) A typical student listening attentively to a physics lecture has a heat output of 100 W. How much heat energy does a class of 90 physics students release into a lecture hall over the course of a 50-min lecture? (b) Assume that all the heat energy in part (a) is transferred to the 3200 m³ of air in the room. The air has specific heat capacity 1020 J/kg · K and density 1.20 kg/m³. If none of the heat escapes and the air conditioning system is off, how much will the temperature of the air in the room rise during the 50-min lecture? (c) If the class is taking an exam, the heat output per student rises to 280 W. What is the temperature rise during 50 min in this case?

17.100. The molar heat capacity of a certain substance varies with temperature according to the empirical equation

$$C = 29.5\ \text{J/mol} \cdot \text{K} + (8.20 \times 10^{-3}\ \text{J/mol} \cdot \text{K}^2)\,T$$

How much heat is necessary to change the temperature of 3.00 mol of this substance from 27°C to 227°C? (*Hint:* Use Eq. (17.18) in the form $dQ = nC\,dT$ and integrate.)

17.101. For your cabin in the wilderness, you decide to build a primitive refrigerator out of Styrofoam, planning to keep the interior cool with a block of ice that has an initial mass of 24.0 kg. The box has dimensions of 0.500 m × 0.800 m × 0.500 m. Water from melting ice collects in the bottom of the box. Suppose the ice block is at 0.00°C and the outside temperature is 21.0°C. If the top of the empty box is never opened and you want the interior of the box to remain at 5.00°C for exactly one week, until all the ice melts, what must be the thickness of the Styrofoam?

17.102. Hot Water Versus Steam Heating. In a household hot-water heating system, water is delivered to the radiators at 70.0°C (158.0°F) and leaves at 28.0°C (82.4°F). The system is to be replaced by a steam system in which steam at atmospheric pressure condenses in the radiators and the condensed steam leaves the radiators at 35.0°C (95.0°F). How many kilograms of steam will supply the same heat as was supplied by 1.00 kg of hot water in the first system?

17.103. A copper calorimeter can with mass 0.446 kg contains 0.0950 kg of ice. The system is initially at 0.0°C. (a) If 0.0350 kg of steam at 100.0°C and 1.00 atm pressure is added to the can, what is the final temperature of the calorimeter can and its contents? (b) At the final temperature, how many kilograms are there of ice, how many of liquid water, and how many of steam?

17.104. A Styrofoam bucket of negligible mass contains 1.75 kg of water and 0.450 kg of ice. More ice, from a refrigerator at -15.0°C, is added to the mixture in the bucket, and when thermal equilibrium has been reached, the total mass of ice in the bucket is 0.778 kg. Assuming no heat exchange with the surroundings, what mass of ice was added?

17.105. In a container of negligible mass, 0.0400 kg of steam at 100°C and atmospheric pressure is added to 0.200 kg of water at 50.0°C. (a) If no heat is lost to the surroundings, what is the final temperature of the system? (b) At the final temperature, how many kilograms are there of steam and how many of liquid water?

17.106. A tube leads from a 0.150-kg calorimeter to a flask in which water is boiling under atmospheric pressure. The calorimeter has specific heat capacity 420 J/kg · K, and it originally contains 0.340 kg of water at 15.0°C. Steam is allowed to condense in the calorimeter at atmospheric pressure until the temperature of the calorimeter and contents reaches 71.0°C, at which point the total mass of the calorimeter and its contents is found to be 0.525 kg. Compute the heat of vaporization of water from these data.

17.107. A worker pours 1.250 kg of molten lead at a temperature of 365.0°C into 0.5000 kg of water at a temperature of 75.00°C in an insulated bucket of negligible mass. Assuming no heat loss to the surroundings, calculate the mass of lead and water remaining in the bucket when the materials have reached thermal equilibrium.

17.108. One experimental method of measuring an insulating material's thermal conductivity is to construct a box of the material and measure the power input to an electric heater inside the box that maintains the interior at a measured temperature above the outside surface. Suppose that in such an apparatus a power input of 180 W is required to keep the interior surface of the box 65.0 C° (about 120 F°) above the temperature of the outer surface. The total area of the box is 2.18 m^2, and the wall thickness is 3.90 cm. Find the thermal conductivity of the material in SI units.

17.109. Effect of a Window in a Door. A carpenter builds a solid wood door with dimensions 2.00 m × 0.95 m × 5.0 cm. Its thermal conductivity is $k = 0.120$ W/m · K. The air films on the inner and outer surfaces of the door have the same combined thermal resistance as an additional 1.8-cm thickness of solid wood. The inside air temperature is 20.0°C, and the outside air temperature is -8.0°C. (a) What is the rate of heat flow through the door? (b) By what factor is the heat flow increased if a window 0.500 m on a side is inserted in the door? The glass is 0.450 cm thick, and the glass has a thermal conductivity of 0.80 W/m · K. The air films on the two sides of the glass have a total thermal resistance that is the same as an additional 12.0 cm of glass.

17.110. A wood ceiling with thermal resistance R_1 is covered with a layer of insulation with thermal resistance R_2. Prove that the effective thermal resistance of the combination is $R = R_1 + R_2$.

17.111. Compute the ratio of the rate of heat loss through a single-pane window with area 0.15 m^2 to that for a double-pane window with the same area. The glass of a single pane is 4.2 mm thick, and the air space between the two panes of the double-pane window is 7.0 mm thick. The glass has thermal conductivity 0.80 W/m · K. The air films on the room and outdoor surfaces of either window have a combined thermal resistance of 0.15 m^2 · K/W.

17.112. Rods of copper, brass, and steel are welded together to form a Y-shaped figure. The cross-sectional area of each rod is 2.00 cm^2. The free end of the copper rod is maintained at 100.0°C, and the free ends of the brass and steel rods at 0.0°C. Assume there is no heat loss from the surfaces of the rods. The lengths of the rods are: copper, 13.0 cm; brass, 18.0 cm; steel, 24.0 cm. (a) What is the temperature of the junction point? (b) What is the heat current in each of the three rods?

17.113. Time Needed for a Lake to Freeze Over. (a) When the air temperature is below 0°C, the water at the surface of a lake freezes to form an ice sheet. Why doesn't freezing occur throughout the entire volume of the lake? (b) Show that the thickness of the ice sheet formed on the surface of a lake is proportional to the square root of the time if the heat of fusion of the water freezing on the underside of the ice sheet is conducted through the sheet. (c) Assuming that the upper surface of the ice sheet is at -10°C and the bottom surface is at 0°C, calculate the time it will take to form an ice sheet 25 cm thick. (d) If the lake in part (c) is uniformly 40 m deep, how long would it take to freeze all the water in the lake? Is this likely to occur?

17.114. A rod is initially at a uniform temperature of 0°C throughout. One end is kept at 0°C, and the other is brought into contact with a steam bath at 100°C. The surface of the rod is insulated so that heat can flow only lengthwise along the rod. The cross-sectional area of the rod is 2.50 cm^2, its length is 120 cm, its thermal conductivity is 380 W/m · K, its density is 1.00×10^4 kg/m^3, and its specific heat is 520 J/kg · K. Consider a short cylindrical element of the rod 1.00 cm in length. (a) If the temperature gradient at the cooler end of this element is 140 C°/m, how many joules of heat energy flow across this end per second? (b) If the average temperature of the element is increasing at the rate of 0.250 C°/s, what is the temperature gradient at the other end of the element?

17.115. A rustic cabin has a floor area of 3.50 m × 3.00 m. Its walls, which are 2.50 m tall, are made of wood (thermal conductivity 0.0600 W/m · K) 1.80 cm thick and are further insulated with 1.50 cm of a synthetic material. When the outside temperature is 2.00°C, it is found necessary to heat the room at a rate of 1.25 kW to maintain its temperature at 19.0°C. Calculate the thermal conductivity of the insulating material. Neglect the heat lost through the ceiling and floor. Assume the inner and outer surfaces of the wall have the same termperature as the air inside and outside the cabin.

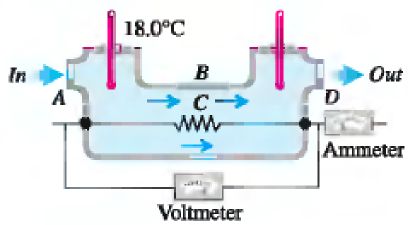
17.116. The rate at which radiant energy from the sun reaches the earth's upper atmosphere is about 1.50 kW/m^2. The distance from the earth to the sun is 1.50×10^{11} m, and the radius of the sun is 6.96×10^8 m. (a) What is the rate of radiation of energy per unit area from the sun's surface? (b) If the sun radiates as an ideal blackbody, what is the temperature of its surface?

17.117. A Thermos for Liquid Helium. A physicist uses a cylindrical metal can 0.250 m high and 0.090 m in diameter to store liquid helium at 4.22 K; at that temperature the heat of vaporization of helium is 2.09×10^4 J/kg. Completely surrounding the metal can are walls maintained at the temperature of liquid nitrogen, 77.3 K, with vacuum between the can and the surrounding walls. How much helium is lost per hour? The emissivity of the metal can is 0.200. The only heat transfer between the metal can and the surrounding walls is by radiation.

17.118. Thermal Expansion of an Ideal Gas. (a) The pressure p, volume V, number of moles n, and Kelvin temperature T of an ideal gas are related by the equation $pV = nRT$, where R is a constant. Prove that the coefficient of volume expansion for an ideal gas is equal to the reciprocal of the Kelvin temperature if the expansion occurs at constant pressure. (b) Compare the coefficients of volume expansion of copper and air at a temperature of 20°C. Assume that air may be treated as an ideal gas and that the pressure remains constant.

17.119. An engineer is developing an electric water heater to provide a continuous supply of hot water. One trial design is shown in Fig. 17.34. Water is flowing at the rate of 0.500 kg/min, the inlet thermometer registers 18.0°C, the voltmeter reads 120 V, and the ammeter reads 15.0 A [corresponding to a power input of $(120\ \text{V}) \times (15.0\ \text{A}) = 1800\ \text{W}$]. (a) When a steady state is finally reached, what is the reading of the outlet thermometer? (b) Why is it unnecessary to take into account the heat capacity mc of the apparatus itself?

Figure 17.34 Problem 17.119.



17.120. Food Intake of a Hamster. The energy output of an animal engaged in an activity is called the basal metabolic rate (BMR) and is a measure of the conversion of food energy into other forms of energy. A simple calorimeter to measure the BMR consists of an insulated box with a thermometer to measure the temperature of the air. The air has density 1.20 kg/m^3 and specific heat 1020 J/kg · K. A 50.0-g hamster is placed in a calorimeter that contains 0.0500 m^3 of air at room temperature. (a) When the hamster is running in a wheel, the temperature of the air in the calorimeter rises 1.60 C° per hour. How much heat does the running hamster generate in an hour? Assume that all this heat goes into the air in the calorimeter. You can ignore the heat that goes into the walls of the box and into the thermometer, and assume that no heat is lost to the surroundings. (b) Assuming that the hamster converts seed into heat with an efficiency of 10% and that hamster seed has a food energy value of 24 J/g, how many grams of seed must the hamster eat per hour to supply this energy?

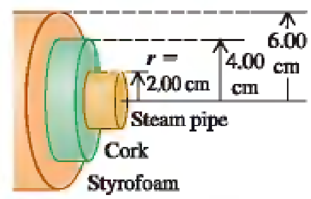
17.121. The icecaps of Greenland and Antarctica contain about 1.75% of the total water (by mass) on the earth's surface; the oceans contain about 97.5%, and the other 0.75% is mainly groundwater. Suppose the icecaps, currently at an average temperature of about −30°C, somehow slid into the ocean and melted. What would be the resulting temperature decrease of the ocean? Assume that the average temperature of ocean water is currently 5.00°C.

Challenge Problems

17.122. (a) A spherical shell has inner and outer radii a and b, respectively, and the temperatures at the inner and outer surfaces are T_2 and T_1. The thermal conductivity of the material of which the shell is made is k. Derive an equation for the total heat current through the shell. (b) Derive an equation for the temperature variation within the shell in part (a); that is, calculate T as a function of r, the distance from the center of the shell. (c) A hollow cylinder has length L, inner radius a, and outer radius b, and the temperatures at the inner and outer surfaces are T_2 and T_1. (The cylinder could represent an insulated hot-water pipe, for example.) The thermal conductivity of the material of which the cylinder is made is k. Derive an equation for the total heat current through the walls of the cylinder. (d) For the cylinder of part (c), derive an equation for the temperature variation inside the cylinder walls. (e) For the spherical shell of part (a) and the hollow cylinder of part (c), show that the equation for the total heat current in each case reduces to Eq. (17.21) for linear heat flow when the shell or cylinder is very thin.

17.123. A steam pipe with a radius of 2.00 cm, carrying steam at 140°C, is surrounded by a cylindrical jacket with inner and outer radii 2.00 cm and 4.00 cm and made of a type of cork with thermal conductivity 4.00×10^{-2} W/m · K. This in turn is surrounded by a cylindrical jacket made of a brand of Styrofoam with thermal conductivity 1.00×10^{-2} W/m · K and having inner and outer radii 4.00 cm and 6.00 cm (Fig. 17.35). The outer surface of the Styrofoam is in contact with air at 15°C. Assume that this outer surface has a temperature of 15°C. (a) What is the temperature at a radius of 4.00 cm, where the two insulating layers meet? (b) What is the total rate of transfer of heat out of a 2.00-m length of pipe? (*Hint:* Use the expression derived in part (c) of Challenge Problem 17.122.)

Figure 17.35 Challenge Problem 17.123.



17.124. Suppose that both ends of the rod in Fig. 17.23 are kept at a temperature of 0°C, and that the initial temperature distribution along the rod is given by $T = (100°\text{C}) \sin \pi x/L$, where x is measured from the left end of the rod. Let the rod be copper, with length $L = 0.100$ m and cross-sectional area 1.00 cm^2. (a) Show the initial temperature distribution in a diagram. (b) What is the final temperature distribution after a very long time has elapsed? (c) Sketch curves that you think would represent the temperature distribution at intermediate times. (d) What is the initial temperature gradient at the ends of the rod? (e) What is the initial heat current from the ends of the rod into the bodies making contact with its ends? (f) What is the initial heat current at the center of the rod? Explain. What is the heat current at this point at any later time? (g) What is the value of the *thermal diffusivity* $k/\rho c$ for copper, and in what unit is it expressed? (Here k is the thermal conductivity, $\rho = 8.9 \times 10^3$ kg/m^3 is the density, and c is the specific heat.) (h) What is the initial time rate of change of temperature at the center of the rod? (i) How much time would be required for the center of the rod to reach its final temperature if the temperature continued to decrease at this rate? (This time is called the *relaxation time* of the rod.) (j) From the graphs in part (c), would you expect the magnitude of the rate of temperature change at the midpoint to remain constant, increase, or decrease as a function of time? (k) What is the initial rate of change of temperature at a point in the rod 2.5 cm from its left end?

17.125. Temperature Change in a Clock. A pendulum clock is designed to tick off one second on each side-to-side swing of the pendulum (two ticks per complete period). (a) Will a pendulum clock gain time in hot weather and lose it in cold, or the reverse? Explain your reasoning. (b) A particular pendulum clock keeps correct time at 20.0°C. The pendulum shaft is steel, and its mass can be ignored compared with that of the bob. What is the fractional change in the length of the shaft when it is cooled to 10.0°C? (c) How many seconds per day will the clock gain or lose at 10.0°C? (d) How closely must the temperature be controlled if the clock is not to gain or lose more than 1.00 s a day? Does the answer depend on the period of the pendulum?

17.126. One end of a solid cylindrical copper rod 0.200 m long is maintained at a temperature of 20.00 K. The other end is blackened and exposed to thermal radiation from surrounding walls at 500.0 K. The sides of the rod are insulated, so no energy is lost or gained except at the ends of the rod. When equilibrium is reached, what is the temperature of the blackened end? (*Hint:* Since copper is a very good conductor of heat at low temperature, with $k = 1670\ \mathrm{W/m \cdot K}$ at 20 K, the temperature of the blackened end is only slightly higher than 20.00 K.)

17.127. A Walk in the Sun. Consider a poor lost soul walking at 5 km/h on a hot day in the desert, wearing only a bathing suit. This person's skin temperature tends to rise due to four mechanisms: (i) energy is generated by metabolic reactions in the body at a rate of 280 W, and almost all of this energy is converted to heat that flows to the skin; (ii) heat is delivered to the skin by convection from the outside air at a rate equal to $k'A_{\mathrm{skin}}(T_{\mathrm{air}} - T_{\mathrm{skin}})$, where k' is $54\ \mathrm{J/h \cdot C° \cdot m^2}$, the exposed skin area A_{skin} is $1.5\ \mathrm{m^2}$, the air temperature T_{air} is 47°C, and the skin temperature T_{skin} is 36°C; (iii) the skin absorbs radiant energy from the sun at a rate of $1400\ \mathrm{W/m^2}$; (iv) the skin absorbs radiant energy from the environment, which has temperature 47°C. (a) Calculate the net rate (in watts) at which the person's skin is heated by all four of these mechanisms. Assume that the emissivity of the skin is $e = 1$ and that the skin temperature is initially 36°C. Which mechanism is the most important? (b) At what rate (in L/h) must perspiration evaporate from this person's skin to maintain a constant skin temperature? (The heat of vaporization of water at 36°C is $2.42 \times 10^6\ \mathrm{J/kg}$.) (c) Suppose instead the person is protected by light-colored clothing $(e \approx 0)$ so that the exposed skin area is only $0.45\ \mathrm{m^2}$. What rate of perspiration is required now? Discuss the usefulness of the traditional clothing worn by desert peoples.

18 THERMAL PROPERTIES OF MATTER

LEARNING GOALS

By studying this chapter, you will learn:

- How to relate the pressure, volume, and temperature of a gas.
- How the interactions between the molecules of a substance determine the properties of the substance.
- How the pressure and temperature of a gas are related to the kinetic energy of its molecules.
- How the heat capacities of a gas reveal whether its molecules are rotating or vibrating.
- What determines whether a substance is a gas, a liquid, or a solid.

? The higher the temperature of a gas, the greater the average kinetic energy of its molecules. How much faster are molecules moving in the air above a frying pan (100°C) than in the surrounding kitchen air (25°C)?

The kitchen is a great place to learn about how the properties of matter depend on temperature. When you boil water in a tea kettle, the increase in temperature produces steam that whistles out of the spout at high pressure. If you forget to poke holes in a potato before baking it, the high-pressure steam produced inside the potato can cause it to explode messily. Water vapor in the air can condense into droplets of liquid on the sides of a glass of ice water; if the glass is just out of the freezer, frost will form on the sides as water vapor changes to a solid.

All of these examples show the relationships among between the large-scale or *macroscopic* properties of a substance, such as pressure, volume, temperature, and mass of substance. But we can also describe a substance using a *microscopic* perspective. This means investigating small-scale quantities such as the masses, speeds, kinetic energies, and momenta of the individual molecules that make up a substance.

The macroscopic and microscopic descriptions are intimately related. For example, the (microscopic) collision forces that occur when air molecules strike a solid surface (such as your skin) cause (macroscopic) atmospheric pressure. Standard atmospheric pressure is 1.01×10^5 Pa; to produce this pressure, 10^{32} molecules strike your skin every day with an average speed of over 1700 km/h (1000 mi/h)!

In this chapter we'll begin our study of the thermal properties of matter by looking at some macroscopic aspects of matter in general. We'll pay special attention to the *ideal gas,* one of the simplest types of matter to understand. Using our knowledge of momentum and kinetic energy, we'll relate the macroscopic properties of an ideal gas to the microscopic behavior of its individual molecules. We'll also use microscopic ideas to understand the heat capacities of both gases and solids. Finally, we'll take a look at the various phases of matter—gas, liquid, and solid—and the conditions under which each occurs.

18.1 Equations of State

The conditions in which a particular material exists are described by physical quantities such as pressure, volume, temperature, and amount of substance. For example, a tank of oxygen in a welding outfit has a pressure gauge and a label stating its volume. We could add a thermometer and place the tank on a scale to determine its mass. These variables describe the *state* of the material and are called **state variables.**

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8.4 State Variables and Ideal Gas Law

The volume V of a substance is usually determined by its pressure p, temperature T, and amount of substance, described by the mass m_{total} or number of moles n. (We are calling the total mass of a substance m_{total} because later in the chapter we will use m for the mass of one molecule.) Ordinarily, we can't change one of these variables without causing a change in another. When the tank of oxygen gets hotter, the pressure increases. If the tank gets too hot, it explodes; this happens occasionally with overheated steam boilers.

In a few cases the relationship among p, V, T, and m (or n) is simple enough that we can express it as an equation called the **equation of state.** When it's too complicated for that, we can use graphs or numerical tables. Even then, the relationship among the variables still exists; we call it an equation of state even when we don't know the actual equation.

Here's a simple (though approximate) equation of state for a solid material. The temperature coefficient of volume expansion β (see Section 17.4) is the fractional volume change $\Delta V/V_0$ per unit temperature change, and the compressibility k (see Section 11.4) is the negative of the fractional volume change $\Delta V/V_0$ per unit pressure change. If a certain amount of material has volume V_0 when the pressure is p_0 and the temperature is T_0, the volume V at slightly differing pressure p and temperature T is approximately

$$V = V_0[1 + \beta(T - T_0) - k(p - p_0)] \qquad (18.1)$$

(There is a negative sign in front of the term $k(p - p_0)$ because an *increase* in pressure causes a *decrease* in the volume.) Equation (18.1) is called an *equation of state* for the material.

The Ideal-Gas Equation

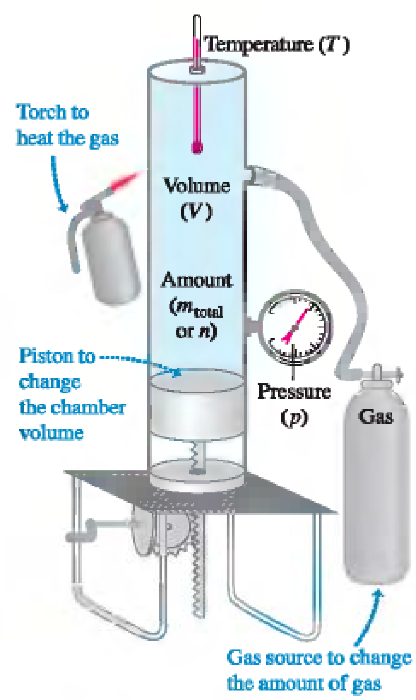
Another simple equation of state is the one for an *ideal gas.* Figure 18.1 shows an experimental setup to study the behavior of a gas. The cylinder has a movable piston to vary the volume, the temperature can be varied by heating, and we can pump any desired amount of any gas into the cylinder. We then measure the pressure, volume, temperature, and amount of gas. Note that *pressure* refers both to the force per unit area exerted by the cylinder on the gas and to the force per unit area exerted by the gas on the cylinder; by Newton's third law, these must be equal.

It is usually easiest to describe the amount of gas in terms of the number of moles n, rather than the mass. We did this when we defined molar heat capacity in Section 17.5; you may want to review that section. The **molar mass** M of a compound (sometimes called *molecular weight*) is the mass per mole, and the total mass m_{total} of a given quantity of that compound is the number of moles n times the mass per mole M:

$$m_{total} = nM \qquad \text{(total mass, number of moles, and molar mass)} \qquad (18.2)$$

Hence if we know the number of moles of gas in the cylinder, we can determine the mass of gas using Eq. (18.2).

18.1 A hypothetical setup for studying the behavior of gases. By heating the gas, varying the volume with a movable piston, and adding more gas, we can control the gas pressure p, volume V, temperature T, and number of moles n.



Measurements of the behavior of various gases lead to three conclusions:

1. The volume V is proportional to the number of moles n. If we double the number of moles, keeping pressure and temperature constant, the volume doubles.
2. The volume varies *inversely* with the absolute pressure p. If we double the pressure while holding the temperature T and number of moles n constant, the gas compresses to one-half of its initial volume. In other words, $pV =$ constant when n and T are constant.
3. The pressure is proportional to the *absolute* temperature. If we double the absolute temperature, keeping the volume and number of moles constant, the pressure doubles. In other words, $p =$ (constant)T when n and V are constant.

These three relationships can be combined neatly into a single equation, called the **ideal-gas equation:**

$$pV = nRT \qquad \text{(ideal-gas equation)} \qquad (18.3)$$

18.2 The ideal-gas equation $pV = nRT$ gives a good description of the air inside an inflated vehicle tire, where the pressure is about 3 atmospheres and the temperature is much too high for nitrogen or oxygen to liquefy. As the tire warms (T increases), the volume V changes only slightly but the pressure p increases.

where R is a proportionality constant. An **ideal gas** is one for which Eq. (18.3) holds precisely for *all* pressures and temperatures. This is an idealized model; it works best at very low pressures and high temperatures, when the gas molecules are far apart and in rapid motion. It is reasonably good (within a few percent) at moderate pressures (such as a few atmospheres) and at temperatures well above those at which the gas liquefies (Fig. 18.2).

We might expect that the constant R in the ideal-gas equation would have different values for different gases, but it turns out to have the same value for *all* gases, at least at sufficiently high temperature and low pressure. It is called the **gas constant** (or *ideal-gas constant*). The numerical value of R depends on the units of p, V, and T. In SI units, in which the unit of p is Pa $(1\ \text{Pa} = 1\text{N/m}^2)$ and the unit of V is m^3, the current best numerical value of R is

$$R = 8.314472(15)\ \text{J/mol} \cdot \text{K}$$

or $R = 8.314\ \text{J/mol} \cdot \text{K}$ to four significant figures. Note that the units of pressure times volume are the same as the units of work or energy (for example, N/m^2 times m^3); that's why R has units of energy per mole per unit of absolute temperature. In chemical calculations, volumes are often expressed in liters (L) and pressures in atmospheres (atm). In this system, to four significant figures,

$$R = 0.08206\ \frac{\text{L} \cdot \text{atm}}{\text{mol} \cdot \text{K}}$$

We can express the ideal-gas equation, Eq. (18.3), in terms of the mass m_{total} of gas, using $m_{\text{total}} = nM$ from Eq. (18.2):

$$pV = \frac{m_{\text{total}}}{M}RT \qquad (18.4)$$

From this we can get an expression for the density $\rho = m_{\text{total}}/V$ of the gas:

$$\rho = \frac{pM}{RT} \qquad (18.5)$$

CAUTION **Density vs. pressure** When using Eq. (18.5), be certain that you distinguish between the Greek letter ρ (rho) for density and the letter p for pressure.

For a *constant mass* (or constant number of moles) of an ideal gas the product nR is constant, so the quantity pV/T is also constant. If the subscripts 1 and 2 refer to any two states of the same mass of a gas, then

$$\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2} = \text{constant} \qquad \text{(ideal gas, constant mass)} \qquad (18.6)$$

Notice that you don't need the value of R to use this equation.

The proportionality of pressure to absolute temperature is familiar; in fact, in Chapter 17 we *defined* a temperature scale in terms of pressure in a constant-volume gas thermometer. That may make it seem that the pressure–temperature relationship in the ideal-gas equation, Eq. (18.3), is just a result of the way we define temperature. But the equation also tells us what happens when we change the volume or the amount of substance. Also, the gas-thermometer scale turns out to correspond closely to a temperature scale that does *not* depend on the properties of any particular material. We'll define this scale in Chapter 20. For now, consider this equation as being based on this genuinely material-independent temperature scale.

Problem-Solving Strategy 18.1 Ideal Gases

IDENTIFY *the relevant concepts:* Unless the problem explicitly states otherwise, you can use the ideal-gas equation for any situation in which you need to find the state (pressure, volume, temperature, and/or number of moles) of a gas.

SET UP *the problem* using the following steps:

1. Identify the target variables.
2. In some problems you will be concerned with only one state of the system, in which case Eq. (18.3) is the relationship to use. Some of the quantities in this equation will be known; others will be unknown. Make a list of what you know and what you have to find.
3. In other problems you will compare two different states of the same amount of gas. Decide which is state 1 and which is state 2, and make a list of the quantities for each: p_1, p_2, V_1, V_2, T_1, T_2. If all but one of these quantities are known, you can use Eq. (18.6). Otherwise, use Eq. (18.3). For example, if p_1, V_1, and n are given, you can't use Eq. (18.6) because you don't know T_1.
4. Some problems involve the density ρ (mass per volume) rather than the number of moles n and the volume V. In this case it's most convenient to use Eq. (18.5), $\rho = pM/RT$.

EXECUTE *the solution* as follows:

1. Use a consistent set of units. You may have to convert atmospheres to pascals or liters to cubic meters ($1\ \mathrm{m^3} = 10^3\ \mathrm{L} = 10^6\ \mathrm{cm^3}$). Sometimes the problem statement will make one system of units clearly more convenient than others. Decide on your system and stick to it.
2. Don't forget that T must always be an *absolute* temperature. If you are given temperatures in °C, be sure to convert to Kelvin temperatures by adding 273.15 (to three significant figures, 273). Likewise, p is always the absolute pressure, never the gauge pressure.
3. You may sometimes have to convert between mass m_{total} and number of moles n. The relationship is $m_{\text{total}} = Mn$, where M is the molar mass. Here's a tricky point: If you use Eq. (18.4), you *must* use the same mass units for m_{total} and M. So if M is in grams per mole (the usual units for molar mass), then m_{total} must also be in grams. If you want to use m_{total} in kilograms, then you must convert M to kg/mol. For example, the molar mass of oxygen is 32 g/mol or 32×10^{-3} kg/mol. Be careful!
4. Once you have taken care of steps 1–3, solve for the target variables.

EVALUATE *your answer:* Look carefully at your results and see whether they make physical sense. For example, we'll find in Example 18.1 that a mole of gas at 1 atmosphere pressure and 0°C occupies a volume of 22.4 liters. If you do a calculation of the amount of air inside a 1-liter volume and get a fantastically large answer like $n = 5000$ moles, you probably converted units incorrectly or made an algebraic error.

Example 18.1 Volume of a gas at STP

The condition called **standard temperature and pressure** (STP) for a gas is defined to be a temperature of $0°\mathrm{C} = 273.15\ \mathrm{K}$ and a pressure of $1\ \mathrm{atm} = 1.013 \times 10^5\ \mathrm{Pa}$. If you want to keep a mole of an ideal gas in your room at STP, how big a container do you need?

SOLUTION

IDENTIFY: This problem involves the properties of an ideal gas. We are given the pressure and temperature, and our target variable is the volume.

SET UP: We are asked about the properties of a single state of the system, so we use Eq. (18.3).

EXECUTE: From Eq. (18.3), using R in J/mol · K,

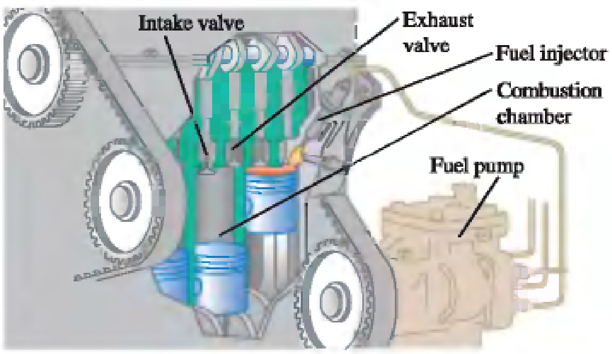
$$V = \frac{nRT}{p} = \frac{(1\ \mathrm{mol})(8.314\ \mathrm{J/mol\cdot K})(273.15\ \mathrm{K})}{1.013 \times 10^5\ \mathrm{Pa}}$$
$$= 0.0224\ \mathrm{m^3} = 22.4\ \mathrm{L}$$

EVALUATE: You may be familiar with this result from your study of chemistry. Note that 22.4 L is almost exactly the volume of three basketballs. A cube 0.282 m on a side would also do the job.

Example 18.2 Compressing gas in an automobile engine

In an automobile engine, a mixture of air and gasoline is compressed in the cylinders before being ignited. A typical engine has a compression ratio of 9.00 to 1; this means that the gas in the cylinders is compressed to $1/(9.00)$ of its original volume (Fig. 18.3). The initial pressure is 1.00 atm and the initial temperature is 27°C. If the pressure after compression is 21.7 atm, find the temperature of the compressed gas.

18.3 Cutaway of an automobile engine. While the air–gasoline mixture is being compressed prior to ignition, the intake and exhaust valves are both in the closed (up) position.



SOLUTION

IDENTIFY: In this problem we are asked to compare two states of the same quantity of ideal gas. The target variable is the temperature in the compressed state. The intake and exhaust valves at the top of the cylinder in Fig. 18.3 stay closed during the compression, so the quantity of gas is constant.

SET UP: Let state 1 be the uncompressed gas, and let state 2 be the fully compressed gas. Then $p_1 = 1.00$ atm, $p_2 = 21.7$ atm, and $V_1 = 9.00\ V_2$. Converting temperature to the Kelvin scale by adding 273, we get $T_1 = 300$ K; the final temperature T_2 is the target variable. The number of moles of gas n is constant, so we can use Eq. (18.6).

EXECUTE: Solving Eq. (18.6) for the temperature T_2 of the compressed gas, we get

$$T_2 = T_1 \frac{p_2 V_2}{p_1 V_1} = (300\text{ K}) \frac{(21.7\text{ atm})\ V_2}{(1.00\text{ atm})(9.00\ V_2)}$$
$$= 723\text{ K} = 450°\text{C}$$

We didn't need to know the values of V_1 and V_2, only their ratio.

EVALUATE: Note that T_2 is the temperature of the air–gasoline mixture *before* the mixture is ignited; when burning starts, the temperature becomes higher still.

Example 18.3 Mass of air in a scuba tank

A typical tank used for scuba diving has a volume of 11.0 L (about 0.4 ft^3) and a gauge pressure, when full, of 2.10×10^7 Pa (about 3000 psig). The "empty" tank contains 11.0 L of air at 21°C and 1 atm $(1.013 \times 10^5\text{ Pa})$. When the tank is filled with hot air from a compressor, the temperature is 42°C and the gauge pressure is 2.11×10^7 Pa. What mass of air was added? (Air is a mixture of gases, about 78% nitrogen, 21% oxygen, and 1% miscellaneous; its average molar mass is 28.8 g/mol = 28.8×10^{-3} kg/mol.)

SOLUTION

IDENTIFY: Our target variable is the *difference* between the mass present at the beginning (state 1) and at the end (state 2).

SET UP: We are given the molar mass of air, so we can use Eq. (18.2) to find the target variable if we know the number of moles present in states 1 and 2. We determine n_1 and n_2 by applying Eq. (18.3) to each state individually.

EXECUTE: We must remember to convert the temperatures to the Kelvin scale by adding 273 and to convert the pressure to absolute by adding 1.013×10^5 Pa. From Eq. (18.3), the number of moles n_1 in the "empty" tank is

$$n_1 = \frac{p_1 V_1}{RT_1} = \frac{(1.013 \times 10^5\text{ Pa})(11.0 \times 10^{-3}\text{ m}^3)}{(8.314\text{ J/mol}\cdot\text{K})(294\text{ K})} = 0.46\text{ mol}$$

The volume of the metal tank is hardly affected by the increased pressure, so $V_1 = V_2$. The number of moles in the full tank is

$$n_2 = \frac{p_2 V_2}{RT_2} = \frac{(2.11 \times 10^7\text{ Pa})(11.0 \times 10^{-3}\text{ m}^3)}{(8.314\text{ J/mol}\cdot\text{K})(315\text{ K})} = 88.6\text{ mol}$$

We added $n_2 - n_1 = 88.6\text{ mol} - 0.46\text{ mol} = 88.1$ mol to the tank. From Eq. (18.2), the added mass is $M(n_2 - n_1) = (28.8 \times 10^{-3}\text{ kg/mol})(88.1\text{ mol}) = 2.54$ kg.

EVALUATE: The added mass is not insubstantial: You could certainly use a scale to determine whether the tank was empty or full.

Could this problem have been solved in the same way as Example 18.2? The volume is constant, so $p/nT = R/V$ is constant and $p_1/n_1 T_1 = p_2/n_2 T_2$; this can be solved for n_2/n_1, the ratio of the final and initial numbers of moles. But we need the *difference* of these two numbers, not the ratio, so this equation by itself isn't enough to solve the problem.

Example 18.4 Variation of atmospheric pressure with elevation

Find the variation of atmospheric pressure with elevation in the earth's atmosphere, assuming that the temperature is 0°C at all elevations. Ignore the variation of g with elevation.

SOLUTION

IDENTIFY: As the elevation increases, both the atmospheric pressure and the density decrease. Hence we have *two* unknown functions of elevation; to solve for them, we need two separate relationships. One of these is the ideal-gas equation, which we can write in terms of pressure and density; the other is the relationship between pressure and density in a fluid in equilibrium, discussed in Section 14.2.

SET UP: In Section 14.2, we found the general equation $dp/dy = -\rho g$, [Eq. (14.4)], for the variation of pressure p with elevation y as a function of density ρ. Equation (18.5), $\rho = pM/RT$, states the ideal-gas equation in terms of density. We are told to assume that g and T are the same at all elevations; we also assume that the atmosphere has the same chemical composition, and hence the same molar mass M, at all heights. We then combine the two expressions and solve for $p(y)$.

EXECUTE: We substitute $\rho = pM/RT$ into $dp/dy = -\rho g$, separate variables, and integrate, letting p_1 be the pressure at elevation y_1 and p_2 be the pressure at y_2:

$$\frac{dp}{dy} = -\frac{pM}{RT}g$$

$$\int_{p_1}^{p_2} \frac{dp}{p} = -\frac{Mg}{RT}\int_{y_1}^{y_2} dy$$

$$\ln\frac{p_2}{p_1} = -\frac{Mg}{RT}(y_2 - y_1)$$

$$\frac{p_2}{p_1} = e^{-Mg(y_2-y_1)/RT}$$

Now let $y_1 = 0$ be at sea level and let the pressure at that point be $p_0 = 1.013 \times 10^5$ Pa. Then our final expression for the pressure p at any height y is

$$p = p_0\, e^{-Mgy/RT}$$

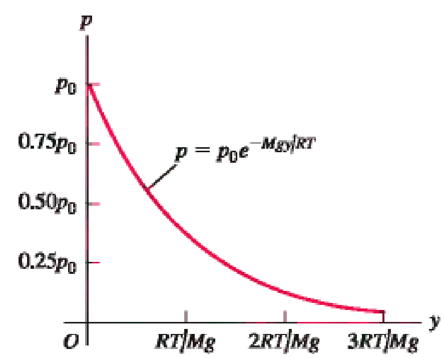
EVALUATE: According to our calculation, the pressure decreases exponentially with elevation. The graph in Fig. 18.4 shows that the slope dp/dy becomes less negative with greater elevation. That result makes sense, since $dp/dy = -\rho g$ and the density also decreases with elevation. At the summit of Mount Everest, where $y = 8863$ m,

$$\frac{Mgy}{RT} = \frac{(28.8 \times 10^{-3}\,\text{kg/mol})(9.80\,\text{m/s}^2)(8863\,\text{m})}{(8.314\,\text{J/mol}\cdot\text{K})(273\,\text{K})} = 1.10$$

$$p = (1.013 \times 10^5\,\text{Pa})e^{-1.10} = 0.337 \times 10^5\,\text{Pa}$$
$$= 0.33\,\text{atm}$$

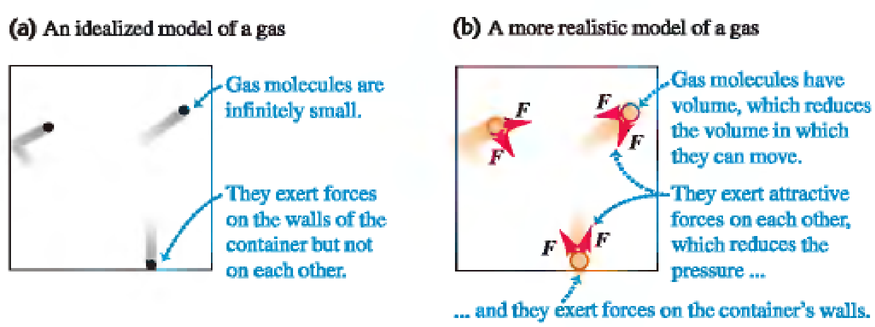
The assumption of constant temperature isn't realistic, and g decreases a little with increasing elevation (see Challenge Problem 18.92). Even so, this example shows why mountaineers need to carry oxygen on Mount Everest. It also shows why jet airliners, which typically fly at altitudes of 8000 to 12,000 m, *must* have pressurized cabins for passenger comfort and health.

18.4 The variation of atmospheric pressure p with elevation y, assuming a constant temperature T.



The van der Waals Equation

The ideal-gas equation, Eq. (18.3), can be obtained from a simple molecular model that ignores the volumes of the molecules themselves and the attractive forces between them (Fig. 18.5a). We'll examine that model in Section 18.3.



18.5 A gas as modeled by (a) the ideal-gas equation and (b) the van der Waals equation.

Meanwhile, we mention another equation of state, the **van der Waals equation,** that makes approximate corrections for these two omissions (Fig. 18.5b). This equation was developed by the 19th-century Dutch physicist J. D. van der Waals; the interaction between atoms that we discussed in Section 13.4 was named the *van der Waals interaction* after him. The van der Waals equation is

$$\left(p + \frac{an^2}{V^2}\right)(V - nb) = nRT \qquad (18.7)$$

The constants a and b are empirical constants, different for different gases. Roughly speaking, b represents the volume of a mole of molecules; the total volume of the molecules is then nb, and the net volume available for the molecules to move around in is $V - nb$. The constant a depends on the attractive intermolecular forces, which reduce the pressure of the gas for given values of n, V, and T by *pulling* the molecules together as they *push* on the walls of the container. The decrease in pressure is proportional to the number of molecules per unit volume in a layer near the wall (which are exerting the pressure on the wall) and is also proportional to the number per unit volume in the next layer beyond the wall (which are doing the attracting). Hence the decrease in pressure due to intermolecular forces is proportional to n^2/V^2.

When n/V is small (that is, when the gas is *dilute*), the average distance between molecules is large, the corrections in the van der Waals equation become insignificant, and Eq. (18.7) reduces to the ideal-gas equation. As an example, for carbon dioxide gas (CO_2) the constants in the van der Waals equation are $a = 0.364\ \text{J} \cdot \text{m}^3/\text{mol}^2$ and $b = 4.27 \times 10^{-5}\ \text{m}^3/\text{mol}$. We found in Example 18.1 that 1 mole of an ideal gas at $T = 0°\text{C} = 273.15\ \text{K}$ and $p = 1\ \text{atm} = 1.013 \times 10^5\ \text{Pa}$ occupies a volume $V = 0.0224\ \text{m}^3$; according to Eq. (18.7), 1 mole of CO_2 occupying this volume at this temperature would be at a pressure 532 Pa less than 1 atm, a difference of only 0.5% from the ideal-gas value.

pV-Diagrams

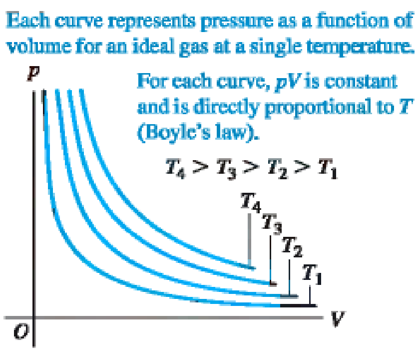
We could in principle represent the *p-V-T* relationship graphically as a *surface* in a three-dimensional space with coordinates p, V, and T. This representation sometimes helps us grasp the overall behavior of the substance, but ordinary two-dimensional graphs are usually more convenient. One of the most useful of these is a set of graphs of pressure as a function of volume, each for a particular constant temperature. Such a diagram is called a ***pV*-diagram.** Each curve, representing behavior at a specific temperature, is called an **isotherm,** or a *pV-isotherm.*

Figure 18.6 shows *pV*-isotherms for a constant amount of an ideal gas. The highest temperature is T_4; the lowest is T_1. This is a graphical representation of the ideal-gas equation of state. We can read off the volume V corresponding to any given pressure p and temperature T in the range shown.

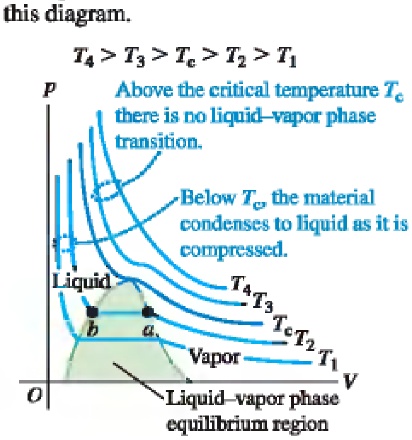
Figure 18.7 shows a *pV*-diagram for a material that *does not* obey the ideal-gas equation. At temperatures below T_c the isotherms develop flat regions in which we can compress the material without an increase in pressure. Observation of the gas shows that it is *condensing* from the vapor (gas) to the liquid phase. The flat parts of the isotherms in the shaded area of Fig. 18.7 represent conditions of liquid-vapor *phase equilibrium.* As the volume decreases, more and more material goes from vapor to liquid, but the pressure does not change. (To keep the temperature constant during condensation, we have to remove the heat of vaporization, discussed in Section 17.6.)

When we compress such a gas at a constant temperature T_2 in Fig. 18.7, it is vapor until point a is reached. Then it begins to liquefy; as the volume decreases further, more material liquefies, and *both* the pressure and the temperature remain constant. At point b, all the material is in the liquid state. After this, any further compression results in a very rapid rise of pressure, because liquids are in general much less compressible than gases. At a lower constant temperature T_1, similar behavior occurs, but the condensation begins at lower pressure and greater volume

18.6 Isotherms, or constant-temperature curves, for a constant amount of an ideal gas.



18.7 A *pV*-diagram for a nonideal gas, showing isotherms for temperatures above and below the critical temperature T_c. The liquid–vapor equilibrium region is shown as a green shaded area. At still lower temperatures the material might undergo phase transitions from liquid to solid or from gas to solid; these are not shown in this diagram.



than at the constant temperature T_2. At temperatures greater than T_c, *no* phase transition occurs as the material is compressed; at the highest temperatures, such as T_4, the curves resemble the ideal-gas curves of Fig. 18.6. We call T_c the *critical temperature* for this material. In Section 18.6 we'll discuss what happens to the phase of the gas above the critical temperature.

We will use pV-diagrams often in the next two chapters. We will show that the *area* under a pV-curve (whether or not it is an isotherm) represents the *work* done by the system during a volume change. This work, in turn, is directly related to heat transfer and changes in the *internal energy* of the system, which we'll get to in Chapter 19.

Test Your Understanding of Section 18.1 Rank the following ideal gases in order from highest to lowest number of moles: (i) pressure 1 atm, volume 1 L, and temperature 300 K; (ii) pressure 2 atm, volume 1 L, and temperature 300 K; (iii) pressure 1 atm, volume 2 L, and temperature 300 K; (iv) pressure 1 atm, volume 1 L, and temperature 600 K; (v) pressure 2 atm, volume 1 L, and temperature 600 K.

18.2 Molecular Properties of Matter

We have studied several properties of matter in bulk, including elasticity, density, surface tension, heat capacities, and equations of state, with only passing references to molecular structure. Now we want to look in more detail at the relationship of bulk behavior to microscopic structure. We begin with a general discussion of the molecular structure of matter. Then in the next two sections we develop the kinetic-molecular model of an ideal gas, obtaining from this molecular model the equation of state and an expression for heat capacity.

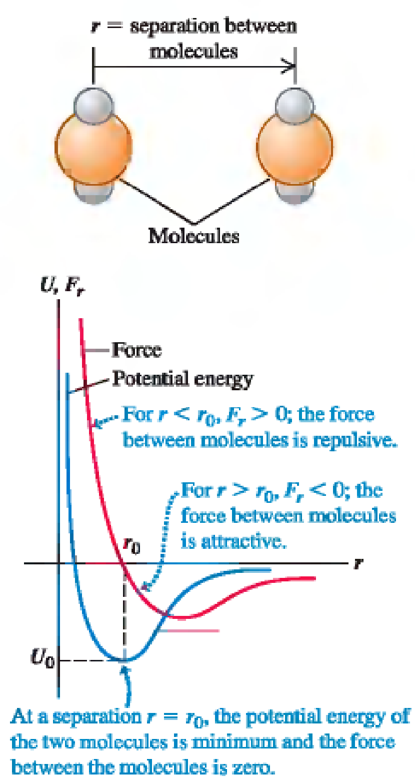
Molecules and Intermolecular Forces

All familiar matter is made up of **molecules.** For any specific chemical compound, all the molecules are identical. The smallest molecules contain one atom each and are of the order of 10^{-10} m in size; the largest contain many atoms and are at least 10,000 times larger. In gases the molecules move nearly independently; in liquids and solids they are held together by intermolecular forces that are electrical in nature, arising from interactions of the electrically charged particles that make up the molecules. Gravitational forces between molecules are negligible in comparison with electrical forces.

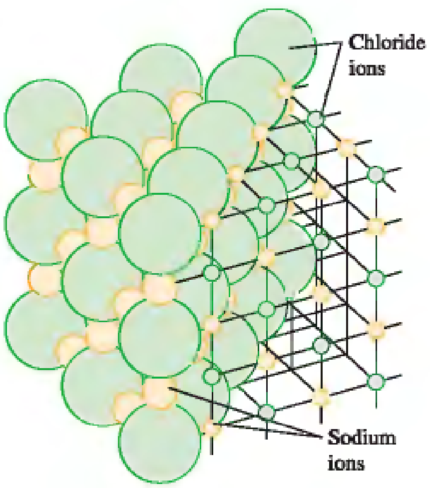
The interaction of two *point* electric charges is described by a force (repulsive for like charges, attractive for unlike charges) with a magnitude proportional to $1/r^2$, where r is the distance between the points. We will study this relationship, called *Coulomb's law,* in Chapter 21. Molecules are *not* point charges but complex structures containing both positive and negative charge, and their interactions are more complex. The force between molecules in a gas varies with the distance r between molecules somewhat as shown in Fig. 18.8, where a positive F_r corresponds to a repulsive force and a negative F_r to an attractive force. When molecules are far apart, the intermolecular forces are very small and usually attractive. As a gas is compressed and its molecules are brought closer together, the attractive forces increase. The intermolecular force becomes zero at an equilibrium spacing r_0, corresponding roughly to the spacing between molecules in the liquid and solid states. In liquids and solids, relatively large pressures are needed to compress the substance appreciably. This shows that at molecular distances slightly *less* than the equilibrium spacing, the forces become *repulsive* and relatively large.

Figure 18.8 also shows the potential energy as a function of r. This function has a *minimum* at r_0, where the force is zero. The two curves are related by $F_r(r) = -dU/dr$, as we showed in Section 7.4. Such a potential energy function is often called a **potential well.** A molecule at rest at a distance r_0 from a second molecule would need an additional energy $|U_0|$, the "depth" of the potential well, to "escape" to an indefinitely large value of r.

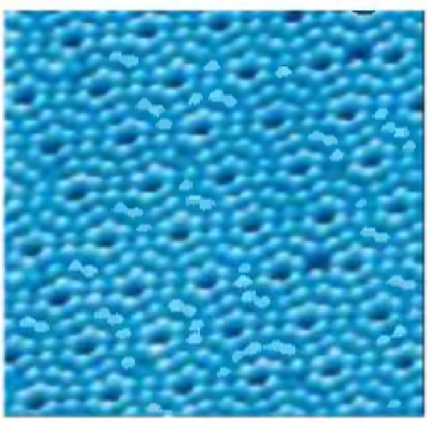
18.8 How the force between molecules and their potential energy of interaction depend on their separation r.



18.9 Schematic representation of the cubic crystal structure of sodium chloride.



18.10 A scanning tunneling microscope image of the surface of a silicon crystal. The area shown is only 9.0 nm (9.0×10^{-9} m) across. Each blue "bead" is an individual silicon atom; you can clearly see how these atoms are arranged in a (nearly) perfect array of hexagons.



Molecules are always in motion; their kinetic energies usually increase with temperature. At very low temperatures the average kinetic energy of a molecule may be much *less* than the depth of the potential well. The molecules then condense into the liquid or solid phase with average intermolecular spacings of about r_0. But at higher temperatures the average kinetic energy becomes larger than the depth $|U_0|$ of the potential well. Molecules can then escape the intermolecular force and become free to move independently, as in the gaseous phase of matter.

In *solids,* molecules vibrate about more or less fixed points. In a crystalline solid these points are arranged in a recurring *crystal lattice.* Figure 18.9 shows the cubic crystal structure of sodium chloride (ordinary salt). A scanning tunneling microscope image of individual silicon atoms on the surface of a crystal is shown in Fig. 18.10.

The vibration of molecules in a solid about their equilibrium positions may be nearly simple harmonic if the potential well is approximately parabolic in shape at distances close to r_0. (We discussed this kind of simple harmonic motion in Section 13.4.) But if the potential-energy curve rises more gradually for $r > r_0$ than for $r < r_0$, as in Fig. 18.8, the average position shifts to larger r with increasing amplitude. As we pointed out in Section 17.4, this is the basis of thermal expansion.

In a *liquid,* the intermolecular distances are usually only slightly greater than in the solid phase of the same substance, but the molecules have much greater freedom of movement. Liquids show regularity of structure only in the immediate neighborhood of a few molecules. This is called *short-range order,* in contrast with the *long-range order* of a solid crystal.

The molecules of a *gas* are usually widely separated and so have only very small attractive forces. A gas molecule moves in a straight line until it collides with another molecule or with a wall of the container. In molecular terms, an *ideal gas* is a gas whose molecules exert *no* attractive forces on each other (Fig. 18.5a) and therefore have no *potential* energy.

At low temperatures, most common substances are in the solid phase. As the temperature rises, a substance melts and then vaporizes. From a molecular point of view, these transitions are in the direction of increasing molecular kinetic energy. Thus temperature and molecular kinetic energy are closely related.

Moles and Avogadro's Number

We have used the mole as a measure of quantity of substance. One **mole** of any pure chemical element or compound contains a definite number of molecules, the same number for all elements and compounds. The official SI definition is

One mole is the amount of substance that contains as many elementary entities as there are atoms in 0.012 kilogram of carbon-12.

In our discussion, the "elementary entities" are molecules. (In a monatomic substance such as carbon or helium, each molecule is a single atom, but we'll still call it a molecule here.) Note that atoms of a given element may occur in any of several isotopes, which are chemically identical but have different atomic masses; "carbon-12" refers to a specific isotope of carbon.

The number of molecules in a mole is called **Avogadro's number,** denoted by N_A. The current best numerical value of N_A is

$$N_A = 6.02214199(47) \times 10^{23} \text{ molecules/mol} \qquad \text{(Avogadro's number)}$$

The *molar mass M* of a compound is the mass of 1 mole. It is equal to the mass m of a single molecule multiplied by Avogadro's number:

$$M = N_A m \qquad \text{(molar mass, Avogadro's number, and mass of a molecule)} \tag{18.8}$$

When the molecule consists of a single atom, the term *atomic mass* is often used instead of molar mass or molecular weight.

Example 18.5 Atomic and molecular mass

Find the mass of a single hydrogen atom and the mass of an oxygen molecule.

SOLUTION

IDENTIFY: This problem involves the relationship between the mass of a molecule or atom (our target variable) and the corresponding molar mass.

SET UP: We use Eq. (18.8) in the form $m = M/N_A$ and the values of the molar masses from the periodic table of the elements (See Appendix D).

EXECUTE: The mass per mole of atomic hydrogen (that is, the atomic mass) is 1.008 g/mol. Therefore the mass m_H of a single hydrogen atom is

$$m_H = \frac{1.008 \text{ g/mol}}{6.022 \times 10^{23} \text{ atoms/mol}} = 1.674 \times 10^{-24} \text{ g/atom}$$

From Appendix D, the atomic mass of oxygen is 16.0 g/mol, so the molar mass of oxygen, which has diatomic (two-atom) molecules, is 32.0 g/mol. The mass of a single molecule of O_2 is

$$m_{O_2} = \frac{32.0 \text{ g/mol}}{6.022 \times 10^{23} \text{ molecules/mol}} = 53.1 \times 10^{-24} \text{ g/molecule}$$

EVALUATE: We note that the values in Appendix D are for the *average* atomic masses of a natural sample of each element. Such a sample may contain several different isotopes of the element, each with a different atomic mass. Natural samples of hydrogen and oxygen are almost entirely made up of just one isotope; this is not the case for all elements, however.

Test Your Understanding of Section 18.2 Suppose you could adjust the value of r_0 for the molecules of a certain chemical compound (Fig. 18.8) by turning a dial. If you doubled the value of r_0, the density of the solid form of this compound would become (i) twice as great; (ii) four times as great; (iii) eight times as great; (iv) $\frac{1}{2}$ as great; (v) $\frac{1}{4}$ as great; (vi) $\frac{1}{8}$ as great.

18.3 Kinetic-Molecular Model of an Ideal Gas

ActivPhysics ONLINE
8.1 Characteristics of a Gas

The goal of any molecular theory of matter is to understand the *macroscopic* properties of matter in terms of its atomic or molecular structure and behavior. Such theories are of tremendous practical importance; once we have this understanding, we can design materials to have specific desired properties. Such analysis has led to the development of high-strength steels, glasses with special optical properties, semiconductor materials for electronic devices, and countless other materials essential to contemporary technology.

In this and the following sections we will consider a simple molecular model of an ideal gas. This *kinetic-molecular model* represents the gas as a large number of particles bouncing around in a closed container. In this section we use the kinetic-molecular model to understand how the ideal-gas equation of state, Eq. (18.3), is related to Newton's laws. In the following section we use the kinetic-molecular model to predict the molar heat capacity of an ideal gas. We'll go on to elaborate the model to include "particles" that are not points but have a finite size. We will be able to see why polyatomic gases have larger molar heat capacities than monatomic gases.

The following discussion of the kinetic-molecular model has several steps, and you may need to go over them several times to grasp how they all go together. Don't get discouraged!

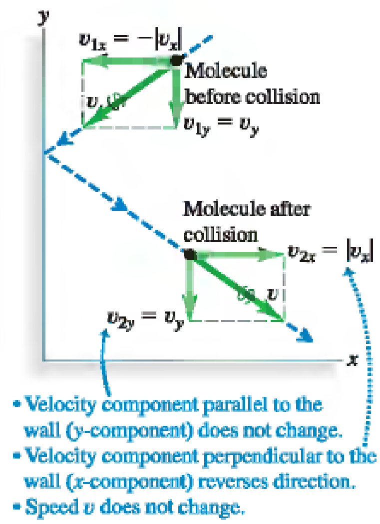
Here are the assumptions of our model:

1. A container with volume V contains a very large number N of identical molecules, each with mass m.
2. The molecules behave as point particles; their size is small in comparison to the average distance between particles and to the dimensions of the container.
3. The molecules are in constant motion; they obey Newton's laws of motion. Each molecule collides occasionally with a wall of the container. These collisions are perfectly elastic.
4. The container walls are perfectly rigid and infinitely massive and do not move.

CAUTION **Molecules vs. moles** Make sure you don't confuse N, the number of *molecules* in the gas, with n, the number of *moles*. The number of molecules is equal to the number of moles multiplied by Avogadro's number: $N = nN_A$.

Collisions and Gas Pressure

18.11 Elastic collision of a molecule with an idealized container wall.



During collisions the molecules exert *forces* on the walls of the container; this is the origin of the *pressure* that the gas exerts. In a typical collision (Fig. 18.11) the velocity component parallel to the wall is unchanged, and the component perpendicular to the wall reverses direction but does not change in magnitude.

Our program is first to determine the *number* of collisions that occur per unit time for a certain area A of wall. Then we find the total momentum change associated with these collisions and the force needed to cause this momentum change. From this we can determine the pressure, which is force per unit area, and compare the result to the ideal-gas equation. We'll find a direct connection between the temperature of the gas and the kinetic energy of the gas molecules.

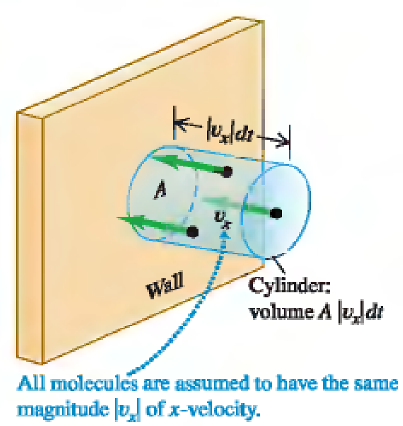
To begin, we will assume that all molecules in the gas have the same *magnitude* of x-velocity, $|v_x|$. This isn't right, but making this temporary assumption helps to clarify the basic ideas. We will show later that this assumption isn't really necessary.

As shown in Fig. 18.11, for each collision the x-component of velocity changes from $-|v_x|$ to $+|v_x|$. So the x-component of momentum changes from $-m|v_x|$ to $+m|v_x|$, and the *change* in the x-component of momentum is $m|v_x| - (-m|v_x|) = 2m|v_x|$.

If a molecule is going to collide with a given wall area A during a small time interval dt, then at the beginning of dt it must be within a distance $|v_x|\,dt$ from the wall (Fig. 18.12) and it must be headed toward the wall. So the number of molecules that collide with A during dt is equal to the number of molecules within a cylinder with base area A and length $|v_x|\,dt$ that have their x-velocity aimed toward the wall. The volume of such a cylinder is $A|v_x|\,dt$. Assuming that the number of molecules per unit volume (N/V) is uniform, the *number* of molecules in this cylinder is $(N/V)(A|v_x|\,dt)$. On the average, half of these molecules are moving toward the wall and half are moving away from it. So the number of collisions with A during dt is

$$\frac{1}{2}\left(\frac{N}{V}\right)(A|v_x|\,dt)$$

18.12 For a molecule to strike the wall in area A during a time interval dt, the molecule must be headed for the wall and be within the shaded cylinder of length $|v_x|\,dt$ at the beginning of the interval.



For the system of all molecules in the gas, the total momentum change dP_x during dt is the *number* of collisions multiplied by $2m|v_x|$:

$$dP_x = \frac{1}{2}\left(\frac{N}{V}\right)(A|v_x|\,dt)(2m|v_x|) = \frac{NAmv_x^2\,dt}{V} \tag{18.9}$$

(We are using capital P for total momentum and small p for pressure. Be careful!) We wrote v_x^2 rather than $|v_x|^2$ in the final expression because the square of the absolute value of a number is equal to the square of that number. The *rate* of change of momentum component P_x is

$$\frac{dP_x}{dt} = \frac{NAmv_x^2}{V} \tag{18.10}$$

According to Newton's second law, this rate of change of momentum equals the force exerted by the wall area A on the gas molecules. From Newton's *third* law this is equal and opposite to the force exerted *on* the wall *by* the molecules. Pressure p is the magnitude of the force exerted on the wall per unit area, and we obtain

$$p = \frac{F}{A} = \frac{Nmv_x^2}{V} \tag{18.11}$$

The pressure exerted by the gas depends on the number of molecules per volume (N/V), the mass m per molecule, and the speed of the molecules.

Pressure and Molecular Kinetic Energies

We mentioned that $|v_x|$ is really *not* the same for all the molecules. But we could have sorted the molecules into groups having the same $|v_x|$ within each group, then added up the resulting contributions to the pressure. The net effect of all this is just to replace v_x^2 in Eq. (18.11) by the *average* value of v_x^2, which we denote by $(v_x^2)_{av}$. Furthermore, $(v_x^2)_{av}$ is related simply to the *speeds* of the molecules. The speed v of any molecule is related to the velocity components v_x, v_y, and v_z by

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

We can average this relation over all molecules:

$$(v^2)_{av} = (v_x^2)_{av} + (v_y^2)_{av} + (v_z^2)_{av}$$

But there is no real difference in our model between the x-, y-, and z-directions. (Molecular speeds are very fast in a typical gas, so the effects of gravity are negligibly small.) It follows that $(v_x^2)_{av}$, $(v_y^2)_{av}$, and $(v_z^2)_{av}$ must all be *equal*. Hence $(v^2)_{av}$ is equal to $3(v_x^2)_{av}$ and

$$(v_x^2)_{av} = \frac{1}{3}(v^2)_{av}$$

so Eq. (18.11) becomes

$$pV = \frac{1}{3}Nm(v^2)_{av} = \frac{2}{3}N\left[\frac{1}{2}m(v^2)_{av}\right] \qquad (18.12)$$

18.13 Summer air (top) is warmer than winter air (bottom); that is, the average translational kinetic energy of air molecules is greater in summer.

We notice that $\frac{1}{2}m(v^2)_{av}$ is the average translational kinetic energy of a single molecule. The product of this and the total number of molecules N equals the total random kinetic energy K_{tr} of translational motion of all the molecules. (The notation K_{tr} reminds us that this energy is associated with *translational* motion. There may be additional energies associated with rotational and vibrational motion of molecules.) The product pV equals two-thirds of the total translational kinetic energy:

$$pV = \frac{2}{3}K_{tr} \qquad (18.13)$$

Now we compare this with the ideal-gas equation,

$$pV = nRT$$

which is based on experimental studies of gas behavior. For the two equations to agree, we must have

$$K_{tr} = \frac{3}{2}nRT \qquad \text{(average translational kinetic energy of } n \text{ moles of ideal gas)} \qquad (18.14)$$

This remarkably simple result shows that K_{tr} is *directly proportional* to the absolute temperature T (Fig. 18.13). We will use this important result several times in the following discussion.

The average translational kinetic energy of a single molecule is the total translational kinetic energy K_{tr} of all molecules divided by the number of molecules, N:

$$\frac{K_{tr}}{N} = \frac{1}{2}m(v^2)_{av} = \frac{3nRT}{2N}$$

Also, the total number of molecules N is the number of moles n multiplied by Avogadro's number N_A, so

$$N = nN_A \qquad \frac{n}{N} = \frac{1}{N_A}$$

and

$$\frac{K_{tr}}{N} = \frac{1}{2}m(v^2)_{av} = \frac{3}{2}\left(\frac{R}{N_A}\right)T \qquad (18.15)$$

The ratio R/N_A occurs frequently in molecular theory. It is called the **Boltzmann constant,** k:

$$k = \frac{R}{N_A} = \frac{8.314\ \text{J/mol}\cdot\text{K}}{6.022 \times 10^{23}\ \text{molecules/mol}}$$
$$= 1.381 \times 10^{-23}\ \text{J/molecule}\cdot\text{K}$$

(The current best numerical value of k is $1.3806503(24) \times 10^{-23}$ J/molecule · K) In terms of k we can rewrite Eq. (18.15) as

$$\frac{1}{2}m(v^2)_{\text{av}} = \frac{3}{2}kT \quad \text{(average translational kinetic energy of a gas molecule)} \tag{18.16}$$

This shows that the average translational kinetic energy *per molecule* depends only on the temperature, not on the pressure, volume, or kind of molecule. We can obtain the average translational kinetic energy *per mole* by multiplying Eq. (18.16) by Avogadro's number and using the relation $M = N_A m$:

$$N_A\frac{1}{2}m(v^2)_{\text{av}} = \frac{1}{2}M(v^2)_{\text{av}} = \frac{3}{2}RT \quad \text{(average translational kinetic energy per mole of gas)} \tag{18.17}$$

The translational kinetic energy of a mole of ideal-gas molecules depends only on T.

Finally, it is sometimes convenient to rewrite the ideal-gas equation on a molecular basis. We use $N = N_A n$ and $R = N_A k$ to obtain the alternative form of the ideal-gas equation:

$$pV = NkT \tag{18.18}$$

This shows that we can think of the Boltzmann constant k as a gas constant on a "per-molecule" basis instead of the usual "per-mole" basis for R.

Molecular Speeds

From Eqs. (18.16) and (18.17) we can obtain expressions for the square root of $(v^2)_{\text{av}}$, called the **root-mean-square speed** (or **rms speed**) v_{rms}:

$$v_{\text{rms}} = \sqrt{(v^2)_{\text{av}}} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3RT}{M}} \quad \text{(root-mean-square speed of a gas molecule)} \tag{18.19}$$

It might seem more natural to characterize molecular speeds by their *average* value rather than by v_{rms}, but we see that v_{rms} follows more directly from Eqs. (18.16) and (18.17). To compute the rms speed, we square each molecular speed, add, divide by the number of molecules, and take the square root; v_{rms} is the *root* of the *mean* of the *squares*. Example 18.7 illustrates this procedure.

Equations (18.16) and (18.19) show that at a given temperature T, gas molecules of different mass m have the same average kinetic energy but different root-mean-square speeds. On average, the nitrogen molecules $(M = 28\ \text{g/mol})$ in the air around you are moving faster than are the oxygen molecules $(M = 32\ \text{g/mol})$. Hydrogen molecules $(M = 2\ \text{g/mol})$ are fastest of all; this is why there is hardly any hydrogen in the earth's atmosphere, despite its being the most common element in the universe (Fig. 18.14). A sizable fraction of any H_2 molecules in the atmosphere would have speeds greater than the earth's escape speed of 1.12×10^4 m/s (calculated in Example 12.5 in Section 12.3) and would escape into space. The heavier, slower-moving gases cannot escape so easily, which is why they predominate in our atmosphere.

18.14 While hydrogen is a desirable fuel for vehicles, it is only a trace constituent of our atmosphere (0.00005% by volume). Hence hydrogen fuel has to be generated by electrolysis of water, which is itself an energy-intensive process.

The assumption that individual molecules undergo perfectly elastic collisions with the container wall is actually a little too simple. More detailed investigation has shown that in most cases, molecules actually adhere to the wall for a short time and then leave again with speeds that are characteristic of the temperature *of the wall.* However, the gas and the wall are ordinarily in thermal equilibrium and have the same temperature. So there is no net energy transfer between gas and wall, and this discovery does not alter the validity of our conclusions.

Problem-Solving Strategy 18.2 Kinetic-Molecular Theory

IDENTIFY *the relevant concepts:* Use the results of the kinetic-molecular model whenever you are asked to relate macroscopic properties of a gas, such as temperature and pressure, to microscopic properties, such as molecular speeds.

SET UP *the problem* using the following steps:
1. Identify which variables are known and which are the unknown target variables.
2. Choose the equation(s) to be used from among Eqs. (18.14), (18.16), and (18.19).

EXECUTE *the solution* as follows: As you solve for the target variable, be on your guard for inconsistency in units. Special caution is needed in the following places:
1. The usual units for molar mass M are grams per mole; the molar mass of oxygen (O_2) is 32 g/mol, for example. These units are often omitted in tables. In equations such as Eq. (18.19), when you use SI units you *must* express M in kilograms per mole by multiplying the table value by $(1\ \text{kg}/10^3\ \text{g})$. Thus in SI units, M for oxygen is 32×10^{-3} kg/mol.
2. Are you working on a "per-molecule" basis or a "per-mole" basis? Remember that m is the mass of a single molecule and M is the mass of a mole of molecules; N is the number of molecules and n is the number of moles; k is the gas constant per molecule and R is the gas constant per mole. You can do a complete unit check if you think of N as having units of "molecules"; then m has units of "mass per molecule," and k has units of "joules per molecule per kelvin."
3. Remember that T is always *absolute* (Kelvin) temperature.

EVALUATE *your answer:* Are your answers reasonable? Keep in mind that typical molecular speeds at room temperature are several hundred meters per second. If your answer seems dramatically different, recheck your calculations.

Example 18.6 Calculating molecular kinetic energy and v_{rms}

(a) What is the average translational kinetic energy of a molecule of an ideal gas at a temperature of 27°C? (b) What is the total random translational kinetic energy of the molecules in 1 mole of this gas? (c) What is the root-mean-square speed of oxygen molecules at this temperature?

SOLUTION

IDENTIFY: This problem involves the translational kinetic energy of an ideal gas on a per-molecule basis and a per-mole basis, as well as the rms speed of molecules in the gas.

SET UP: We are given temperature $T = 27°\text{C}$ and number of moles $n = 1$ mol, and the molecular mass m is that for oxygen. We use Eq. (18.16) to determine the average kinetic energy of a molecule, Eq. (18.14) to find the total molecular kinetic energy, and Eq. (18.19) to find the rms speed of a molecule.

EXECUTE: (a) To use Eq. (18.16), we first convert the temperature to the Kelvin scale: 27°C = 300 K. Then

$$\frac{1}{2}m(v^2)_{\text{av}} = \frac{3}{2}kT = \frac{3}{2}(1.38 \times 10^{-23}\ \text{J/K})(300\ \text{K})$$
$$= 6.21 \times 10^{-21}\ \text{J}$$

This answer does not depend on the mass of the molecule.

(b) From Eq. (18.14), the total translational kinetic energy of a mole of molecules is

$$K_{\text{tr}} = \frac{3}{2}nRT = \frac{3}{2}(1\ \text{mol})(8.314\ \text{J/mol}\cdot\text{K})(300\ \text{K})$$
$$= 3740\ \text{J}$$

This is about the same kinetic energy as that of a sprinter in a 100-m dash.

(c) From Example 18.5 (Section 18.2), the mass of an oxygen molecule is

$$m_{O_2} = (53.1 \times 10^{-24}\ \text{g})(1\ \text{kg}/10^3\ \text{g}) = 5.31 \times 10^{-26}\ \text{kg}$$

From Eq. (18.19),

$$v_{\text{rms}} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3(1.38 \times 10^{-23}\ \text{J/K})(300\ \text{K})}{5.31 \times 10^{-26}\ \text{kg}}}$$
$$= 484\ \text{m/s}$$

This is 1740 km/h, or 1080 mi/h! Alternatively,

$$v_{\text{rms}} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3(8.314\ \text{J/mol}\cdot\text{K})(300\ \text{K})}{32.0 \times 10^{-3}\ \text{kg/mol}}}$$
$$= 484\ \text{m/s}$$

EVALUATE: We can check our result in part (b) by noting that the translational kinetic energy per mole must be equal to the average translational kinetic energy per molecule from part (a) multiplied by Avogadro's number N_A: $K_{\text{tr}} = (6.022 \times 10^{23}\ \text{molecules})(6.21 \times 10^{-21}\ \text{J/molecule}) = 3740\ \text{J}$.

In part (c), note that when we use Eq. (18.19) with R in SI units, we must express M in *kilograms* per mole, not grams per mole. In this example we use $M = 32.0 \times 10^{-3}$ kg/mol, *not* 32.0 g/mol.

Example 18.7 rms and average speeds

Five gas molecules chosen at random are found to have speeds of 500, 600, 700, 800, and 900 m/s. Find the rms speed. Is it the same as the *average* speed?

SOLUTION

IDENTIFY: To solve this problem, we must use the definitions of the root mean square and the average of a collection of numbers.

SET UP: To find the *root-mean-square* value, we square each speed, find the average (mean) of the squares, and then take the square root of the result.

EXECUTE: The average value of v^2 for the five molecules is

$$(v^2)_{av} = \frac{\left[\begin{array}{c}(500\ \text{m/s})^2 + (600\ \text{m/s})^2 + (700\ \text{m/s})^2 \\ + (800\ \text{m/s})^2 + (900\ \text{m/s})^2\end{array}\right]}{5}$$
$$= 5.10 \times 10^5\ \text{m}^2/\text{s}^2$$

The square root of this is v_{rms}:

$$v_{rms} = 714\ \text{m/s}$$

The *average* speed v_{av} is given by

$$v_{av} = \frac{500\ \text{m/s} + 600\ \text{m/s} + 700\ \text{m/s} + 800\ \text{m/s} + 900\ \text{m/s}}{5}$$
$$= 700\ \text{m/s}$$

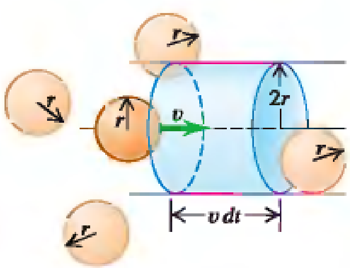
EVALUATE: We see that in general, v_{rms} and v_{av} are *not* the same. Roughly speaking, v_{rms} gives greater weight to the higher speeds than does v_{av}.

Collisions Between Molecules

We have ignored the possibility that two gas molecules might collide. If they are really points, they *never* collide. But consider a more realistic model in which the molecules are rigid spheres with radius r. How often do they collide with other molecules? How far do they travel, on average, between collisions? We can get approximate answers from the following rather primitive model.

Consider N spherical molecules with radius r in a volume V. Suppose only one molecule is moving. When it collides with another molecule, the distance between centers is $2r$. Suppose we draw a cylinder with radius $2r$, with its axis parallel to the velocity of the molecule (Fig. 18.15). The moving molecule collides with any other molecule whose center is inside this cylinder. In a short time dt a molecule with speed v travels a distance $v\,dt$; during this time it collides with any molecule that is in the cylindrical volume of radius $2r$ and length $v\,dt$. The volume of the cylinder is $4\pi r^2 v\,dt$. There are N/V molecules per unit volume, so the number dN with centers in this cylinder is

18.15 In a time dt a molecule with radius r will collide with any other molecule within a cylindrical volume of radius $2r$ and length $v\,dt$.



$$dN = 4\pi r^2 v\,dt\,N/V$$

Thus the number of collisions *per unit time* is

$$\frac{dN}{dt} = \frac{4\pi r^2 vN}{V}$$

This result assumes that only one molecule is moving. The analysis is quite a bit more involved when all the molecules move at once. It turns out that in this case the collisions are more frequent, and the above equation has to be multiplied by a factor of $\sqrt{2}$:

$$\frac{dN}{dt} = \frac{4\pi\sqrt{2}r^2 vN}{V}$$

The average time t_{mean} between collisions, called the *mean free time,* is the reciprocal of this expression:

$$t_{mean} = \frac{V}{4\pi\sqrt{2}r^2 vN} \tag{18.20}$$

The average distance traveled between collisions is called the **mean free path,** denoted by λ (the Greek letter lambda). In our simple model, this is just the molecule's speed v multiplied by t_{mean}:

$$\lambda = vt_{mean} = \frac{V}{4\pi\sqrt{2}r^2N} \qquad \text{(mean free path of a gas molecule)} \qquad (18.21)$$

The mean free path is inversely proportional to the number of molecules per unit volume (N/V) and inversely proportional to the cross-sectional area πr^2 of a molecule; the more molecules there are and the larger the molecule, the shorter the mean distance between collisions (Fig. 18.16). Note that the mean free path *does not* depend on the speed of the molecule.

We can express Eq. (18.21) in terms of macroscopic properties of the gas, using the ideal-gas equation in the form of Eq. (18.18), $pV = NkT$. We find

$$\lambda = \frac{kT}{4\pi\sqrt{2}r^2p} \qquad (18.22)$$

If the temperature is increased at constant pressure, the gas expands, the average distance between molecules increases, and λ increases. If the pressure is increased at constant temperature, the gas compresses and λ decreases.

18.16 If you try to walk through a crowd, your mean free path—the distance you can travel on average without running into another person—depends on how large the people are and how closely they are spaced.

Example 18.8 Calculating mean free path

(a) Estimate the mean free path of a molecule of air at 27°C and 1 atm. Model the molecules as spheres with radius $r = 2.0 \times 10^{-10}$ m. (b) Estimate the mean free time of an oxygen molecule with $v = v_{rms}$.

SOLUTION

IDENTIFY: This problem uses the concepts of mean free path and mean free time (which are our target variables).

SET UP: We use Eq. (18.21) to determine the mean free path λ. To find the mean free time t_{mean} we could use Eq. (18.20), but it's more convenient to use the basic relationship $\lambda = vt_{mean}$ in Eq. (18.21). For the speed v we use the root-mean-square speed for oxygen calculated in Example 18.6.

EXECUTE: (a) From Eq. (18.22),

$$\begin{aligned}\lambda &= \frac{kT}{4\pi\sqrt{2}r^2p}\\ &= \frac{(1.38 \times 10^{-23}\ \text{J/K})(300\ \text{K})}{4\pi\sqrt{2}(2.0 \times 10^{-10}\ \text{m})^2(1.01 \times 10^5\ \text{Pa})}\\ &= 5.8 \times 10^{-8}\ \text{m}\end{aligned}$$

The molecule doesn't get very far between collisions, but the distance is still several hundred times the radius of the molecule. To get a mean free path of 1 meter, the pressure must be about 5.8×10^{-8} atm. Pressures this low are found 100 km or so above the earth's surface, at the outer fringe of our atmosphere.

(b) From Example 18.6, for oxygen at 27°C the root-mean-square speed is $v_{rms} = 484$ m/s, so the mean free time for a molecule with this speed is

$$t_{mean} = \frac{\lambda}{v} = \frac{5.8 \times 10^{-8}\ \text{m}}{484\ \text{m/s}} = 1.2 \times 10^{-10}\ \text{s}$$

This molecule undergoes about 10^{10} collisions per second!

EVALUATE: Note that the mean free *path* calculated in part (a) doesn't depend on the molecule's speed, but the mean free *time* does. Slower molecules have a longer average time interval t_{mean} between collisions than do fast ones, but the average *distance* λ between collisions is the same no matter what the molecule's speed.

Test Your Understanding of Section 18.3 Rank the following gases in order from (a) highest to lowest rms speed of molecules and (b) highest to lowest average translational kinetic energy of a molecule: (i) oxygen $(M = 32.0\ \text{g/mol})$ at 300 K; (ii) nitrogen $(M = 28.0\ \text{g/mol})$ at 300 K; (iii) oxygen at 330 K; (iv) nitrogen at 330 K.

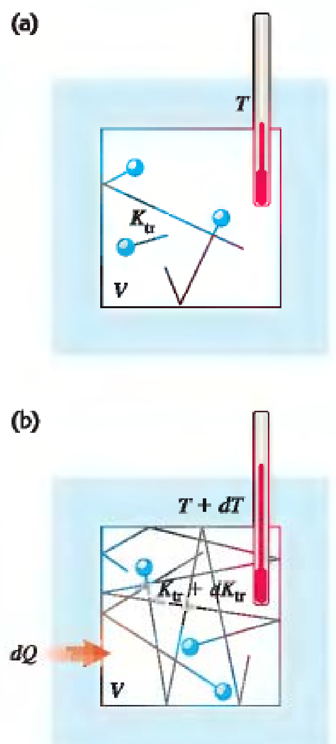
18.4 Heat Capacities

When we introduced the concept of heat capacity in Section 17.5, we talked about ways to *measure* the specific heat or molar heat capacity of a particular material. Now we'll see how these numbers can be *predicted* on theoretical grounds. That's a significant step forward.

Heat Capacities of Gases

The basis of our analysis is that heat is *energy* in transit. When we add heat to a substance, we are increasing its molecular energy. In this discussion we will keep the volume of the gas constant so that we don't have to worry about energy transfer through mechanical work. If we were to let the gas expand, it would do work by pushing on moving walls of its container, and this additional energy transfer would have to be included in our calculations. We'll return to this more general case in Chapter 19. For now, with the volume held constant, we are concerned with C_V, the molar heat capacity *at constant volume.*

18.17 (a) A fixed volume V of a monatomic ideal gas. (b) When an amount of heat dQ is added to the gas, the total translational kinetic energy increases by $dK_{tr} = dQ$ and the temperature increases by $dT = dQ/nC_V$.



In the simple kinetic-molecular model of Section 18.3 the molecular energy consists only of the translational kinetic energy K_{tr} of the pointlike molecules. This energy is directly proportional to the absolute temperature T, as shown by Eq. (18.14), $K_{tr} = \frac{3}{2}nRT$. When the temperature changes by a small amount dT, the corresponding change in kinetic energy is

$$dK_{tr} = \frac{3}{2}nR\,dT \tag{18.23}$$

From the definition of molar heat capacity at constant volume, C_V (see Section 17.5), we also have

$$dQ = nC_V\,dT \tag{18.24}$$

where dQ is the heat input needed for a temperature change dT. Now if K_{tr} represents the total molecular energy, as we have assumed, then dQ and dK_{tr} must be *equal* (Fig. 18.17). Equating the expressions given by Eqs. (18.23) and (18.24), we get

$$nC_V\,dT = \frac{3}{2}nR\,dT$$

$$C_V = \frac{3}{2}R \qquad \text{(ideal gas of point particles)} \tag{18.25}$$

This surprisingly simple result says that the molar heat capacity (at constant volume) of *every* gas whose molecules can be represented as points is equal to $3R/2$.

To see whether this makes sense, let's first check the units. The gas constant *does* have units of energy per mole per kelvin, the correct units for a molar heat capacity. But more important is whether Eq. (18.25) agrees with *measured* values of molar heat capacities. In SI units, Eq. (18.25) gives

$$C_V = \frac{3}{2}(8.314\ \text{J/mol}\cdot\text{K}) = 12.47\ \text{J/mol}\cdot\text{K}$$

Table 18.1 Molar Heat Capacities of Gases

Type of Gas	Gas	C_V (J/mol·K)
Monatomic	He	12.47
	Ar	12.47
Diatomic	H_2	20.42
	N_2	20.76
	O_2	21.10
	CO	20.85
Polyatomic	CO_2	28.46
	SO_2	31.39
	H_2S	25.95

For comparison, Table 18.1 gives measured values of C_V for several gases. We see that for *monatomic* gases our prediction is right on the money, but that it is way off for diatomic and polyatomic gases.

This comparison tells us that our point-molecule model is good enough for monatomic gases but that for diatomic and polyatomic molecules we need something more sophisticated. For example, we can picture a diatomic molecule as *two* point masses, like a little elastic dumbbell, with an interaction force between the atoms of the kind shown in Fig. 18.8. Such a molecule can have additional kinetic energy associated with *rotation* about axes through its center of mass. The atoms may also have *vibrating* motion along the line joining them, with additional kinetic and potential energies. Figure 18.18 shows these possibilities.

When heat flows into a *monatomic* gas at constant volume, *all* of the added energy goes into an increase in random *translational* molecular kinetic energy. Equation (18.23) shows that this gives rise to an increase in temperature. But when the temperature is increased by the same amount in a *diatomic* or *polyatomic* gas, additional heat is needed to supply the increased rotational and vibrational energies. Thus polyatomic gases have *larger* molar heat capacities than monatomic gases, as Table 18.1 shows.

But how do we know how much energy is associated with each additional kind of motion of a complex molecule, compared to the translational kinetic energy? The new principle that we need is called the principle of **equipartition of energy.** It can be derived from sophisticated statistical-mechanics considerations; that derivation is beyond our scope, and we will treat the principle as an axiom.

The principle of equipartition of energy states that each velocity component (either linear or angular) has, on average, an associated kinetic energy per molecule of $\frac{1}{2}kT$, or one-half the product of the Boltzmann constant and the absolute temperature. The number of velocity components needed to describe the motion of a molecule completely is called the number of **degrees of freedom.** For a monatomic gas, there are three degrees of freedom (for the velocity components v_x, v_y, and v_z); this gives a total average kinetic energy per molecule of $3(\frac{1}{2}kT)$, consistent with Eq. (18.16).

For a *diatomic* molecule there are two possible axes of rotation, perpendicular to each other and to the molecule's axis. (We don't include rotation about the molecule's own axis because in ordinary collisions there is no way for this rotational motion to change.) If we assign five degrees of freedom to a diatomic molecule, the average total kinetic energy per molecule is $\frac{5}{2}kT$ instead of $\frac{3}{2}kT$. The total kinetic energy of n moles is $K_{\text{total}} = nN_A(\frac{5}{2}kT) = \frac{5}{2}n(kN_A)T = \frac{5}{2}nRT$, and the molar heat capacity (at constant volume) is

$$C_V = \frac{5}{2}R \qquad \text{(diatomic gas, including rotation)} \qquad (18.26)$$

In SI units,

$$C_V = \frac{5}{2}(8.314 \text{ J/mol} \cdot \text{K}) = 20.79 \text{ J/mol} \cdot \text{K}$$

This agrees within a few percent with the measured values for diatomic gases given in Table 18.1.

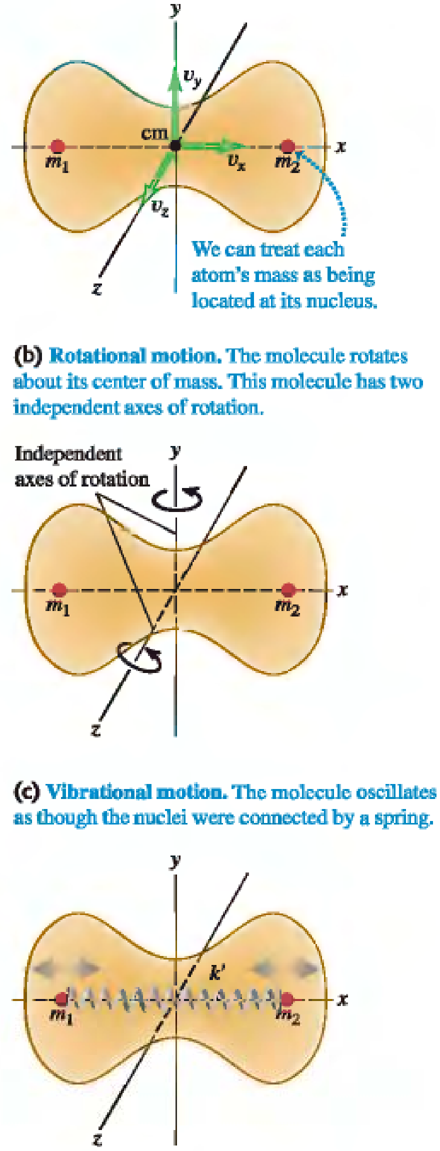
Vibrational motion can also contribute to the heat capacities of gases. Molecular bonds are not rigid; they can stretch and bend, and the resulting vibrations lead to additional degrees of freedom and additional energies. For most diatomic gases, however, vibrational motion does *not* contribute appreciably to heat capacity. The reason for this is a little subtle and involves some concepts of quantum mechanics. Briefly, vibrational energy can change only in finite steps. If the energy change of the first step is much larger than the energy possessed by most molecules, then nearly all the molecules remain in the minimum-energy state of motion. In that case, changing the temperature does not change their average vibrational energy appreciably, and the vibrational degrees of freedom are said to be "frozen out." In more complex molecules the gaps between permitted energy levels are sometimes much smaller, and then vibration *does* contribute to heat capacity. The rotational energy of a molecule also changes by finite steps, but they are usually much smaller; the "freezing out" of rotational degrees of freedom occurs only in rare instances, such as for the hydrogen molecule below about 100 K.

18.18 Motions of a diatomic molecule.

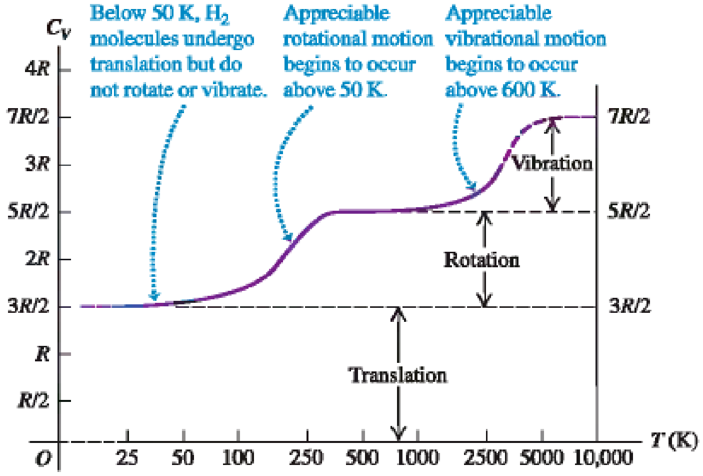
(a) Translational motion. The molecule moves as a whole; its velocity may be described as the x-, y-, and z-velocity components of its center of mass.

(b) Rotational motion. The molecule rotates about its center of mass. This molecule has two independent axes of rotation.

(c) Vibrational motion. The molecule oscillates as though the nuclei were connected by a spring.



18.19 Experimental values of C_V, the molar heat capacity at constant volume, for hydrogen gas (H_2). The temperature is plotted on a logarithmic scale.

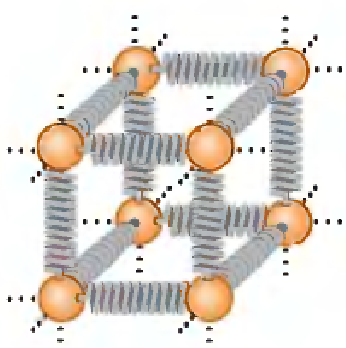


In Table 18.1 the large values of C_V for some polyatomic molecules show the contributions of vibrational energy. In addition, a molecule with three or more atoms that are not in a straight line has three, not two, rotational degrees of freedom.

From this discussion we expect heat capacities to be temperature-dependent, generally increasing with increasing temperature. Figure 18.19 is a graph of the temperature dependence of C_V for hydrogen gas (H_2), showing the temperatures at which the rotational and vibrational energies begin to contribute to the heat capacity.

Heat Capacities of Solids

18.20 To visualize the forces between neighboring atoms in a crystal, envision every atom as being attached to its neighbors by springs.



We can carry out a similar heat-capacity analysis for a crystalline solid. Consider a crystal consisting of N identical atoms (a *monatomic solid*). Each atom is bound to an equilibrium position by interatomic forces. The elasticity of solid materials shows us that these forces must permit stretching and bending of the bonds. We can think of a crystal as an array of atoms connected by little springs (Fig. 18.20). Each atom can *vibrate* about its equilibrium position.

Each atom has three degrees of freedom, corresponding to its three components of velocity. According to the equipartition principle, each atom has an average kinetic energy of $\frac{1}{2}kT$ for each degree of freedom. In addition, each atom has *potential* energy associated with the elastic deformation. For a simple harmonic oscillator (discussed in Chapter 13) it is not hard to show that the average kinetic energy of an atom is *equal* to its average potential energy. In our model of a crystal, each atom is essentially a three-dimensional harmonic oscillator; it can be shown that the equality of average kinetic and potential energies also holds here, provided that the "spring" forces obey Hooke's law.

Thus we expect each atom to have an average kinetic energy $\frac{3}{2}kT$ and an average potential energy $\frac{3}{2}kT$, or an average total energy $3kT$ per atom. If the crystal contains N atoms or n moles, its total energy is

$$E_{\text{total}} = 3NkT = 3nRT \qquad (18.27)$$

From this we conclude that the molar heat capacity of a crystal should be

$$C_V = 3R \qquad \text{(ideal monatomic solid)} \qquad (18.28)$$

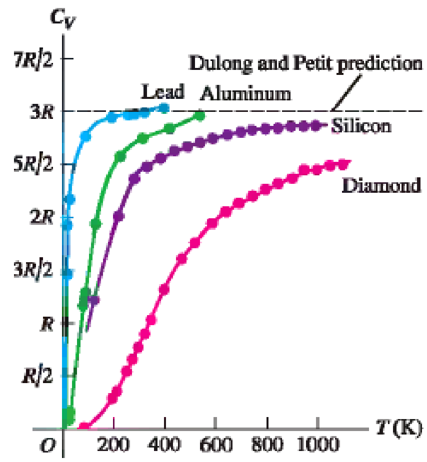
In SI units,

$$C_V = (3)(8.314\ \text{J/mol}\cdot\text{K}) = 24.9\ \text{J/mol}\cdot\text{K}$$

This is the **rule of Dulong and Petit,** which we encountered as an *empirical* finding in Section 17.5: Elemental solids all have molar heat capacities of about 25 J/mol · K. Now we have *derived* this rule from kinetic theory. The agreement is only approximate, to be sure, but considering the very simple nature of our model, it is quite significant.

At low temperatures, the heat capacities of most solids *decrease* with decreasing temperature (Fig. 18.21) for the same reason that vibrational degrees of freedom of molecules are frozen out at low temperatures. At very low temperatures the quantity kT is much *smaller* than the smallest energy step the vibrating atoms can take. Hence most of the atoms remain in their lowest energy states because the next higher energy level is out of reach. The average vibrational energy per atom is then *less* than $3kT$, and the heat capacity per molecule is *less* than $3k$. At higher temperatures when kT is *large* in comparison to the minimum energy step, the equipartition principle holds, and the total heat capacity is $3k$ per molecule or $3R$ per mole as the Dulong and Petit rule predicts. Quantitative understanding of the temperature variation of heat capacities was one of the triumphs of quantum mechanics during its initial development in the 1920s.

18.21 Experimental values of C_V for lead, aluminum, silicon, and diamond. At high temperatures, C_V for each solid approaches about $3R$, in agreement with the rule of Dulong and Petit. At low temperatures, C_V is much less than $3R$.



Test Your Understanding of Section 18.4 A cylinder with a fixed volume contains hydrogen gas (H_2) at 25 K. You then add heat to the gas at a constant rate until its temperature reaches 500 K. Does the temperature of the gas increase at a constant rate? Why or why not? If not, does the temperature increase most rapidly near the beginning or near the end of this process?

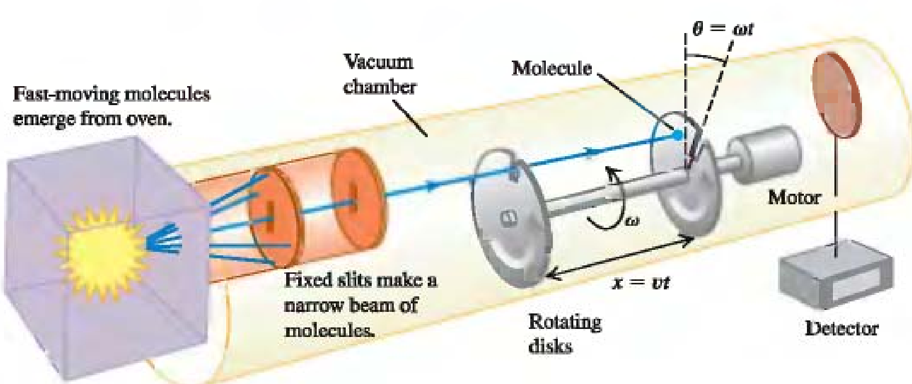
*18.5 Molecular Speeds

As we mentioned in Section 18.3, the molecules in a gas don't all have the same speed. Figure 18.22 shows one experimental scheme for measuring the distribution of molecular speeds. A substance is vaporized in a hot oven; molecules of the vapor escape through an aperture in the oven wall and into a vacuum chamber. A series of slits blocks all molecules except those in a narrow beam, which is aimed at a pair of rotating disks. A molecule passing through the slit in the first disk is blocked by the second disk unless it arrives just as the slit in the second disk is lined up with the beam. The disks function as a speed selector that passes only molecules within a certain narrow speed range. This range can be varied by changing the disk rotation speed, and we can measure how many molecules lie within each of various speed ranges.

To describe the results of such measurements, we define a function $f(v)$ called a *distribution function*. If we observe a total of N molecules, the number dN having speeds in the range between v and $v + dv$ is given by

$$dN = Nf(v)\,dv \qquad (18.29)$$

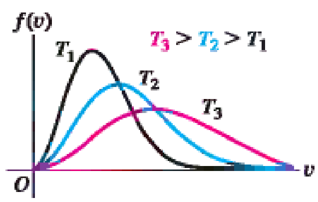
18.22 A molecule with a speed v passes through the slit in the first rotating disk. When the molecule reaches the second rotating disk, the disks have rotated through the offset angle θ. If $v = \omega x/\theta$, the molecule passes through the slit in the second rotating disk and reaches the detector.



We can also say that the *probability* that a randomly chosen molecule will have a speed in the interval v to $v + dv$ is $f(v)\,dv$. Hence $f(v)$ is the probability per unit speed *interval;* it is *not* equal to the probability that a molecule has speed exactly equal to v. Since a probability is a pure number, $f(v)$ has units of reciprocal speed (s/m).

18.23 (a) Curves of the Maxwell–Boltzmann distribution function $f(v)$ for three temperatures. (b) The shaded areas under the curve represent the fractions of molecules within certain speed ranges. The most probable speed v_{mp} for a given temperature is at the peak of the curve.

(a)



As temperature increases:
- the curve flattens.
- the maximum shifts to higher speeds.

(b)

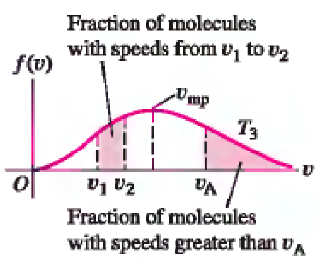


Figure 18.23a shows distribution functions for three different temperatures. At each temperature the height of the curve for any value of v is proportional to the number of molecules with speeds near v. The peak of the curve represents the *most probable speed* v_{mp} for the corresponding temperature. As the temperature increases, the average molecular kinetic energy increases, and so the peak of $f(v)$ shifts to higher and higher speeds.

Figure 18.23b shows that the area under a curve between any two values of v represents the fraction of all the molecules having speeds in that range. Every molecule must have *some* value of v, so the integral of $f(v)$ over all v must be unity for any T.

If we know $f(v)$, we can calculate the most probable speed v_{mp}, the average speed v_{av}, and the rms speed v_{rms}. To find v_{mp}, we simply find the point where $df/dv = 0$; this gives the value of the speed where the curve has its peak. To find v_{av}, we take the number $Nf(v)\,dv$ having speeds in each interval dv, multiply each number by the corresponding speed v, add all these products (by integrating over all v from zero to infinity), and finally divide by N. That is,

$$v_{av} = \int_0^\infty v f(v)\,dv \qquad (18.30)$$

The rms speed is obtained similarly; the average of v^2 is given by

$$(v^2)_{av} = \int_0^\infty v^2 f(v)\,dv \qquad (18.31)$$

and v_{rms} is the square root of this.

The Maxwell–Boltzmann Distribution

The function $f(v)$ describing the actual distribution of molecular speeds is called the **Maxwell–Boltzmann distribution.** It can be derived from statistical-mechanics considerations, but that derivation is beyond our scope. Here is the result:

$$f(v) = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT} \qquad \text{(Maxwell–Boltzmann distribution)} \qquad (18.32)$$

We can also express this function in terms of the translational kinetic energy of a molecule, which we denote by ϵ; that is, $\epsilon = \frac{1}{2}mv^2$. We invite you (see Exercise 18.47) to verify that when this is substituted into Eq. (18.32), the result is

$$f(v) = \frac{8\pi}{m}\left(\frac{m}{2\pi kT}\right)^{3/2} \epsilon e^{-\epsilon/kT} \qquad (18.33)$$

This form shows that the exponent in the Maxwell–Boltzmann distribution function is $-\epsilon/kT$ and that the shape of the curve is determined by the relative magnitude of ϵ and kT at any point. We leave it to you (see Exercise 18.48) to prove that the *peak* of each curve occurs where $\epsilon = kT$, corresponding to a most probable speed v_{mp} given by

$$v_{mp} = \sqrt{\frac{2kT}{m}} \qquad (18.34)$$

To find the average speed, we substitute Eq. (18.32) into Eq. (18.30) and carry out the integration, making a change of variable $v^2 = x$ and then integrating by parts. The result is

$$v_{av} = \sqrt{\frac{8kT}{\pi m}} \qquad (18.35)$$

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8.2 Maxwell-Boltzmann Distribution—Conceptual Analysis

8.3 Maxwell-Boltzmann Distribution—Quantitative Analysis

Finally, to find the rms speed, we substitute Eq. (18.32) into Eq. (18.31). Evaluating the resulting integral takes some mathematical acrobatics, but we can find it in a table of integrals. The result is

$$v_{rms} = \sqrt{\frac{3kT}{m}} \tag{18.36}$$

This result agrees with Eq. (18.19); it *must* agree if the Maxwell–Boltzmann distribution is to be consistent with the equipartition theorem and our other kinetic-theory calculations.

Table 18.2 shows the fraction of all the molecules in an ideal gas that have speeds *less than* various multiples of v_{rms}. These numbers were obtained by numerical integration; they are the same for all ideal gases.

The distribution of molecular speeds in liquids is similar, although not identical, to that for gases. We can understand the vapor pressure of a liquid and the phenomenon of boiling on this basis. Suppose a molecule must have a speed at least as great as v_A in Fig. 18.23b to escape from the surface of a liquid into the adjacent vapor. The number of such molecules, represented by the area under the "tail" of each curve (to the right of v_A), increases rapidly with temperature. Thus the rate at which molecules can escape is strongly temperature-dependent. This process is balanced by another one in which molecules in the vapor phase collide inelastically with the surface and are trapped back into the liquid phase. The number of molecules suffering this fate per unit time is proportional to the pressure in the vapor phase. Phase equilibrium between liquid and vapor occurs when these two competing processes proceed at exactly the same rate. So if the molecular speed distributions are known for various temperatures, we can make a theoretical prediction of vapor pressure as a function of temperature. When liquid evaporates, it's the high-speed molecules that escape from the surface. The ones that are left have less energy on average; this gives us a molecular view of evaporative cooling.

Table 18.2 Fractions of Molecules in an Ideal Gas with Speeds Less than Various Multiples of v/v_{rms}

v/v_{rms}	Fraction
0.20	0.011
0.40	0.077
0.60	0.218
0.80	0.411
1.00	0.608
1.20	0.771
1.40	0.882
1.60	0.947
1.80	0.979
2.00	0.993

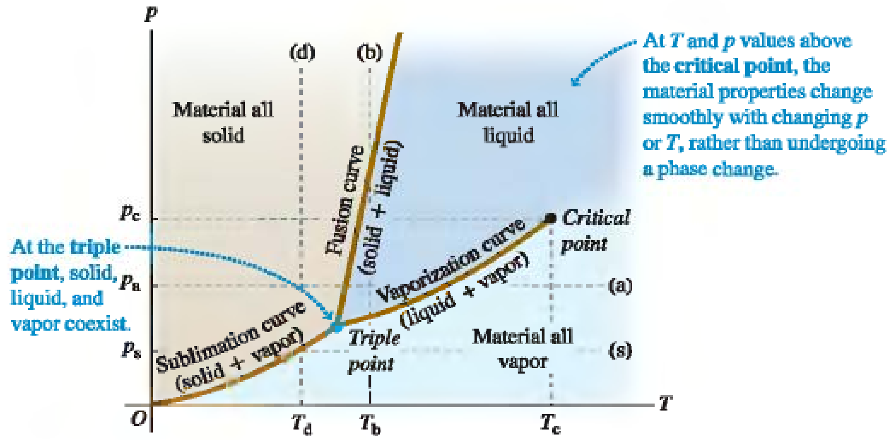
Rates of chemical reactions are often strongly temperature dependent, and the reason is contained in the Maxwell–Boltzmann distribution. When two reacting molecules collide, the reaction can occur only when the molecules are close enough for the electric-charge distributions of their electrons to interact strongly. This requires a minimum energy, called the *activation energy*, and thus a certain minimum molecular speed. Figure 18.23a shows that the number of molecules in the high-speed tail of the curve increases rapidly with temperature. Thus we expect the rate of any reaction that depends on an activation energy to increase rapidly with temperature. Similarly, many plant growth processes have strongly temperature-dependent rates, as can be seen by the rapid and diverse growth in tropical rain forests.

Test Your Understanding of Section 18.5 A quantity of gas containing N molecules has a speed distribution function $f(v)$. How many molecules have speeds between v_1 and $v_2 > v_1$? (i) $\int_0^{v_2} f(v)\,dv - \int_0^{v_1} f(v)\,dv$; (ii) $N\left[\int_0^{v_2} f(v)\,dv - \int_0^{v_1} f(v)\,dv\right]$; (iii) $\int_0^{v_1} f(v)\,dv - \int_0^{v_2} f(v)\,dv$; (iv) $N\left[\int_0^{v_1} f(v)\,dv - \int_0^{v_2} f(v)\,dv\right]$; (v) none of these.

18.6 Phases of Matter

We've talked a lot about ideal gases in the last few sections. An ideal gas is the simplest system to analyze from a molecular viewpoint because we ignore the interactions between molecules. But those interactions are the very thing that makes matter condense into the liquid and solid phases under some conditions. So it's not surprising that theoretical analysis of liquid and solid structure and behavior is a lot more complicated than that for gases. We won't try to go far here with a microscopic picture, but we can talk in general about phases of matter, phase equilibrium, and phase transitions.

18.24 A typical pT phase diagram, showing regions of temperature and pressure at which the various phases exist and where phase changes occur.



In Section 17.6 we learned that each phase is stable only in certain ranges of temperature and pressure. A transition from one phase to another ordinarily takes place under conditions of **phase equilibrium** between the two phases, and for a given pressure this occurs at only one specific temperature. We can represent these conditions on a graph with axes p and T, called a **phase diagram;** Fig. 18.24 shows an example. Each point on the diagram represents a pair of values of p and T. Only a single phase can exist at each point, except for points on the solid lines, where two phases can coexist in phase equilibrium.

These lines separate the diagram into solid, liquid, and vapor regions. For example, the fusion curve separates the solid and liquid areas and represents possible conditions of solid-liquid phase equilibrium. Similarly, the vaporization curve separates the liquid and vapor areas, and the sublimation curve separates the solid and vapor areas. The three curves meet at the **triple point,** the only condition under which all three phases can coexist (Fig. 18.25). In Section 17.3 we used the triple-point temperature of water to define the Kelvin temperature scale. Triple-point data for several substances are given in Table 18.3.

18.25 Atmospheric pressure on earth is higher than the triple-point pressure of water (see line (a) in Fig. 18.24). Depending on the temperature, water can exist as a vapor (in the atmosphere), as a liquid (in the ocean), or as a solid (like the iceberg shown here).

Table 18.3 Triple-Point Data

Substance	Temperature (K)	Pressure (Pa)
Hydrogen	13.80	0.0704×10^5
Deuterium	18.63	0.171×10^5
Neon	24.56	0.432×10^5
Nitrogen	63.18	0.125×10^5
Oxygen	54.36	0.00152×10^5
Ammonia	195.40	0.0607×10^5
Carbon dioxide	216.55	5.17×10^5
Sulfur dioxide	197.68	0.00167×10^5
Water	273.16	0.00610×10^5

If we add heat to a substance at a constant pressure p_a, it goes through a series of states represented by the horizontal line (a) in Figure 18.24. The melting and boiling temperatures at this pressure are the temperatures at which the line intersects the fusion and vaporization curves, respectively. When the pressure is p_s, constant-pressure heating transforms a substance from solid directly to vapor. This process is called *sublimation;* the intersection of line (s) with the sublimation curve gives the temperature T_s at which it occurs for a pressure p_s. At any pressure less than the triple-point pressure, no liquid phase is possible. The triple-point pressure for carbon dioxide is 5.1 atm. At normal atmospheric pressure, solid carbon dioxide ("dry ice") undergoes sublimation; there is no liquid phase at this pressure.

Line (b) in Fig. 18.24 represents compression at a constant temperature T_b. The material passes from vapor to liquid and then to solid at the points where line (b) crosses the vaporization curve and fusion curve, respectively. Line (d) shows constant-temperature compression at a lower temperature T_d; the material passes from vapor to solid at the point where line (d) crosses the sublimation curve.

We saw in the pV-diagram of Fig. 18.7 that a liquid-vapor phase transition occurs only when the temperature and pressure are less than those at the point lying at the top of the green shaded area labeled "Liquid-vapor phase equilibrium region." This point corresponds to the endpoint at the top of the vaporization curve in Fig. 18.24. It is called the **critical point,** and the corresponding values of p and T are called the critical pressure and temperature, p_c and T_c. A gas at a pressure *above* the critical pressure does not separate into two phases when it is cooled at constant pressure (along a horizontal line above the critical point in Fig. 18.24). Instead, its properties change gradually and continuously from those we ordinarily associate with a gas (low density, large compressibility) to those of a liquid (high density, small compressibility) *without a phase transition.*

If this stretches credibility, think about liquid-phase transitions at successively higher points on the vaporization curve. As we approach the critical point, the *differences* in physical properties (such as density, bulk modulus, and viscosity) between the liquid and vapor phases become smaller and smaller. Exactly *at* the critical point they all become zero, and at this point the distinction between liquid and vapor disappears. The heat of vaporization also grows smaller and smaller as we approach the critical point, and it too becomes zero at the critical point.

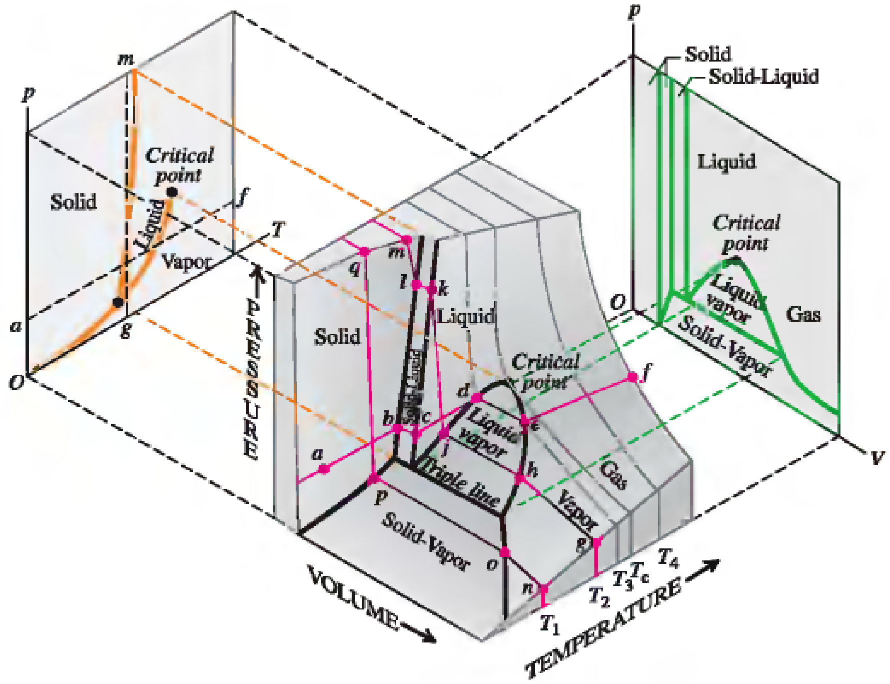
For nearly all familiar materials the critical pressures are much greater than atmospheric pressure, so we don't observe this behavior in everyday life. For example, the critical point for water is at 647.4 K and 221.2×10^5 Pa (about 218 atm or 3210 psi). But high-pressure steam boilers in electric generating plants regularly run at pressures and temperatures well above the critical point.

Many substances can exist in more than one solid phase. A familiar example is carbon, which exists as noncrystalline soot and crystalline graphite and diamond. Water is another example; at least eight types of ice, differing in crystal structure and physical properties, have been observed at very high pressures.

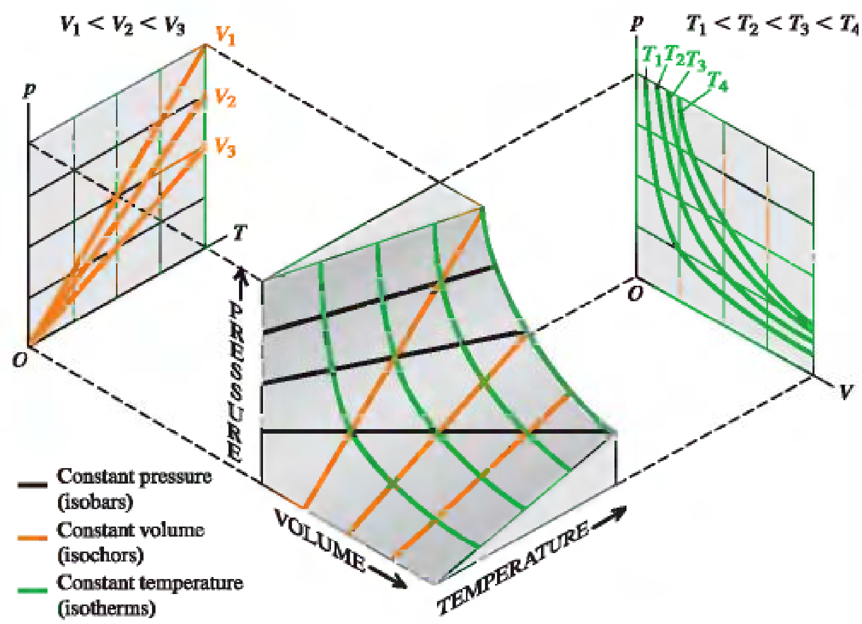
pVT-Surfaces

We remarked in Section 18.1 that the equation of state of any material can be represented graphically as a surface in a three-dimensional space with coordinates p, V, and T. Such a surface is seldom useful in representing detailed quantitative information, but it can add to our general understanding of the behavior of materials at various temperatures and pressure. Figure 18.26 shows a typical pVT-surface. The light lines represent pV-isotherms; projecting them onto the pV-plane would give a diagram similar to Fig. 18.7. The pV-isotherms represent contour lines on the pVT-surface, just as contour lines on a topographic map represent the elevation (the third dimension) at each point. The projections of the edges of the surface onto the pT-plane give the pT phase diagram of Fig. 18.24.

Line *abcdef* in Fig. 18.26 represents constant-pressure heating, with melting along *bc* and vaporization along *de*. Note the volume changes that occur as T increases along this line. Line *ghjklm* corresponds to an isothermal (constant temperature) compression, with liquefaction along *hj* and solidification along *kl*. Between these, segments *gh* and *jk* represent isothermal compression with increase in pressure; the pressure increases are much greater in the liquid region *jk* and the solid region *lm* than in the vapor region *gh*. Finally, line *nopq* represents isothermal solidification directly from the vapor phase; this is the process involved in growth of crystals directly from vapor, as in the formation of snowflakes or frost and in the fabrication of some solid-state electronic devices. These three lines on the pVT-surface are worth careful study.



18.26 A pVT-surface for a substance that expands on melting. Projections of the boundaries on the surface on the pT- and pV-planes are also shown.



18.27 A pVT-surface for an ideal gas. At the left, each red line corresponds to a certain constant volume; at the right, each green line corresponds to a certain constant temperature.

For contrast, Fig. 18.27 shows the much simpler pVT-surface for a substance that obeys the ideal-gas equation of state under all conditions. The projections of the constant-temperature curves onto the pV-plane correspond to the curves of Fig. 18.6, and the projections of the constant-volume curves onto the pT-plane show the direct proportionality of pressure to absolute temperature.

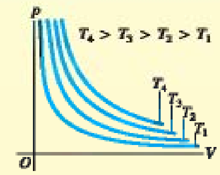
Test Your Understanding of Section 16.6 The average atmospheric pressure on Mars is 6.0×10^2 Pa. Could there be lakes or rivers on Mars today? What about in the past, when the atmospheric pressure is thought to have been substantially greater than today?

CHAPTER 18 SUMMARY

Equations of state: The pressure p, volume V, and absolute temperature T of a given quantity of a substance are called state variables. They are related by an equation of state. This relationship pertains only to equilibrium states, in which p and T are uniform throughout the system. The ideal-gas equation of state relates p, V, T, and the number of moles n through a constant R that is the same for all gases. (See Examples 18.1–18.4.)

A pV-diagram is a set of graphs, called isotherms, each showing pressure as a function of volume for a constant temperature.

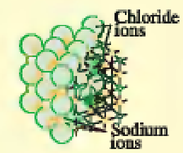
$$pV = nRT \qquad (18.3)$$



Molecular properties of matter: The molar mass M of a pure substance is the mass per mole. The mass m_{total} of a quantity of substance equals M multiplied by the number of moles n. Avogadro's number N_A is the number of molecules in a mole. The mass m of an individual molecule is M divided by N_A. (See Example 18.5.)

$$m_{total} = nM \qquad (18.2)$$

$$M = N_A m \qquad (18.8)$$



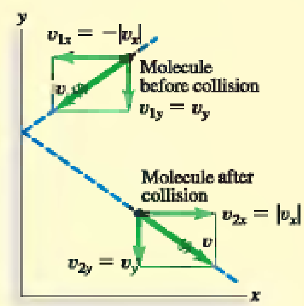
Kinetic-molecular model of an ideal gas: In an ideal gas, the total translational kinetic energy of the gas as a whole (K_{tr}) and the average translational kinetic energy per molecule $[\frac{1}{2}m(v^2)_{av}]$ are proportional to the absolute temperature T. The root-mean-square speed of molecules in an ideal gas is proportional to the square root of T. These expressions involve the Boltzmann constant $k = R/N_A$. (See Examples 18.6 and 18.7.) The mean free path λ of molecules in an ideal gas depends on the number of molecules per volume (N/V) and the molecular radius r. (See Example 18.8.)

$$K_{tr} = \frac{3}{2}nRT \qquad (18.14)$$

$$\frac{1}{2}m(v^2)_{av} = \frac{3}{2}kT \qquad (18.16)$$

$$v_{rms} = \sqrt{(v^2)_{av}} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3RT}{M}} \qquad (18.19)$$

$$\lambda = vt_{mean} = \frac{V}{4\pi\sqrt{2}r^2N} \qquad (18.21)$$

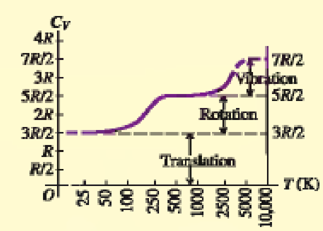


Heat capacities: The molar heat capacity at constant volume C_V can be expressed as a simple multiple of the gas constant R for certain idealized cases: an ideal monatomic gas [Eq. (18.25)]; an ideal diatomic gas including rotation energy [Eq. (18.26)]; and an ideal monatomic solid [Eq. (18.28)]. Many real systems are approximated well by these idealizations.

$$C_V = \frac{3}{2}R \quad \text{(monatomic gas)} \qquad (18.25)$$

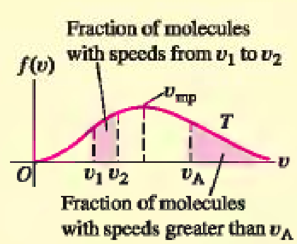
$$C_V = \frac{5}{2}R \quad \text{(diatomic gas)} \qquad (18.26)$$

$$C_V = 3R \quad \text{(monatomic solid)} \qquad (18.28)$$

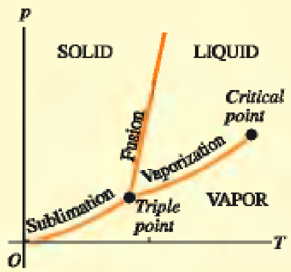


Molecular speeds: The speeds of molecules in an ideal gas are distributed according to the Maxwell–Boltzmann distribution $f(v)$. The quantity $f(v)\,dv$ describes what fraction of the molecules have speeds between v and $v + dv$.

$$f(v) = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT} \qquad (18.32)$$



Phases of matter: Ordinary matter exists in the solid, liquid, and gas phases. A phase diagram shows conditions under which two phases can coexist in phase equilibrium. All three phases can coexist at the triple point. The vaporization curve ends at the critical point, above which the distinction between the liquid and gas phases disappears.



Key Terms

state variables, *611*
equation of state, *611*
molar mass, *611*
ideal-gas equation, *612*
ideal gas, *612*
gas constant, *612*
standard temperature and pressure, *613*
van der Waals equation, *616*
pV-diagram, *616*
isotherm, *616*
molecule, *617*
potential well, *617*
mole, *618*
Avogadro's number, *618*
Boltzmann constant, *622*
root-mean-square speed (rms speed), *622*
mean free path, *625*
equipartition of energy, *627*
degrees of freedom, *627*
rule of Dulong and Petit, *628*
Maxwell–Boltzmann distribution, *630*
phase equilibrium, *632*
phase diagram, *632*
triple point, *632*
critical point, *633*

Answer to Chapter Opening Question ?

From Eq. (18.19), the root-mean-square speed of a gas molecule is proportional to the square root of the absolute temperature T. The temperature range we're considering is from $(25 + 273.15)\ \text{K} = 298\ \text{K}$ to $(100 + 273.15)\ \text{K} = 373\ \text{K}$. Hence the speeds increase by a factor of $\sqrt{(373\ \text{K})/(298\ \text{K})} = 1.12$; that is, there is a 12% increase. While 100°C feels far warmer than 25°C, the difference in molecular speeds is relatively small.

Answers to Test Your Understanding Questions

18.1 Answer: (ii) and (iii) (tie), (i) and (v) (tie), (iv) We can rewrite the ideal-gas equation, Eq. (18.3), as $n = pV/RT$. This tells us that the number of moles n is proportional to the pressure and volume and inversely proportional to the absolute temperature. Hence, compared to (i), the number of moles in each case is (ii) $(2)(1)/(1) = 2$ times as much, (iii) $(1)(2)/(1) = 2$ times as much, (iv) $(1)(1)/(2) = \frac{1}{2}$ as much, and (v) $(2)(1)/(2) = 1$ times as much (that is, equal).

18.2 Answer: (vi) The value of r_0 determines the equilibrium separation of the molecules in the solid phase, so doubling r_0 means that the separation doubles as well. Hence a solid cube of this compound might grow from 1 cm on a side to 2 cm on a side. The volume would then be $2^3 = 8$ times larger, and the density (mass divided by volume) would be $\frac{1}{8}$ as great.

18.3 Answers: (a) (iv), (ii), (iii), (i); (b) (iii) and (iv) (tie), (i) and (ii) (tie) (a) Equation (18.19) tells us that $v_{rms} = \sqrt{3RT/M}$, so the rms speed is proportional to the square root of the ratio of absolute temperature T to molar mass M. Compared to (i) oxygen at 300 K, v_{rms} in the other cases is (ii) $\sqrt{(32.0\ \text{g/mol})/(28.0\ \text{g/mol})} = 1.07$ times faster, (iii) $\sqrt{(330\ \text{K})/(300\ \text{K})} = 1.05$ times faster, and (iv) $\sqrt{(330\ \text{K})(32.0\ \text{g/mol})/(300\ \text{K})(28.0\ \text{g/mol})} = 1.12$ times faster. (b) From Eq. (18.16), the average translational kinetic energy per molecule is $\frac{1}{2}m(v^2)_{av} = \frac{3}{2}kT$, which is directly proportional to T and independent of M. We have $T = 300\ \text{K}$ for cases (i) and (ii) and $T = 330\ \text{K}$ for cases (iii) and (iv), so $\frac{1}{2}m(v^2)_{av}$ has equal values for cases (iii) and (iv) and equal (but smaller) values for cases (i) and (ii).

18.4 Answers: no, near the beginning Adding a small amount of heat dQ to the gas changes the temperature by dT, where $dQ = nC_V\,dT$ from Eq. (18.24). Figure 18.19 shows that C_V for H_2 varies with temperature between 25 K and 500 K, so a given amount of heat gives rise to different amounts of temperature change during the process. Hence the temperature will *not* increase at a constant rate. The temperature change $dT = dQ/nC_V$ is inversely proportional to C_V, so the temperature increases most rapidly at the beginning of the process when the temperature is lowest and C_V is smallest (see Fig. 18.19).

18.5 Answer: (ii) Figure 18.23b shows that the *fraction* of molecules with speeds between v_1 and v_2 equals the area under the curve of $f(v)$ versus v from $v = v_1$ to $v = v_2$. This is equal to the integral $\int_{v_1}^{v_2} f(v)\,dv$, which in turn is equal to the difference between the integrals $\int_0^{v_2} f(v)\,dv$ (the fraction of molecules with speeds from 0 to v_2 and $\int_0^{v_1} f(v)\,dv$ (the fraction of molecules with speeds from 0 to the slower speed v_1). The *number* of molecules with speeds from v_1 to v_2 equals the fraction of molecules in this speed range multiplied by N, the total number of molecules.

18.6 Answers: no, yes The triple-point pressure of water from Table 18.3 is 6.10×10^2 Pa. The present-day pressure on Mars is just less than this value, corresponding to the line labeled p_s in Fig. 18.24. Hence liquid water cannot exist on the present-day Martian surface, and there are no rivers or lakes. Planetary scientists conclude that liquid water could have and almost certainly did exist on Mars in the past, when the atmosphere was thicker.

PROBLEMS

For instructor-assigned homework, go to www.masteringphysics.com

Discussion Questions

Q18.1. Section 18.1 states that ordinarily, pressure, volume, and temperature cannot change individually without one affecting the others. Yet when a liquid evaporates, its volume changes, even though its pressure and temperature are constant. Is this inconsistent? Why or why not?

Q18.2. In the ideal-gas equation, could an equivalent Celsius temperature be used instead of the Kelvin one if an appropriate numerical value of the constant R is used? Why or why not?

Q18.3. On a chilly morning you can "see your breath." Can you really? What are you actually seeing? Does this phenomenon depend on the temperature of the air, the humidity, or both? Explain.

Q18.4. When a car is driven some distance, the air pressure in the tires increases. Why? Should you let out some air to reduce the pressure? Why or why not?

Q18.5. The coolant in an automobile radiator is kept at a pressure higher than atmospheric pressure. Why is this desirable? The radiator cap will release coolant when the gauge pressure of the coolant reaches a certain value, typically 15 lb/in.^{2} or so. Why not just seal the system completely?

Q18.6. Unwrapped food placed in a freezer experiences dehydration, known as "freezer burn." Why?

Q18.7. "Freeze-drying" food involves the same process as "freezer burn," referred to Question 18.6. For freeze-drying, the food is usually frozen first, and then placed in a vacuum chamber and irradiated with infrared radiation. What is the purpose of the vacuum? The radiation? What advantages might freeze-drying have in comparison to ordinary drying?

Q18.8. A group of students drove from their university (near sea level) up into the mountains for a skiing weekend. Upon arriving at the slopes, they discovered that the bags of potato chips they had brought for snacks had all burst open. What caused this to happen?

Q18.9. How does evaporation of perspiration from your skin cool your body?

Q18.18. A rigid, perfectly insulated container has a membrane dividing its volume in half. One side contains a gas at an absolute temperature T_0 and pressure p_0, while the other half is completely empty. Suddenly a small hole develops in the membrane, allowing the gas to leak out into the other half until it eventually occupies twice its original volume. In terms of T_0 and p_0, what will be the new temperature and pressure of the gas when it is distributed equally in both halves of the container? Explain your reasoning.

Q18.11. (a) Which has more atoms: a kilogram of hydrogen or a kilogram of lead? Which has more mass? (b) Which has more atoms: a mole of hydrogen or a mole of lead? Which has more mass? Explain your reasoning.

Q18.12. Use the concepts of the kinetic-molecular model to explain: (a) why the pressure of a gas in a rigid container increases as heat is added to the gas and; (b) why the pressure of a gas increases as we compress it, even if we do not change its temperature.

Q18.13. The proportion of various gases in the earth's atmosphere changes somewhat with altitude. Would you expect the proportion of oxygen at high altitude to be greater or less than at sea level compared to the proportion of nitrogen? Why?

Q18.14. Comment on the following statement: *When two gases are mixed, if they are to be in thermal equilibrium, they must have the same average molecular speed.* Is the statement correct? Why or why not?

Q18.15. The kinetic-molecular model contains a hidden assumption about the temperature of the container walls. What is this assumption? What would happen if this assumption were not valid?

Q18.16. The temperature of an ideal gas is directly proportional to the average kinetic energy of its molecules. If a container of ideal gas is moving past you at 2000 m/s, is the temperature of the gas higher than if the container was at rest? Explain your reasoning.

Q18.17. If the pressure of an ideal monatomic gas is increased while the number of moles is kept constant, what happens to the average translational kinetic energy of one atom of the gas? Is it possible to change *both* the volume and the pressure of an ideal gas and keep the average translational kinetic energy of the atoms constant? Explain.

Q18.18. In deriving the ideal-gas equation from the kinetic-molecular model, we ignored potential energy due to the earth's gravity. Is this omission justified? Why or why not?

Q18.19. The derivation of the ideal-gas equation included the assumption that the number of molecules is very large, so that we could compute the average force due to many collisions. However, the ideal-gas equation holds accurately only at low pressures, where the molecules are few and far between. Is this inconsistent? Why or why not?

Q18.20. A gas storage tank has a small leak. The pressure in the tank drops more quickly if the gas is hydrogen or helium than if it is oxygen. Why?

Q18.21. Consider two specimens of ideal gas at the same temperature. Specimen A has the same total mass as specimen B, but the molecules in specimen A have greater molar mass than they do in specimen B. In which specimen is the total kinetic energy of the gas greater? Does your answer depend on the molecular structure of the gases? Why or why not?

Q18.22. The temperature of an ideal monatomic gas is increased from 25°C to 50°C. Does the average translational kinetic energy of each gas atom double? Explain. If your answer is no, what would the final temperature be if the average translational kinetic energy was doubled?

Q18.23. If the root-mean-square speed of the atoms of an ideal gas is to be doubled, by what factor must the Kelvin temperature of the gas be increased? Explain.

Q18.24. (a) If you apply the same amount of heat to 1.00 mol of an ideal monatomic gas and 1.00 mol of an ideal diatomic gas, which one (if any) will increase more in temperature? (b) Physically, *why* do diatomic gases have a greater molar heat capacity than monatomic gases?

Q18.25. The discussion in Section 18.4 concluded that all ideal diatomic gases have the same heat capacity C_V. Does this mean that it takes the same amount of heat to raise the temperature of 1.0 g of each one by 1.0 K? Explain your reasoning.

***Q18.26.** In a gas that contains N molecules, is it accurate to say that the number of molecules with speed v is equal to $f(v)$? Is it accurate to say that this number is given by $Nf(v)$? Explain your answers.

***Q18.27.** Imagine a special air filter placed in a window of a house. The tiny holes in the filter allow only air molecules moving faster than a certain speed to exit the house, and allow only air molecules moving slower than that speed to enter the house from outside. What effect would this filter have on the temperature inside the house? (It turns out that the second law of thermodynamics—which we will discuss in Chapter 20—tells us that such a wonderful air filter would be impossible to make.)

Q18.28. A beaker of water at room temperature is placed in an enclosure, and the air pressure in the enclosure is slowly reduced. When the air pressure is reduced sufficiently, the water begins to boil. The temperature of the water does not rise when it boils; in fact, the temperature *drops* slightly. Explain these phenomena.

Q18.29. Ice is slippery to walk on, and especially slippery if you wear ice skates. What does this tell you about how the melting temperature of ice depends on pressure? Explain.

Q18.30. Hydrothermal vents are openings in the ocean floor that discharge very hot water. The water emerging from one such vent off the Oregon coast, 2400 m below the surface, has a temperature of 279°C. Despite its high temperature, the water doesn't boil. Why not?

Q18.31. The dark areas on the moon's surface are called *maria,* Latin for "seas," and were once thought to be bodies of water. In fact, the maria are not "seas" at all, but plains of solidified lava. Given that there is no atmosphere on the moon, how can you explain the absence of liquid water on the moon's surface?

Q18.32. In addition to the normal cooking directions printed on the back of a box of rice, there are also "high-altitude directions." The only difference is that the "high-altitude directions" suggest increasing the cooking time and using a greater volume of boiling water in which to cook the rice. Why should the directions depend on the altitude in this way?

Exercises

Section 18.1 Equations of State

18.1. A 20.0-L tank contains 0.225 kg of helium at 18.0°C. The molar mass of helium is 4.00 g/mol. (a) How many moles of helium are in the tank? (b) What is the pressure in the tank, in pascals and in atmospheres?

18.2. Helium gas with a volume of 2.60 L, under a pressure of 1.30 atm and at a temperature of 41.0°C, is warmed until both pressure and volume are doubled. (a) What is the final temperature? (b) How many grams of helium are there? The molar mass of helium is 4.00 g/mol.

18.3. A cylindrical tank has a tight-fitting piston that allows the volume of the tank to be changed. The tank originally contains $0.110\ \text{m}^3$ of air at a pressure of 3.40 atm. The piston is slowly pulled out until the volume of the gas is increased to $0.390\ \text{m}^3$. If the temperature remains constant, what is the final value of the pressure?

18.4. A 3.00-L tank contains air at 3.00 atm and 20.0°C. The tank is sealed and cooled until the pressure is 1.00 atm. (a) What is the temperature then in degrees Celsius? Assume that the volume of the tank is constant. (b) If the temperature is kept at the value found in part (a) and the gas is compressed, what is the volume when the pressure again becomes 3.00 atm?

18.5. (a) Use the ideal-gas law to estimate the number of air molecules in your physics lab room, assuming all the air is N_2. (b) Calculate the particle density in the lab (that is, the number of molecules per cubic centimeter).

18.6. You have several identical balloons. You experimentally determine that a balloon will break if its volume exceeds 0.900 L. The pressure of the gas inside the balloon equals air pressure (1.00 atm. (a) If the air inside the balloon is at a constant temperature of 22.0°C and behaves as an ideal gas, what mass of air can you blow into one of the balloons before it bursts? (b) Repeat part (a) if the gas is helium rather than air.

18.7. A Jaguar XK8 convertible has an eight-cylinder engine. At the beginning of its compression stroke, one of the cylinders contains $499\ \text{cm}^3$ of air at atmospheric pressure $(1.01 \times 10^5\ \text{Pa})$ and a temperature of 27.0°C. At the end of the stroke, the air has been compressed to a volume of $46.2\ \text{cm}^3$ and the gauge pressure has increased to 2.72×10^6 Pa. Compute the final temperature.

18.8. A welder using a tank of volume $0.0750\ \text{m}^3$ fills it with oxygen (molar mass 32.0 g/mol) at a gauge pressure of 3.00×10^5 Pa and temperature of 37.0°C. The tank has a small leak, and in time some of the oxygen leaks out. On a day when the temperature is 22.0°C, the gauge pressure of the oxygen in the tank is 1.80×10^5 Pa. Find (a) the initial mass of oxygen and (b) the mass of oxygen that has leaked out.

18.9. A large cylindrical tank contains $0.750\ \text{m}^3$ of nitrogen gas at 27°C and 1.50×10^5 Pa (absolute pressure). The tank has a tight-fitting piston that allows the volume to be changed. What will be the pressure if the volume is decreased to $0.480\ \text{m}^3$ and the temperature is increased to 157°C?

18.10. An empty cylindrical canister 1.50 m long and 90.0 cm in diameter is to be filled with pure oxygen at 22.0°C to store in a space station. To hold as much gas as possible, the absolute pressure of the oxygen will be 21.0 atm. The molar mass of oxygen is 32.0 g/mol. (a) How many moles of oxygen does this canister hold? (b) For someone lifting this canister, by how many kilograms does this gas increase the mass to be lifted?

18.11. The gas inside a balloon will always have a pressure nearly equal to atmospheric pressure, since that is the pressure applied to the outside of the balloon. You fill a balloon with helium (a nearly ideal gas) to a volume of 0.600 L at a temperature of 19.0°C. What is the volume of the balloon if you cool it to the boiling point of liquid nitrogen (77.3 K)?

18.12. Deviations from the Ideal-Gas Equation. For carbon dioxide gas (CO_2), the constants in the van der Waals equation are $a = 0.364\ \text{J} \cdot \text{m}^3/\text{mol}^2$ and $b = 4.27 \times 10^{-5}\ \text{m}^3/\text{mol}$. (a) If 1.00 mol of CO_2 gas at 350 K is confined to a volume of $400\ \text{cm}^3$, find the pressure of the gas using the ideal-gas equation and the van der Waals equation. (b) Which equation gives a lower pressure? Why? What is the percentage difference of the van der Waals equation result from the ideal-gas equation result? (c) The gas is kept at the same temperature as it expands to a volume of $4000\ \text{cm}^3$. Repeat the calculations of parts (a) and (b). (d) Explain how your calculations show that the van der Waals equation is equivalent to the ideal-gas equation if n/V is small.

18.13. The total lung volume for a typical physics student is 6.00 L. A physics student fills her lungs with air at an absolute pressure of 1.00 atm. Then, holding her breath, she compresses her chest cavity, decreasing her lung volume to 5.70 L. What is the pressure of the air in her lungs then? Assume that the temperature of the air remains constant.

18.14. A diver observes a bubble of air rising from the bottom of a lake (where the absolute pressure is 3.50 atm) to the surface (where the pressure is 1.00 atm). The temperature at the bottom is

4.0°C, and the temperature at the surface is 23.0°C. (a) What is the ratio of the volume of the bubble as it reaches the surface to its volume at the bottom? (b) Would it be safe for the diver to hold his breath while ascending from the bottom of the lake to the surface? Why or why not?

18.15. A metal tank with volume 3.10 L will burst if the absolute pressure of the gas it contains exceeds 100 atm. (a) If 11.0 mol of an ideal gas is put into the tank at a temperature of 23.0°C, to what temperature can the gas be warmed before the tank ruptures? You can ignore the thermal expansion of the tank. (b) Based on your answer to part (a), is it reasonable to ignore the thermal expansion of the tank? Explain.

18.16. Three moles of an ideal gas are in a rigid cubical box with sides of length 0.200 m. (a) What is the force that the gas exerts on each of the six sides of the box when the gas temperature is 20.0°C? (b) What is the force when the temperature of the gas is increased to 100.0°C?

18.17. With the assumptions of Example 18.4 (Section 18.1), at what altitude above sea level is air pressure 90% of the pressure at sea level?

18.16. Make the same assumptions as in Example 18.4 (Section 18.1). How does the percentage decrease in air pressure in going from sea level to an altitude of 100 m compare to that when going from sea level to an altitude of 1000 m? If your second answer is not 10 times your first answer, explain why.

18.19. With the assumptions of Example 18.4 (Section 18.1), how does the density of air at sea level compare to the density at an altitude of 100 m above sea level?

18.20. With the assumption that the air temperature is a uniform 0.0°C (as in Example 18.4), what is the density of the air at an altitude of 1.00 km as a percentage of the density at the surface?

18.21. At an altitude of 11,000 m (a typical cruising altitude for a jet airliner), the air temperature is $-56.5°C$ and the air density is $0.364\ kg/m^3$. What is the pressure of the atmosphere at that altitude? (*Note:* The temperature at this altitude is not the same as at the surface of the earth, so the calculation of Example 18.4 in Section 18.1 doesn't apply.)

Section 18.2 Molecular Properties of Matter

18.22. A large organic molecule has a mass of 1.41×10^{-21} kg. What is the molar mass of this compound?

18.23. Suppose you inherit 3.00 mol of gold from your uncle (an eccentric chemist) at a time when this metal is selling for \$14.75 per gram. Consult the periodic table in Appendix D and Table 14.1. (a) To the nearest dollar, what is this gold worth? (b) If you have your gold formed into a spherical nugget, what is its diameter?

18.24. Modern vacuum pumps make it easy to attain pressures of the order of 10^{-13} atm in the laboratory. (a) At a pressure of 9.00×10^{-14} atm and an ordinary temperature of 300.0 K, how many molecules are present in a volume of $1.00\ cm^3$? (b) How many molecules would be present at the same temperature but at 1.00 atm instead?

18.25. The Lagoon Nebula (Fig. 18.28) is a cloud of hydrogen gas located 3900 light-years from the earth. The cloud is about 45 light-years in diameter and glows because of its high temperature of 7500 K. (The gas is raised to this temperature by the stars that lie within the nebula.) The cloud is also very thin; there are only 80 molecules per cubic centimeter. (a) Find the gas pressure (in atmospheres) in the Lagoon Nebula. Compare it to the laboratory pressure referred to in Exercise 18.24. (b) Science fiction films

Figure 18.28 Exercise 18.25.

sometimes show starships being buffeted by turbulence as they fly through gas clouds such as the Lagoon Nebula. Does this seem realistic? Why or why not?

18.26. In a gas at standard conditions, what is the length of the side of a cube that contains a number of molecules equal to the population of the earth (about 6×10^9 people)?

18.27. How many moles are in a 1.00-kg bottle of water? How many molecules? The molar mass of water is 18.0 g/mol.

18.28. How Close Together Are Gas Molecules? Consider an ideal gas at 27°C and 1.00 atm pressure. To get some idea how close these molecules are to each other, on the average, imagine them to be uniformly spaced, with each molecule at the center of a small cube. (a) What is the length of an edge of each cube if adjacent cubes touch but do not overlap? (b) How does this distance compare with the diameter of a typical molecule? (c) How does their separation compare with the spacing of atoms in solids, which typically are about 0.3 nm apart?

18.28. Consider 5.00 mol of liquid water. (a) What volume is occupied by this amount of water? The molar mass of water is 18.0 g/mol. (b) Imagine the molecules to be, on average, uniformly spaced, with each molecule at the center of a small cube. What is the length of an edge of each small cube if adjacent cubes touch but don't overlap? (c) How does this distance compare with the diameter of a molecule?

Section 18.3 Kinetic-Molecular Model of an Ideal Gas

18.30. A flask contains a mixture of neon (Ne), krypton (Kr), and radon (Rn) gases. Compare (a) the average kinetic energies of the three types of atoms and; (b) the root-mean-square speeds. (*Hint:* The periodic table in Appendix D shows the molar mass (in g/mol) of each element under the chemical symbol for that element.)

18.31. Gaseous Diffusion of Uranium. (a) A process called *gaseous diffusion* is often used to separate isotopes of uranium—that is, atoms of the elements that have different masses, such as ^{235}U and ^{238}U. The only gaseous compound of uranium at ordinary temperatures is uranium hexafluoride, UF_6. Speculate on how $^{235}UF_6$ and $^{238}UF_6$ molecules might be separated by diffusion. (b) The molar masses for $^{235}UF_6$ and $^{238}UF_6$ molecules are 0.349 kg/mol and 0.352 kg/mol, respectively. If uranium hexafluoride acts as an ideal gas, what is the ratio of the root-mean-square speed of $^{235}UF_6$ molecules to that of $^{238}UF_6$ molecules if the temperature is uniform?

18.32. The ideas of average and root-mean-square value can be applied to any distribution. A class of 150 students had the following scores on a 100-point quiz:

Score	Number of Students
10	11
20	12
30	24
40	15
50	19
60	10
70	12
80	20
90	17
100	10

(a) Find the average score for the class. (b) Find the root-mean-square score for the class.

18.33. We have two equal-size boxes, A and B. Each box contains gas that behaves as an ideal gas. We insert a thermometer into each box and find that the gas in box A is at a temperature of 50°C while the gas in box B is at 10°C. This is all we know about the gas in the boxes. Which of the following statements *must* be true? Which *could* be true? (a) The pressure in A is higher than in B. (b) There are more molecules in A than in B. (c) A and B cannot contain the same type of gas. (d) The molecules in A have more average kinetic energy per molecule than those in B. (e) The molecules in A are moving faster than those in B. Explain the reasoning behind your answers.

18.34. STP. The conditions of standard temperature and pressure (STP) are a temperature of 0.00°C and a pressure of 1.00 atm. (a) How many liters does 1.00 mol of any ideal gas occupy at STP? (b) For a scientist on Venus, an absolute pressure of 1 Venusian-atmosphere is 92 Earth-atmospheres. Of course she would use the Venusian-atmosphere to define STP. Assuming she kept the same temperature, how many liters would 1 mole of ideal gas occupy on Venus?

18.35. (a) A deuteron, ${}_1^2H$, is the nucleus of a hydrogen isotope and consists of one proton and one neutron. The plasma of deuterons in a nuclear fusion reactor must be heated to about 300 million K. What is the rms speed of the deuterons? Is this a significant fraction of the speed of light $(c = 3.0 \times 10^8\ \text{m/s})$? (b) What would the temperature of the plasma be if the deuterons had an rms speed equal to $0.10c$?

18.36. Martian Climate. The atmosphere of Mars is mostly CO_2 (molar mass 44.0 g/mol) under a pressure of 650 Pa, which we shall assume remains constant. In many places the temperature varies from 0.0°C in summer to −100°C in winter. Over the course of a martian year, what are the ranges of (a) the rms speeds of the CO_2 molecules, and (b) the density (in mol/m^3) of the atmosphere?

18.37. (a) Oxygen (O_2) has a molar mass of 32.0 g/mol. What is the average translational kinetic energy of an oxygen molecule at a temperature of 300 K? (b) What is the average value of the square of its speed? (c) What is the root-mean-square speed? (d) What is the momentum of an oxygen molecule traveling at this speed? (e) Suppose an oxygen molecule traveling at this speed bounces back and forth between opposite sides of a cubical vessel 0.10 m on a side. What is the average force the molecule exerts on one of the walls of the container? (Assume that the molecule's velocity is perpendicular to the two sides that it strikes.) (f) What is the average force per unit area? (g) How many oxygen molecules traveling at this speed are necessary to produce an average pressure of 1 atm? (h) Compute the number of oxygen molecules that are actually contained in a vessel of this size at 300 K and atmospheric pressure. (i) Your answer for part (h) should be three times as large as the answer for part (g). Where does this discrepancy arise?

18.38. Calculate the mean free path of air molecules at a pressure of 3.50×10^{-13} atm and a temperature of 300 K. (This pressure is readily attainable in the laboratory; see Exercise 18.24.) As in Example 18.8, model the air molecules as spheres of radius 2.0×10^{-10} m.

18.39. At what temperature is the root-mean-square speed of nitrogen molecules equal to the root-mean-square speed of hydrogen molecules at 20.0°C? (*Hint:* The periodic table in Appendix D shows the molar mass (in g/mol) of each element under the chemical symbol for that element. The molar mass of H_2 is twice the molar mass of hydrogen atoms, and similarly for N_2.)

18.40. Smoke particles in the air typically have masses of the order of 10^{-16} kg. The Brownian motion (rapid, irregular movement) of these particles, resulting from collisions with air molecules, can be observed with a microscope. (a) Find the root-mean-square speed of Brownian motion for a particle with a mass of 3.00×10^{-16} kg in air at 300 K. (b) Would the root-mean-square speed be different if the particle were in hydrogen gas at the same temperature? Explain.

Section 18.4 Heat Capacities

18.41. (a) How much heat does it take to increase the temperature of 2.50 mol of a diatomic ideal gas by 30.0 K near room temperature if the gas is held at constant volume? (b) What is the answer to the question in part (a) if the gas is monatomic rather than diatomic?

18.42. Perfectly rigid containers each hold n moles of ideal gas, one being hydrogen (H_2) and other being neon (Ne). If it takes 100 J of heat to increase the temperature of the hydrogen by 2.50°C, by how many degrees will the same amount of heat raise the temperature of the neon?

18.43. (a) Compute the specific heat capacity at constant volume of nitrogen (N_2) gas, and compare with the specific heat capacity of liquid water. The molar mass of N_2 is 28.0 g/mol. (b) You warm 1.00 kg of water at a constant volume of 1.00 L from 20.0°C to 30.0°C in a kettle. For the same amount of heat, how many kilograms of 20.0°C air would you be able to warm to 30.0°C? What volume (in liters) would this air occupy at 20.0°C and a pressure of 1.00 atm? Make the simplifying assumption that air is 100% N_2.

18.44. (a) Calculate the specific heat capacity at constant volume of water vapor, assuming the nonlinear triatomic molecule has three translational and three rotational degrees of freedom and that vibrational motion does not contribute. The molar mass of water is 18.0 g/mol. (b) The actual specific heat capacity of water vapor at low pressures is about 2000 J/kg · K. Compare this with your calculation and comment on the actual role of vibrational motion.

18.45. (a) Use Eq. 18.28 to calculate the heat capacity at constant volume of aluminum in units of J/kg · K. Consult the periodic table in Appendix D. (b) Compare the answer in part (a) with the value given in Table 17.3. Try to explain any disagreement between these two values.

*Section 18.5 Molecular Speeds

***18.46.** For a gas of nitrogen molecules (N_2), what must the temperature be if 94.7% of all the molecules have speeds less than (a) 1500 m/s; (b) 1000 m/s; (c) 500 m/s? Use Table 18.2. The molar mass of N_2 is 28.0 g/mol.

***18.47.** Derive Eq. (18.33) from Eq. (18.32).

***18.48.** Prove that $f(v)$ as given by Eq. (18.33) is maximum for $\epsilon = kT$. Use this result to obtain Eq. (18.34).

***18.48.** For diatomic carbon dioxide gas (CO_2, molar mass 44.0 g/mol) at $T = 300$ K, calculate (a) the most probable speed v_{mp}; (b) the average speed v_{av}; (c) the root-mean-square speed v_{rms}.

Section 18.6 Phases of Matter

18.50. Puffy cumulus clouds, which are made of water droplets, occur at lower altitudes in the atmosphere. Wispy cirrus clouds, which are made of ice crystals, occur only at higher altitudes. Find the altitude y (measured from sea level) above which only cirrus clouds can occur. On a typical day and at altitudes less than 11 km, the temperature at an altitude y is given by $T = T_0 - \alpha y$, where $T_0 = 15.0°C$ and $\alpha = 6.0\ C°/1000$ m.

18.51. Solid water (ice) is slowly warmed from a very low temperature. (a) What minimum external pressure p_1 must be applied to the solid if a melting phase transition is to be observed? Describe the sequence of phase transitions that occur if the applied pressure p is such that $p < p_1$. (b) Above a certain maximum pressure p_2, no boiling transition is observed. What is this pressure? Describe the sequence of phase transitions that occur if $p_1 < p < p_2$.

18.52. A physicist places a piece of ice at 0.00°C and a beaker of water at 0.00°C inside a glass box and closes the lid of the box. All the air is then removed from the box. If the ice, water, and beaker are all maintained at a temperature of 0.00°C, describe the final equilibrium state inside the box.

18.53. The atmosphere of the planet Mars is 95.3% carbon dioxide (CO_2) and about 0.03% water vapor. The atmospheric pressure is only about 600 Pa, and the surface temperature varies from −30°C to −100°C. The polar ice caps contain both CO_2 ice and water ice. Could there be *liquid* CO_2 on the surface of Mars? Could there be liquid water? Why or why not?

Problems

18.54. (a) Use Eq. (18.1) to estimate the change in the volume of a solid steel sphere of volume 11 L when the temperature and pressure increase from 21°C and 1.013×10^5 Pa to 42°C and 2.10×10^7 Pa. (*Hint:* Consult Chapters 11 and 17 to determine the values of β and k.) (b) In Example 18.3 the change in volume of an 11-L steel scuba tank was ignored. Was this a good approximation? Explain.

18.55. A cylinder 1.00 m tall with inside diameter 0.120 m is used to hold propane gas (molar mass 44.1 g/mol) for use in a barbecue. It is initially filled with gas until the gauge pressure is 1.30×10^6 Pa and the temperature is 22.0°C. The temperature of the gas remains constant as it is partially emptied out of the tank, until the gauge pressure is 2.50×10^5 Pa. Calculate the mass of propane that has been used.

18.56. During a test dive in 1939, prior to being accepted by the U.S. Navy, the submarine *Squalus* sank at a point where the depth of water was 73.0 m. The temperature at the surface was 27.0°C, and at the bottom it was 7.0°C. The density of seawater is 1030 kg/m³. (a) A diving bell was used to rescue 33 trapped crewmen from the *Squalus.* The diving bell was in the form of a circular cylinder 2.30 m high, open at the bottom and closed at the top. When the diving bell was lowered to the bottom of the sea, to what height did water rise within the diving bell? (*Hint:* You may ignore the relatively small variation in water pressure between the bottom of the bell and the surface of the water within the bell.) (b) At what gauge pressure must compressed air have been supplied to the bell while on the bottom to expel all the water from it?

18.57. Atmosphere of Titan. Titan, the largest satellite of Saturn, has a thick nitrogen atmosphere. At its surface, the pressure is 1.5 Earth-atmospheres and the temperature is 94 K. (a) What is the surface temperature in °C? (b) Calculate the surface density in Titan's atmosphere in molecules per cubic meter. (c) Compare the density of Titan's surface atmosphere to the density of Earth's atmosphere at 22°C. Which body has denser atmosphere?

18.58. Pressure on Venus. At the surface of Venus the average temperature is a balmy 460°C due to the greenhouse effect (global warming!), the pressure is 92 Earth-atmospheres, and the acceleration due to gravity is $0.894g_{Earth}$. The atmosphere is nearly all CO_2 (molar mass 44.0 g/mol) and the temperature remains remarkably constant. We shall assume that the temperature does not change at all with altitude. (a) What is the atmospheric pressure 1.00 km above the surface of Venus? Express your answer in Venus-atmospheres and Earth-atmospheres. (b) What is the root-mean-square speed of the CO_2 molecules at the surface of Venus and at an altitude of 1.00 km?

18.58. An automobile tire has a volume of 0.0150 m³ on a cold day when the temperature of the air in the tire is 5.0°C and atmospheric pressure is 1.02 atm. Under these conditions the gauge pressure is measured to be 1.70 atm (about 25 lb/in.²). After the car is driven on the highway for 30 min, the temperature of the air in the tires has risen to 45.0°C and the volume has risen to 0.0159 m³. What then is the gauge pressure?

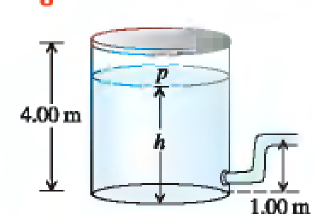
18.60. A flask with a volume of 1.50 L, provided with a stopcock, contains ethane gas (C_2H_6) at 300 K and atmospheric pressure (1.013×10^5 Pa). The molar mass of ethane is 30.1 g/mol. The system is warmed to a temperature of 380 K, with the stopcock open to the atmosphere. The stopcock is then closed, and the flask is cooled to its original temperature. (a) What is the final pressure of the ethane in the flask? (b) How many grams of ethane remain in the flask?

18.61. A balloon whose volume is 750 m³ is to be filled with hydrogen at atmospheric pressure (1.01×10^5 Pa). (a) If the hydrogen is stored in cylinders with volumes of 1.90 m³ at a gauge pressure of 1.20×10^6 Pa, how many cylinders are required? Assume that the temperature of the hydrogen remains constant. (b) What is the total weight (in addition to the weight of the gas) that can be supported by the balloon if the gas in the balloon and the surrounding air are both at 15.0°C? The molar mass of hydrogen (H_2) is 2.02 g/mol. The density of air at 15.0°C and atmospheric pressure is 1.23 kg/m³. See Chapter 14 for a discussion of buoyancy. (c) What weight could be supported if the balloon were filled with helium (molar mass 4.00 g/mol) instead of hydrogen, again at 15.0°C?

18.62. A vertical cylindrical tank contains 1.80 mol of an ideal gas under a pressure of 1.00 atm at 20.0°C. The round part of the tank has a radius of 10.0 cm, and the gas is supporting a piston that can move up and down in the cylinder without friction. (a) What is the mass of this piston? (b) How tall is the column of gas that is supporting the piston?

18.63. A large tank of water has a hose connected to it, as shown in Fig. 18.29. The tank is sealed at the top and has compressed air between the water surface and the top. When the water height h has the value 3.50 m, the absolute pressure p of the compressed air

Figure 18.29 Problem 18.63.



is 4.20×10^5 Pa. Assume that the air above the water expands at constant temperature, and take the atmospheric pressure to be 1.00×10^5 Pa. (a) What is the speed with which water flows out of the hose when $h = 3.50$ m? (b) As water flows out of the tank, h decreases. Calculate the speed of flow for $h = 3.00$ m and for $h = 2.00$ m. (c) At what value of h does the flow stop?

18.64. A person at rest inhales 0.50 L of air with each breath at a pressure of 1.00 atm and a temperature of 20.0°C. The inhaled air is 21.0% oxygen. (a) How many oxygen molecules does this person inhale with each breath? (b) Suppose this person is now resting at an elevation of 2,000 m but the temperature is still 20.0°C. Assuming that the oxygen percentage and volume per inhalation are the same as stated above, how many oxygen molecules does this person now inhale with each breath? (c) Given that the body still requires the same number of oxygen molecules per second as at sea level to maintain its functions, explain why some people report "shortness of breath" at high elevations.

18.65. How Many Atoms Are You? Estimate the number of atoms in the body of a 50-kg physics student. Note that the human body is mostly water, which has molar mass 18.0 g/mol, and that each water molecule contains three atoms.

18.66. The size of an oxygen molecule is about 2.0×10^{-10} m. Make a rough estimate of the pressure at which the finite volume of the molecules should cause noticeable deviations from ideal-gas behavior at ordinary temperatures ($T = 300$ K).

18.67. You have two identical containers, one containing gas A and the other gas B. The masses of these molecules are $m_A = 3.34 \times 10^{-27}$ kg and $m_B = 5.34 \times 10^{-26}$ kg. Both gases are under the same pressure and are at 10.0°C. (a) Which molecules (A or B) have greater translational kinetic energy per molecule and rms speeds? Now you want to raise the temperature of only one of these containers so that both gases will have the same rms speed. (b) For which gas should you raise the temperature? (c) At what temperature will you accomplish your goal? (d) Once you have accomplished your goal, which molecules (A or B) now have greater average translational kinetic energy per molecule?

18.68. Insect Collisions. A cubical cage 1.25 m on each side contains 2500 angry bees, each flying randomly at 1.10 m/s. We can model these insects as spheres 1.50 cm in diameter. On the average, (a) how far does a typical bee travel between collisions, (b) what is the average time between collisions, and (c) how many collisions per second does a bee make?

18.69. Successive Approximations and the van der Waals Equation. In the ideal-gas equation, the number of moles per volume n/V is simply equal to p/RT. In the van der Waals equation, solving for n/V in terms of the pressure p and temperature T is somewhat more involved. (a) Show that the van der Waals equation can be written as

$$\frac{n}{V} = \left(\frac{p + an^2/V^2}{RT}\right)\left(1 - \frac{bn}{V}\right)$$

(b) The van der Waals parameters for hydrogen sulfide gas (H_2S) are $a = 0.448\ \text{J} \cdot \text{m}^3/\text{mol}^2$ and $b = 4.29 \times 10^{-5}\ \text{m}^3/\text{mol}$. Determine the number of moles per volume of H_2S gas at 127°C and an absolute pressure of 9.80×10^5 Pa as follows: (i) Calculate a first approximation using the ideal-gas equation, $n/V = p/RT$. (ii) Substitute this approximation for n/V into the right-hand side of the equation in part (a). The result is a new, improved approximation for n/V. (iii) Substitute the new approximation for n/V into the right-hand side of the equation in (a). The result is a further improved approximation for n/V. (iv) Repeat step (iii) until successive approximations agree to the desired level of accuracy (in this case, to three significant figures). (c) Compare your final result in part (b) to the result p/RT obtained using the ideal-gas equation. Which result gives a larger value of n/V? Why?

18.70. Gas on Europa. A canister of 1.20 mol of nitrogen gas (28.0 g/mol) at 25.0°C is left on Jupiter's satellite after completion of a future space mission. Europa has no appreciable atmosphere, and the acceleration due to gravity at its surface is 1.30 m/s^2. After some time, the canister springs a small leak, allowing molecules to escape through a small hole. What is the maximum height (in km) above Europa's surface that a N_2 molecule having speed equal to the rms speed will reach if it is shot straight up out of the hole in the canister? Ignore the variation in g with altitude.

18.71. You blow up a spherical balloon to a diameter of 50.0 cm until the absolute pressure inside is 1.25 atm and the temperature is 22.0°C. Assume that all the gas in N_2 is of molar mass 28.0 g/mol. (a) Find the mass of a single N_2 molecule. (b) How much translational kinetic energy does an average N_2 molecule have? (c) How many N_2 molecules are in this balloon? (d) What is the *total* translational kinetic energy of all the molecules in the balloon?

18.72. (a) Compute the increase in gravitational potential energy for a nitrogen molecule (molar mass 28.0 g/mol) for an increase in elevation of 400 m near the earth's surface. (b) At what temperature is this equal to the average kinetic energy of a nitrogen molecule? (c) Is it possible that a nitrogen molecule near sea level where $T = 15.0$°C could rise to an altitude of 400 m? Is it likely that it could do so without hitting any other molecules along the way? Explain.

18.73. The Lennard-Jones Potential. A commonly used potential-energy function for the interaction of two molecules (see Fig. 18.8) is the Lennard-Jones 6-12 potential

$$U(r) = U_0\left[\left(\frac{R_0}{r}\right)^{12} - 2\left(\frac{R_0}{r}\right)^{6}\right]$$

where r is the distance between the centers of the molecules and U_0 and R_0 are positive constants. The corresponding force $F(r)$ is given in Eq. (13.26). (a) Graph $U(r)$ and $F(r)$ versus r. (b) Let r_1 be the value of r at which $U(r) = 0$, and let r_2 be the value of r at which $F(r) = 0$. Show the locations of r_1 and r_2 on your graphs of $U(r)$ and $F(r)$. Which of these values represents the equilibrium separation between the molecules? (c) Find the values of r_1 and r_2 in terms of R_0, and find the ratio r_1/r_2. (d) If the molecules are located a distance r_2 apart [as calculated in part (c)], how much work must be done to pull them apart so that $r \to \infty$?

18.74. (a) What is the total random translational kinetic energy of 5.00 L of hydrogen gas (molar mass 2.016 g/mol) with pressure 1.01×10^5 Pa and temperature 300 K? (*Hint:* Use the procedure of Problem 18.71 as a guide.) (b) If the tank containing the gas is placed on a swift jet moving at 300.0 m/s, by what percentage is the *total* kinetic energy of the gas increased? (c) Since the kinetic energy of the gas molecules is greater when it is on the jet, does this mean that its temperature has gone up? Explain.

18.75. The speed of propagation of a sound wave in air at 27°C is about 350 m/s. Calculate, for comparison, (a) v_{rms} for nitrogen molecules and (b) the rms value of v_x at this temperature. The molar mass of nitrogen (N_2) is 28.0 g/mol.

18.76. Hydrogen on the Sun. The surface of the sun has a temperature of about 5800 K and consists largely of hydrogen atoms. (a) Find the rms speed of a hydrogen atom at this temperature. (The mass of a single hydrogen atom is 1.67×10^{-27} kg.) (b) The escape speed for a particle to leave the gravitational influence of the sun is given by $(2GM/R)^{1/2}$, where M is the sun's mass, R its radius, and G the gravitational constant (see Example 12.5 of Section 12.3). Use the data in Appendix F to calculate this escape speed. (c) Can appreciable quantities of hydrogen escape from the sun? Can *any* hydrogen escape? Explain.

18.77. (a) Show that a projectile with mass m can "escape" from the surface of a planet if it is launched vertically upward with a kinetic energy greater than mgR_p, where g is the acceleration due to gravity at the planet's surface and R_p is the planet's radius. Ignore air resistance. (See Problem 18.76.) (b) If the planet in question is the earth, at what temperature does the average translational kinetic energy of a nitrogen molecule (molar mass 28.0 g/mol) equal that required to escape? What about a hydrogen molecule (molar mass 2.02 g/mol)? (c) Repeat part (b) for the moon, for which $g = 1.63\ \text{m/s}^2$ and $R_p = 1740$ km. (d) While the earth and the moon have similar average surface temperatures, the moon has essentially no atmosphere. Use your results from parts (b) and (c) to explain why.

18.78. Planetary Atmospheres. (a) The temperature near the top of Jupiter's multicolored cloud layer is about 140 K. The temperature at the top of the earth's troposphere, at an altitude of about 20 km, is about 220 K. Calculate the rms speed of hydrogen molecules in both these environments. Give your answers in m/s and as a fraction of the escape speed from the respective planet (see Problem 18.76). (b) Hydrogen gas (H_2) is a rare element in the earth's atmosphere. In the atmosphere of Jupiter, by contrast, 89% of all molecules are H_2. Explain why, using your results from part (a). (c) Suppose an astronomer claims to have discovered an oxygen (O_2) atmosphere on the asteroid Ceres. How likely is this? Ceres has a mass equal to 0.014 times the mass of the moon, a density of 2400 kg/m^3, and a surface temperature of about 200 K.

18.78. (a) For what mass of molecule or particle is v_{rms} equal to 1.00 mm/s at 300 K? (b) If the particle is an ice crystal, how many molecules does it contain? The molar mass of water is 18.0 g/mol. (c) Calculate the diameter of the particle if it is a spherical piece of ice. Would it be visible to the naked eye?

18.60. In describing the heat capacities of solids in Section 18.4, we stated that the potential energy $U = \frac{1}{2}kx^2$ of a harmonic oscillator averaged over one period of the motion is equal to the kinetic energy $K = \frac{1}{2}mv^2$ averaged over one period. Prove this result using Eqs. (13.13) and (13.15) for the position and velocity of a simple harmonic oscillator. For simplicity, assume that the initial position and velocity make the phase angle ϕ equal to zero. (*Hint:* Use the trigonometric identities $\cos^2(\theta) = [1 + \cos(2\theta)]/2$ and $\sin^2(\theta) = [1 - \cos(2\theta)]/2$. What is the average value of $\cos(2\omega t)$ over one period?)

18.81. It is possible to make crystalline solids that are only one layer of atoms thick. Such "two-dimensional" crystals can be created by depositing atoms on a very flat surface. (a) If the atoms in such a two-dimensional crystal can move only within the plane of the crystal, what will be its molar heat capacity near room temperature? Give your answer as a multiple of R and in J/mol · K. (b) At very low temperatures, will the molar heat capacity of a two-dimensional crystal be greater than, less than, or equal to the result you found in part (a)? Explain why.

18.62. (a) Calculate the total *rotational* kinetic energy of the molecules in 1.00 mol of a diatomic gas at 300 K. (b) Calculate the moment of inertia of an oxygen molecule (O_2) for rotation about either the y- or z-axis shown in Fig. 18.18. Treat the molecule as two massive points (representing the oxygen atoms) separated by a distance of 1.21×10^{-10} m. The molar mass of oxygen *atoms* is 16.0 g/mol. (c) Find the rms angular velocity of rotation of an oxygen molecule about either the y- or z-axis shown in Fig. 18.15. How does your answer compare to the angular velocity of a typical piece of rapidly rotating machinery (10,000 rev/min)?

18.63. For each polyatomic gas in Table 18.1, compute the value of the molar heat capacity at constant volume, C_V, on the assumption that there is no vibrational energy. Compare with the measured values in the table, and compute the fraction of the total heat capacity that is due to vibration for each of the three gases. (*Note:* CO_2 is linear; SO_2 and H_2S are not. Recall that a linear polyatomic molecule has two rotational degrees of freedom, and a nonlinear molecule has three.)

***18.64.** (a) Show that $\int_0^\infty f(v)\,dv = 1$, where $f(v)$ is the Maxwell–Boltzmann distribution of Eq. (18.32). (b) In terms of the physical definition of $f(v)$, explain why the integral in part (a) *must* have this value.

***18.65.** Calculate the integral in Eq. (18.31), $\int_0^\infty v^2 f(v)\,dv$, and compare this result to $(v^2)_{av}$ as given by Eq. (18.16). (*Hint:* You may use the tabulated integral

$$\int_0^\infty x^{2n} e^{-\alpha x^2}\,dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1}\alpha^n}\sqrt{\frac{\pi}{\alpha}}$$

where n is a positive integer and α is a positive constant.)

***18.66.** Calculate the integral in Eq. (18.30), $\int_0^\infty v f(v)\,dv$, and compare this result to v_{av} as given by Eq. (18.35). (*Hint:* Make the change of variable $v^2 = x$ and use the tabulated integral

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

where n is a positive integer and α is a positive constant.)

***18.67.** (a) Explain why in a gas of N molecules, the number of molecules having speeds in the *finite* interval v to $v + \Delta v$ is $\Delta N = N\int_v^{v+\Delta v} f(v)\,dv$. (b) If Δv is small, then $f(v)$ is approximately constant over the interval and $\Delta N \approx Nf(v)\,\Delta v$. For oxygen gas ($O_2$, molar mass 32.0 g/mol) at $T = 300$ K, use this approximation to calculate the number of molecules with speeds within $\Delta v = 20$ m/s of v_{mp}. Express your answer as a multiple of N. (c) Repeat part (b) for speeds within $\Delta v = 20$ m/s of $7v_{mp}$. (d) Repeat parts (b) and (c) for a temperature of 600 K. (e) Repeat parts (b) and (c) for a temperature of 150 K. (f) What do your results tell you about the shape of the distribution as a function of temperature? Do your conclusions agree with what is shown in Fig. 18.26?

18.60. Meteorology. The *vapor pressure* is the pressure of the vapor phase of a substance when it is in equilibrium with the solid or liquid phase of the substance. The *relative humidity* is the partial pressure of water vapor in the air divided by the vapor pressure of water at that same temperature, expressed as a percentage. The air is saturated when the humidity is 100%. (a) The vapor pressure of water at 20.0°C is 2.34×10^3 Pa. If the air temperature is 20.0°C and the relative humidity is 60%, what is the partial pressure of water vapor in the atmosphere (that is, the pressure due to water vapor alone)? (b) Under the conditions of part (a), what is the mass of water in 1.00 m^3 of air? (The molar mass of water is 18.0 g/mol. Assume that water vapor can be treated as an ideal gas.)

18.89. The Dew Point. The vapor pressure of water (see Problem 18.88) decreases as the temperature decreases. If the amount of water vapor in the air is kept constant as the air is cooled, a temperature is reached, called the *dew point,* at which the partial pressure and vapor pressure coincide and the vapor is saturated. If the air is cooled further, vapor condenses to liquid until the partial pressure again equals the vapor pressure at that temperature. The temperature in a room is 30.0°C. A meteorologist cools a metal can by gradually adding cold water. When the can temperature reaches 16.0°C, water droplets form on its outside surface. What is the relative humidity of the 30.0°C air in the room? The table lists the vapor pressure of water at various temperatures:

Temperature (°C)	Vapor Pressure (Pa)
10.0	1.23×10^3
12.0	1.40×10^3
14.0	1.60×10^3
16.0	1.81×10^3
18.0	2.06×10^3
20.0	2.34×10^3
22.0	2.65×10^3
24.0	2.99×10^3
26.0	3.36×10^3
28.0	3.78×10^3
30.0	4.25×10^3

18.90. Altitude at Which Clouds Form. On a spring day in the midwestern United States, the air temperature at the surface is 28.0°C. Puffy cumulus clouds form at an altitude where the air temperature equals the dew point (see Problem 18.89). If the air temperature decreases with altitude at a rate of 0.6 C°/100 m, at approximately what height above the ground will clouds form if the relative humidity at the surface is 35% and 80%? (*Hint:* Use the table in Problem 18.89.)

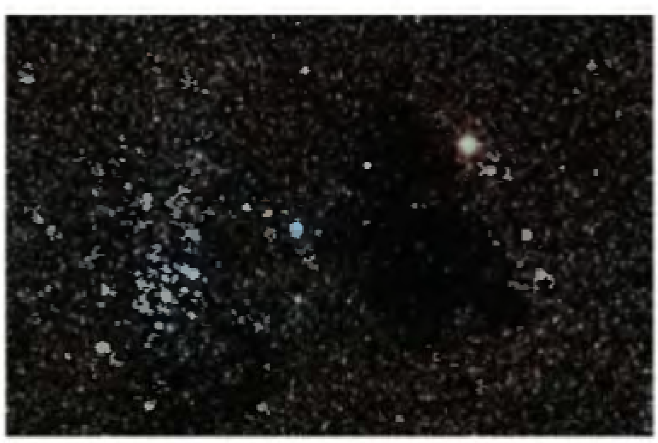
Challenge Problems

18.91. Dark Nebulae and the Interstellar Medium. The dark area in Fig. 18.30 that appears devoid of stars is a *dark nebula,* a cold gas cloud in interstellar space that contains enough material to block out light from the stars behind it. A typical dark nebula is about 20 light-years in diameter and contains about 50 hydrogen atoms per cubic centimeter (monatomic hydrogen, *not* H_2) at a temperature of about 20 K. (A light-year is the distance light travels in vacuum in one year and is equal to 9.46×10^{15} m.) (a) Estimate the mean free path for a hydrogen atom in a dark nebula. The radius of a hydrogen atom is 5.0×10^{-11} m. (b) Estimate the rms speed of a hydrogen atom and the mean free time (the average time between collisions for a given atom). Based on this result, do you think that atomic collisions, such as those leading to H_2 molecule formation, are very important in determining the composition of the nebula? (c) Estimate the pressure inside a dark nebula. (d) Compare the rms speed of a hydrogen atom to the escape speed at the surface of the nebula (assumed spherical). If the space around the nebula were a vacuum, would such a cloud be stable or would it tend to evaporate? (e) The stability of dark nebulae is explained by the presence of the *interstellar medium* (ISM), an even thinner gas that permeates space and in which the dark nebulae are embedded. Show that for dark nebulae to be in equilibrium with the ISM, the numbers of atoms per volume (N/V) and the temperatures (T) of dark nebulae and the ISM must be related by

$$\frac{(N/V)_{\text{nebula}}}{(N/V)_{\text{ISM}}} = \frac{T_{\text{ISM}}}{T_{\text{nebula}}}$$

(f) In the vicinity of the sun, the ISM contains about 1 hydrogen atom per 200 cm^3. Estimate the temperature of the ISM in the vicinity of the sun. Compare to the temperature of the sun's surface, about 5800 K. Would a spacecraft coasting through interstellar space burn up? Why or why not?

Figure 18.30 Challenge Problem 18.91.



18.92. Earth's Atmosphere. In the *troposphere,* the part of the atmosphere that extends from earth's surface to an altitude of about 11 km, the temperature is not uniform but decreases with increasing elevation. (a) Show that if the temperature variation is approximated by the linear relationship

$$T = T_0 - \alpha y$$

where T_0 is the temperature at the earth's surface and T is the temperature at height y, the pressure p at height y is given by

$$\ln\left(\frac{p}{p_0}\right) = \frac{Mg}{R\alpha}\ln\left(\frac{T_0 - \alpha y}{T_0}\right)$$

where p_0 is the pressure at the earth's surface and M is the molar mass for air. The coefficient α is called the lapse rate of temperature. It varies with atmospheric conditions, but an average value is about 0.6 C°/100 m. (b) Show that the above result reduces to the result of Example 18.4 (Section 18.1) in the limit that $\alpha \to 0$. (c) With $\alpha = 0.6$ C°/100 m, calculate p for $y = 8863$ m and compare your answer to the result of Example 18.4. Take $T_0 = 288$ K and $p_0 = 1.00$ atm.

18.93. Van der Waals Equation and Critical Points. (a) In pV-diagrams the slope $\partial p/\partial V$ along an isotherm is never positive. Explain why. (b) Regions where $\partial p/\partial V = 0$ represent equilibrium between two phases; volume can change with no change in pressure, as when water boils at atmospheric pressure. We can use this to determine the temperature, pressure, and volume per mole at the critical point using the equation of state $p = p(V, T, n)$. If $T > T_c$, then $p(V)$ has no maximum along an isotherm, but if $T < T_c$, then $p(V)$ has a maximum. Explain how this leads to the following condition for determining the critical point:

$$\frac{\partial p}{\partial V} = 0 \quad \text{and} \quad \frac{\partial^2 p}{\partial V^2} = 0 \quad \text{at the critical point}$$

(c) Solve the van der Waals equation (Eq. 18.7) for p; that is, find $p(V, T, n)$. Find $\partial p/\partial V$ and $\partial^2 p/\partial V^2$. Set these equal to zero to obtain two equations for V, T, and n. (d) Simultaneous solution of the two equations obtained in part (c) gives the temperature and volume per mole at the critical point, T_c and $(V/n)_c$. Find these constants in terms of a and b. (*Hint:* Divide one equation by the other to eliminate T.) (e) Substitute these values into the equation of state to find p_c, the pressure at the critical point. (f) Use the results from parts (d) and (e) to find the ratio $RT_c/p_c(V/n)_c$. This should not contain either a or b and so should have the same value for all gases. (g) Compute the ratio $RT_c/p_c(V/n)_c$ for the gases H_2, N_2, and H_2O using the critical point data given in the table.

Gas	$T_c(\mathrm{K})$	$p_c(\mathrm{Pa})$	$(V/n)_c(\mathrm{m^3/mol})$
H_2	33.3	13.0×10^5	65.0×10^{-6}
N_2	126.2	33.9×10^5	90.1×10^{-6}
H_2O	647.4	221.2×10^5	56.0×10^{-6}

(h) Discuss how well the results of part (g) compare to the prediction of part (f) based on the van der Waals equation. What do you conclude about the accuracy of the van der Waals equation as a description of the behavior of gases near the critical point?

18.94. *In Example 18.7 (Section 18.3) we saw that $v_{rms} > v_{av}$. It is not difficult to show that this is *always* the case. (The only exception is when the particles have the same speed, in which case $v_{rms} = v_{av}$.) (a) For two particles with speeds v_1 and v_2, show that $v_{rms} \geq v_{av}$, regardless of the numerical values of v_1 and v_2. Then show that $v_{rms} > v_{av}$ if $v_1 \neq v_2$. (b) Suppose that for a collection of N particles you know that $v_{rms} > v_{av}$. Another particle, with speed u, is added to the collection of particles. If the new rms and average speeds are denoted as v'_{rms} and v'_{av}, show that

$$v'_{rms} = \sqrt{\frac{Nv_{rms}^2 + u^2}{N+1}} \qquad \text{and} \qquad v'_{av} = \frac{Nv_{av} + u}{N+1}$$

(c) Use the expressions in part (b) to show that $v'_{rms} > v'_{av}$ regardless of the numerical value of u. (d) Explain why your results for (a) and (c) together show that $v_{rms} > v_{av}$ for any collection of particles if the particles do not all have the same speed.

19 THE FIRST LAW OF THERMODYNAMICS

LEARNING GOALS

By studying this chapter, you will learn:

- How to represent heat transfer and work done in a thermodynamic process.
- How to calculate the work done by a thermodynamic system when its volume changes.
- What is meant by a path between thermodynamic states.
- How to use the first law of thermodynamics to relate heat transfer, work done, and internal energy change.
- How to distinguish among adiabatic, isochoric, isobaric, and isothermal processes.
- How we know that the internal energy of an ideal gas depends only on its temperature.
- The difference between molar heat capacities at constant volume and at constant pressure, and how to use these quantities in calculations.
- How to analyze adiabatic processes in an ideal gas.

? A steam locomotive operates using the first law of thermodynamics: Water is heated and boils, and the expanding steam does work to propel the locomotive. Would it be possible for the steam to propel the locomotive by doing work as it *condenses?*

Every time you drive a car, turn on an air conditioner, or cook a meal, you reap the practical benefits of *thermodynamics,* the study of relationships involving heat, mechanical work, and other aspects of energy and energy transfer. For example, in a car engine heat is generated by the chemical reaction of oxygen and vaporized gasoline in the engine's cylinders. The heated gas pushes on the pistons within the cylinders, doing mechanical work that is used to propel the car. This is an example of a *thermodynamic process.*

The first law of thermodynamics, central to the understanding of such processes, is an extension of the principle of conservation of energy. It broadens this principle to include energy exchange by both heat transfer and mechanical work and introduces the concept of the *internal energy* of a system. Conservation of energy plays a vital role in every area of physical science, and the first law has extremely broad usefulness. To state energy relationships precisely, we need the concept of a *thermodynamic system.* We'll discuss *heat* and *work* as two means of transferring energy into or out of such a system.

19.1 The popcorn in the pot is a thermodynamic system. In the thermodynamic process shown here, heat is added to the system, and the system does work on its surroundings to lift the lid of the pot.

19.1 Thermodynamic Systems

We have studied energy transfer through mechanical work (Chapter 6) and through heat transfer (Chapters 17 and 18). Now we are ready to combine and generalize these principles.

We always talk about energy transfer to or from some specific *system.* The system might be a mechanical device, a biological organism, or a specified quantity of material, such as the refrigerant in an air conditioner or steam expanding in a turbine. In general, a **thermodynamic system** is any collection of objects that is convenient to regard as a unit, and that may have the potential to exchange energy with its surroundings. A familiar example is a quantity of popcorn kernels in a pot with a lid. When the pot is placed on a stove, energy is added to the popcorn by conduction of heat. As the popcorn pops and expands, it does work as it exerts an upward force on the lid and moves it through a displacement (Fig. 19.1). The *state* of the popcorn changes in this process, since the volume, temperature, and pressure of the popcorn all change as it pops. A process such as this one, in which there are changes in the state of a thermodynamic system, is called a **thermodynamic process.**

In mechanics we used the concept of *system* on a regular basis in connection with free-body diagrams and conservation of energy and momentum. With *thermodynamic* systems, as with all others, it is essential to define clearly at the start exactly what is and is not included in the system. Only then can we describe unambiguously the energy transfers into and out of that system. For instance, in our popcorn example we defined the system to include the popcorn but not the pot, lid, or stove.

Thermodynamics has its roots in many practical problems other than popping popcorn (Fig. 19.2). The gasoline engine in an automobile, the jet engines in an airplane, and the rocket engines in a launch vehicle use the heat of combustion of their fuel to perform mechanical work in propelling the vehicle. Muscle tissue in living organisms metabolizes chemical energy in food and performs mechanical work on the organism's surroundings. A steam engine or steam turbine uses the heat of combustion of coal or other fuel to perform mechanical work such as driving an electric generator or pulling a train.

19.2 (a) A rocket engine uses the heat of combustion of its fuel to do work propelling the launch vehicle. (b) Humans and other biological organisms are more complicated systems than we can analyze fully in this book, but the same basic principles of thermodynamics apply to them.

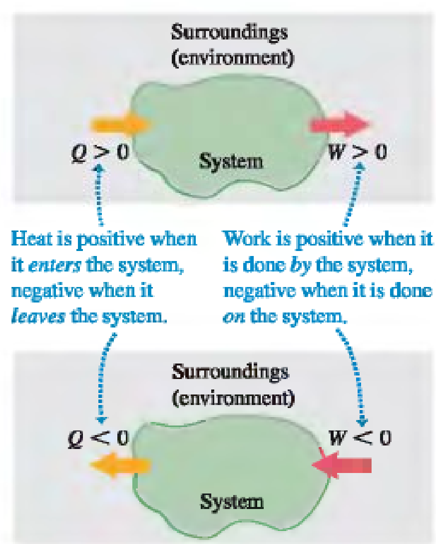
(a) (b)

Signs for Heat and Work in Thermodynamics

We describe the energy relationships in any thermodynamic process in terms of the quantity of heat Q added *to* the system and the work W done *by* the system. Both Q and W may be positive, negative, or zero (Fig. 19.3). A positive value of Q represents heat flow *into* the system, with a corresponding input of energy to it; negative Q represents heat flow *out of* the system. A positive value of W represents work done *by* the system against its surroundings, such as work done by an expanding gas, and hence corresponds to energy *leaving* the system. Negative W, such as work done during compression of a gas in which work is done *on the gas* by its surroundings, represents energy *entering* the system. We will use these conventions consistently in the examples in this chapter and the next.

CAUTION **Be careful with the sign of work W** Note that our sign rule for work is *opposite* to the one we used in mechanics, in which we always spoke of the work done by the forces acting *on* a body. In thermodynamics it is usually more convenient to call W the work done *by* the system so that when a system expands, the pressure, volume change, and work are all positive. Take care to use the sign rules for work and heat consistently!

19.3 A thermodynamic system may exchange energy with its surroundings (environment) by means of heat, work, or both. Note the sign conventions for Q and W.



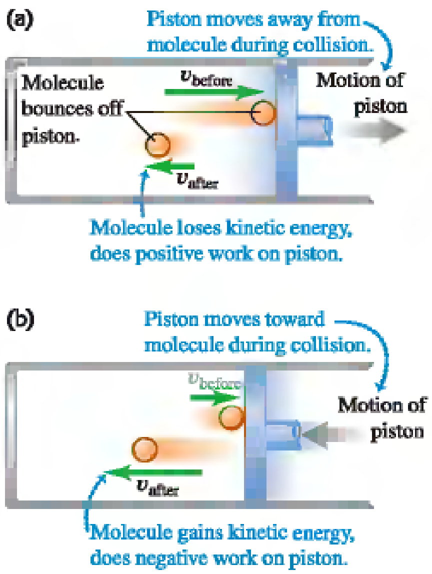
Test Your Understanding of Section 19.1 In Example 17.8 (Section 17.6), what is the sign of Q for the coffee? For the aluminum cup? If a block slides along a horizontal surface with friction, what is the sign of W for the block?

19.2 Work Done During Volume Changes

A simple but common example of a thermodynamic system is a quantity of gas enclosed in a cylinder with a movable piston. Internal-combustion engines, steam engines, and compressors in refrigerators and air conditioners all use some version of such a system. In the next several sections we will use the gas-in-cylinder system to explore several kinds of processes involving energy transformations.

We'll use a microscopic viewpoint, based on the kinetic and potential energies of individual molecules in a material, to develop intuition about thermodynamic quantities. But it is important to understand that the central principles of thermodynamics can be treated in a completely *macroscopic* way, without reference to microscopic models. Indeed, part of the great power and generality of thermodynamics is that it does *not* depend on details of the structure of matter.

19.4 A molecule striking a piston (a) does positive work if the piston is moving away from the molecule and (b) does negative work if the piston is moving toward the molecule. Hence a gas does positive work when it expands as in (a) but does negative work when it compresses as in (b).



First we consider the *work* done by the system during a volume change. When a gas expands, it pushes outward on its boundary surfaces as they move outward. Hence an expanding gas always does positive work. The same thing is true of any solid or fluid material that expands under pressure, such as the popcorn in Fig. 19.1.

We can understand the work done by a gas in a volume change by considering the molecules that make up the gas. When one such molecule collides with a stationary surface, it exerts a momentary force on the wall but does no work because the wall does not move. But if the surface is moving, like a piston in a gasoline engine, the molecule *does* do work on the surface during the collision. If the piston in Fig. 19.4a moves to the right, so that the volume of the gas increases, the molecules that strike the piston exert a force through a distance and do *positive* work on the piston. If the piston moves toward the left as in Fig. 19.4b, so the volume of the gas decreases, then positive work is done *on* the molecule during the collision. Hence the gas molecules do *negative* work on the piston.

Figure 19.5 shows a system whose volume can change (a gas, liquid, or solid) in a cylinder with a movable piston. Suppose that the cylinder has cross-sectional area A and that the pressure exerted by the system at the piston face is p. The total force F exerted by the system on the piston is $F = pA$. When the piston moves out an infinitesimal distance dx, the work dW done by this force is

$$dW = F\,dx = pA\,dx$$

But

$$A\,dx = dV$$

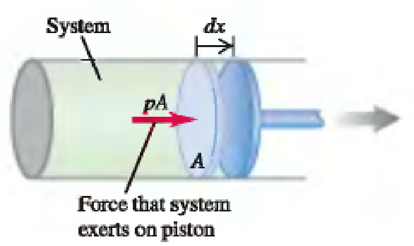
where dV is the infinitesimal change of volume of the system. Thus we can express the work done by the system in this infinitesimal volume change as

$$dW = p\,dV \tag{19.1}$$

In a finite change of volume from V_1 to V_2,

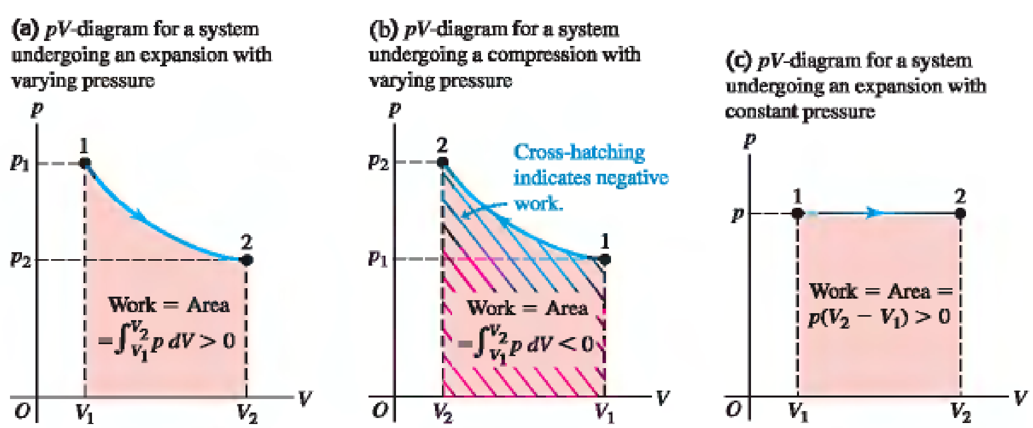
$$W = \int_{V_1}^{V_2} p\,dV \qquad \text{(work done in a volume change)} \tag{19.2}$$

19.5 The infinitesimal work done by the system during the small expansion dx is $dW = pA\,dx$.



In general, the pressure of the system may vary during the volume change. For example, this is the case in the cylinders of an antomobile engine as the pistons move back and forth. To evaluate the integral in Eq. (19.2), we have to know how the pressure varies as a function of volume. We can represent this relationship as a graph of p as a function of V (a pV-diagram, described at the end of Section 18.1). Figure 19.6 a shows a simple example. In this figure, Eq. (19.2) is repre-

19.6 The work done equals the area under the curve on a pV-diagram.



sented graphically as the *area* under the curve of p versus V between the limits V_1 and V_2. (In Section 6.3 we used a similar interpretation of the work done by a force F as the area under the curve of F versus x between the limits x_1 and x_2.)

According to the rule we stated in Section 19.1, work is *positive* when a system *expands*. In an expansion from state 1 to state 2 in Fig. 19.6a the area under the curve and the work are positive. A *compression* from 1 to 2 in Fig. 19.6b gives a *negative* area; when a system is compressed, its volume decreases and it does *negative* work on its surroundings (see also Fig. 19.4b).

CAUTION **Be careful with subscripts 1 and 2** When using Eq. (19.2), always remember that V_1 is the *initial* volume and V_2 is the *final* volume. That's why the labels 1 and 2 are reversed in Fig. 19.6b compared to Fig. 19.6a, even though both processes move between the same two thermodynamic states.

If the pressure p remains constant while the volume changes from V_1 to V_2 (Fig. 19.6c), the work done by the system is

$$W = p(V_2 - V_1) \quad \text{(work done in a volume change at constant pressure)} \tag{19.3}$$

ActivPhysics ONLINE 8.5 Work Done By a Gas

In any process in which the volume is *constant*, the system does no work because there is no displacement.

Example 19.1 Isothermal expansion of an ideal gas

An ideal gas undergoes an *isothermal* (constant-temperature) *expansion* at temperature T, during which its volume changes from V_1 to V_2. How much work does the gas do?

SOLUTION

IDENTIFY: The ideal-gas law tells us that if the temperature of an ideal gas remains constant, the quantity $pV = nRT$ also remains constant. If the volume V changes, the pressure p must change as well. Hence this problem asks for the work done by a gas that changes volume with varying pressure.

SET UP: Although it may be tempting to do so, we *cannot* use Eq. (19.3) to calculate the work done because the temperature, not the pressure, is constant. Instead we must use Eq. (19.2). To evaluate the integral in this equation we need to know the pressure as a function of volume; for this we use the ideal-gas law, Eq. (18.3).

EXECUTE: From Eq. (19.2),

$$W = \int_{V_1}^{V_2} p\,dV$$

From Eq. (18.3) the pressure p of n moles of ideal gas occupying volume V at absolute temperature T is

$$p = \frac{nRT}{V}$$

where R is the gas constant. We substitute this into the integral, take the constants n, R, and T outside, and evaluate the integral:

$$W = nRT\int_{V_1}^{V_2} \frac{dV}{V} = nRT\ln\frac{V_2}{V_1} \quad \text{(ideal gas, isothermal process)}$$

Also, when T is constant,

$$p_1V_1 = p_2V_2 \quad \text{or} \quad \frac{V_2}{V_1} = \frac{p_1}{p_2}$$

so the isothermal work may also be expressed as

$$W = nRT\ln\frac{p_1}{p_2} \quad \text{(ideal gas, isothermal process)}$$

EVALUATE: We check our result by noting that in an expansion $V_2 > V_1$ and the ratio V_2/V_1 is greater than 1. The logarithm of a number greater than 1 is positive, so $W > 0$, as it should be. As an additional check, look at our second expression for W: In an isothermal expansion the volume increases and the pressure drops, so $p_2 < p_1$, the ratio $p_1/p_2 > 1$, and $W = nRT\ln(p_1/p_2)$ is again positive.

These results also apply to an isothermal *compression* of a gas, for which $V_2 < V_1$ and $p_2 > p_1$.

Test Your Understanding of Section 19.2 A quantity of ideal gas undergoes an expansion that increases its volume from V_1 to $V_2 = 2V_1$. The final pressure of the gas is p_2. Does the gas do more work on its surroundings if the expansion is at constant *pressure* or at constant *temperature*? (i) constant pressure; (ii) constant temperature; (iii) the same amount of work is done in both cases; (iv) not enough information is given to decide.

19.3 Paths Between Thermodynamic States

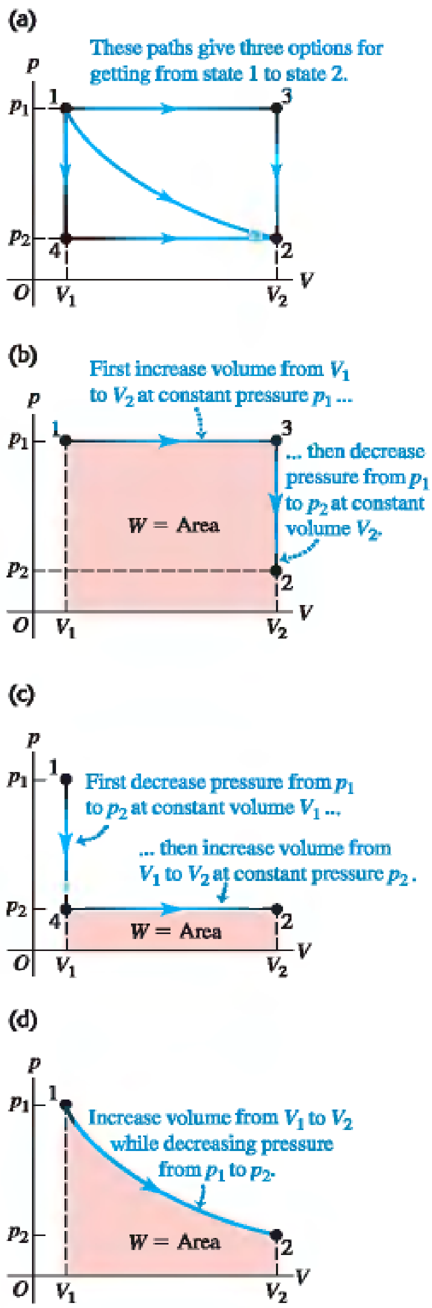
We've seen that if a thermodynamic process involves a change in volume, the system undergoing the process does work (either positive or negative) on its surroundings. Heat also flows into or out of the system during the process if there is a temperature difference between the system and its surroundings. Let's now examine how the work done by and the heat added to the system during a thermodynamic process depend on the details of how the process takes place.

Work Done in a Thermodynamic Process

When a thermodynamic system changes from an initial state to a final state, it passes through a series of intermediate states. We call this series of states a **path.** There are always infinitely many different possibilities for these intermediate states. When they are all equilibrium states, the path can be plotted on a pV-diagram (Fig. 19.7a). Point 1 represents an initial state with pressure p_1 and volume V_1, and point 2 represents a final state with pressure p_2 and volume V_2. To pass from state 1 to state 2, we could keep the pressure constant at p_1 while the system expands to volume V_2 (point 3 in Fig. 19.7b), then reduce the pressure to p_2 (probably by decreasing the temperature) while keeping the volume constant at V_2 (to point 2 on the diagram). The work done by the system during this process is the area under the line $1 \rightarrow 3$; no work is done during the constant-volume process $3 \rightarrow 2$. Or the system might traverse the path $1 \rightarrow 4 \rightarrow 2$ (Fig. 19.7c); in that case the work is the area under the line $4 \rightarrow 2$, since no work is done during the constant-volume process $1 \rightarrow 4$. The smooth curve from 1 to 2 is another possibility (Fig. 19.7d), and the work for this path is different from that for either of the other paths.

We conclude that *the work done by the system depends not only on the initial and final states, but also on the intermediate states—that is, on the path.* Furthermore, we can take the system through a series of states forming a closed loop, such as $1 \rightarrow 3 \rightarrow 2 \rightarrow 4 \rightarrow 1$. In this case the final state is the same as the initial state, but the total work done by the system is *not* zero. (In fact, it is represented on the graph by the area enclosed by the loop; can you prove that? See Exercise 19.7.) It follows that it doesn't make sense to talk about the amount of work *contained in* a system. In a particular state, a system may have definite values of the state coordinates p, V, and T, but it wouldn't make sense to say that it has a definite value of W.

19.7 The work done by a system during a transition between two states depends on the path chosen.



Heat Added in a Thermodynamic Process

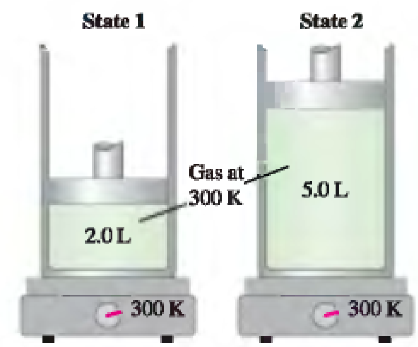
Like work, the *heat* added to a thermodynamic system when it undergoes a change of state depends on the path from the initial state to the final state. Here's an example. Suppose we want to change the volume of a certain quantity of an ideal gas from 2.0 L to 5.0 L while keeping the temperature constant at $T = 300$ K. Figure 19.8 shows two different ways in which we can do this. In Fig. 19.8a the gas is contained in a cylinder with a piston, with an initial volume of 2.0 L. We let the gas expand slowly, supplying heat from the electric heater to keep the temperature at 300 K. After expanding in this slow, controlled, isothermal manner, the gas reaches its final volume of 5.0 L; it absorbs a definite amount of heat in the process.

Figure 19.8b shows a different process leading to the same final state. The container is surrounded by insulating walls and is divided by a thin, breakable partition into two compartments. The lower part has volume 2.0 L and the upper part has volume 3.0 L. In the lower compartment we place the same amount of the same gas as in Fig. 19.8a, again at $T = 300$ K. The initial state is the same as before. Now we break the partition; the gas undergoes a rapid, uncontrolled expansion, with no heat passing through the insulating walls. The final volume is

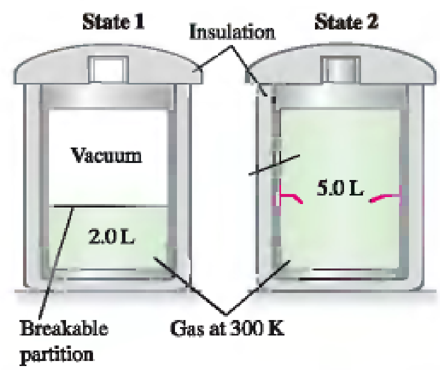
5.0 L, the same as in Fig. 19.8a. The gas does no work during this expansion because it doesn't push against anything that moves. This uncontrolled expansion of a gas into vacuum is called a **free expansion;** we will discuss it further in Section 19.6.

19.8 (a) Slow, controlled isothermal expansion of a gas from an initial state 1 to a final state 2 with the same temperature but lower pressure. (b) Rapid, uncontrolled expansion of the same gas starting at the same state 1 and ending at the same state 2.

(a) System does work on piston; hot plate adds heat to system ($W > 0$ and $Q > 0$).



(b) System does no work; no heat enters or leaves system ($W = 0$ and $Q = 0$).



Experiments have shown that when an ideal gas undergoes a free expansion, there is no temperature change. Therefore the final state of the gas is the same as in Fig. 19.8a. The intermediate states (pressures and volumes) during the transition from state 1 to state 2 are entirely different in the two cases; Figs. 19.8a and 19.8b represent *two different paths* connecting the *same states* 1 and 2. For the path in Fig. 19.8b, *no* heat is transferred into the system, and the system does no work. Like work, *heat depends not only on the initial and final states but also on the path.*

Because of this path dependence, it would not make sense to say that a system "contains" a certain quantity of heat. To see this, suppose we assign an arbitrary value to the "heat in a body" in some reference state. Then presumably the "heat in the body" in some other state would equal the heat in the reference state plus the heat added when the body goes to the second state. But that's ambiguous, as we have just seen; the heat added depends on the *path* we take from the reference state to the second state. We are forced to conclude that there is *no* consistent way to define "heat in a body"; it is not a useful concept.

While it doesn't make sense to talk about "work in a body" or "heat in a body," it *does* make sense to speak of the amount of *internal energy* in a body. This important concept is our next topic.

Test Your Understanding of Section 19.3 The system described in Fig. 19.7a undergoes four different thermodynamic processes. Each process is represented in a pV-diagram as a straight line from the initial state to the final state. (These processes are different from those shown in the pV-diagrams of Fig. 19.7.) Rank the processes in order of the amount of work done by the system, from the most positive to the most negative. (i) $1 \rightarrow 2$; (ii) $2 \rightarrow 1$; (iii) $3 \rightarrow 4$; (iv) $4 \rightarrow 3$.

19.4 Internal Energy and the First Law of Thermodynamics

Internal energy is one of the most important concepts in thermodynamics. In Section 7.3, when we discussed energy changes for a body sliding with friction, we stated that warming a body increased its internal energy and that cooling the body decreased its internal energy. But what *is* internal energy? We can look at it in various ways; let's start with one based on the ideas of mechanics. Matter consists of atoms and molecules, and these are made up of particles having kinetic and potential energies. We *tentatively* define the **internal energy** of a system as the sum of the kinetic energies of all of its constituent particles, plus the sum of all the potential energies of interaction among these particles.

CAUTION **Is it internal?** Note that internal energy does *not* include potential energy arising from the interaction between the system and its surroundings. If the system is a glass of water, placing it on a high shelf increases the gravitational potential energy arising from the interaction between the glass and the earth. But this has no effect on the interaction between the molecules of the water, and so the internal energy of the water does not change.

We use the symbol U for internal energy. (We used this same symbol in our study of mechanics to represent potential energy. You may have to remind yourself occasionally that U has a different meaning in thermodynamics.) During a change of state of the system the internal energy may change from an initial value U_1 to a final value U_2. We denote the change in internal energy as $\Delta U = U_2 - U_1$.

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8.6 Heat, Internal Energy, and First Law of Thermodynamics

We know that heat transfer is energy transfer. When we add a quantity of heat Q to a system and the system does no work during the process, the internal energy increases by an amount equal to Q; that is, $\Delta U = Q$. When a system does work W by expanding against its surroundings and no heat is added during the process, energy leaves the system and the internal energy decreases. That is, when W is positive, ΔU is negative, and vice versa. So $\Delta U = -W$. When *both* heat transfer and work occur, the *total* change in internal energy is

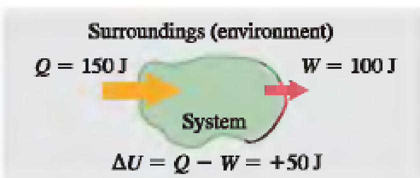
$$U_2 - U_1 = \Delta U = Q - W \qquad \text{(first law of thermodynamics)} \qquad (19.4)$$

We can rearrange this to the form

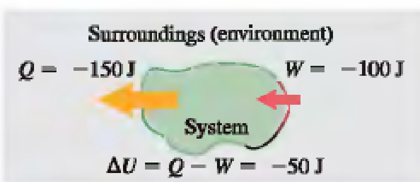
$$Q = \Delta U + W \qquad (19.5)$$

19.9 In a thermodynamic process, the internal energy U of a system may (a) increase ($\Delta U > 0$), (b) decrease ($\Delta U < 0$), or (c) remain the same ($\Delta U = 0$).

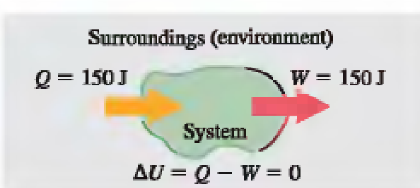
(a) More heat is added to system than system does work: Internal energy of system increases.



(b) More heat flows out of system than work is done: Internal energy of system decreases.



(c) Heat added to system equals work done by system: Internal energy of system unchanged.



The message of Eq. (19.5) is that in general, when heat Q is added to a system, some of this added energy remains within the system, changing its internal energy by an amount ΔU; the remainder leaves the system again as the system does work W against its surroundings. Because W and Q may be positive, negative, or zero, ΔU can be positive, negative, or zero for different processes (Fig. 19.9).

Equation (19.4) or (19.5) is the **first law of thermodynamics.** It is a generalization of the principle of conservation of energy to include energy transfer through heat as well as mechanical work. As you will see in later chapters, this principle can be extended to ever-broader classes of phenomena by identifying additional forms of energy and energy transfer. In every situation in which it seems that the total energy in all known forms is not conserved, it has been possible to identify a new form of energy such that the total energy, including the new form, *is* conserved. There is energy associated with electric fields, with magnetic fields, and, according to the theory of relativity, even with mass itself.

Understanding the First Law of Thermodynamics

At the beginning of this discussion we tentatively defined internal energy in terms of microscopic kinetic and potential energies. This has drawbacks, however. Actually *calculating* internal energy in this way for any real system would be hopelessly complicated. Furthermore, this definition isn't an *operational* one because it doesn't describe how to determine internal energy from physical quantities that we can measure directly.

So let's look at internal energy in another way. Starting over, we define the *change* in internal energy ΔU during any change of a system as the quantity given by Eq. (19.4), $\Delta U = Q - W$. This *is* an operational definition because we can measure Q and W. It does not define U itself, only ΔU. This is not a shortcoming because we can *define* the internal energy of a system to have a specified value in some reference state, and then use Eq. (19.4) to define the internal energy in any other state. This is analogous to our treatment of potential energy in Chapter 7, in which we arbitrarily defined the potential energy of a mechanical system to be zero at a certain position.

This new definition trades one difficulty for another. If we define ΔU by Eq. (19.4), then when the system goes from state 1 to state 2 by two different paths, how do we know that ΔU is the same for the two paths? We have already seen that Q and W are, in general, *not* the same for different paths. If ΔU, which equals $Q - W$, is also path dependent, then ΔU is ambiguous. If so, the concept of internal energy of a system is subject to the same criticism as the erroneous concept of quantity of heat in a system, as we discussed at the end of Section 19.3.

The only way to answer this question is through *experiment.* For various materials we measure Q and W for various changes of state and various paths to learn whether ΔU is or is not path dependent. The results of many such investigations are clear and unambiguous: While Q and W depend on the path, $\Delta U = Q - W$ *is independent of path. The change in internal energy of a system*

during any thermodynamic process depends only on the initial and final states, not on the path leading from one to the other.

Experiment, then, is the ultimate justification for believing that a thermodynamic system in a specific state has a unique internal energy that depends only on that state. An equivalent statement is that the internal energy U of a system is a function of the state coordinates p, V, and T (actually, any two of these, since the three variables are related by the equation of state).

To say that the first law of thermodynamics, given by Eq. (19.4) or (19.5), represents conservation of energy for thermodynamic processes is correct, as far as it goes. But an important *additional* aspect of the first law is the fact that internal energy depends only on the state of a system (Fig. 19.10). In changes of state, the change in internal energy is independent of the path.

All this may seem a little abstract if you are satisfied to think of internal energy as microscopic mechanical energy. There's nothing wrong with that view, and we will make use of it at various times during our discussion. But in the interest of precise *operational* definitions, internal energy, like heat, can and must be defined in a way that is independent of the detailed microscopic structure of the material.

19.10 The internal energy of a cup of coffee depends on just its thermodynamic state—how much water and ground coffee it contains, and what its temperature is. It does not depend on the history of how the coffee was prepared—that is, the thermodynamic path that led to its current state.

Cyclic Processes and Isolated Systems

Two special cases of the first law of thermodynamics are worth mentioning. A process that eventually returns a system to its initial state is called a *cyclic process.* For such a process, the final state is the same as the initial state, and so the *total* internal energy change must be zero. Then

$$U_2 = U_1 \qquad \text{and} \qquad Q = W$$

If a net quantity of work W is done by the system during this process, an equal amount of energy must have flowed into the system as heat Q. But there is no reason either Q or W individually has to be zero (Fig. 19.11).

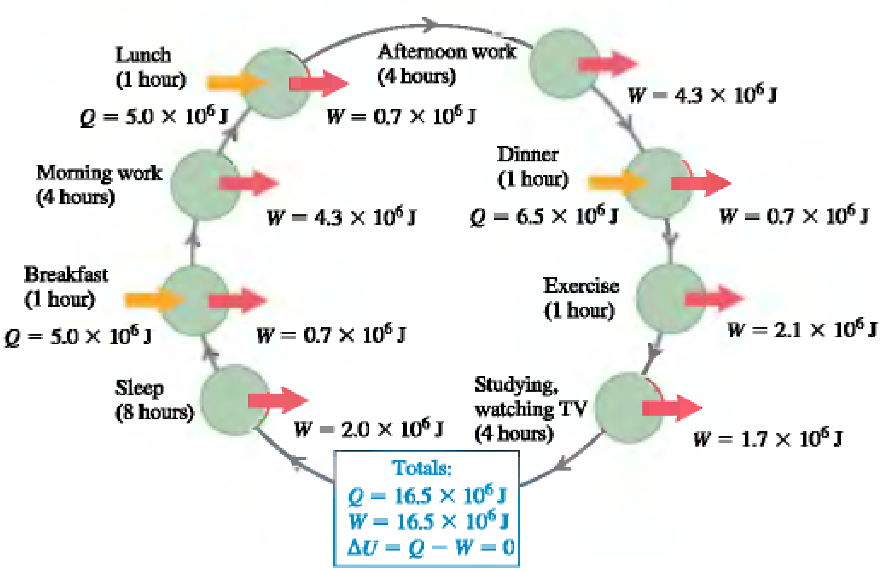
Another special case occurs in an *isolated system,* one that does no work on its surroundings and has no heat flow to or from its surroundings. For any process taking place in an isolated system,

$$W = Q = 0$$

and therefore

$$U_2 = U_1 = \Delta U = 0$$

In other words, *the internal energy of an isolated system is constant.*



19.11 Every day, your body (a thermodynamic system) goes through a cyclic thermodynamic process like this one. Heat Q is added by metabolizing food, and your body does work W in breathing, walking, and other activities. If you return to the same state at the end of the day, $Q = W$ and the net change in your internal energy is zero.

Problem-Solving Strategy 19.1 The First Law of Thermodynamics

IDENTIFY *the relevant concepts:* The first law of thermodynamics is the statement of the law of conservation of energy in its most general form. You can apply it to *any* situation in which you are concerned with changes in the internal energy of a system, with heat flow into or out of a system, and/or with work done by or on a system.

SET UP *the problem* using the following steps:

1. Carefully define what the thermodynamic system is.
2. The first law of thermodynamics focuses on systems that go through thermodynamic processes. Some problems involve processes with more than one step, so make sure that you identify the initial and final states for each step.
3. Identify the known quantities and the target variables.
4. Check whether you have enough equations. The first law, $\Delta U = Q - W$, can be applied just once to each step in a thermodynamic process, so you will often need additional equations. These often include Eq. (19.2) for the work done in a volume change and the equation of state of the material that makes up the thermodynamic system (for an ideal gas, $pV = nRT$).

EXECUTE *the solution* as follows:

1. You shouldn't be surprised to be told that consistent units are essential. If p is in Pa and V in m^3, then W is in joules. Otherwise, you may want to convert the pressure and volume units into units of Pa and m^3. If a heat capacity is given in terms of calories, usually the simplest procedure is to convert it to joules. Be especially careful with moles. When you use $n = m_{total}/M$ to convert between total mass and number of moles, remember that if m_{total} is in kilograms, M must be in *kilograms* per mole. The usual units for M are *grams* per mole; be careful!
2. The internal energy change ΔU in any thermodynamic process or series of processes is independent of the path, whether the substance is an ideal gas or not. This point is of the utmost importance in the problems in this chapter and the next. Sometimes you will be given enough information about one path between the given initial and final states to calculate ΔU for that path. Since ΔU is the same for every possible path between the same two states, you can then relate the various energy quantities for other paths.
3. When a process consists of several distinct steps, it often helps to make a table showing Q, W, and ΔU for each step. Put these quantities for each step on a different line, and arrange them so the Q's, W's, and ΔU's form columns. Then you can apply the first law to each line; in addition, you can add each column and apply the first law to the sums. Do you see why?
4. Using steps 1–3, solve for the target variables.

EVALUATE *your answer:* Check your results for reasonableness. In particular, make sure that each of your answers has the correct algebraic sign. Remember that a positive Q means that heat flows *into* the system, and a negative Q means that heat flows *out of* the system. A positive W means that work is done *by* the system on its environment, while a negative W means that work is done *on* the system by its environment.

Example 19.2 Working off your dessert

You propose to eat a 900-calorie hot fudge sundae (with whipped cream) and then run up several flights of stairs to work off the energy you have taken in. How high do you have to climb? Assume that your mass is 60.0 kg.

SOLUTION

IDENTIFY: Eating the hot fudge sundae corresponds to a heat flow into your body, and running up stairs means that you do work. We can relate these quantities using the first law of thermodynamics.

SET UP: The system consists of your body. We are given that 900 food calories of heat flow into your body. The purpose of running up the stairs is to make sure that the final state of the system is the same as the initial state (no fatter, no leaner), so there is no net change in internal energy: $\Delta U = 0$. The work you must do to raise your mass m a height h is $W = mgh$; our target variable is h.

EXECUTE: Using the first law of thermodynamics, $\Delta U = 0 = Q - W$, so $W = Q$: The work that you do running up the stairs must just equal the heat input from the sundae. From $W = mgh$, the height that you climb is $h = Q/mg$. Before substituting values into this equation, we first convert units: One food-value calorie is $1\ \text{kcal} = 1000\ \text{cal} = 4190\ \text{J}$ (to three significant figures), so

$$Q = 900\ \text{kcal}\,(4190\ \text{J}/1\ \text{kcal}) = 3.77 \times 10^6\ \text{J}$$

Then

$$h = \frac{Q}{mg} = \frac{3.77 \times 10^6\ \text{J}}{(60.0\ \text{kg})(9.80\ \text{m/s}^2)} = 6410\ \text{m} \qquad (\text{about } 21{,}000\ \text{ft})$$

EVALUATE: Good luck! We have assumed 100% efficiency in the conversion of food energy into mechanical work; this isn't very realistic. As a result, the actual distance you would have to climb is quite a bit less than we have calculated. We'll talk more about efficiency later.

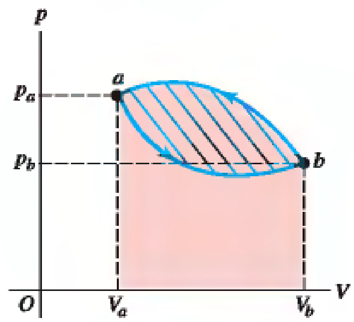
Example 19.3 A cyclic process

Figure 19.12 shows a pV-diagram for a *cyclic* process, one in which the initial and final states are the same. It starts at point a and proceeds counterclockwise in the pV-diagram to point b, then back to a, and the total work is $W = -500\ \text{J}$. (a) Why is the work negative? (b) Find the change in internal energy and the heat added during this process.

SOLUTION

IDENTIFY: This problem asks us to relate the change in internal energy, the heat added, and the work done in a thermodynamic process. Hence we can apply the first law of thermodynamics.

19.12 The net work done by the system in the process *aba* is −500 J. What would it have been if the process had proceeded clockwise in this *pV*-diagram?



SET UP: The thermodynamic process here has two steps: $a \to b$ via the lower curve in Fig. 19.12 and $b \to a$ via the upper curve. But note that the questions in (a) and (b) are about the *entire* cyclic process $a \to b \to a$ (around the loop in Fig. 19.12).

EXECUTE: (a) The work done equals the area under the curve, with the area taken as positive for increasing volume and negative for decreasing volume. The area under the lower curve from *a* to *b* is positive, but it is smaller than the absolute value of the negative area under the upper curve from *b* back to *a*. Therefore the net area (the area enclosed by the path, shown with red stripes) and the work are negative. In other words, 500 more joules of work is done *on* the system than *by* the system.

(b) For this and any other cyclic process (in which the beginning and end points are the same), $\Delta U = 0$, so $Q = W = -500$ J. That is, 500 joules of heat must come *out of* the system.

EVALUATE: This example illustrates a general principle about *pV*-diagrams of cyclic processes: The total work is positive if the process goes around the cycle in a clockwise direction, and the total work is negative if the process goes around the cycle in a counterclockwise direction (as in Fig. 19.12).

Example 19.4 Comparing thermodynamic processes

A series of thermodynamic processes is shown in the *pV*-diagram of Fig. 19.13. In process *ab*, 150 J of heat is added to the system, and in process *bd*, 600 J of heat is added. Find (a) the internal energy change in process *ab*; (b) the internal energy change in process *abd* (shown in light blue); and (c) the total heat added in process *acd* (shown in dark blue).

SOLUTION

IDENTIFY: In each process we use $\Delta U = Q - W$ to determine the desired quantity.

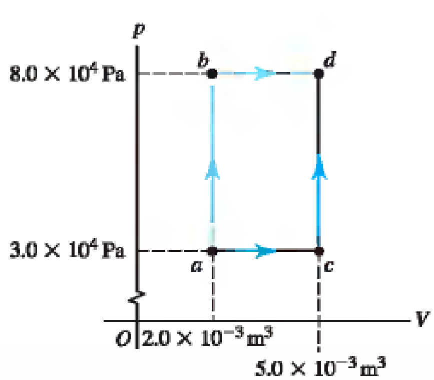
SET UP: We are given $Q_{ab} = +150$ J and $Q_{bd} = +600$ J (both values are positive because heat is *added* to the system). Our target variables are (a) ΔU_{ab}, (b) ΔU_{abd}, and (c) Q_{acd}.

EXECUTE: (a) No volume change occurs during process *ab*, so $W_{ab} = 0$ and $\Delta U_{ab} = Q_{ab} = 150$ J.

(b) Process *bd* occurs at constant pressure, so the work done by the system during this expansion is

$$\begin{aligned} W_{bd} &= p(V_2 - V_1) \\ &= (8.0 \times 10^4\,\text{Pa})(5.0 \times 10^{-3}\,\text{m}^3 - 2.0 \times 10^{-3}\,\text{m}^3) \\ &= 240\,\text{J} \end{aligned}$$

19.13 A *pV*-diagram showing the various thermodynamic processes.



The total work for process *abd* is

$$W_{abd} = W_{ab} + W_{bd} = 0 + 240\,\text{J} = 240\,\text{J}$$

and the total heat is

$$Q_{abd} = Q_{ab} + Q_{bd} = 150\,\text{J} + 600\,\text{J} = 750\,\text{J}$$

Applying Eq. (19.4) to *abd*, we find

$$\Delta U_{abd} = Q_{abd} - W_{abd} = 750\,\text{J} - 240\,\text{J} = 510\,\text{J}$$

(c) Because ΔU is independent of path, the internal energy change is the same for path *acd* as for path *abd*; that is,

$$\Delta U_{acd} = \Delta U_{abd} = 510\,\text{J}$$

The total work for the path *acd* is

$$\begin{aligned} W_{acd} &= W_{ac} + W_{cd} = p(V_2 - V_1) + 0 \\ &= (3.0 \times 10^4\,\text{Pa})(5.0 \times 10^{-3}\,\text{m}^3 - 2.0 \times 10^{-3}\,\text{m}^3) \\ &= 90\,\text{J} \end{aligned}$$

Now we apply Eq. (19.5) to process *acd*:

$$Q_{acd} = \Delta U_{acd} + W_{acd} = 510\,\text{J} + 90\,\text{J} = 600\,\text{J}$$

Here is a tabulation of the various quantities:

Step	Q	W	$\Delta U = Q - W$	Step	Q	W	$\Delta U = Q - W$
ab	150 J	0 J	150 J	*ac*	?	90 J	?
bd	600 J	240 J	360 J	*cd*	?	0 J	?
abd	750 J	240 J	510 J	*acd*	600 J	90 J	510 J

EVALUATE: We see that although ΔU is the same (510 J) for *abd* and *acd*, W (240 J versus 90 J) and Q (750 J versus 600 J) are quite different for the two processes.

Notice that we don't have enough information to find Q or ΔU for the processes *ac* and *cd*. We were nonetheless able to analyze the composite process *acd* by comparing it to the process *abd*, which has the same initial and final states and for which we have more complete information.

Example 19.5 Thermodynamics of boiling water

One gram of water $(1\ \mathrm{cm}^3)$ becomes $1671\ \mathrm{cm}^3$ of steam when boiled at a constant pressure of 1 atm $(1.013 \times 10^5\ \mathrm{Pa})$. The heat of vaporization at this pressure is $L_v = 2.256 \times 10^6\ \mathrm{J/kg}$. Compute (a) the work done by the water when it vaporizes and (b) its increase in internal energy.

SOLUTION

IDENTIFY: The new feature of this problem is that the heat added causes the system (water) to change phase from liquid to vapor. We can nonetheless apply the first law of thermodynamics, which is true for thermodynamic processes of all kinds.

SET UP: The water is boiled at a constant pressure, so we can calculate the work W done by the water using Eq. (19.3). We can calculate the heat Q added to the water from the mass and the heat of vaporization, and we can then find the internal energy change using $\Delta U = Q - W$.

EXECUTE: (a) From Eq. (19.3), the work done by the vaporizing water is

$$\begin{aligned} W &= p(V_2 - V_1) \\ &= (1.013 \times 10^5\ \mathrm{Pa})(1671 \times 10^{-6}\ \mathrm{m}^3 - 1 \times 10^{-6}\ \mathrm{m}^3) \\ &= 169\ \mathrm{J} \end{aligned}$$

(b) From Eq. (17.20), the heat added to the water to vaporize it is

$$Q = mL_v = (10^{-3}\ \mathrm{kg})(2.256 \times 10^6\ \mathrm{J/kg}) = 2256\ \mathrm{J}$$

From the first law of thermodynamics, Eq. (19.4), the change in internal energy is

$$\Delta U = Q - W = 2256\ \mathrm{J} - 169\ \mathrm{J} = 2087\ \mathrm{J}$$

EVALUATE: To vaporize 1 gram of water, we have to add 2256 J of heat. Most (2087 J) of this added energy remains in the system as an increase in internal energy. The remaining 169 J leaves the system again as it does work against the surroundings while expanding from liquid to vapor. The increase in internal energy is associated mostly with the intermolecular forces that hold the molecules together in the liquid state. These forces are attractive, so the associated potential energies are greater after work has been done to pull the molecules apart, forming the vapor state. It's like increasing gravitational potential energy by pulling an elevator farther from the center of the earth.

Infinitesimal Changes of State

In the preceding examples the initial and final states differ by a finite amount. Later we will consider *infinitesimal* changes of state in which a small amount of heat dQ is added to the system, the system does a small amount of work dW, and its internal energy changes by an amount dU. For such a process we state the first law in differential form as

$$dU = dQ - dW \quad \text{(first law of thermodynamics, infinitesimal process)} \tag{19.6}$$

For the systems we will discuss, the work dW is given by $dW = p\,dV$, so we can also state the first law as

$$dU = dQ - p\,dV \tag{19.7}$$

Test Your Understanding of Section 19.4 Rank the following thermodynamic processes according to the change in internal energy in each process, from most positive to most negative. (i) As you do 250 J of work on a system, it transfers 250 J of heat to its surroundings; (ii) as you do 250 J of work on a system, it absorbs 250 J of heat from its surroundings; (iii) as a system does 250 J of work on you, it transfers 250 J of heat to its surroundings; (iv) as a system does 250 J of work on you, it absorbs 250 J of heat from its surroundings.

19.5 Kinds of Thermodynamic Processes

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8.4 State Variables and Ideal Gas Law

In this section we describe four specific kinds of thermodynamic processes that occur often in practical situations. These can be summarized briefly as "no heat transfer" or *adiabatic*, "constant volume" or *isochoric*, "constant pressure" or *isobaric*, and "constant temperature" or *isothermal*. For some of these processes we can use a simplified form of the first law of thermodynamics.

Adiabatic Process

An **adiabatic process** (pronounced "ay-dee-ah-*bat*-ic") is defined as one with no heat transfer into or out of a system; $Q = 0$. We can prevent heat flow either by surrounding the system with thermally insulating material or by carrying out the process so quickly that there is not enough time for appreciable heat flow. From the first law we find that for every adiabatic process,

$$U_2 - U_1 = \Delta U = -W \qquad \text{(adiabatic process)} \qquad (19.8)$$

When a system expands adiabatically, W is positive (the system does work on its surroundings), so ΔU is negative and the internal energy decreases. When a system is *compressed* adiabatically, W is negative (work is done on the system by its surroundings) and U increases. In many (but not all) systems an increase of internal energy is accompanied by a rise in temperature, and a decrease in internal energy with a drop in temperature (Fig. 19.14).

The compression stroke in an internal-combustion engine is an approximately adiabatic process. The temperature rises as the air–fuel mixture in the cylinder is compressed. The expansion of the burned fuel during the power stroke is also an approximately adiabatic expansion with a drop in temperature. In Section 19.8 we'll consider adiabatic processes in an ideal gas.

19.14 When the cork is popped on a bottle of champagne, the pressurized gases inside the bottle expand into the outside air so rapidly that there is no time for them to exchange heat with their surroundings. Hence the expansion is adiabatic. As the expanding gases do work on their surroundings, their internal energy and temperature both drop; the lowered temperature makes water vapor condense and form a miniature cloud.

Isochoric Process

An **isochoric process** (pronounced "eye-so-*kor*-ic") is a *constant-volume* process. When the volume of a thermodynamic system is constant, it does no work on its surroundings. Then $W = 0$ and

$$U_2 - U_1 = \Delta U = Q \qquad \text{(isochoric process)} \qquad (19.9)$$

In an isochoric process, all the energy added as heat remains in the system as an increase in internal energy. Heating a gas in a closed constant-volume container is an example of an isochoric process. The processes ab and cd in Example 19.4 are also examples of isochoric processes. (Note that there are types of work that do not involve a volume change. For example, we can do work on a fluid by stirring it. In some literature, "isochoric" is used to mean that no work of any kind is done.)

Isobaric Process

An **isobaric process** (pronounced "eye-so-*bear*-ic") is a *constant-pressure* process. In general, none of the three quantities ΔU, Q, and W is zero in an isobaric process, but calculating W is easy nonetheless. From Eq. (19.3),

$$W = p(V_2 - V_1) \qquad \text{(isobaric process)} \qquad (19.10)$$

Example 19.5 concerns an isobaric process, boiling water at constant pressure (Fig. 19.15).

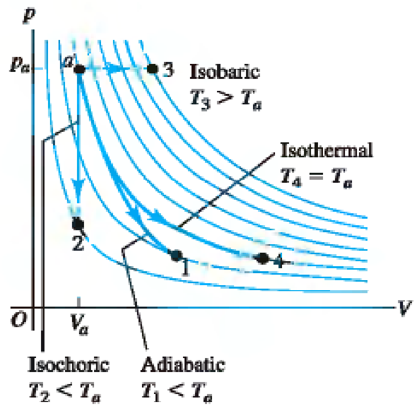
19.15 Most cooking involves isobaric processes. That's because the air pressure above a saucepan or frying pan, or inside a microwave oven, remains essentially constant while the food is being heated.

Isothermal Process

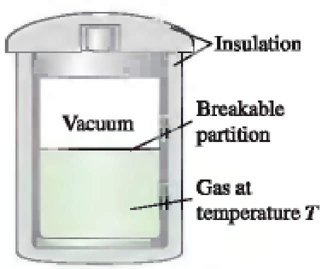
An **isothermal process** is a *constant-temperature* process. For a process to be isothermal, any heat flow into or out of the system must occur slowly enough that thermal equilibrium is maintained. In general, none of the quantities ΔU, Q, or W is zero in an isothermal process.

In some special cases the internal energy of a system depends *only* on its temperature, not on its pressure or volume. The most familiar system having this special property is an ideal gas, as we'll discuss in the next section. For such systems, if the temperature is constant, the internal energy is also constant; $\Delta U = 0$ and $Q = W$. That is, any energy entering the system as heat Q must leave it again as work W done by the system. Example 19.1, involving an ideal gas, is an example of an isothermal process in which U is also constant. For most systems other than

19.16 Four different processes for a constant amount of an ideal gas, all starting at state a. For the adiabatic process, $Q = 0$; for the isochoric process, $W = 0$; and for the isothermal process, $\Delta U = 0$. The temperature increases only during the isobaric expansion.



19.17 The partition is broken (or removed) to start the free expansion of gas into the vacuum region.



ideal gases, the internal energy depends on pressure as well as temperature, so U may vary even when T is constant.

Figure 19.16 shows a pV-diagram for these four processes for a constant amount of an ideal gas. The path followed in an adiabatic process (a to 1) is called an **adiabat.** A vertical line (constant volume) is an **isochor,** a horizontal line (constant pressure) is an **isobar,** and a curve of constant temperature (shown as light blue lines in Fig. 19.16) is an **isotherm.**

Test Your Understanding of Section 19.5 Which of the processes in Fig. 19.7 are isochoric? Which are isobaric? Is it possible to tell if any of the processes are isothermal or adiabatic?

19.6 Internal Energy of an Ideal Gas

We now show that for an ideal gas, the internal energy U depends only on temperature, not on pressure or volume. Let's think again about the free-expansion experiment described in Section 19.3. A thermally insulated container with rigid walls is divided into two compartments by a partition (Fig. 19.17). One compartment has a quantity of an ideal gas and the other is evacuated.

When the partition is removed or broken, the gas expands to fill both parts of the container. The gas does no work on its surroundings because the walls of the container don't move, and there is no heat flow through the insulation. So both Q and W are zero and the internal energy U is constant. This is true of any substance, whether it is an ideal gas or not.

Does the *temperature* change during a free expansion? Suppose it *does* change, while the internal energy stays the same. In that case we have to conclude that the internal energy depends on both the temperature and the volume or on both the temperature and the pressure, but certainly not on the temperature alone. But if T is constant during a free expansion, for which we know that U is constant even though both p and V change, then we have to conclude that U depends only on T, not on p or V.

Many experiments have shown that when a low-density gas undergoes a free expansion, its temperature *does not* change. Such a gas is essentially an ideal gas. The conclusion is:

The internal energy of an ideal gas depends only on its temperature, not on its pressure or volume.

This property, in addition to the ideal-gas equation of state, is part of the ideal-gas model. Make sure you understand that U depends only on T for an ideal gas, for we will make frequent use of this fact.

For nonideal gases, some temperature change occurs during free expansions, even though the internal energy is constant. This shows that the internal energy cannot depend *only* on temperature; it must depend on pressure as well. From the microscopic viewpoint, in which internal energy U is the sum of the kinetic and potential energies for all the particles that make up the system, this is not surprising. Nonideal gases usually have attractive intermolecular forces, and when molecules move farther apart, the associated potential energies increase. If the total internal energy is constant, the kinetic energies must decrease. Temperature is directly related to molecular *kinetic* energy, and for such a gas a free expansion is usually accompanied by a *drop* in temperature.

Test Your Understanding of Section 19.6 Is the internal energy of a solid likely to be independent of its volume, as is the case for an ideal gas? Explain your reasoning. (*Hint:* See Fig. 18.20.)

19.7 Heat Capacities of an Ideal Gas

We defined specific heat and molar heat capacity in Section 17.5. We also remarked at the end of that section that the specific heat or molar heat capacity of a substance depends on the conditions under which the heat is added. It is usually easiest to measure the heat capacity of a gas in a closed container under constant-volume conditions. The corresponding heat capacity is the **molar heat capacity at constant volume,** denoted by C_V. Heat capacity measurements for solids and liquids are usually carried out in the atmosphere under constant atmospheric pressure, and we call the corresponding heat capacity the **molar heat capacity at constant pressure,** C_p. If neither p nor V is constant, we have an infinite number of possible heat capacities.

Let's consider C_V and C_p for an ideal gas. To measure C_V, we raise the temperature of an ideal gas in a rigid container with constant volume, neglecting its thermal expansion (Fig. 19.18a). To measure C_p, we let the gas expand just enough to keep the pressure constant as the temperature rises (Fig. 19.18b).

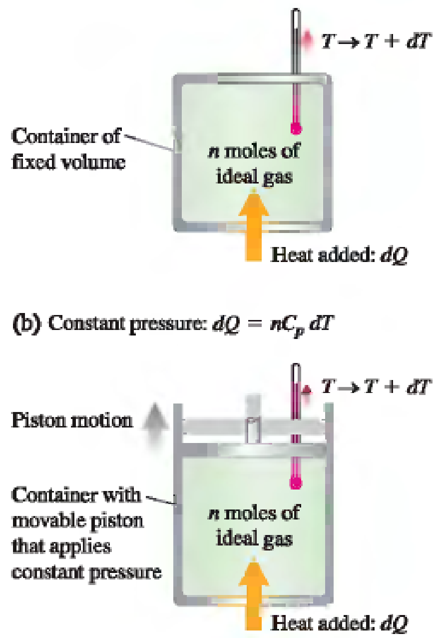
Why should these two molar heat capacities be different? The answer lies in the first law of thermodynamics. In a constant-volume temperature increase, the system does no work, and the change in internal energy ΔU equals the heat added Q. In a constant-pressure temperature increase, on the other hand, the volume *must* increase; otherwise, the pressure (given by the ideal-gas equation of state $p = nRT/V$) could not remain constant. As the material expands, it does an amount of work W. According to the first law,

$$Q = \Delta U + W \tag{19.11}$$

For a given temperature increase, the internal energy change ΔU of an ideal gas has the same value no matter what the process (remember that the internal energy of an ideal gas depends only on temperature, not on pressure or volume). Equation (19.11) then shows that the heat input for a constant-pressure process must be *greater* than that for a constant-volume process because additional energy must be supplied to account for the work W done during the expansion. So C_p is greater than C_V for an ideal gas. The pV-diagram in Fig. 19.19 shows this relationship. For air, C_p is 40% greater than C_V.

For a very few substances (one of which is water between 0°C and 4°C) the volume *decreases* during heating. In this case, W is negative, the heat input is *less* than in the constant-volume case, and C_p is *less* than C_V.

19.18 Measuring the molar heat capacity of an ideal gas (a) at constant volume and (b) at constant pressure.



Relating C_p and C_V for an Ideal Gas

We can derive a simple relationship between C_p and C_V for an ideal gas. First consider the constant-*volume* process. We place n moles of an ideal gas at temperature T in a constant-volume container. We place it in thermal contact with a hotter body; an infinitesimal quantity of heat dQ flows into the gas, and its temperature increases by an infinitesimal amount dT. By the definition of C_V, the molar heat capacity at constant volume,

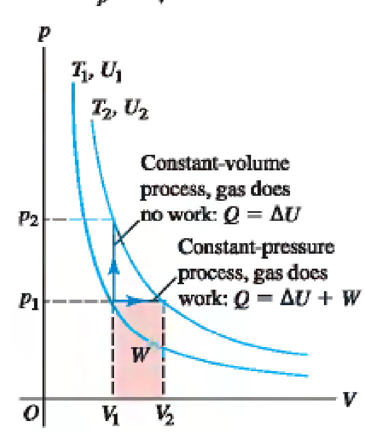
$$dQ = nC_V\,dT \tag{19.12}$$

The pressure increases during this process, but the gas does no work $(dW = 0)$ because the volume is constant. The first law in differential form, Eq. (19.6), is $dQ = dU + dW$. Since $dW = 0$, $dQ = dU$ and Eq. (19.12) can also be written as

$$dU = nC_V\,dT \tag{19.13}$$

Now consider a constant-*pressure* process with the same temperature change dT. We place the same gas in a cylinder with a piston that we can allow to move just enough to maintain constant pressure, as shown in Fig. 19.18b. Again we bring the system into contact with a hotter body. As heat flows into the gas, it

19.19 Raising the temperature of an ideal gas from T_1 to T_2 by a constant-volume or a constant-pressure process. For an ideal gas, U depends only on T, so ΔU is the same for both processes. But for the constant-pressure process, more heat Q must be added to both increase U and do work W. Hence $C_p > C_V$.



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8.7 Heat Capacity
8.8 Isochoric Process
8.9 Isobaric Process
8.10 Isothermmal Process

expands at constant pressure and does work. By the definition of C_p, the molar heat capacity at constant pressure, the amount of heat dQ entering the gas is

$$dQ = nC_p\,dT \tag{19.14}$$

The work dW done by the gas in this constant-pressure process is

$$dW = p\,dV$$

We can also express dW in terms of the temperature change dT by using the ideal-gas equation of state, $pV = nRT$. Because p is constant, the change in V is proportional to the change in T:

$$dW = p\,dV = nR\,dT \tag{19.15}$$

Now we substitute Eqs. (19.14) and (19.15) into the first law, $dQ = dU + dW$. We obtain

$$nC_p\,dT = dU + nR\,dT \tag{19.16}$$

Now here comes the crux of the calculation. The internal energy change dU for the constant-pressure process is again given by Eq. (19.13), $dU = nC_V\,dT$, *even though now the volume is not constant.* Why is this so? Recall the discussion of Section 19.6; one of the special properties of an ideal gas is that its internal energy depends *only* on temperature. Thus the *change* in internal energy during any process must be determined only by the temperature change. If Eq. (19.13) is valid for an ideal gas for one particular kind of process, it must be valid for an ideal gas for *every* kind of process with the same dT. So we may replace dU in Eq. (19.16) by $nC_V\,dT$:

$$nC_p\,dT = nC_V\,dT + nR\,dT$$

When we divide each term by the common factor $n\,dT$, we get

$$C_p = C_V + R \qquad \text{(molar heat capacities of an ideal gas)} \tag{19.17}$$

As we predicted, the molar heat capacity of an ideal gas at constant pressure is *greater* than the molar heat capacity at constant volume; the difference is the gas constant R. (Of course, R must be expressed in the same units as C_p and C_V, such as $\text{J/mol}\cdot\text{K}$.)

We have used the ideal-gas model to derive Eq. (19.17), but it turns out to be obeyed to within a few percent by many real gases at moderate pressures. Measured values of C_p and C_V are given in Table 19.1 for several real gases at low pressures; the difference in most cases is approximately $R = 8.314\ \text{J/mol}\cdot\text{K}$.

The table also shows that the molar heat capacity of a gas is related to its molecular structure, as we discussed in Section 18.4. In fact, the first two columns of Table 19.1 are the same as Table 18.1.

Table 19.1 Molar Heat Capacities of Gases at Low Pressure

Type of Gas	Gas	C_V (J/mol·K)	C_p (J/mol·K)	$C_p - C_V$ (J/mol·K)	$\gamma = C_p/C_V$
Monatomic	He	12.47	20.78	8.31	1.67
	Ar	12.47	20.78	8.31	1.67
Diatomic	H_2	20.42	28.74	8.32	1.41
	N_2	20.76	29.07	8.31	1.40
	O_2	20.85	29.17	8.31	1.40
	CO	20.85	29.16	8.31	1.40
Polyatomic	CO_2	28.46	36.94	8.48	1.30
	SO_2	31.39	40.37	8.98	1.29
	H_2S	25.95	34.60	8.65	1.33

The Ratio of Heat Capacities

The last column of Table 19.1 lists the values of the dimensionless **ratio of heat capacities, C_p/C_V,** denoted by γ (the Greek letter gamma):

$$\gamma = \frac{C_p}{C_V} \quad \text{(ratio of heat capacities)} \tag{19.18}$$

(This is sometimes called the "ratio of specific heats.") For gases, C_p is always greater than C_V and γ is always greater than unity. This quantity plays an important role in *adiabatic* processes for an ideal gas, which we will study in the next section.

We can use our kinetic-theory discussion of the molar heat capacity of an ideal gas (see Section 18.4) to predict values of γ. As an example, an ideal monatomic gas has $C_V = \frac{3}{2}R$. From Eq. (19.17),

$$C_p = C_V + R = \frac{3}{2}R + R = \frac{5}{2}R$$

so

$$\gamma = \frac{C_p}{C_V} = \frac{\frac{5}{2}R}{\frac{3}{2}R} = \frac{5}{3} = 1.67$$

As Table 19.1 shows, this agrees well with values of γ computed from measured heat capacities. For most diatomic gases near room temperature, $C_V = \frac{5}{2}R$, $C_p = C_V + R = \frac{7}{2}R$, and

$$\gamma = \frac{C_p}{C_V} = \frac{\frac{7}{2}R}{\frac{5}{2}R} = \frac{7}{5} = 1.40$$

also in good agreement with measured values.

Here's a final reminder: For an ideal gas the internal energy change in *any* process is given by $\Delta U = nC_V\,\Delta T$, *whether the volume is constant or not.* This relationship, which comes in handy in the following example, holds for other substances *only* when the volume is constant.

Example 19.6 Cooling your room

A typical dorm room or bedroom contains about 2500 moles of air. Find the change in the internal energy of this much air when it is cooled from 23.9°C to 11.6°C at a constant pressure of 1.00 atm. Treat the air as an ideal gas with $\gamma = 1.400$.

SOLUTION

IDENTIFY: Our target variable is the change in the internal energy ΔU of an ideal gas in a constant-pressure process. We are given the number of moles and the temperature change.

SET UP: Your first impulse may be to find C_p and then calculate Q from $Q = nC_p\,\Delta T$; find the volume change and find the work done by the gas from $W = p\,\Delta V$; then finally use the first law to find ΔU. This would be perfectly correct, but there's a much easier way. For an ideal gas the internal energy change is $\Delta U = nC_V\,\Delta T$ for *every* process, *whether the volume is constant or not.* So all we have to do is find C_V and use this expression for ΔU.

EXECUTE: We are given the value of γ for air, so we use Eqs. (19.17) and (19.18) to determine C_V:

$$\gamma = \frac{C_p}{C_V} = \frac{C_V + R}{C_V} = 1 + \frac{R}{C_V}$$

$$C_V = \frac{R}{\gamma - 1} = \frac{8.314\ \text{J/mol}\cdot\text{K}}{1.400 - 1} = 20.79\ \text{J/mol}\cdot\text{K}$$

Then

$$\begin{aligned}\Delta U &= nC_V\,\Delta T \\ &= (2500\ \text{mol})(20.79\ \text{J/mol}\cdot\text{K})(11.6^\circ\text{C} - 23.9^\circ\text{C}) \\ &= -6.39 \times 10^5\ \text{J}\end{aligned}$$

EVALUATE: A room air conditioner must extract this much internal energy from the air in your room and transfer it to the air outside. We'll discuss how this is done in Chapter 20.

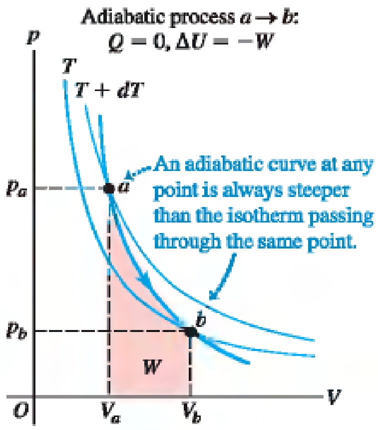
Test Your Understanding of Section 19.7 You want to cool a storage cylinder containing 10 moles of compressed gas from 30°C to 20°C. For which kind of gas would this be easiest? (i) a monatomic gas; (ii) a diatomic gas; (iii) a polyatomic gas; (iv) it would be equally easy for all of these.

19.8 Adiabatic Processes for an Ideal Gas

An adiabatic process, defined in Section 19.5, is a process in which no heat transfer takes place between a system and its surroundings. Zero heat transfer is an idealization, but a process is approximately adiabatic if the system is well insulated or if the process takes place so quickly that there is not enough time for appreciable heat flow to occur.

In an adiabatic process, $Q = 0$, so from the first law, $\Delta U = -W$. An adiabatic process for an ideal gas is shown in the pV-diagram of Fig. 19.20. As the gas expands from volume V_a to V_b, it does positive work, so its internal energy decreases and its temperature drops. If point a, representing the initial state, lies on an isotherm at temperature $T + dT$, then point b for the final state is on a different isotherm at a lower temperature T. For an ideal gas an adiabatic curve (adiabat) at any point is always *steeper* than the isotherm passing through the same point. For an adiabatic *compression* from V_b to V_a the situation is reversed and the temperature rises.

19.20 A pV-diagram of an adiabatic ($Q = 0$) process for an ideal gas. As the gas expands from V_a to V_b, it does positive work W on its environment, its internal energy decreases ($\Delta U = -W < 0$), and its temperature drops from $T + dT$ to T. (An adiabatic process is also shown in Fig. 19.16.)



The air in the output hoses of air compressors used in gasoline stations, in paint-spraying equipment, and to fill scuba tanks is always warmer than the air entering the compressor; this is because the compression is rapid and hence approximately adiabatic. Adiabatic *cooling* occurs when you open a bottle of your favorite carbonated beverage. The gas just above the beverage surface expands rapidly in a nearly adiabatic process; the temperature of the gas drops so much that water vapor in the gas condenses, forming a miniature cloud (see Fig. 19.14).

CAUTION **"Heating" and "cooling" without heat** Keep in mind that when we talk about "adiabatic heating" and "adiabatic cooling," we really mean "raising the temperature" and "lowering the temperature," respectively. In an adiabatic process, the temperature change is due to work done by or on the system; there is *no* heat flow at all.

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8.11 Adiabatic Process

Adiabatic Ideal Gas: Relating V, T, and p

We can derive a relationship between volume and temperature changes for an infinitesimal adiabatic process in an ideal gas. Equation (19.13) gives the internal energy change dU for *any* process for an ideal gas, adiabatic or not, so we have $dU = nC_V\,dT$. Also, the work done by the gas during the process is given by $dW = p\,dV$. Then, since $dU = -dW$ for an adiabatic process, we have

$$nC_V\,dT = -p\,dV \tag{19.19}$$

To obtain a relationship containing only the volume V and temperature T, we eliminate p using the ideal-gas equation in the form $p = nRT/V$. Substituting this into Eq.(19.19) and rearranging, we get

$$nC_V\,dT = -\frac{nRT}{V}dV$$

$$\frac{dT}{T} + \frac{R}{C_V}\frac{dV}{V} = 0$$

The coefficient R/C_V can be expressed in terms of $\gamma = C_p/C_V$. We have

$$\frac{R}{C_V} = \frac{C_p - C_V}{C_V} = \frac{C_p}{C_V} - 1 = \gamma - 1$$

$$\frac{dT}{T} + (\gamma - 1)\frac{dV}{V} = 0 \tag{19.20}$$

Because γ is always greater than unity for a gas, $(\gamma - 1)$ is always positive. This means that in Eq. (19.20), dV and dT always have opposite signs. An adiabatic *expansion* of an ideal gas $(dV > 0)$ always occurs with a *drop* in temperature $(dT < 0)$, and an adiabatic *compression* $(dV < 0)$ always occurs with a *rise* in temperature $(dT > 0)$; this confirms our earlier prediction.

For finite changes in temperature and volume we integrate Eq. (19.20), obtaining

$$\ln T + (\gamma - 1)\ln V = \text{constant}$$
$$\ln T + \ln V^{\gamma-1} = \text{constant}$$
$$\ln(TV^{\gamma-1}) = \text{constant}$$

and finally,

$$TV^{\gamma-1} = \text{constant} \tag{19.21}$$

Thus for an initial state (T_1, V_1) and a final state (T_2, V_2),

$$T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1} \quad \text{(adiabatic process, ideal gas)} \tag{19.22}$$

Because we have used the ideal-gas equation in our derivation of Eqs. (19.21) and (19.22), the T's must always be *absolute* (Kelvin) temperatures.

We can also convert Eq. (19.21) into a relationship between pressure and volume by eliminating T, using the ideal-gas equation in the form $T = pV/nR$. Substituting this into Eq. (19.21), we find

$$\frac{pV}{nR}V^{\gamma-1} = \text{constant}$$

or, because n and R are constant,

$$pV^{\gamma} = \text{constant} \tag{19.23}$$

For an initial state (p_1, V_1) and a final state (p_2, V_2), Eq. (19.23) becomes

$$p_1V_1^{\gamma} = p_2V_2^{\gamma} \quad \text{(adiabatic process, ideal gas)} \tag{19.24}$$

We can also calculate the *work* done by an ideal gas during an adiabatic process. We know that $Q = 0$ and $W = -\Delta U$ for *any* adiabatic process. For an ideal gas, $\Delta U = nC_V(T_2 - T_1)$. If the number of moles n and the initial and final temperatures T_1 and T_2 are known, we have simply

$$W = nC_V(T_1 - T_2) \quad \text{(adiabatic process, ideal gas)} \tag{19.25}$$

We may also use $pV = nRT$ in this equation to obtain

$$W = \frac{C_V}{R}(p_1V_1 - p_2V_2) = \frac{1}{\gamma - 1}(p_1V_1 - p_2V_2) \quad \text{(adiabatic process, ideal gas)} \tag{19.26}$$

(We used the result $C_V = R/(\gamma - 1)$ from Example 19.6.) If the process is an expansion, the temperature drops, T_1 is greater than T_2, p_1V_1 is greater than p_2V_2, and the work is *positive*, as we should expect. If the process is a compression, the work is negative.

Throughout this analysis of adiabatic processes we have used the ideal-gas equation of state, which is valid only for *equilibrium* states. Strictly speaking, our results are valid only for a process that is fast enough to prevent appreciable heat exchange with the surroundings (so that $Q = 0$ and the process is adiabatic), yet slow enough that the system does not depart very much from thermal and mechanical equilibrium. Even when these conditions are not strictly satisfied, though, Eqs. (19.22), (19.24), and (19.26) give useful approximate results.

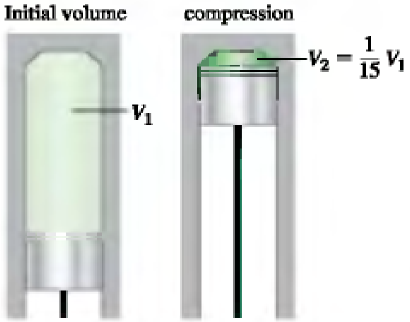
Example 19.7 Adiabatic compression in a diesel engine

The compression ratio of a diesel engine is 15 to 1; this means that air in the cylinders is compressed to $\frac{1}{15}$ of its initial volume (Fig. 19.21). If the initial pressure is 1.01×10^5 Pa and the initial temperature is 27°C (300 K), find the final pressure and the temperature after compression. Air is mostly a mixture of diatomic oxygen and nitrogen; treat it as an ideal gas with $\gamma = 1.40$.

SOLUTION

IDENTIFY: Since this problem involves the adiabatic compression of an ideal gas, we can use the ideas of this section.

19.21 Adiabatic compression of air in a cylinder of a diesel engine.



SET UP: We are given the initial pressure $p_1 = 1.01 \times 10^5$ Pa and the initial temperature $T_1 = 300$ K, and we are told that the ratio of initial and final volumes is $V_1/V_2 = 15$. We can find the final temperature T_2 using Eq. (19.22) and the final pressure p_2 using Eq. (19.24).

EXECUTE: From Eq. (19.22),

$$T_2 = T_1\left(\frac{V_1}{V_2}\right)^{\gamma-1} = (300\text{ K})(15)^{0.40} = 886\text{ K} = 613°\text{C}$$

From Eq. (19.24),

$$p_2 = p_1\left(\frac{V_1}{V_2}\right)^{\gamma} = (1.01 \times 10^5\text{ Pa})(15)^{1.40}$$
$$= 44.8 \times 10^5\text{ Pa} = 44\text{ atm}$$

EVALUATE: If the compression had been isothermal, the final pressure would have been 15 atm, but because the temperature also increases during an adiabatic compression, the final pressure is much greater. When fuel is injected into the cylinders near the end of the compression stroke, the high temperature of the air attained during compression causes the fuel to ignite spontaneously without the need for spark plugs.

Example 19.8 Work done in an adiabatic process

In Example 19.7, how much work does the gas do during the compression if the initial volume of the cylinder is $1.00\text{ L} = 1.00 \times 10^{-3}\text{ m}^3$? Assume that C_V for air is 20.8 J/mol · K and $\gamma = 1.40$.

SOLUTION

IDENTIFY: Our target variable is the work done *by* the gas during the adiabatic compression. We are given the initial volume of the gas, and we know (from Example 19.7) the initial and final values of temperature and pressure.

SET UP: We use Eq. (19.25) to determine the work done. We are not given the number of moles n, but we can calculate it from the given information using the ideal-gas law $pV = nRT$.

EXECUTE: The number of moles is

$$n = \frac{p_1V_1}{RT_1} = \frac{(1.01 \times 10^5\text{ Pa})(1.00 \times 10^{-3}\text{ m}^3)}{(8.314\text{ J/mol}\cdot\text{K})(300\text{ K})}$$
$$= 0.0405\text{ mol}$$

and Eq. (19.25) gives

$$W = nC_V(T_1 - T_2)$$
$$= (0.0405\text{ mol})(20.8\text{ J/mol}\cdot\text{K})(300\text{ K} - 886\text{ K})$$
$$= -494\text{ J}$$

EVALUATE: We can check our result using Eq. (19.26), the alternative expression for work done by an ideal gas in an adiabatic process:

$$W = \frac{1}{\gamma - 1}(p_1V_1 - p_2V_2)$$
$$= \frac{1}{1.40 - 1}\left[\begin{array}{l}(1.01 \times 10^5\text{ Pa})(1.00 \times 10^{-3}\text{ m}^3) \\ -(44.8 \times 10^5\text{ Pa})\left(\dfrac{1.00 \times 10^{-3}\text{ m}^3}{15}\right)\end{array}\right]$$
$$= -494\text{ J}$$

The work is negative because the gas is compressed.

Test Your Understanding of Section 19.8 You have four samples of ideal gas, each of which contains the same number of moles of gas and has the same initial temperature, volume, and pressure. You compress each sample to one-half of its initial volume. Rank the four samples in order from highest to lowest value of the final pressure. (i) a monatomic gas compressed isothermally; (ii) a monatomic gas compressed adiabatically; (iii) a diatomic gas compressed isothermally; (iv) a diatomic gas compressed adiabatically.

CHAPTER 19 SUMMARY

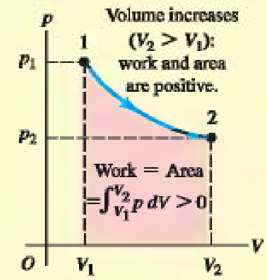
Heat and work in thermodynamic processes: A thermodynamic system has the potential to exchange energy with its surroundings by heat transfer or by mechanical work. When a system at pressure p changes volume from V_1 to V_2, it does an amount of work W given by the integral of p with respect to volume. If the pressure is constant, the work done is equal to p times the change in volume. A negative value of W means that work is done on the system. (See Example 19.1.)

In any thermodynamic process, the heat added to the system and the work done by the system depend not only on the initial and final states, but also on the path (the series of intermediate states through which the system passes).

$$W = \int_{V_1}^{V_2} p\, dV \tag{19.2}$$

$$W = p(V_2 - V_1) \tag{19.3}$$

(constant pressure only)



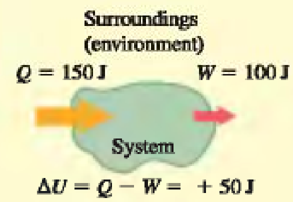
The first law of thermodynamics: The first law of thermodynamics states that when heat Q is added to a system while the system does work W, the internal energy U changes by an amount equal to $Q - W$. This law can also be expressed for an infinitesimal process. (See Examples 19.2, 19.3, and 19.5.)

The internal energy of any thermodynamic system depends only on its state. The change in internal energy in any process depends only on the initial and final states, not on the path. The internal energy of an isolated system is constant. (See Example 19.4.)

$$\Delta U = Q - W \tag{19.4}$$

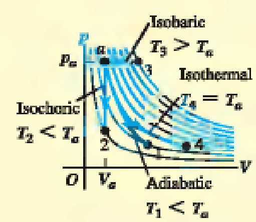
$$dU = dQ - dW \tag{19.6}$$

(infinitesimal process)



Important types of thermodynamic processes:

- Adiabatic process: No heat transfer into or out of a system; $Q = 0$.
- Isochoric process: Constant volume; $W = 0$.
- Isobaric process: Constant pressure; $W = p(V_2 - V_1)$.
- Isothermal process: Constant temperature.

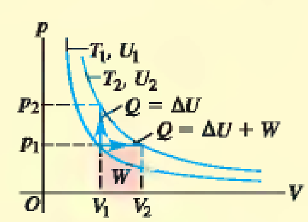


Thermodynamics of ideal gases: The internal energy of an ideal gas depends only on its temperature, not its pressure or volume. For other substances the internal energy generally depends on both pressure and temperature.

The molar heat capacities C_V and C_p of an ideal gas differ by R, the ideal gas constant. The dimensionless ratio of heat capacities, C_p/C_V, is denoted by γ. (See Example 19.6.)

$$C_p = C_V + R \tag{19.17}$$

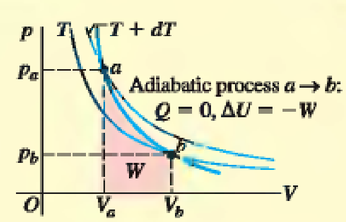
$$\gamma = \frac{C_p}{C_V} \tag{19.18}$$



Adiabatic processes in ideal gases: For an adiabatic process for an ideal gas, the quantities $TV^{\gamma-1}$ and pV^{γ} are constant. The work done by an ideal gas during an adiabatic expansion can be expressed in terms of the initial and final values of temperature, or in terms of the initial and final values of pressure and volume. (See Examples 19.7 and 19.8.)

$$W = nC_V(T_1 - T_2) = \frac{C_V}{R}(p_1V_1 - p_2V_2) \tag{19.25}$$

$$= \frac{1}{\gamma - 1}(p_1V_1 - p_2V_2) \tag{19.26}$$



Key Terms

Answer to Chapter Opening Question ?

No. The work done by a gas as its volume changes from V_1 to V_2 is equal to the integral $\int p\,dV$ between those two volume limits. If the volume of the gas contracts, the final volume V_2 is less than the initial volume V_1 and the gas does negative work. Propelling the locomotive requires that the gas do positive work, so the gas doesn't contribute to propulsion while contracting.

Answers to Test Your Understanding Questions

19.1 Answers: negative, positive, positive Heat flows out of the coffee, so $Q_{\text{coffee}} < 0$; heat flows into the aluminum cup, so $Q_{\text{aluminum}} > 0$. In mechanics, we would say that negative work is done *on* the block, since the surface exerts a force on the block that opposes the block's motion. But in thermodynamics we use the opposite convention and say that $W > 0$, which means that positive work is done *by* the block on the surface.

19.2 Answer: (ii) The work done in an expansion is represented by the area under the curve of pressure p versus volume V. In an isothermal expansion the pressure decreases as the volume increases, so the pV-diagram looks like Fig. 19.6a and the work done equals the shaded area under the blue curve from point 1 to point 2. If, however, the expansion is at constant pressure, the curve of p versus V would be the same as the dashed horizontal line at pressure p_2 in Fig. 19.6a. The area under this dashed line is smaller than the area under the blue curve for an isothermal expansion, so less work is done in the constant-pressure expansion than in the isothermal expansion.

19.3 Answer: (i) and (iv) (tie), (ii) and (iii) (tie) The accompanying figure shows the pV-diagrams for each of the four processes. The trapezoidal area under the curve, and hence the absolute value of the work, is the same in all four cases. In cases (i) and (iv) the volume increases, so the system does positive work as it expands against its surroundings. In cases (ii) and (iii) the volume decreases, so the system does negative work (shown by cross-hatching) as the surroundings push inward on it.

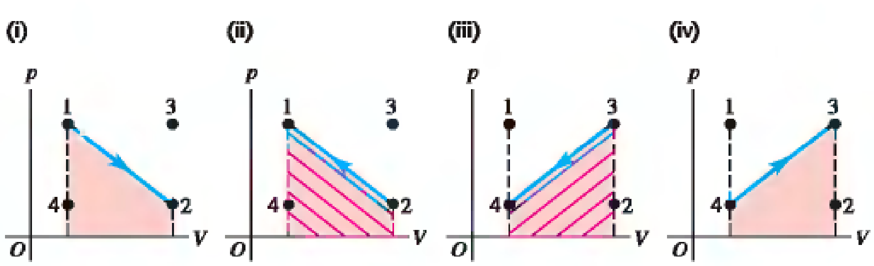
19.4 Answer: (ii), (i) and (iv) (tie), (iii) In the expression $\Delta U = Q - W$, Q is the heat *added* to the system and W is the work done *by* the system. If heat is transferred from the system to its surroundings, Q is negative; if work is done on the system, W is negative. Hence we have (i) $Q = -250\text{ J}$, $W = -250\text{ J}$, $\Delta U = -250\text{ J} - (-250\text{ J}) = 0$; (ii) $Q = 250\text{ J}$, $W = -250\text{ J}$, $\Delta U = 250\text{ J} - (-250\text{ J}) = 500\text{ J}$; (iii) $Q = -250\text{ J}$, $W = 250\text{ J}$, $\Delta U = -250\text{ J} - 250\text{ J} = -500\text{ J}$; and (iv) $Q = 250\text{ J}$, $W = 250\text{ J}$, $\Delta U = 250\text{ J} - 250\text{ J} = 0$.

19.5 Answers: 1 → 4 and 3 → 2 are isochoric; 1 → 3 and 4 → 2 are isobaric; no In a pV-diagram like those shown in Fig. 19.7, isochoric processes are represented by vertical lines (lines of constant volume) and isobaric processes are represented by horizontal lines (lines of constant pressure). The process 1 → 2 in Fig. 19.7 is shown as a curved line, which superficially resembles the adiabatic and isothermal processes for an ideal gas in Fig. 19.16. Without more information we can't tell whether process 1 → 2 is isothermal, adiabatic, or neither.

19.6 Answer: no Using the model of a solid in Fig. 18.20, we can see that the internal energy of a solid *does* depend on its volume. Compressing the solid means compressing the "springs" between the atoms, thereby increasing their stored potential energy and hence the internal energy of the solid.

19.7 Answer: (i) For a given number of moles n and a given temperature change ΔT, the amount of heat that must be transferred out of a fixed volume of air is $Q = nC_V\Delta T$. Hence the amount of heat transfer required is least for the gas with the smallest value of C_V. From Table 19.1, the value of C_V is smallest for monatomic gases.

19.6 Answer: (ii), (iv), (i) and (iii) (tie) Samples (i) and (iii) are compressed isothermally, so $pV = \text{constant}$. The volume of each sample decreases to one-half of its initial value, so the final pressure is twice the initial pressure. By contrast, samples (ii) and (iv) are compressed adiabatically, so $pV^\gamma = \text{constant}$ and the pressure increases by a factor of 2^γ. Sample (ii) is a monatomic gas for which $\gamma = \frac{5}{3}$, so its final pressure is $2^{5/3} = 3.17$ times greater than the initial pressure. Sample (iv) is a diatomic gas for which $\gamma = \frac{7}{5}$, so its final pressure is greater than the initial pressure by a factor of $2^{7/5} = 2.64$.



PROBLEMS

For instructor-assigned homework, go to www.masteringphysics.com

Discussion Questions

Q19.1. For the following processes, is the work done by the system (defined as the expanding or contracting gas) on the environment positive or negative? (a) expansion of the burned gasoline–air mixture in the cylinder of an automobile engine; (b) opening a bottle of champagne; (c) filling a scuba tank with compressed air; (d) partial crumpling of a sealed, empty water bottle, as you drive from the mountains down to sea level.

Q19.2. It is not correct to say that a body contains a certain amount of heat, yet a body can transfer heat to another body. How can a body give away something it does not have in the first place?

Q19.3. In which situation must you do more work: inflating a balloon at sea level or inflating the same balloon to the same volume at the summit of Mt. McKinley? Explain in terms of pressure and volume change.

Q19.4. If you are told the initial and final states of a system and the associated change in internal energy, can you determine whether the internal energy change was due to work or to heat transfer? Explain.

Q19.5. Discuss the application of the first law of thermodynamics to a mountaineer who eats food, gets warm and perspires a lot during a climb, and does a lot of mechanical work in raising herself to the summit. The mountaineer also gets warm during the descent. Is the source of this energy the same as the source during the ascent?

Q19.6. When ice melts at 0°C, its volume decreases. Is the internal energy change greater than, less than, or equal to the heat added? How can you tell?

Q19.7. You hold an inflated balloon over a hot air vent in your house and watch it slowly expand. You then remove it and let it cool back to room temperature. During the expansion, which was larger: the heat added to the balloon or the work done by the air inside it? Explain. (Assume that air is an ideal gas.) Once the balloon has returned to room temperature, how does the net heat gained or lost by the air inside it compare to the net work done on or by the surrounding air?

Q19.6. You bake chocolate chip cookies and put them, still warm, in a container with a loose (not airtight) lid. What kind of process does the air inside the container undergo as the cookies gradually cool to room temperature (isothermal, isochoric, adiabatic, isobaric, or some combination)? Explain your answer.

Q19.9. Imagine a gas made up entirely of negatively charged electrons. Like charges repel, so the electrons exert repulsive forces on each other. Would you expect that the temperature of such a gas would rise, fall, or stay the same in a free expansion? Why?

Q19.19. There are a few materials that contract when their temperature is increased, such as water between 0°C and 4°C. Would you expect C_p for such materials to be greater or less than C_V? Explain?

Q19.11. When you blow on the back of your hand with your mouth wide open, your breath feels warm. But if you partially close your mouth to form an "o" and then blow on your hand, your breath feels cool. Why?

Q19.12. In hot-air balloons, the air in the balloon envelope is heated through a hole in the bottom by a propane burner. The hot air inside the envelope stays at atmospheric pressure because of the hole in the bottom, and the volume of the envelope is essentially constant. Thus, when the pilot fires up the burner to heat the air, the volume of the envelope and the pressure inside it are constant, but the temperature rises. The ideal-gas law seems to forbid this. What's going on?

Q19.13. On a warm summer day, a large cylinder of compressed gas (propane or butane) is used to supply several large gas burners at a cookout. After a while, frost forms on the outside of the tank. Why?

Q19.14. When you use a hand pump to inflate the tires of your bicycle, the pump gets warm after a while. Why? What happens to the temperature of the air in the pump as you compress it? Why does this happen? When you raise the pump handle to draw outside air into the pump, what happens to the temperature of the air taken in? Again, why does this happen?

Q19.15. In the carburetor of an aircraft or automobile engine, air flows through a relatively small aperture and then expands. In cool, foggy weather, ice sometimes forms in this aperture even though the outside air temperature is above freezing. Why?

Q19.16. On a sunny day, large "bubbles" of air form on the sun-warmed earth, gradually expand, and finally break free to rise through the atmosphere. Soaring birds and glider pilots are fond of using these "thermals" to gain altitude easily. This expansion is essentially an adiabatic process. Why?

Q19.17. The prevailing winds on the Hawaiian island of Kauai blow from the northeast. The winds cool as they go up the slope of Mt. Waialeale (elevation 1523 m), causing water vapor to condense and rain to fall. There is much more precipitation at the summit than at the base of the mountain. In fact, Mt. Waialeale is the rainiest spot on earth, averaging 11.7 m of rainfall a year. But what makes the winds cool?

Q19.18. Applying the same considerations as in Question 19.17, explain why the island of Niihau, a few kilometers to the southwest of Kauai, is almost a desert and farms there need to be irrigated.

Q19.19. In a constant-volume process, $dU = nC_V dT$. But in a constant-pressure process, it is *not* true that $dU = nC_p dT$. Why not?

Q19.20. When a gas surrounded by air is compressed adiabatically, its temperature rises even though there is no heat input to the gas. Where does the energy come from to raise the temperature?

Q19.21. When a gas expands adiabatically, it does work on its surroundings. But if there is no heat input to the gas, where does the energy come from to do the work?

Q19.22. The gas used in separating the two uranium isotopes ^{235}U and ^{238}U has the formula UF_6. If you added heat at equal rates to a mole of UF_6 gas and a mole of H_2 gas, which one's temperature would you expect to rise faster? Explain.

Exercises

Section 19.2 Work Done During Volume Changes and Section 19.3 Paths Between Thermodynamic States

19.1. Two moles of an ideal gas are heated at constant pressure from $T = 27°C$ to $T = 107°C$. (a) Draw a pV-diagram for this process. (b) Calculate the work done by the gas.

19.2. Six moles of an ideal gas are in a cylinder fitted at one end with a movable piston. The initial temperature of the gas is 27.0°C and the pressure is constant. As part of a machine design project, calculate the final temperature of the gas after it has done 1.75×10^3 J of work.

19.3. Two moles of an ideal gas are compressed in a cylinder at a constant temperature of 85.0°C until the original pressure has tripled. (a) Sketch a pV-diagram for this process. (b) Calculate the amount of work done.

19.4. A metal cylinder with rigid walls contains 2.50 mol of oxygen gas. The gas is cooled until the pressure decreases to 30.0% of its original value. You can ignore the thermal contraction of the cylinder. (a) Draw a pV-diagram for this process. (b) Calculate the work done by the gas.

19.5. During the time 0.305 mol of an ideal gas undergoes an isothermal compression at 22.0°C, 518 J of work is done on it by the surroundings. (a) If the final pressure is 1.76 atm, what was the initial pressure? (b) Sketch a pV-diagram for the process.

19.6. A gas undergoes two processes. In the first, the volume remains constant at $0.200\ \text{m}^3$ and the pressure increases from 2.00×10^5 Pa to 5.00×10^5 Pa. The second process is a compression to a volume of $0.120\ \text{m}^3$ at a constant pressure of 5.00×10^5 Pa. (a) In a pV-diagram, show both processes. (b) Find the total work done by the gas during both processes.

19.7. Work Done in a Cyclic Process. (a) In Fig. 19.7a, consider the closed loop $1 \rightarrow 3 \rightarrow 2 \rightarrow 4 \rightarrow 1$. This is a *cyclic* process in which the initial and final states are the same. Find the total work done by the system in this cyclic process, and show that it is equal to the area enclosed by the loop. (b) How is the work done for the process in part (a) related to the work done if the loop is traversed in the opposite direction, $1 \rightarrow 4 \rightarrow 2 \rightarrow 3 \rightarrow 1$? Explain.

Section 19.4 Internal Energy and the First Law of Thermodynamics

19.8. You close off the nozzle of a bicycle tire pump and very slowly depress the plunger so that the air inside is compressed to half its original volume. Assume the air behaves like an ideal gas. If you do this so slowly that the temperature of the air inside the pump never changes: (a) Is the work done by the air in the pump positive or negative? (b) Is the heat flow to the air positive or negative? (c) What can you say about the relative *magnitudes* of the heat flow and the work? Explain.

19.9. A gas in a cylinder expands from a volume of $0.110\ \text{m}^3$ to $0.320\ \text{m}^3$. Heat flows into the gas just rapidly enough to keep the pressure constant at 1.80×10^5 Pa during the expansion. The total heat added is 1.15×10^5 J. (a) Find the work done by the gas. (b) Find the change in internal energy of the gas. (c) Does it matter whether the gas is ideal? Why or why not?

19.10. Five moles of an ideal monatomic gas with an initial temperature of 127°C expand and, in the process, absorb 1200 J of heat and do 2100 J of work. What is the final temperature of the gas?

19.11. You kick a soccer ball, compressing it suddenly to $\frac{2}{3}$ of its original volume. In the process, you do 410 J of work on the air (assumed to be an ideal gas) inside the ball. (a) What is the change in internal energy of the air inside the ball due to being compressed? (b) Does the temperature of the air inside the ball rise or fall due to being compressed? Explain.

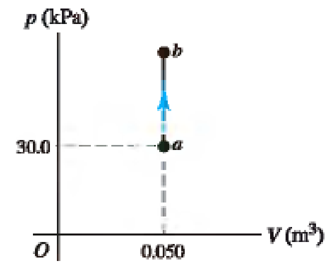
19.12. A gas in a cylinder is held at a constant pressure of 2.30×10^5 Pa and is cooled and compressed from $1.70\ \text{m}^3$ to $1.20\ \text{m}^3$. The internal energy of the gas decreases by 1.40×10^5 J. (a) Find the work done by the gas. (b) Find the absolute value $|Q|$ of the heat flow into or out of the gas, and state the direction of the heat flow. (c) Does it matter whether the gas is ideal? Why or why not?

19.13. Doughnuts: Breakfast of Champions! A typical doughnut contains 2.0 g of protein, 17.0 g of carbohydrates, and 7.0 g of fat. The average food energy values of these substances are 4.0 kcal/g for protein and carbohydrates and 9.0 kcal/g for fat. (a) During heavy exercise, an average person uses energy at a rate of 510 kcal/h. How long would you have to exercise to "work off" one doughnut? (b) If the energy in the doughnut could somehow be converted into the kinetic energy of your body as a whole, how fast could you move after eating the doughnut? Take your mass to be 60 kg, and express your answer in m/s and km/h.

19.14. A liquid is irregularly stirred in a well-insulated container and thereby undergoes a rise in temperature. Regard the liquid as the system. (a) Has heat been transferred? How can you tell? (b) Has work been done? How can you tell? Why is it important that the stirring is irregular? (c) What is the sign of ΔU? How can you tell?

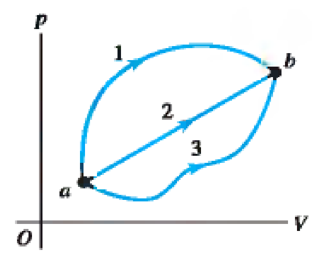
19.15. An ideal gas is taken from a to b on the pV-diagram shown in Fig. 19.22. During this process, 400 J of heat is added and the pressure doubles. (a) How much work is done by or on the gas? Explain. (b) How does the temperature of the gas at a compare to its temperature at b? Be specific. (c) How does the internal energy of the gas at a compare to the internal energy at b? Again, be specific and explain.

Figure 19.22 Exercise 19.15.



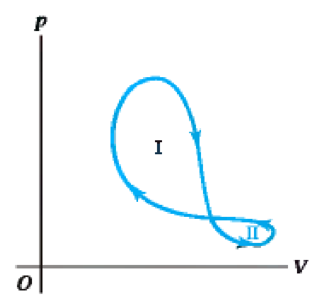
19.16. A system is taken from state a to state b along the three paths shown in Fig. 19.23. (a) Along which path is the work done by the system the greatest? The least? (b) If $U_b > U_a$, along which path is the absolute value $|Q|$ of the heat transfer the greatest? For this path, is heat absorbed or liberated by the system?

Figure 19.23 Exercise 19.16.



19.17. A thermodynamic system undergoes a cyclic process as shown in Fig. 19.24. The cycle consists of two closed loops: I and II. (a) Over one complete cycle, does the system do positive or negative work? (b) In each of loops I and II, is the net work done by the system positive or negative? (c) Over one complete cycle, does heat flow into or out of the system? (d) In each of loops I and II, does heat flow into or out of the system?

Figure 19.24 Exercise 19.17.



19.18. A student performs a combustion experiment by burning a mixture of fuel and oxygen in a constant-volume metal can surrounded by a water bath. During the experiment the temperature of the water is observed to rise. Regard the mixture of fuel and oxygen as the system. (a) Has heat been transferred? How can you tell? (b) Has work been done? How can you tell? (c) What is the sign of ΔU? How can you tell?

19.19. Boiling Water at High Pressure. When water is boiled at a pressure of 2.00 atm, the heat of vaporization is 2.20×10^6 J/kg and the boiling point is 120°C. At this pressure, 1.00 kg of water has a volume of $1.00 \times 10^{-3}\ \text{m}^3$, and 1.00 kg of steam has a volume of $0.824\ \text{m}^3$. (a) Compute the work done when 1.00 kg of steam is formed at this temperature. (b) Compute the increase in internal energy of the water.

Section 19.5 Kinds of Thermodynamic Processes, Section 19.6 Internal Energy of an Ideal Gas, and Section 19.7 Heat Capacities of an Ideal Gas

19.20. During an isothermal compression of an ideal gas, 335 J of heat must be removed from the gas to maintain constant temperature. How much work is done by the gas during the process?

19.21. A cylinder contains 0.250 mol of carbon dioxide (CO_2) gas at a temperature of 27.0°C. The cylinder is provided with a frictionless piston, which maintains a constant pressure of 1.00 atm on the gas. The gas is heated until its temperature increases to 127.0°C. Assume that the CO_2 may be treated as an ideal gas. (a) Draw a pV-diagram for this process. (b) How much work is done by the gas in this process? (c) On what is this work done? (d) What is the change in internal energy of the gas? (e) How much heat was supplied to the gas? (f) How much work would have been done if the pressure had been 0.50 atm?

19.22. A cylinder contains 0.0100 mol of helium at $T = 27.0°C$. (a) How much heat is needed to raise the temperature to 67.0°C while keeping the volume constant? Draw a pV-diagram for this process. (b) If instead the pressure of the helium is kept constant, how much heat is needed to raise the temperature from 27.0°C to 67.0°C? Draw a pV-diagram for this process. (c) What accounts for the difference between your answers to parts (a) and (b)? In which case is more heat required? What becomes of the additional heat? d) If the gas is ideal, what is the change in its internal energy in part (a)? In part (b)? How do the two answers compare? Why?

19.23. In an experiment to simulate conditions inside an automobile engine, 0.185 mol of air at a temperature of 780 K and a pressure of 3.00×10^6 Pa is contained in a cylinder of volume 40.0 cm^3. Then 645 J of heat is transferred to the cylinder. (a) If the volume of the cylinder is constant while the heat is added, what is the final temperature of the air? Assume that the air is essentially nitrogen gas, and use the data in Table 19.1 even though the pressure is not low. Draw a pV-diagram for this process. (b) If instead the volume of the cylinder is allowed to increase while the pressure remains constant, find the final temperature of the air. Draw a pV-diagram for this process.

19.24. An ideal gas expands while the pressure is kept constant. During this process, does heat flow into the gas or out of the gas? Justify your answer.

19.25. Heat Q flows into a monatomic ideal gas, and the volume increases while the pressure is kept constant. What fraction of the heat energy is used to do the expansion work of the gas?

19.26. When a quantity of monatomic ideal gas expands at a constant pressure of 4.00×10^4 Pa, the volume of the gas increases from 2.00×10^{-3} m^3 to 8.00×10^{-3} m^3. What is the change in the internal energy of the gas?

19.27. A cylinder with a movable piston contains 3.00 mol of N_2 gas (assumed to behave like an ideal gas). (a) The N_2 is heated at constant volume until 1557 J of heat have been added. Calculate the change in temperature. (b) Suppose the same amount of heat is added to the N_2, but this time the gas is allowed to expand while remaining at constant pressure. Calculate the temperature change. (c) In which case, (a) or (b), is the final internal energy of the N_2 higher? How do you know? What accounts for the difference between the two cases?

19.28. Three moles of an ideal monatomic gas expands at a constant pressure of 2.50 atm; the volume of the gas changes from 3.20×10^{-2} m^3 to 4.50×10^{-2} m^3. (a) Calculate the initial and final temperatures of the gas. (b) Calculate the amount of work the gas does in expanding. (c) Calculate the amount of heat added to the gas. (d) Calculate the change in internal energy of the gas.

19.29. The temperature of 0.150 mol of an ideal gas is held constant at 77.0°C while its volume is reduced to 25.0% of its initial volume. The initial pressure of the gas is 1.25 atm. (a) Determine the work done by the gas. (b) What is the change in its internal energy? (c) Does the gas exchange heat with its surroundings? If so, how much? Does the gas absorb or liberate heat?

19.30. Propane gas (C_3H_8) behaves like an ideal gas with $\gamma = 1.127$. Determine the molar heat capacity at constant volume and the molar heat capacity at constant pressure.

19.31. An experimenter adds 970 J of heat to 1.75 mol of an ideal gas to heat it from 10.0°C to 25.0°C at constant pressure. The gas does +223 J of work during the expansion. (a) Calculate the change in internal energy of the gas. (b) Calculate γ for the gas.

Section 19.8 Adiabatic Processes for an Ideal Gas

19.32. In an adiabatic process for an ideal gas, the pressure decreases. In this process does the internal energy of the gas increase or decrease? Explain your reasoning.

19.33. A monatomic ideal gas that is initially at a pressure of 1.50×10^5 Pa and has a volume of 0.0800 m^3 is compressed adiabatically to a volume of 0.0400 m^3. (a) What is the final pressure? (b) How much work is done by the gas? (c) What is the ratio of the final temperature of the gas to its initial temperature? Is the gas heated or cooled by this compression?

19.34. The engine of a Ferrari F355 F1 sports car takes in air at 20.0°C and 1.00 atm and compresses it adiabatically to 0.0900 times the original volume. The air may be treated as an ideal gas with $\gamma = 1.40$. (a) Draw a pV-diagram for this process. (b) Find the final temperature and pressure.

19.35. Two moles of carbon monoxide (CO) start at a pressure of 1.2 atm and a volume of 30 liters. The gas is then compressed adiabatically to $\frac{1}{3}$ this volume. Assume that the gas may be treated as ideal. What is the change in the internal energy of the gas? Does the internal energy increase or decrease? Does the temperature of the gas increase or decrease during this process? Explain.

19.36. A player bounces a basketball on the floor, compressing it to 80.0% of its original volume. The air (assume it is essentially N_2 gas) inside the ball is originally at a temperature of 20.0°C and a pressure of 2.00 atm. The ball's diameter is 23.9 cm. (a) What temperature does the air in the ball reach at its maximum compression? (b) By how much does the internal energy of the air change between the ball's original state and its maximum compression?

19.37. During an adiabatic expansion the temperature of 0.450 mol of argon (Ar) drops from 50.0°C to 10.0°C. The argon may be treated as an ideal gas. (a) Draw a pV-diagram for this process. (b) How much work does the gas do? (c) What is the change in internal energy of the gas?

19.38. A cylinder contains 0.100 mol of an ideal monatomic gas. Initially the gas is at a pressure of 1.00×10^5 Pa and occupies a volume of 2.50×10^{-3} m^3. (a) Find the initial temperature of the gas in kelvins. (b) If the gas is allowed to expand to twice the initial volume, find the final temperature (in kelvins) and pressure of the gas if the expansion is (i) isothermal; (ii) isobaric; (iii) adiabatic.

19.38. On a warm summer day, a large mass of air (atmospheric pressure 1.01×10^5 Pa) is heated by the ground to a temperature of 26.0°C and then begins to rise through the cooler surrounding air. (This can be treated approximately as an adiabatic process; why?) Calculate the temperature of the air mass when it has risen to a level at which atmospheric pressure is only 0.850×10^5 Pa. Assume that air is an ideal gas, with $\gamma = 1.40$. (This rate of cooling for dry, rising air, corresponding to roughly 1°C per 100 m of altitude, is called the *dry adiabatic lapse rate*.)

Problems

19.40. Figure 19.25 shows the pV-diagram for an isothermal expansion of 1.50 mol of an ideal gas, at a temperature of 15.0°C. (a) What is the change in internal energy of the gas? Explain. (b) Calculate the work done by (or on) the gas and the heat absorbed (or released) by the gas during the expansion.

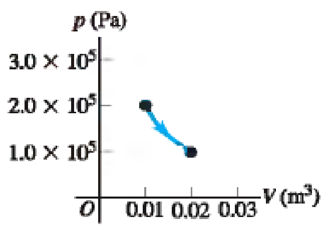


Figure 19.25 Problem 19.40.

19.41. A quantity of air is taken from state a to state b along a path that is a straight line in the pV-diagram (Fig. 19.26). (a) In this process, does the temperature of the gas increase, decrease, or stay the same? Explain. (b) If $V_a = 0.0700\ \text{m}^3$, $V_b = 0.1100\ \text{m}^3$, $p_a = 1.00 \times 10^5$ Pa, and $p_b = 1.40 \times 10^5$ Pa, what is the work W done by the gas in this process? Assume that the gas may be treated as ideal.

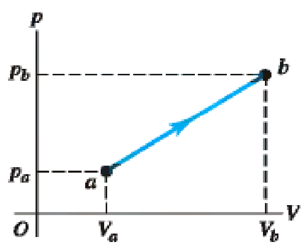
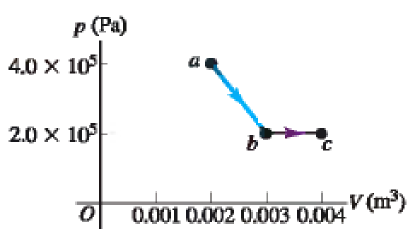


Figure 19.26 Problem 19.41.

19.42. One-half mole of an ideal gas is taken from state a to state c, as shown in Fig. 19.27. (a) Calculate the final temperature of the gas. (b) Calculate the work done on (or by) the gas as it moves from state a to state c. (c) Does heat leave the system or enter the system during this process? How much heat? Explain.

Figure 19.27 Problem 19.42.



19.43. When a system is taken from state a to state b in Fig. 19.28 along the path acb, 90.0 J of heat flows into the system and 60.0 J of work is done by the system. (a) How much heat flows into the system along path adb if the work done by the system is 15.0 J? (b) When the system is returned from b to a along the curved path, the absolute value of the work done by the system is 35.0 J. Does the system absorb or liberate heat? How much heat? (c) If $U_a = 0$ and $U_d = 8.0$ J, find the heat absorbed in the processes ad and db.

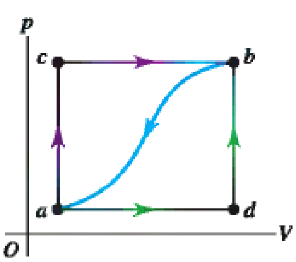


Figure 19.28 Problem 19.43.

19.44. A thermodynamic system is taken from state a to state c in Fig. 19.29 along either path abc or path adc. Along path abc, the work W done by the system is 450 J. Along path adc, W is 120 J. The internal energies of each of the four states shown in the figure are $U_a = 150$ J, $U_b = 240$ J, $U_c = 680$ J, and $U_d = 330$ J. Calculate the heat flow Q for each of the four processes ab, bc, ad, and dc. In each process, does the system absorb or liberate heat?

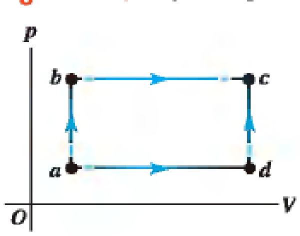


Figure 19.29 Problem 19.44.

19.45. A volume of air (assumed to be an ideal gas) is first cooled without changing its volume and then expanded without changing its pressure, as shown by the path abc in Fig. 19.30. (a) How does the final temperature of the gas compare with its initial temperature? (b) How much heat does the air exchange with its surroundings during the process abc? Does the air absorb heat or release heat during this process? Explain. (c) If the air instead expands from state a to state c by the straight-line path shown, how much heat does it exchange with its surroundings?

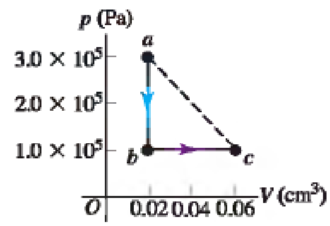


Figure 19.30 Problem 19.45.

19.46. Three moles of argon gas (assumed to be an ideal gas) originally at a pressure of 1.50×10^4 Pa and a volume of $0.0280\ \text{m}^3$ are first heated and expanded at constant pressure to a volume of $0.0435\ \text{m}^3$, then heated at constant volume until the pressure reaches 3.50×10^4 Pa, then cooled and compressed at constant pressure until the volume is again $0.0280\ \text{m}^3$, and finally cooled at constant volume until the pressure drops to its original value of 1.50×10^4 Pa. (a) Draw the pV-diagram for this cycle. (b) Calculate the total work done by (or on) the gas during the cycle. (c) Calculate the net heat exchanged with the surroundings. Does the gas gain or lose heat overall?

19.47. Two moles of an ideal monatomic gas go through the cycle abc. For the complete cycle, 800 J of heat flows out of the gas. Process ab is at constant pressure, and process bc is at constant volume. States a and b have temperatures $T_a = 200$ K and $T_b = 300$ K. (a) Sketch the pV-diagram for the cycle. (b) What is the work W for the process ca?

19.48. Three moles of an ideal gas are taken around the cycle abc shown in Fig. 19.31. For this gas, $C_p = 29.1\ \text{J/mol} \cdot \text{K}$. Process ac is at constant pressure, process ba is at constant volume, and process cb is adiabatic. The temperatures of the gas in states a, c, and b are $T_a = 300$ K, $T_c = 492$ K, and $T_b = 600$ K. Calculate the total work W for the cycle.

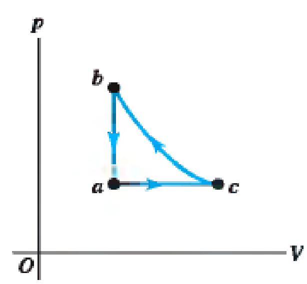


Figure 19.31 Problem 19.48.

19.46. Starting with 2.50 mol of N_2 gas (assumed to be ideal) in a cylinder at 1.00 atm and 20.0°C, a chemist first heats the gas at constant volume, adding 1.52×10^4 J of heat, then continues heating and allows the gas to expand at constant pressure to twice its original volume. (a) Calculate the final temperature of the gas. (b) Calculate the amount of work done by the gas. (c) Calculate the amount of heat added to the gas while it was expanding. (d) Calculate the change in internal energy of the gas for the whole process.

19.50. Nitrogen gas in an expandable container is cooled from 50.0°C to 10.0°C with the pressure held constant at 3.00×10^5 Pa. The total heat liberated by the gas is 2.50×10^4 J. Assume that the gas may be treated as ideal. (a) Find the number of moles of gas. (b) Find the change in internal energy of the gas. (c) Find the work done by the gas. (d) How much heat would be liberated by the gas for the same temperature change if the volume were constant?

19.51. In a certain process, 2.15×10^5 J of heat is liberated by a system, and at the same time the system contracts under a constant external pressure of 9.50×10^5 Pa. The internal energy of the system is the same at the beginning and end of the process. Find the change in volume of the system. (The system is *not* an ideal gas.)

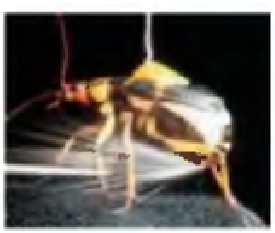
19.52. A cylinder with a frictionless, movable piston like that shown in Fig. 19.5 contains a quantity of helium gas. Initially the gas is at a pressure of 1.00×10^5 Pa, has a temperature of 300 K, and occupies a volume of 1.50 L. The gas then undergoes two processes. In the first, the gas is heated and the piston is allowed to move to keep the temperature equal to 300 K. This continues until the pressure reaches 2.50×10^4 Pa. In the second process, the gas is compressed at constant pressure until it returns to its original volume of 1.50 L. Assume that the gas may be treated as ideal. (a) In a pV-diagram, show both processes. (b) Find the volume of the gas at the end of the first process, and find the pressure and temperature at the end of the second process. (c) Find the total work done by the gas during both processes. (d) What would you have to do to the gas to return it to its original pressure and temperature?

19.53. A Thermodynamic Process in a Liquid. A chemical engineer is studying the properties of liquid methanol (CH_3OH). She uses a steel cylinder with a cross-sectional area of 0.0200 m^2 and containing 1.20×10^{-2} m^3 of methanol. The cylinder is equipped with a tightly fitting piston that supports a load of 3.00×10^4 N. The temperature of the system is increased from 20.0°C to 50.0°C. For methanol, the coefficient of volume expansion is 1.20×10^{-3} K^{-1}, the density is 791 kg/m^3, and the specific heat capacity at constant pressure is $c_p = 2.51 \times 10^3$ J/kg · K. You can ignore the expansion of the steel cylinder. Find (a) the increase in volume of the methanol; (b) the mechanical work done by the methanol against the 3.00×10^4 N force; (c) the amount of heat added to the methanol; (d) the change in internal energy of the methanol. (e) Based on your results, explain whether there is any substantial difference between the specific heat capacities c_p (at constant pressure) and c_V (at constant volume) for methanol under these conditions.

19.54. A Thermodynamic Process in a Solid. A cube of copper 2.00 cm on a side is suspended by a string. (The physical properties of copper are given in Tables 14.1, 17.2, and 17.3.) The cube is heated with a burner from 20.0°C to 90.0°C. The air surrounding the cube is at atmospheric pressure (1.01×10^5 Pa). Find (a) the increase in volume of the cube; (b) the mechanical work done by the cube to expand against the pressure of the surrounding air; (c) the amount of heat added to the cube; (d) the change in internal energy of the cube. (e) Based on your results, explain whether there is any substantial difference between the specific heat capacities c_p (at constant pressure) and c_V (at constant volume) for copper under these conditions.

19.55. A Thermodynamic Process in an Insect. The African bombardier beetle *Stenaptinus insignis* can emit a jet of defensive spray from the movable tip of its abdomen (Fig. 19.32). The beetle's body has reservoirs of two different chemicals; when the beetle is disturbed, these chemicals are combined in a reaction chamber, producing a compound that is warmed from 20°C to 100°C by the heat of reaction. The high pressure produced allows the compound to be sprayed out at speeds up to 19 m/s (68 km/h), scaring away predators of all kinds. (The beetle shown in the figure is 2 cm long.) Calculate the heat of reaction of the two chemicals (in J/kg). Assume that the specific heat capacity of the two chemicals and the spray is the same as that of water, 4.19×10^3 J/kg · K, and that the initial temperature of the chemicals is 20°C.

Figure 19.32 Problem 19.55.



19.56. High-Altitude Research. A large research balloon containing 2.00×10^3 m^3 of helium gas at 1.00 atm and a temperature of 15.0°C rises rapidly from ground level to an altitude at which the atmospheric pressure is only 0.900 atm (Fig. 19.33). Assume the helium behaves like an ideal gas and the balloon's ascent is too rapid to permit much heat exchange with the surrounding air. (a) Calculate the volume of the gas at the higher altitude. (b) calculate the temperature of the gas at the higher altitude. (c) What is the change in internal energy of the helium as the balloon rises to the higher altitude?

Figure 19.33 Problem 19.56.

19.57. Chinook. During certain seasons strong winds called chinooks blow from the west across the eastern slopes of the Rockies and downhill into Denver and nearby areas. Although the mountains are cool, the wind in Denver is very hot; within a few minutes after the chinook wind arrives, the temperature can climb 20 C° ("chinook" is a Native American word meaning "snow eater"). Similar winds occur in the Alps (called foehns) and in southern California (called Santa Anas). (a) Explain why the temperature of the chinook wind rises as it descends the slopes. Why is it important that the wind be fast moving? (b) Suppose a strong wind is blowing toward Denver (elevation 1630 m) from Grays Peak (80 km west of Denver, at an elevation of 4350 m), where the air pressure is 5.60×10^4 Pa and the air temperature is −15.0°C. The temperature and pressure in Denver before the wind arrives are 2.0°C and 8.12×10^4 Pa. By how many Celsius degrees will the temperature in Denver rise when the chinook arrives?

19.56. A certain ideal gas has molar heat capacity at constant volume C_V. A sample of this gas initially occupies a volume V_0 at pressure p_0 and absolute temperature T_0. The gas expands isobarically to a volume $2V_0$ and then expands further adiabatically to a final volume of $4V_0$. (a) Draw a pV-diagram for this sequence of processes. (b) Compute the total work done by the gas for this sequence of processes. (c) Find the final temperature of the gas. (d) Find the absolute value $|Q|$ of the total heat flow into or out of the gas for this sequence of processes, and state the direction of heat flow.

19.56. An air pump has a cylinder 0.250 m long with a movable piston. The pump is used to compress air from the atmosphere (at absolute pressure 1.01×10^3 Pa) into a very large tank at 4.20×10^5 Pa gauge pressure. (For air, $C_V = 20.8$ J/mol · K.) (a) The piston begins the compression stroke at the open end of the cylinder. How far down the length of the cylinder has the piston moved when air first begins to flow from the cylinder into the tank? Assume that the compression is adiabatic. (b) If the air is taken into the pump at 27.0°C, what is the temperature of the compressed air? (c) How much work does the pump do in putting 20.0 mol of air into the tank?

19.60. Engine Turbochargers and Intercoolers. The power output of an automobile engine is directly proportional to the mass of air that can be forced into the volume of the engine's cylinders to react chemically with gasoline. Many cars have a *turbocharger*, which compresses the air before it enters the engine, giving a greater mass of air per volume. This rapid, essentially adiabatic compression also heats the air. To compress it further, the air then passes through an *intercooler* in which the air exchanges heat with its surroundings at essentially constant pressure. The air is then drawn into the cylinders. In a typical installation, air is taken into the turbocharger at atmospheric pressure $(1.01 \times 10^5\ \mathrm{Pa})$, density $\rho = 1.23\ \mathrm{kg/m^3}$, and temperature 15.0°C. It is compressed adiabatically to 1.45×10^5 Pa. In the intercooler, the air is cooled to the original temperature of 15.0°C at a constant pressure of 1.45×10^5 Pa. (a) Draw a pV-diagram for this sequence of processes. b) If the volume of one of the engine's cylinders is 575 cm^3, what mass of air exiting from the intercooler will fill the cylinder at 1.45×10^5 Pa? Compared to the power output of an engine that takes in air at 1.01×10^5 Pa at 15.0°C, what percentage increase in power is obtained by using the turbocharger and intercooler? (c) If the intercooler is not used, what mass of air exiting from the turbocharger will fill the cylinder at 1.45×10^5 Pa? Compared to the power output of an engine that takes in air at 1.01×10^5 Pa at 15.0°C, what percentage increase in power is obtained by using the turbocharger alone?

19.61. A monatomic ideal gas expands slowly to twice its original volume, doing 300 J of work in the process. Find the heat added to the gas and the change in internal energy of the gas if the process is (a) isothermal; (b) adiabatic; (c) isobaric.

19.62. A cylinder with a piston contains 0.250 mol of oxygen at 2.40×10^5 Pa and 355 K. The oxygen may be treated as an ideal gas. The gas first expands isobarically to twice its original volume. It is then compressed isothermally back to its original volume, and finally it is cooled isochorically to its original pressure. (a) Show the series of processes on a pV-diagram. (b) Compute the temperature during the isothermal compression. (c) Compute the maximum pressure. (d) Compute the total work done by the piston on the gas during the series of processes.

19.63. Use the conditions and processes of Problem 19.62 to compute (a) the work done by the gas, the heat added to it, and its internal-energy change during the initial expansion; (b) the work done, the heat added, and the internal-energy change during the final cooling; (c) the internal-energy change during the isothermal compression.

19.64. A cylinder with a piston contains 0.150 mol of nitrogen at 1.80×10^5 Pa and 300 K. The nitrogen may be treated as an ideal gas. The gas is first compressed isobarically to half its original volume. It then expands adiabatically back to its original volume, and finally it is heated isochorically to its original pressure. (a) Show the series of processes in a pV-diagram. (b) Compute the temperatures at the beginning and end of the adiabatic expansion. (c) Compute the minimum pressure.

19.65. Use the conditions and processes of Problem 19.64 to compute (a) the work done by the gas, the heat added to it, and its internal-energy change during the initial compression; (b) the work done by the gas, the heat added to it, and its internal-energy change during the adiabatic expansion; (c) the work done, the heat added, and the internal-energy change during the final heating.

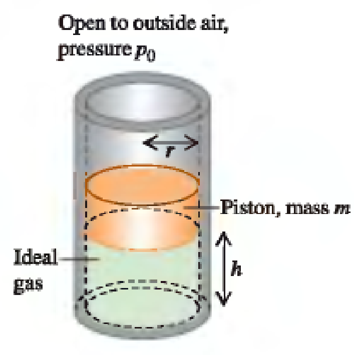
19.66. Comparing Thermodynamic Processes. In a cylinder, 1.20 mol of an ideal monatomic gas, initially at 3.60×10^5 Pa and 300 K, expands until its volume triples. Compute the work done by the gas if the expansion is (a) isothermal; (b) adiabatic; (c) isobaric. (d) Show each process in a pV-diagram. In which case is the absolute value of the work done by the gas greatest? Least? (e) In which case is the absolute value of the heat transfer greatest? Least? (f) In which case is the absolute value of the change in internal energy of the gas greatest? Least?

19.67. In a cylinder sealed with a piston, you rapidly compress 3.00 L of N_2 gas initially at 1.00 atm pressure and 0.00°C to half its original volume. Assume the N_2 behaves like an ideal gas. (a) Calculate the final temperature and pressure of the gas. (b) If you now cool the gas back to 0.00°C without changing the pressure, what is its final volume?

Challenge Problems

19.68. Oscillations of a Piston. A vertical cylinder of radius r contains a quantity of ideal gas and is fitted with a piston with mass m that is free to move (Fig. 19.34). The piston and the walls of the cylinder are frictionless and the entire cylinder is placed in a constant-temperature bath. The outside air pressure is p_0. In equilibrium, the piston sits at a height h above the bottom of the cylinder. (a) Find the absolute pressure of the gas trapped below the piston when in equilibrium. (b) The piston is pulled up by a small distance and released. Find the net force acting on the piston when its base is a distance $h + y$ above the bottom of the cylinder, where y is much less than h. (c) After the piston is displaced from equilibrium and released, it oscillates up and down. Find the frequency of these small oscillations. If the displacement is not small, are the oscillations simple harmonic? How can you tell?

Figure 19.34 Challenge Problem 19.68.



19.69. The van der Waals equation of state, an approximate representation of the behavior of gases at high pressure, is given by Eq. (18.7):

$$\left(p + \frac{an^2}{V^2}\right)(V - nb) = nRT$$

where a and b are constants having different values for different gases. (In the special case of $a = b = 0$, this is the ideal-gas equation.) (a) Calculate the work done by a gas with this equation of state in an isothermal expansion from V_1 to V_2. Show that your answer agrees with the ideal-gas result found in Example 19.1 (Section 19.2) when you set $a = b = 0$. (b) For ethane gas (C_2H_6), $a = 0.554\ \mathrm{J \cdot m^3/mol^2}$ and $b = 6.38 \times 10^{-5}\ \mathrm{m^3/mol}$. Calculate the work W done by 1.80 mol of ethane when it expands from $2.00 \times 10^{-3}\ \mathrm{m^3}$ to $4.00 \times 10^{-3}\ \mathrm{m^3}$ at a constant temperature of 300 K. Do the calculation using (i) the van der Waals equation of state and (ii) the ideal-gas equation of state. (c) How large is the difference between the two results for W in part (b)? For which equation of state is W larger? Use the interpretation of the terms a and b given in Section 18.1 to explain why this should be so. Are the differences between the two equations of state important in this case?

THE SECOND LAW OF THERMODYNAMICS

20

? The second law of thermodynamics tells us that heat naturally flows from a hot body (such as a freshly cooked ear of corn) to a cold one (such as a pat of butter). Is it *ever* possible for heat to flow from a cold body to a hot one?

LEARNING GOALS

By studying this chapter, you will learn:

- What determines whether a thermodynamic process is reversible or irreversible.
- What a heat engine is, and how to calculate its efficiency.
- The physics of internal-combustion engines.
- How refrigerators and heat engines are related, and how to analyze the performance of a refrigerator.
- How the second law of thermodynamics sets limits on the efficiency of engines and the performance of refrigerators.
- How to do calculations involving the idealized Carnot cycle for engines and refrigerators.
- What is meant by entropy, and how to use this concept to analyze thermodynamic processes.

Many thermodynamic processes proceed naturally in one direction but not the opposite. For example, heat by itself always flows from a hot body to a cooler body, never the reverse. Heat flow from a cool body to a hot body would not violate the first law of thermodynamics; energy would be conserved. But it doesn't happen in nature. Why not? As another example, note that it is easy to convert mechanical energy completely into heat; this happens every time we use a car's brakes to stop it. In the reverse direction, there are plenty of devices that convert heat *partially* into mechanical energy. (An automobile engine is an example.) But even the cleverest would-be inventors have never succeeded in building a machine that converts heat *completely* into mechanical energy. Again, why not?

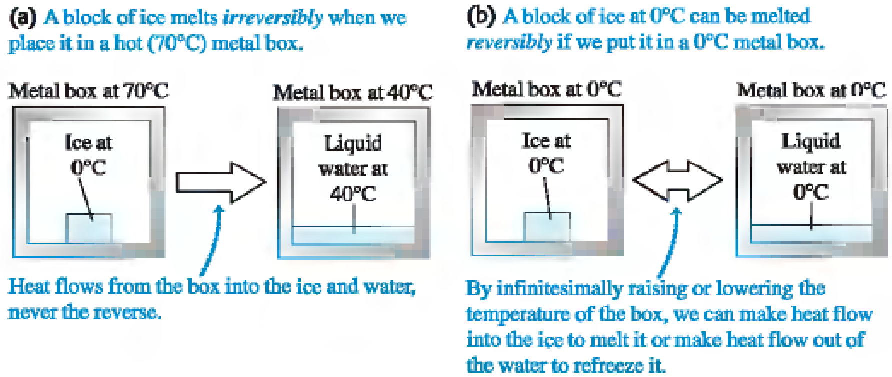
The answer to both of these questions has to do with the *directions* of thermodynamic processes and is called the *second law of thermodynamics.* This law places fundamental limitations on the efficiency of an engine or a power plant. It also places limitations on the minimum energy input needed to operate a refrigerator. So the second law is directly relevant for many important practical problems.

We can also state the second law in terms of the concept of *entropy,* a quantitative measure of the degree of disorder or randomness of a system. The idea of entropy helps explain why ink mixed with water never spontaneously unmixes and why we never observe a host of other seemingly possible processes.

20.1 Directions of Thermodynamic Processes

Thermodynamic processes that occur in nature are all **irreversible processes.** These are processes that proceed spontaneously in one direction but not the other (Fig. 20.1a). The flow of heat from a hot body to a cooler body is irreversible, as is the free expansion of a gas discussed in Sections 19.3 and 19.6. Sliding a book across a table converts mechanical energy into heat by friction; this process is

20.1 Reversible and irreversible processes.



irreversible, for no one has ever observed the reverse process (in which a book initially at rest on the table would spontaneously start moving and the table and book would cool down). Our main topic for this chapter is the *second law of thermodynamics,* which determines the preferred direction for such processes.

Despite this preferred direction for every natural process, we can think of a class of idealized processes that *would* be reversible. A system that undergoes such an idealized **reversible process** is always very close to being in thermodynamic equilibrium within itself and with its surroundings. Any change of state that takes place can then be reversed (made to go the other way) by making only an infinitesimal change in the conditions of the system. For example, we can reverse heat flow between two bodies whose temperatures differ only infinitesimally by making only a very small change in one temperature or the other (Fig. 20.1b).

Reversible processes are thus **equilibrium processes,** with the system always in thermodynamic equilibrium. Of course, if a system were *truly* in thermodynamic equilibrium, no change of state would take place. Heat would not flow into or out of a system with truly uniform temperature throughout, and a system that is truly in mechanical equilibrium would not expand and do work against its surroundings. A reversible process is an idealization that can never be precisely attained in the real world. But by making the temperature gradients and the pressure differences in the substance very small, we can keep the system very close to equilibrium states and make the process nearly reversible. That's why we call a reversible process a *quasi-equilibrium process.*

By contrast, heat flow with finite temperature difference, free expansion of a gas, and conversion of work to heat by friction are all *irreversible* processes; no small change in conditions could make any of them go the other way. They are also all *nonequilibrium* processes, in that the system is not in thermodynamic equilibrium at any point until the end of the process.

Disorder and Thermodynamic Processes

There is a relationship between the direction of a process and the *disorder* or *randomness* of the resulting state. For example, imagine a tedious sorting job, such as alphabetizing a thousand book titles written on file cards. Throw the alphabetized stack of cards into the air. Do they come down in alphabetical order? Alas, no: their tendency is to come down in a random or disordered state. In the free expansion of a gas discussed in Sections 19.3 and 19.6, the air is more disordered after it has expanded into the entire box than when it was confined in one side, just as your clothes are more disordered when scattered all over your floor than when confined to your closet.

Similarly, macroscopic kinetic energy is energy associated with organized, coordinated motions of many molecules, but heat transfer involves changes in energy of random, disordered molecular motion. Therefore conversion of mechanical energy into heat involves an increase of randomness or disorder.

In the following sections we will introduce the second law of thermodynamics by considering two broad classes of devices: *heat engines,* which are partly successful in converting heat into work, and *refrigerators,* which are partly successful in transporting heat from cooler to hotter bodies.

Test Your Understanding of Section 20.1 Your left and right hands are normally at the same temperature, just like the metal box and ice in Fig. 20.1b. Is rubbing your hands together to warm them (i) a reversible process or (ii) an irreversible process?

20.2 Heat Engines

ActivPhysics ONLINE
8.12 Cyclic Process–Strategies
8.13 Cyclic Process–Problems

The essence of our technological society is the ability to use sources of energy other than muscle power. Sometimes, mechanical energy is directly available; water power and wind power are examples. But most of our energy comes from the burning of fossil fuels (coal, oil, and gas) and from nuclear reactions. They supply energy that is transferred as *heat.* This is directly useful for heating buildings, for cooking, and for chemical processing, but to operate a machine or propel a vehicle, we need *mechanical* energy.

Thus it's important to know how to take heat from a source and convert as much of it as possible into mechanical energy or work. This is what happens in gasoline engines in automobiles, jet engines in airplanes, steam turbines in electric power plants, and many other systems. Closely related processes occur in the animal kingdom; food energy is "burned" (that is, carbohydrates combine with oxygen to yield water, carbon dioxide, and energy) and partly converted to mechanical energy as an animal's muscles do work on its surroundings.

Any device that transforms heat partly into work or mechanical energy is called a **heat engine** (Fig. 20.2). Usually, a quantity of matter inside the engine undergoes inflow and outflow of heat, expansion and compression, and sometimes change of phase. We call this matter the **working substance** of the engine. In internal-combustion engines, such as those used in automobiles, the working substance is a mixture of air and fuel; in a steam turbine it is water.

20.2 All motorized vehicles other than purely electric vehicles use heat engines for propulsion. (Hybrid vehicles use their internal-combustion engine to help charge the batteries for the electric motor.)

The simplest kind of engine to analyze is one in which the working substance undergoes a **cyclic process,** a sequence of processes that eventually leaves the substance in the same state in which it started. In a steam turbine the water is recycled and used over and over. Internal-combustion engines do not use the same air over and over, but we can still analyze them in terms of cyclic processes that approximate their actual operation.

Hot and Cold Reservoirs

All heat engines *absorb* heat from a source at a relatively high temperature, perform some mechanical work, and *discard* or *reject* some heat at a lower temperature. As far as the engine is concerned, the discarded heat is wasted. In internal-combustion engines the waste heat is that discarded in the hot exhaust gases and the cooling system; in a steam turbine it is the heat that must flow out of the used steam to condense and recycle the water.

When a system is carried through a cyclic process, its initial and final internal energies are equal. For any cyclic process, the first law of thermodynamics requires that

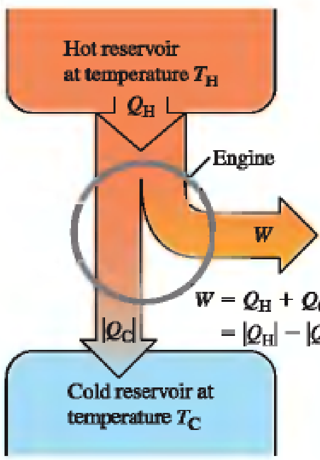
$$U_2 - U_1 = 0 = Q - W \qquad \text{so} \qquad Q = W$$

That is, the net heat flowing into the engine in a cyclic process equals the net work done by the engine.

When we analyze heat engines, it helps to think of two bodies with which the working substance of the engine can interact. One of these, called the *hot reservoir,* represents the heat source; it can give the working substance large amounts of heat at a constant temperature T_H without appreciably changing its own temperature. The other body, called the *cold reservoir,* can absorb large amounts of discarded heat from the engine at a constant lower temperature T_C. In a steam-turbine system the flames and hot gases in the boiler are the hot reservoir, and the cold water and air used to condense and cool the used steam are the cold reservoir.

We denote the quantities of heat transferred from the hot and cold reservoirs as Q_H and Q_C, respectively. A quantity of heat Q is positive when heat is transferred *into* the working substance and is negative when heat leaves the working substance. Thus in a heat engine, Q_H is positive but Q_C is negative, representing heat *leaving* the working substance. This sign convention is consistent with the rules we stated in Section 19.1; we will continue to use those rules here. Frequently, it clarifies the relationships to state them in terms of the absolute values of the Q's and W's because absolute values are always positive. When we do this, our notation will show it explicitly.

Energy-Flow Diagrams and Efficiency



20.3 Schematic energy-flow diagram for a heat engine.

We can represent the energy transformations in a heat engine by the *energy-flow diagram* of Fig. 20.3. The engine itself is represented by the circle. The amount of heat Q_H supplied to the engine by the hot reservoir is proportional to the width of the incoming "pipeline" at the top of the diagram. The width of the outgoing pipeline at the bottom is proportional to the magnitude $|Q_C|$ of the heat rejected in the exhaust. The branch line to the right represents the portion of the heat supplied that the engine converts to mechanical work, W.

When an engine repeats the same cycle over and over, Q_H and Q_C represent the quantities of heat absorbed and rejected by the engine *during one cycle;* Q_H is positive, and Q_C is negative. The *net* heat Q absorbed per cycle is

$$Q = Q_H + Q_C = |Q_H| - |Q_C| \tag{20.1}$$

The useful output of the engine is the net work W done by the working substance. From the first law,

$$W = Q = Q_H + Q_C = |Q_H| - |Q_C| \tag{20.2}$$

Ideally, we would like to convert *all* the heat Q_H into work; in that case we would have $Q_H = W$ and $Q_C = 0$. Experience shows that this is impossible; there is always some heat wasted, and Q_C *is never zero.* We define the **thermal efficiency** of an engine, denoted by e, as the quotient

$$e = \frac{W}{Q_H} \tag{20.3}$$

The thermal efficiency e represents the fraction of Q_H that *is* converted to work. To put it another way, e is what you get divided by what you pay for. This is always less than unity, an all-too-familiar experience! In terms of the flow diagram of Fig. 20.3, the most efficient engine is one for which the branch pipeline representing the work output is as wide as possible and the exhaust pipeline representing the heat thrown away is as narrow as possible.

When we substitute the two expressions for W given by Eq. (20.2) into Eq. (20.3), we get the following equivalent expressions for e:

$$e = \frac{W}{Q_H} = 1 + \frac{Q_C}{Q_H} = 1 - \left|\frac{Q_C}{Q_H}\right| \quad \text{(thermal efficiency of an engine)} \tag{20.4}$$

Note that e is a quotient of two energy quantities and thus is a pure number, without units. Of course, we must always express W, Q_H, and Q_C in the same units.

Problem-Solving Strategy 20.1 Heat Engines

Problems involving heat engines are, first and foremost, problems in the first law of thermodynamics. Hence Problem-Solving Strategy 19.1 (Section 19.4) is equally useful throughout the present chapter, and we suggest that you reread it.

IDENTIFY *the relevant concepts:* A heat engine is any device that converts heat partially to work, as shown schematically in Fig. 20.3. We will see in Section 20.4 that a refrigerator is essentially a heat engine running in reverse, so many of the same concepts apply.

SET UP *the problem* as suggested in Problem-Solving Strategy 19.1. Equation (20.4) is useful in situations for which the thermal efficiency of the engine is relevant. It's helpful to sketch an energy-flow diagram like Fig. 20.3.

EXECUTE *the solution* as follows:

1. Be very careful with the sign conventions for W and the various Q's. W is positive when the system expands and does work; W is negative when the system is compressed. Each Q is positive if it represents heat entering the system and is negative if it represents heat leaving the system. When you know that a quantity is negative, such as Q_C in the above discussion, it sometimes helps to write it as $Q_C = -|Q_C|$.
2. Some problems deal with power rather than energy quantities. Power is work per unit time ($P = W/t$), and rate of heat transfer (heat current) H is heat transfer per unit time ($H = Q/t$). In such problems it helps to ask, "What is W or Q in one second (or one hour)?"
3. Keeping steps 1 and 2 in mind, solve for the target variables.

EVALUATE *your answer:* Use the first law of thermodynamics to check your results, paying particular attention to algebraic signs.

Example 20.1 Analyzing a heat engine

A gasoline engine in a large truck takes in 10,000 J of heat and delivers 2000 J of mechanical work per cycle. The heat is obtained by burning gasoline with heat of combustion $L_c = 5.0 \times 10^4$ J/g. (a) What is the thermal efficiency of this engine? (b) How much heat is discarded in each cycle? (c) How much gasoline is burned in each cycle? (d) If the engine goes through 25 cycles per second, what is its power output in watts? In horsepower? (e) How much gasoline is burned per second? Per hour?

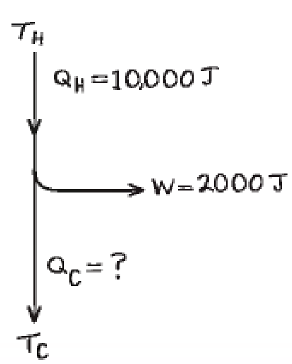
SOLUTION

IDENTIFY: This problem is about a heat engine, so we can use the ideas of this section.

SET UP: Figure 20.4 is our sketch of the energy-flow diagram for one engine cycle. We are given the amount of work done by the engine per cycle ($W = 2000$ J) and the amount of heat taken in by the engine per cycle ($Q_H = 10{,}000$ J).

Hence we use the first form of Eq. (20.4) to find the thermal efficiency. The first law of thermodynamics tells us the amount of heat rejected per cycle, and the heat of combustion tells us how much gasoline must be burned per cycle and hence per unit time.

20.4 Our sketch for this problem.



EXECUTE: (a) From the first expression in Eq. (20.4), the thermal efficiency is

$$e = \frac{W}{Q_H} = \frac{2000\ \text{J}}{10{,}000\ \text{J}} = 0.20 = 20\%$$

This is a fairly typical figure for cars and trucks if W includes only the work actually delivered to the wheels.

(b) From Eq. (20.2), $W = Q_H + Q_C$ so

$$Q_C = W - Q_H = 2000\ \text{J} - 10{,}000\ \text{J}$$
$$= -8000\ \text{J}$$

That is, 8000 J of heat leaves the engine during each cycle.

(c) Let m be the mass of gasoline burned during each cycle. Then

$$Q_H = mL_c$$
$$m = \frac{Q_H}{L_c} = \frac{10{,}000\ \text{J}}{5.0 \times 10^4\ \text{J/g}} = 0.20\ \text{g}$$

(d) The power P (rate of doing work) is the work per cycle multiplied by the number of cycles per second:

$$P = (2000\ \text{J/cycle})(25\ \text{cycles/s}) = 50{,}000\ \text{W} = 50\ \text{kW}$$
$$= (50{,}000\ \text{W})\frac{1\ \text{hp}}{746\ \text{W}} = 67\ \text{hp}$$

(e) The mass of gasoline burned per second is the mass per cycle multiplied by the number of cycles per second:

$$(0.20\ \text{g/cycle})(25\ \text{cycles/s}) = 5.0\ \text{g/s}$$

The mass burned per hour is

$$(5.0\ \text{g/s})\frac{3600\ \text{s}}{1\ \text{h}} = 18{,}000\ \text{g/h} = 18\ \text{kg/h}$$

EVALUATE: We can check our result in part (e) by converting it to a more familiar quantity, the amount of fuel consumed per unit distance. The density of gasoline is about 0.70 g/cm^3, so this is about 25,700 cm^3, 25.7 L, or 6.8 gallons of gasoline per hour. If the truck is traveling at 55 mi/h (88 km/h), this represents fuel consumption of 8.1 miles/gallon (3.4 km/L). This is substantially greater fuel consumption than a passenger car, but fairly typical of large trucks.

Test Your Understanding of Section 20.2 Rank the following heat engines in order from highest to lowest thermal efficiency. (i) an engine that in one cycle absorbs 5000 J of heat and rejects 4500 J of heat; (ii) an engine that in one cycle absorbs 25,000 J of heat and does 2000 J of work; (iii) an engine that in one cycle does 400 J of work and rejects 2800 J of heat.

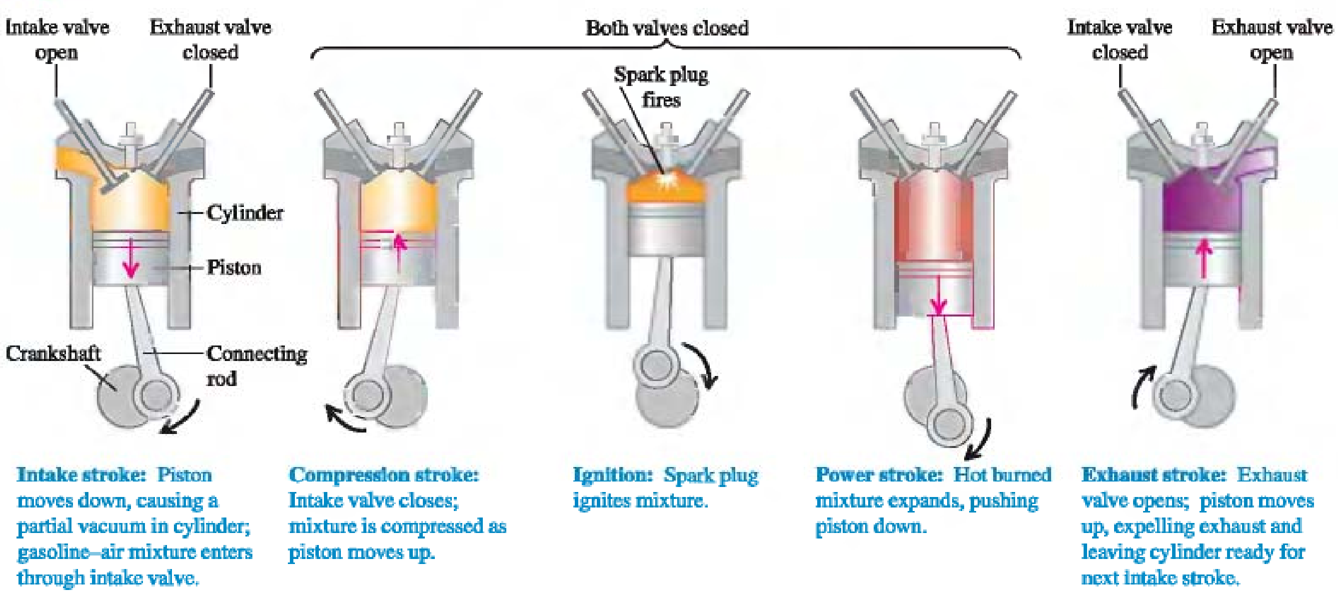
20.3 Internal-Combustion Engines

The gasoline engine, used in automobiles and many other types of machinery, is a familiar example of a heat engine. Let's look at its thermal efficiency. Figure 20.5 shows the operation of one type of gasoline engine. First a mixture of air and gasoline vapor flows into a cylinder through an open intake valve while the piston descends, increasing the volume of the cylinder from a minimum of V (when the piston is all the way up) to a maximum of rV (when it is all the way down). The quantity r is called the **compression ratio;** for present-day automobile engines its value is typically 8 to 10. At the end of this *intake stroke,* the intake valve closes and the mixture is compressed, approximately adiabatically, to volume V during the *compression stroke.* The mixture is then ignited by the spark plug, and the heated gas expands, approximately adiabatically, back to volume rV, pushing on the piston and doing work; this is the *power stroke.* Finally, the exhaust valve opens, and the combustion products are pushed out (during the *exhaust stroke*), leaving the cylinder ready for the next intake stroke.

The Otto Cycle

Figure 20.6 is a pV-diagram for an idealized model of the thermodynamic processes in a gasoline engine. This model is called the **Otto cycle.** At point a the gasoline–air mixture has entered the cylinder. The mixture is compressed adiabatically to point b and is then ignited. Heat Q_H is added to the system by the burning gasoline along line bc, and the power stroke is the adiabatic expansion to d. The gas is cooled to the temperature of the outside air along line da; during this process, heat $|Q_C|$ is rejected. In practice, this gas leaves the engine as exhaust and does not enter the engine again. But since an equivalent amount of gasoline and air enters, we may consider the process to be cyclic.

20.5 Cycle of a four-stroke internal-combustion engine.



We can calculate the efficiency of this idealized cycle. Processes bc and da are constant-volume, so the heats Q_H and Q_C are related simply to the temperatures:

$$Q_H = nC_V(T_c - T_b) > 0$$
$$Q_C = nC_V(T_a - T_d) < 0$$

The thermal efficiency is given by Eq. (20.4). Inserting the above expressions and cancelling out the common factor nC_V, we find

$$e = \frac{Q_H + Q_C}{Q_H} = \frac{T_c - T_b + T_a - T_d}{T_c - T_b} \qquad (20.5)$$

To simplify this further, we use the temperature–volume relationship for adiabatic processes for an ideal gas, Eq. (19.22). For the two adiabatic processes ab and cd,

$$T_a(rV)^{\gamma-1} = T_b V^{\gamma-1} \qquad \text{and} \qquad T_d(rV)^{\gamma-1} = T_c V^{\gamma-1}$$

We divide each of these equations by the common factor $V^{\gamma-1}$ and substitute the resulting expressions for T_b and T_c back into Eq. (20.5). The result is

$$e = \frac{T_d r^{\gamma-1} - T_a r^{\gamma-1} + T_a - T_d}{T_d r^{\gamma-1} - T_a r^{\gamma-1}} = \frac{(T_d - T_a)(r^{\gamma-1} - 1)}{(T_d - T_a) r^{\gamma-1}}$$

Dividing out the common factor $(T_d - T_a)$, we get

$$e = 1 - \frac{1}{r^{\gamma-1}} \qquad \text{(thermal efficiency in Otto cycle)} \qquad (20.6)$$

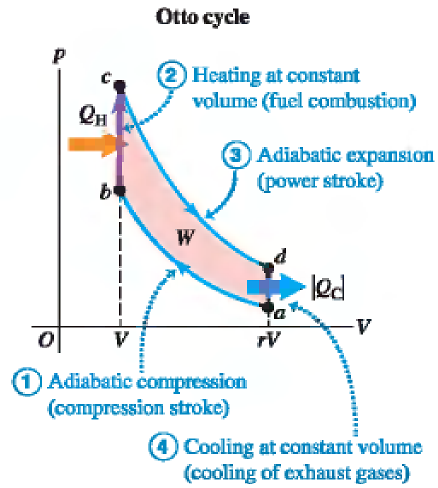
The thermal efficiency given by Eq. (20.6) is always less than unity, even for this idealized model. With $r = 8$ and $\gamma = 1.4$ (the value for air) the theoretical efficiency is $e = 0.56$, or 56%. The efficiency can be increased by increasing r. However, this also increases the temperature at the end of the adiabatic compression of the air–fuel mixture. If the temperature is too high, the mixture explodes spontaneously during compression instead of burning evenly after the spark plug ignites it. This is called *pre-ignition* or *detonation;* it causes a knocking sound and can damage the engine. The octane rating of a gasoline is a measure of its antiknock qualities. The maximum practical compression ratio for high-octane, or "premium," gasoline is about 10 to 13. Higher ratios can be used with more exotic fuels.

The Otto cycle, which we have just described, is a highly idealized model. It assumes that the mixture behaves as an ideal gas; it neglects friction, turbulence, loss of heat to cylinder walls, and many other effects that combine to reduce the efficiency of a real engine. Another source of inefficiency is incomplete combustion. A mixture of gasoline vapor with just enough air for complete combustion of the hydrocarbons to H_2O and CO_2 does not ignite readily. Reliable ignition requires a mixture that is "richer" in gasoline. The resulting incomplete combustion leads to CO and unburned hydrocarbons in the exhaust. The heat obtained from the gasoline is then less than the total heat of combustion; the difference is wasted, and the exhaust products contribute to air pollution. Efficiencies of real gasoline engines are typically around 35%.

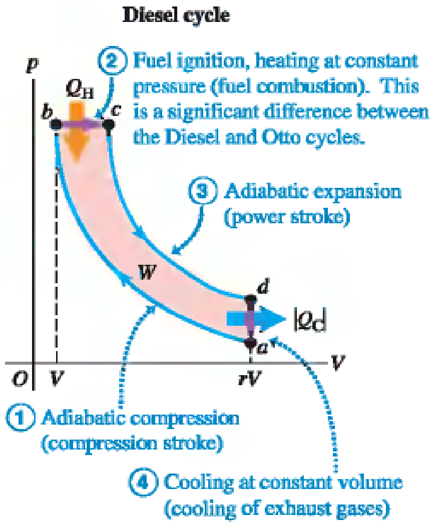
The Diesel Cycle

The Diesel engine is similar in operation to the gasoline engine. The most important difference is that there is no fuel in the cylinder at the beginning of the compression stroke. A little before the beginning of the power stroke, the injectors start to inject fuel directly into the cylinder, just fast enough to keep the pressure approximately constant during the first part of the power stroke. Because of the

20.6 The pV-diagram for the Otto cycle, an idealized model of the thermodynamic processes in a gasoline engine.



20.7 The pV-diagram for the idealized Diesel cycle.



high temperature developed during the adiabatic compression, the fuel ignites spontaneously as it is injected; no spark plugs are needed.

Fig. 20.7 shows the idealized **Diesel cycle.** Starting at point a, air is compressed adiabatically to point b, heated at constant pressure to point c, expanded adiabatically to point d, and cooled at constant volume to point a. Because there is no fuel in the cylinder during most of the compression stroke, pre-ignition cannot occur, and the compression ratio r can be much higher than for a gasoline engine. This improves efficiency and ensures reliable ignition when the fuel is injected (because of the high temperature reached during the adiabatic compression). Values of r of 15 to 20 are typical; with these values and $\gamma = 1.4$, the theoretical efficiency of the idealized Diesel cycle is about 0.65 to 0.70. As with the Otto cycle, the efficiency of any actual engine is substantially less than this. While Diesel engines are very efficient, they must be built to much tighter tolerances than gasoline engines and the fuel-injection system requires careful maintenance.

Test Your Understanding of Section 20.3 For an Otto-cycle engine with cylinders of a fixed size and a fixed compression ratio, which of the following aspects of the pV-diagram in Fig. 20.6 would change if you doubled the amount of fuel burned per cycle? (There may be more than one correct answer.) (i) the vertical distance between points b and c; (ii) the vertical distance between points a and d; (iii) the horizontal distance between points b and a.

20.4 Refrigerators

We can think of a **refrigerator** as a heat engine operating in reverse. A heat engine takes heat from a hot place and gives off heat to a colder place. A refrigerator does the opposite; it takes heat from a cold place (the inside of the refrigerator) and gives it off to a warmer place (usually the air in the room where the refrigerator is located). A heat engine has a net *output* of mechanical work; the refrigerator requires a net *input* of mechanical work. Using the sign conventions from Section 20.2, for a refrigerator Q_C is positive but both W and Q_H are negative; hence $|W| = -W$ and $|Q_H| = -Q_H$.

20.8 Schematic energy-flow diagram of a refrigerator.

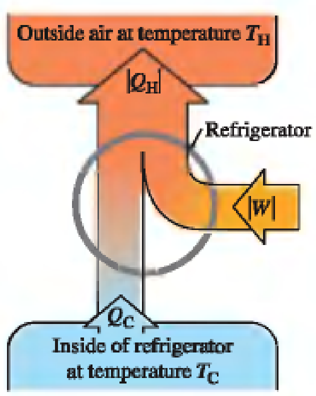


Fig. 20.8 shows an energy-flow diagram for a refrigerator. From the first law for a cyclic process,

$$Q_H + Q_C - W = 0 \qquad \text{or} \qquad -Q_H = Q_C - W$$

or, because both Q_H and W are negative,

$$|Q_H| = Q_C + |W| \tag{20.7}$$

Thus, as the diagram shows, the heat $|Q_H|$ leaving the working substance and given to the hot reservoir is always *greater* than the heat Q_C taken from the cold reservoir. Note that the absolute-value relationship

$$|Q_H| = |Q_C| + |W| \tag{20.8}$$

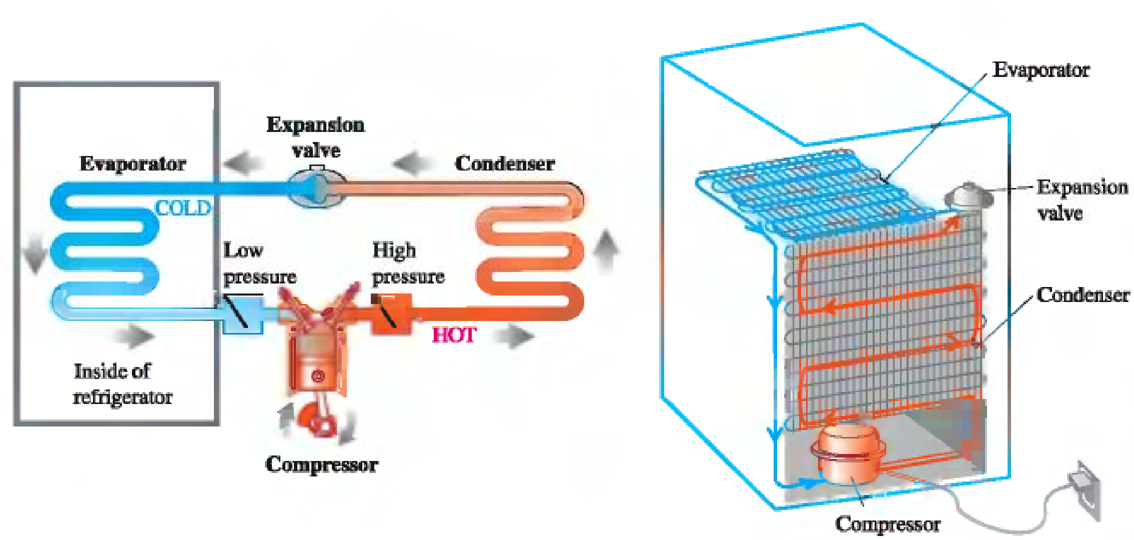
is valid for both heat engines and refrigerators.

From an economic point of view, the best refrigeration cycle is one that removes the greatest amount of heat $|Q_C|$ from the inside of the refrigerator for the least expenditure of mechanical work, $|W|$. The relevant ratio is therefore $|Q_C|/|W|$; the larger this ratio, the better the refrigerator. We call this ratio the **coefficient of performance,** denoted by K. From Eq. (20.8), $|W| = |Q_H| - |Q_C|$, so

$$K = \frac{|Q_C|}{|W|} = \frac{|Q_C|}{|Q_H| - |Q_C|} \qquad \text{(coefficient of performance of a refrigerator)} \tag{20.9}$$

As always, we measure Q_H, Q_C, and W all in the same energy units; K is then a dimensionless number.

20.9 (a) Principle of the mechanical refrigeration cycle. (b) How the key elements are arranged in a practical refrigerator.

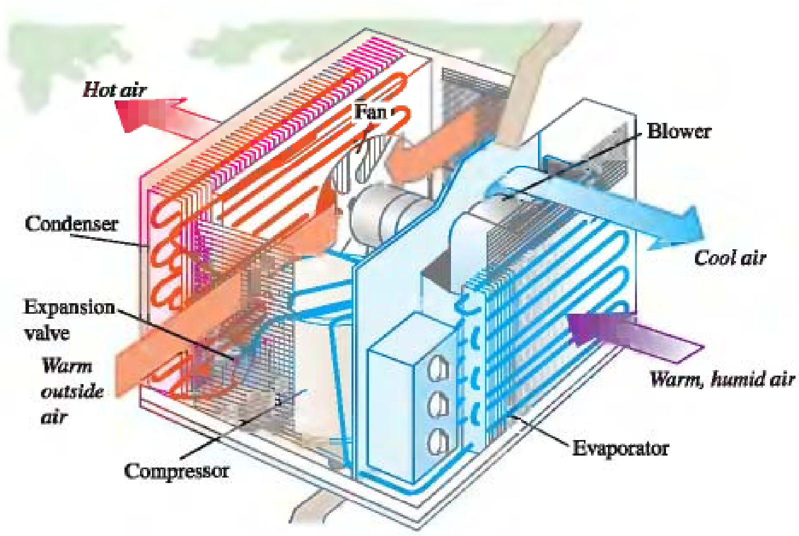


Practical Refrigerators

The principles of the common refrigeration cycle are shown schematically in Fig. 20.9a. The fluid "circuit" contains a refrigerant fluid (the working substance). The left side of the circuit (including the cooling coils inside the refrigerator) is at low temperature and low pressure; the right side (including the condenser coils outside the refrigerator) is at high temperature and high pressure. Ordinarily, both sides contain liquid and vapor in phase equilibrium.

The compressor takes in fluid, compresses it adiabatically, and delivers it to the condenser coil at high pressure. The fluid temperature is then higher than that of the air surrounding the condenser, so the refrigerant gives off heat $|Q_H|$ and partially condenses to liquid. The fluid then expands adiabatically into the evaporator at a rate controlled by the expansion valve. As the fluid expands, it cools considerably, enough that the fluid in the evaporator coil is colder than its surroundings. It absorbs heat $|Q_C|$ from its surroundings, cooling them and partially vaporizing. The fluid then enters the compressor to begin another cycle. The compressor, usually driven by an electric motor (Fig. 20.9b), requires energy input and does work $|W|$ *on* the working substance during each cycle.

An air conditioner operates on exactly the same principle. In this case the refrigerator box becomes a room or an entire building. The evaporator coils are inside, the condenser is outside, and fans circulate air through these (Fig. 20.10).



20.10 An air conditioner works on the same principle as a refrigerator.

In large installations the condenser coils are often cooled by water. For air conditioners the quantities of greatest practical importance are the *rate* of heat removal (the heat current H from the region being cooled) and the *power* input $P = W/t$ to the compressor. If heat $|Q_C|$ is removed in time t, then $H = |Q_C|/t$. Then we can express the coefficient of performance as

$$K = \frac{|Q_C|}{|W|} = \frac{Ht}{Pt} = \frac{H}{P}$$

Typical room air conditioners have heat removal rates H of 5000 to 10,000 Btu/h, or about 1500–3000 W, and require electric power input of about 600 to 1200 W. Typical coefficients of performance are about 3; the actual values depend on the inside and outside temperatures.

Unfortunately, K is often expressed commercially in mixed units, with H in Btu per hour and P in watts. In these units, H/P is called the **energy efficiency rating** (EER); the units, customarily omitted, are (Btu/h)/W. Because 1 W = 3.413 Btu/h, the EER is numerically 3.413 times as large as the dimensionless K. Room air conditioners typically have an EER of about 10.

A variation on this theme is the **heat pump,** used to heat buildings by cooling the outside air. It functions like a refrigerator turned inside out. The evaporator coils are outside, where they take heat from cold air, and the condenser coils are inside, where they give off heat to the warmer air. With proper design, the heat $|Q_H|$ delivered to the inside per cycle can be considerably greater than the work $|W|$ required to get it there.

Work is *always* needed to transfer heat from a colder to a hotter body. Heat flows spontaneously from hotter to colder, and to reverse this flow requires the addition of work from the outside. Experience shows that it is impossible to make a refrigerator that transports heat from a colder body to a hotter body without the addition of work. If no work were needed, the coefficient of performance would be infinite. We call such a device a *workless refrigerator;* it is a mythical beast, like the unicorn and the free lunch. ?

Test Your Understanding of Section 20.4 Can you cool your house by leaving the refrigerator door open?

20.5 The Second Law of Thermodynamics

Experimental evidence suggests strongly that it is *impossible* to build a heat engine that converts heat completely to work—that is, an engine with 100% thermal efficiency. This impossibility is the basis of one statement of the **second law of thermodynamics,** as follows:

It is impossible for any system to undergo a process in which it absorbs heat from a reservoir at a single temperature and converts the heat completely into mechanical work, with the system ending in the same state in which it began.

We will call this the "engine" statement of the second law. (It is also known to physicists as the *Kelvin–Planck statement* of this law.)

The basis of the second law of thermodynamics is the difference between the nature of internal energy and that of macroscopic mechanical energy. In a moving body the molecules have random motion, but superimposed on this is a coordinated motion of every molecule in the direction of the body's velocity. The kinetic energy associated with this *coordinated* macroscopic motion is what we call the kinetic energy of the moving body. The kinetic and potential energies associated with the *random* motion constitute the internal energy.

When a body sliding on a surface comes to rest as a result of friction, the organized motion of the body is converted to random motion of molecules in the body and in the surface. Since we cannot control the motions of individual molecules, we cannot convert this random motion completely back to organized motion. We can convert *part* of it, and this is what a heat engine does.

If the second law were *not* true, we could power an automobile or run a power plant by cooling the surrounding air. Neither of these impossibilities violates the *first* law of thermodynamics. The second law, therefore, is not a deduction from the first but stands by itself as a separate law of nature. The first law denies the possibility of creating or destroying energy; the second law limits the *availability* of energy and the ways in which it can be used and converted.

Restating the Second Law

Our analysis of refrigerators in Section 20.4 forms the basis for an alternative statement of the second law of thermodynamics. Heat flows spontaneously from hotter to colder bodies, never the reverse. A refrigerator does take heat from a colder to a hotter body, but its operation requires an input of mechanical energy or work. Generalizing this observation, we state:

It is impossible for any process to have as its sole result the transfer of heat from a cooler to a hotter body.

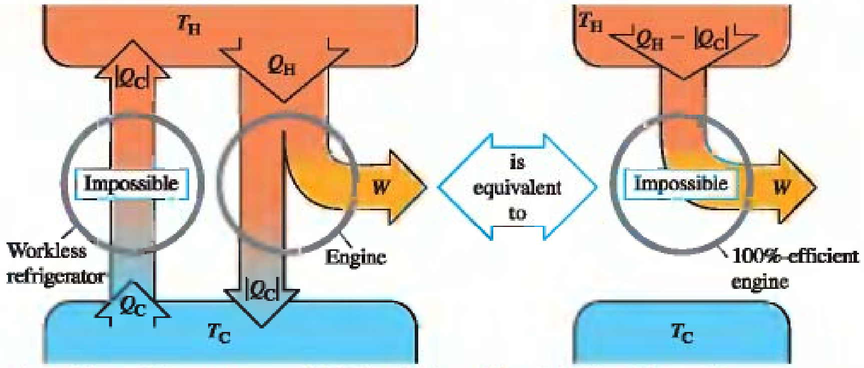
We'll call this the "refrigerator" statement of the second law. (It is also known as the *Clausius statement.*) It may not seem to be very closely related to the "engine" statement. In fact, though, the two statements are completely equivalent. For example, if we could build a workless refrigerator, violating the second or "refrigerator" statement of the second law, we could use it in conjunction with a heat engine, pumping the heat rejected by the engine back to the hot reservoir to be reused. This composite machine (Fig. 20.11a) would violate the "engine" statement of the second law because its net effect would be to take a net quantity of heat $Q_H - |Q_C|$ from the hot reservoir and convert it completely to work W.

Alternatively, if we could make an engine with 100% thermal efficiency, in violation of the first statement, we could run it using heat from the hot reservoir and use the work output to drive a refrigerator that pumps heat from the cold reservoir to the hot (Fig. 20.11b). This composite device would violate the "refrigerator" statement because its net effect would be to take heat Q_C from the cold reservoir and deliver it to the hot reservoir without requiring any input of work. Thus any device that violates one form of the second law can be used to make a device that violates the other form. If violations of the first form are impossible, so are violations of the second!

The conversion of work to heat, as in friction or viscous fluid flow, and heat flow from hot to cold across a finite temperature gradient, are *irreversible* processes. The "engine" and "refrigerator" statements of the second law state that these processes can be only partially reversed. We could cite other examples. Gases always seep spontaneously through an opening from a region of high pressure to a region of low pressure; gases and miscible liquids left by themselves always tend to mix, not to unmix. The second law of thermodynamics is an expression of the inherent one-way aspect of these and many other irreversible processes. Energy conversion is an essential aspect of all plant and animal life and of human technology, so the second law of thermodynamics is of the utmost fundamental importance in the world we live in.

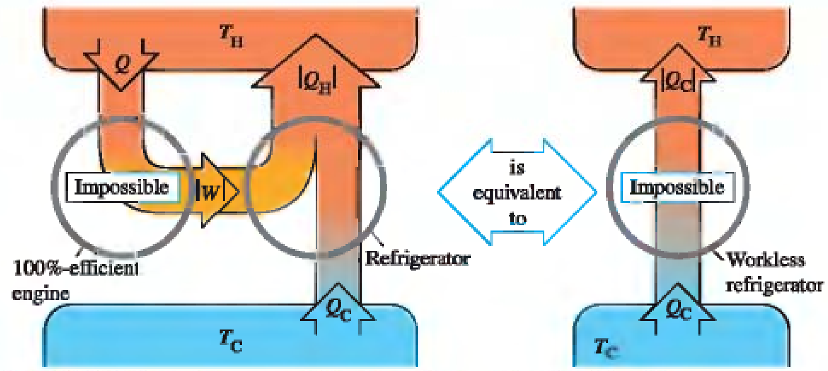
20.11 Energy-flow diagrams showing that the two forms of the second law are equivalent.

(a) The "engine" statement of the second law of thermodynamics



If a workless refrigerator were possible, it could be used in conjunction with an ordinary heat engine to form a 100%-efficient engine, converting heat $Q_H - |Q_C|$ completely to work.

(b) The "refrigerator" statement of the second law of thermodynamics



If a 100%-efficient engine were possible, it could be used in conjunction with an ordinary refrigerator to form a workless refrigerator, transferring heat Q_C from the cold to the hot reservoir with no input of work.

Test Your Understanding of Section 20.5 Would a 100%-efficient engine (Fig. 20.11a) violate the *first* law of thermodynamics? What about a workless refrigerator (Fig. 20.11b)?

20.12 The temperature of the firebox of a steam engine is much higher than the temperature of water in the boiler, so heat flows irreversibly from firebox to water. Carnot's quest to understand the efficiency of steam engines led him to the idea that an ideal engine would involve only *reversible* processes.

20.6 The Carnot Cycle

According to the second law, no heat engine can have 100% efficiency. How great an efficiency *can* an engine have, given two heat reservoirs at temperatures T_H and T_C? This question was answered in 1824 by the French engineer Sadi Carnot (1796–1832), who developed a hypothetical, idealized heat engine that has the maximum possible efficiency consistent with the second law. The cycle of this engine is called the **Carnot cycle.**

To understand the rationale of the Carnot cycle, we return to a recurrent theme in this chapter: *reversibility* and its relationship to directions of thermodynamic processes. Conversion of work to heat is an irreversible process; the purpose of a heat engine is a *partial* reversal of this process, the conversion of heat to work with as great an efficiency as possible. For maximum heat-engine efficiency, therefore, *we must avoid all irreversible processes* (Fig. 20.12). This requirement turns out to be enough to determine the basic sequence of steps in the Carnot cycle, as we will show next.

Heat flow through a finite temperature drop is an irreversible process. Therefore, during heat transfer in the Carnot cycle there must be *no* finite temperature difference. When the engine takes heat from the hot reservoir at temperature T_H, the working substance of the engine must also be at T_H; otherwise, irreversible heat flow would occur. Similarly, when the engine discards heat to the cold reservoir at T_C, the engine itself must be at T_C. That is, every process that involves heat transfer must be *isothermal* at either T_H or T_C.

Conversely, in any process in which the temperature of the working substance of the engine is intermediate between T_H and T_C, there must be *no* heat transfer between the engine and either reservoir because such heat transfer could not be reversible. Therefore any process in which the temperature T of the working substance changes must be *adiabatic.*

The bottom line is that every process in our idealized cycle must be either isothermal or adiabatic. In addition, thermal and mechanical equilibrium must be maintained at all times so that each process is completely reversible.

Steps of the Carnot Cycle

The Carnot cycle consists of two reversible isothermal and two reversible adiabatic processes. Fig. 20.13 shows a Carnot cycle using as its working substance an ideal gas in a cylinder with a piston. It consists of the following steps:

8.14 Carnot Cycle

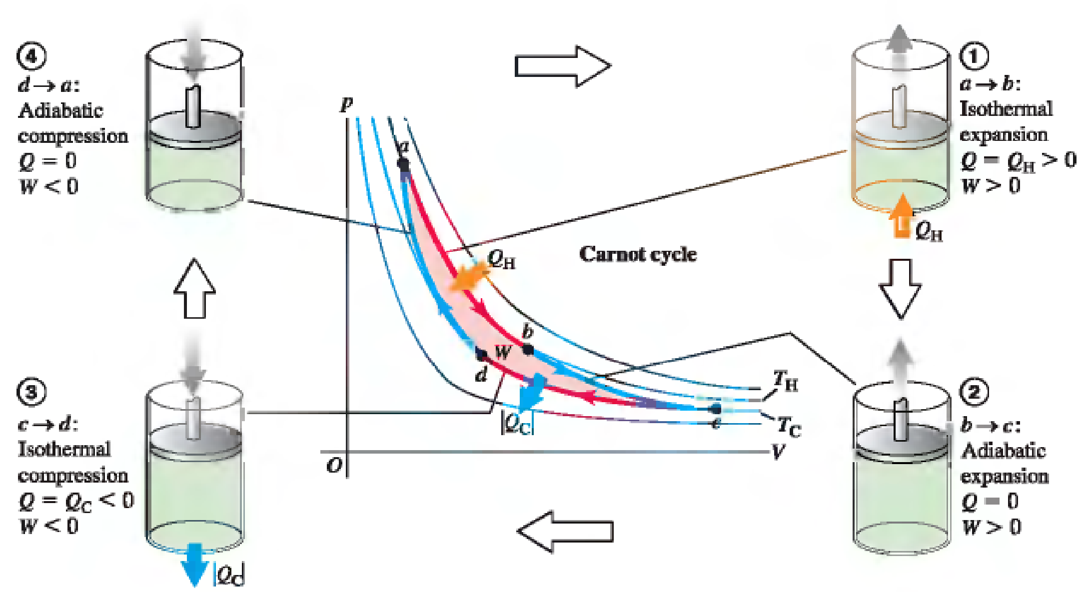
1. The gas expands isothermally at temperature T_H, absorbing heat Q_H (ab).
2. It expands adiabatically until its temperature drops to T_C (bc).
3. It is compressed isothermally at T_C, rejecting heat $|Q_C|$ (cd).
4. It is compressed adiabatically back to its initial state at temperature T_H (da).

We can calculate the thermal efficiency e of a Carnot engine in the special case shown in Fig.20.13 in which the working substance is an *ideal gas.* To carry out this calculation, we will first find the ratio Q_C/Q_H of the quantities of heat transferred in the two isothermal processes and then use Eq. (20.4) to find e.

For an ideal gas the internal energy U depends only on temperature and is thus constant in any isothermal process. For the isothermal expansion ab, $\Delta U_{ab} = 0$ and Q_H is equal to the work W_{ab} done by the gas during its isothermal expansion at temperature T_H. We calculated this work in Example 19.1 (Section 19.2); using that result, we have

$$Q_H = W_{ab} = nRT_H \ln \frac{V_b}{V_a} \tag{20.10}$$

20.13 The Carnot cycle for an ideal gas. The light blue lines in the pV-diagram are isotherms (curves of constant temperature) and the dark blue lines are adiabats (curves of zero heat flow).



Similarly,

$$Q_C = W_{cd} = nRT_C \ln \frac{V_d}{V_c} = -nRT_C \ln \frac{V_c}{V_d} \tag{20.11}$$

Because V_d is less than V_c, Q_C is negative $(Q_C = -|Q_C|)$; heat flows out of the gas during the isothermal compression at temperature T_C.

The ratio of the two quantities of heat is thus

$$\frac{Q_C}{Q_H} = -\left(\frac{T_C}{T_H}\right) \frac{\ln(V_c/V_d)}{\ln(V_b/V_a)} \tag{20.12}$$

This can be simplified further by use of the temperature–volume relationship for an adiabatic process, Eq. (19.22). We find for the two adiabatic processes:

$$T_H V_b^{\gamma-1} = T_C V_c^{\gamma-1} \quad \text{and} \quad T_H V_a^{\gamma-1} = T_C V_d^{\gamma-1}$$

Dividing the first of these by the second, we find

$$\frac{V_b^{\gamma-1}}{V_a^{\gamma-1}} = \frac{V_c^{\gamma-1}}{V_d^{\gamma-1}} \quad \text{and} \quad \frac{V_b}{V_a} = \frac{V_c}{V_d}$$

Thus the two logarithms in Eq. (20.12) are equal, and that equation reduces to

$$\frac{Q_C}{Q_H} = -\frac{T_C}{T_H} \quad \text{or} \quad \frac{|Q_C|}{|Q_H|} = \frac{T_C}{T_H} \quad \text{(heat transfer in a Carnot engine)} \tag{20.13}$$

The ratio of the heat rejected at T_C to the heat absorbed at T_H is just equal to the ratio T_C/T_H. Then from Eq. (20.4) the efficiency of the Carnot engine is

$$e_{\text{Carnot}} = 1 - \frac{T_C}{T_H} = \frac{T_H - T_C}{T_H} \quad \text{(efficiency of a Carnot engine)} \tag{20.14}$$

This simple result says that the efficiency of a Carnot engine depends only on the temperatures of the two heat reservoirs. The efficiency is large when the temperature *difference* is large, and it is very small when the temperatures are nearly equal. The efficiency can never be exactly unity unless $T_C = 0$; we'll see later that this, too, is impossible.

CAUTION **Use Kelvin temperature in Carnot calculations** In all calculations involving the Carnot cycle, you must make sure that you use *absolute* (Kelvin) temperatures only. That's because Eqs. (20.10) through (20.14) come from the ideal-gas equation $pV = nRT$, in which T is absolute temperature.

Example 20.2 Analyzing a Carnot engine I

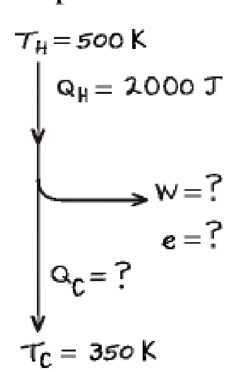
A Carnot engine takes 2000 J of heat from a reservoir at 500 K, does some work, and discards some heat to a reservoir at 350 K. How much work does it do, how much heat is discarded, and what is the efficiency?

SOLUTION

IDENTIFY: This problem involves a Carnot engine, so we can use the ideas of this section as well as the concepts from Section 20.2 (which apply to heat engines of all kinds).

SET UP: Figure 20.14 shows the energy-flow diagram for this problem. For this Carnot engine we are given $Q_H = 2000$ J, the amount of heat absorbed, and the temperatures $T_H = 500$ K and $T_C = 350$ K of the hot and cold reservoirs, respectively. We find

20.14 Our sketch for this problem.



the amount of heat discarded using Eq. (20.13), and then calculate the amount of work done using the first law of thermodynamics: The work done in a complete cycle is the sum of the heat absorbed and the (negative) heat discarded [see Eq. (20.2)]. We find the efficiency from the two temperatures using Eq. (20.14).

EXECUTE: From Eq. (20.13) the heat Q_C discarded by the engine is

$$Q_C = -Q_H \frac{T_C}{T_H} = -(2000\text{ J})\frac{350\text{ K}}{500\text{ K}} = -1400\text{ J}$$

Then from the first law, the work W done by the engine is

$$W = Q_H + Q_C = 2000\text{ J} + (-1400\text{ J}) = 600\text{ J}$$

From Eq. (20.14) the thermal efficiency is

$$e = 1 - \frac{T_C}{T_H} = 1 - \frac{350\text{ K}}{500\text{ K}} = 0.30 = 30\%$$

EVALUATE: The negative sign of Q_C is correct: it shows that heat flows *out* of the engine and into the cold reservoir. Note that we can check our result for e by using the basic definition of thermal efficiency:

$$e = \frac{W}{Q_H} = \frac{600\text{ J}}{2000\text{ J}} = 0.30 = 30\%$$

Example 20.3 Analyzing a Carnot engine II

Suppose 0.200 mol of an ideal diatomic gas ($\gamma = 1.40$) undergoes a Carnot cycle with temperatures 227°C and 27°C. The initial pressure is $p_a = 10.0 \times 10^5$ Pa, and during the isothermal expansion at the higher temperature the volume doubles. (a) Find the pressure and volume at each of points a, b, c, and d in the pV-diagram of Fig.20.13. (b) Find Q, W, and ΔU for each step and for the entire cycle. (c) Determine the efficiency directly from the results of part (b), and compare it with the result from Eq. (20.14).

SOLUTION

IDENTIFY: This problem involves the properties of the Carnot cycle as well as those of an ideal gas.

SET UP: We are given the number of moles and the pressure and temperature at point a (which is at the higher of the two reservoir temperatures), so we can find the volume at a using the ideal-gas equation. We then find the pressure and volume at the other points from the equations given in this section in combination with the ideal-gas equation. Next, for each step in the cycle, we use Eqs. (20.10) and (20.11) to find the heat flow and work done, and we use Eq. (19.13) to calculate the internal energy change. As in Example 20.2, we find the efficiency using Eq. (20.14).

EXECUTE: (a) We first remember to convert the Celsius temperatures to absolute temperatures: The higher temperature is $T_H = (227 + 273.15)\text{ K} = 500\text{ K}$ and the lower temperature is $T_C = (27 + 273.15)\text{ K} = 300\text{ K}$. We then use the ideal-gas equation to find V_a:

$$V_a = \frac{nRT_H}{p_a} = \frac{(0.200\text{ mol})(8.314\text{ J/mol}\cdot\text{K})(500\text{ K})}{10.0 \times 10^5\text{ Pa}} = 8.31 \times 10^{-4}\text{ m}^3$$

The volume doubles during the isothermal expansion $a \to b$, so

$$V_b = 2V_a = 2(8.31 \times 10^{-4}\text{ m}^3) = 16.6 \times 10^{-4}\text{ m}^3$$

Also, during the isothermal expansion $a \to b$, $p_aV_a = p_bV_b$, so

$$p_b = \frac{p_aV_a}{V_b} = 5.00 \times 10^5\text{ Pa}$$

For the adiabatic expansion $b \to c$, $T_H V_b^{\gamma-1} = T_C V_c^{\gamma-1}$, so

$$V_c = V_b\left(\frac{T_H}{T_C}\right)^{1/(\gamma-1)} = (16.6 \times 10^{-4}\text{ m}^3)\left(\frac{500\text{ K}}{300\text{ K}}\right)^{2.5} = 59.6 \times 10^{-4}\text{ m}^3$$

Using the ideal-gas equation again for point c, we find

$$p_c = \frac{nRT_C}{V_c} = \frac{(0.200\text{ mol})(8.314\text{ J/mol}\cdot\text{K})(300\text{ K})}{59.6 \times 10^{-4}\text{ m}^3} = 0.837 \times 10^5\text{ Pa}$$

For the adiabatic compression $d \to a$, $T_C V_d^{\gamma-1} = T_H V_a^{\gamma-1}$, and

$$V_d = V_a\left(\frac{T_H}{T_C}\right)^{1/(\gamma-1)} = (8.31 \times 10^{-4}\text{ m}^3)\left(\frac{500\text{ K}}{300\text{ K}}\right)^{2.5} = 29.8 \times 10^{-4}\text{ m}^3$$

$$p_d = \frac{nRT_C}{V_d} = \frac{(0.200\text{ mol})(8.314\text{ J/mol}\cdot\text{K})(300\text{ K})}{29.8 \times 10^{-4}\text{ m}^3} = 1.67 \times 10^5\text{ Pa}$$

(b) For the isothermal expansion $a \to b$, $\Delta U_{ab} = 0$. To find W_{ab} $(= Q_H)$, we use Eq. (20.10):

$$W_{ab} = Q_H = nRT_H \ln\frac{V_b}{V_a} = (0.200\text{ mol})(8.314\text{ J/mol}\cdot\text{K})(500\text{ K})(\ln 2) = 576\text{ J}$$

For the adiabatic expansion $b \to c$, $Q_{bc} = 0$. From the first law of thermodynamics, $\Delta U_{bc} = Q_{bc} - W_{bc} = -W_{bc}$; hence the work W_{bc} done by the gas in this process equals the negative of the change in internal energy of the gas. From Eq. (19.13) we have $\Delta U = nC_V\Delta T$, where $\Delta T = T_C - T_H$ (final temperature minus initial temperature). Using $C_V = 20.8\text{ J/mol}\cdot\text{K}$ for an ideal diatomic gas, we find

$$W_{bc} = -\Delta U_{bc} = -nC_V(T_C - T_H) = nC_V(T_H - T_C) = (0.200\text{ mol})(20.8\text{ J/mol}\cdot\text{K})(500\text{ K} - 300\text{ K}) = 832\text{ J}$$

Continued

For the isothermal compression $c \rightarrow d$, $\Delta U_{cd} = 0$; Eq. (20.11) gives

$$W_{cd} = Q_C = nRT_C \ln\frac{V_d}{V_c}$$
$$= (0.200\,\text{mol})(8.314\,\text{J/mol}\cdot\text{K})(300\,\text{K})\left(\ln\frac{29.8\times10^{-4}\,\text{m}^3}{59.6\times10^{-4}\,\text{m}^3}\right)$$
$$= -346\,\text{J}$$

For the adiabatic compression $d \rightarrow a$, $Q_{da} = 0$, and

$$W_{da} = -\Delta U_{da} = -nC_V(T_H - T_C) = nC_V(T_C - T_H)$$
$$= (0.200\,\text{mol})(20.8\,\text{J/mol}\cdot\text{K})(300\,\text{K} - 500\,\text{K})$$
$$= -832\,\text{J}$$

We can tabulate the results as follows:

Process	Q	W	ΔU
$a \rightarrow b$	576 J	576 J	0
$b \rightarrow c$	0	832 J	−832 J
$c \rightarrow d$	−346 J	−346 J	0
$d \rightarrow a$	0	−832 J	832 J
Total	230 J	230 J	0

(c) From the table, $Q_H = 576$ J and the total work is 230 J. Thus

$$e = \frac{W}{Q_H} = \frac{230\,\text{J}}{576\,\text{J}} = 0.40 = 40\%$$

We can compare this with the result from Eq. (20.14):

$$e = \frac{T_H - T_C}{T_H} = \frac{500\,\text{K} - 300\,\text{K}}{500\,\text{K}} = 0.40 = 40\%$$

EVALUATE: In the table of results in part (b), note that for the entire cycle $Q = W$ and $\Delta U = 0$. These results are just what we would expect: In a complete cycle, the net heat input is used to do work with zero net change in the internal energy of the system. Note also that the quantities of work in the two adiabatic processes are negatives of each other. Can you show from the analysis leading to Eq. (20.13) that this must *always* be the case in a Carnot cycle?

Note that the efficiency in this example is greater than that obtained in Example 20.2. That's because the ratio of the high and low temperatures is higher, $(500\,\text{K})/(300\,\text{K})$ as compared to $(500\text{K})/(350\,\text{K})$.

The Carnot Refrigerator

Because each step in the Carnot cycle is reversible, the *entire cycle* may be reversed, converting the engine into a refrigerator. The coefficient of performance of the Carnot refrigerator is obtained by combining the general definition of K, Eq. (20.9), with Eq. (20.13) for the Carnot cycle. We first rewrite Eq. (20.9) as

$$K = \frac{|Q_C|}{|Q_H| - |Q_C|} = \frac{|Q_C|/|Q_H|}{1 - |Q_C|/|Q_H|}$$

Then we substitute Eq. (20.13), $|Q_C|/|Q_H| = T_C/T_H$, into this expression. The result is

$$K_{\text{Carnot}} = \frac{T_C}{T_H - T_C} \quad \text{(coefficient of performance of a Carnot refrigerator)} \qquad (20.15)$$

When the temperature difference $T_H - T_C$ is small, K is much larger than unity; in this case a lot of heat can be "pumped" from the lower to the higher temperature with only a little expenditure of work. But the greater the temperature difference, the smaller the value of K and the more work is required to transfer a given quantity of heat.

Example 20.4 Analyzing a Carnot refrigerator

If the cycle described in Example 20.3 is run backward as a refrigerator, what is its coefficient of performance?

SOLUTION

IDENTIFY: This problem uses the ideas of Section 20.3 (for refrigerators in general) as well as the above discussion of Carnot refrigerators.

SET UP: Equation (20.9) gives the coefficient of performance of *any* refrigerator in terms of the heat extracted from the cold reservoir per cycle and the work that must be done per cycle.

EXECUTE: In Example 20.3 we found that in one cycle the Carnot engine rejects heat $Q_C = -346$ J to the cold reservoir and does work $W = 230$ J. Hence, when run in reverse as a refrigerator, the system extracts heat $Q_C = +346$ J from the cold reservoir while requiring a work input of $W = -230$ J. From Eq. (20.9),

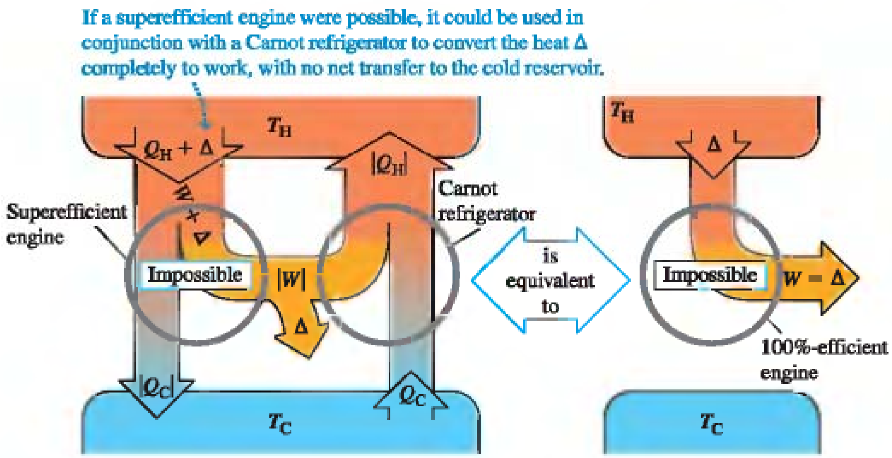
$$K = \frac{|Q_C|}{|W|} = \frac{346\,\text{J}}{230\,\text{J}} = 1.50$$

Because the cycle is a Carnot cycle, we may also use Eq. (20.15):

$$K = \frac{T_C}{T_H - T_C} = \frac{300\,\text{K}}{500\,\text{K} - 300\,\text{K}} = 1.50$$

EVALUATE: For a Carnot cycle, e and K depend only on the temperatures, as shown by Eqs. (20.14) and (20.15), and we don't need to calculate Q and W. For cycles containing irreversible processes, however, these two equations are not valid, and more detailed calculations are necessary.

20.15 Proving that the Carnot engine has the highest possible efficiency. A "superefficient" engine (more efficient than a Carnot engine) combined with a Carnot refrigerator could convert heat completely into work with no net heat transfer to the cold reservoir. This would violate the second law of thermodynamics.



The Carnot Cycle and the Second Law

We can prove that **no engine can be more efficient than a Carnot engine operating between the same two temperatures.** The key to the proof is the above observation that since each step in the Carnot cycle is reversible, the *entire cycle* may be reversed. Run backward, the engine becomes a refrigerator. Suppose we have an engine that is more efficient than a Carnot engine (Fig. 20.15). Let the Carnot engine, run backward as a refrigerator by negative work $-|W|$, take in heat Q_C from the cold reservoir and expel heat $|Q_H|$ to the hot reservoir. The superefficient engine expels heat $|Q_C|$, but to do this, it takes in a greater amount of heat $Q_H + \Delta$. Its work output is then $W + \Delta$, and the net effect of the two machines together is to take a quantity of heat Δ and convert it completely into work. This violates the engine statement of the second law. We could construct a similar argument that a superefficient engine could be used to violate the refrigerator statement of the second law. Note that we don't have to assume that the superefficient engine is reversible. In a similar way we can show that *no refrigerator can have a greater coefficient of performance than a Carnot refrigerator operating between the same two temperatures.*

Thus the statement that no engine can be more efficient than a Carnot engine is yet another equivalent statement of the second law of thermodynamics. It also follows directly that **all Carnot engines operating between the same two temperatures have the same efficiency, irrespective of the nature of the working substance.** Although we derived Eq. (20.14) for a Carnot engine using an ideal gas as its working substance, it is in fact valid for *any* Carnot engine, no matter what its working substance.

Equation (20.14), the expression for the efficiency of a Carnot engine, sets an upper limit to the efficiency of a real engine such as a steam turbine. To maximize this upper limit and the actual efficiency of the real engine, the designer must make the intake temperature T_H as high as possible and the exhaust temperature T_C as low as possible (Fig. 20.16).

The exhaust temperature cannot be lower than the lowest temperature available for cooling the exhaust. For a steam turbine at an electric power plant, T_C may be the temperature of river or lake water; then we want the boiler temperature T_H to be as high as possible. The vapor pressures of all liquids increase rapidly with temperature, so we are limited by the mechanical strength of the boiler. At 500°C the vapor pressure of water is about 240×10^5 Pa (235 atm); this is about the maximum practical pressure in large present-day steam boilers.

20.16 To maximize efficiency, the temperatures inside a jet engine are made as high as possible. Exotic ceramic materials are used that can withstand temperatures in excess of 1000°C without melting or becoming soft.

*The Kelvin Temperature Scale

In Chapter 17 we expressed the need for a temperature scale that doesn't depend on the properties of any particular material. We can now use the Carnot cycle to define such a scale. The thermal efficiency of a Carnot engine operating between two heat reservoirs at temperatures T_H and T_C is independent of the nature of the working substance and depends only on the temperatures. From Eq. (20.4), this thermal efficiency is

$$e = \frac{Q_H + Q_C}{Q_H} = 1 + \frac{Q_C}{Q_H}$$

Therefore the ratio Q_C/Q_H is the same for *all* Carnot engines operating between two given temperatures T_H and T_C.

Kelvin proposed that we *define* the ratio of the temperatures, T_C/T_H, to be equal to the magnitude of the ratio Q_C/Q_H of the quantities of heat absorbed and rejected:

$$\frac{T_C}{T_H} = \frac{|Q_C|}{|Q_H|} = -\frac{Q_C}{Q_H} \quad \text{(definition of Kelvin temperature)} \quad (20.16)$$

Equation (20.16) looks identical to Eq. (20.13), but there is a subtle and crucial difference. The temperatures in Eq. (20.13) are based on an ideal-gas thermometer, as defined in Section 17.3, while Eq. (20.16) *defines* a temperature scale based on the Carnot cycle and the second law of thermodynamics and is independent of the behavior of any particular substance. Thus the **Kelvin temperature scale** is truly *absolute.* To complete the definition of the Kelvin scale, we assign, as in Section 17.3, the arbitrary value of 273.16 K to the temperature of the triple point of water. When a substance is taken around a Carnot cycle, the ratio of the heats absorbed and rejected, $|Q_H|/|Q_C|$, is equal to the ratio of the temperatures of the reservoirs *as expressed on the gas-thermometer scale* defined in Section 17.3. Since the triple point of water is chosen to be 273.16 K in both scales, it follows that *the Kelvin and ideal-gas scales are identical.*

The zero point on the Kelvin scale is called **absolute zero.** Absolute zero can be interpreted on a molecular level; at absolute zero the system has its *minimum* possible total internal energy (kinetic plus potential). Because of quantum effects, however, it is *not* true that at $T = 0$, all molecular motion ceases. There are theoretical reasons for believing that absolute zero cannot be attained experimentally, although temperatures below 10^{-7} K have been achieved. The more closely we approach absolute zero, the more difficult it is to get closer. One statement of the *third law of thermodynamics* is that it is impossible to reach absolute zero in a finite number of thermodynamic steps.

Test Your Understanding of Section 20.6 An inventor looking for financial support comes to you with an idea for a gasoline engine that runs on a novel type of thermodynamic cycle. His design is made entirely of copper and is air-cooled. He claims that the engine will be 85% efficient. Should you invest in this marvelous new engine? (*Hint:* See Table 17.4.)

20.7 Entropy

The second law of thermodynamics, as we have stated it, is rather different in form from many familiar physical laws. It is not an equation or a quantitative relationship but rather a statement of *impossibility.* However, the second law *can* be stated as a quantitative relationship with the concept of *entropy,* the subject of this section.

We have talked about several processes that proceed naturally in the direction of increasing disorder. Irreversible heat flow increases disorder because the molecules are initially sorted into hotter and cooler regions; this sorting is lost when the system comes to thermal equilibrium. Adding heat to a body increases its disorder because it increases average molecular speeds and therefore the randomness of molecular motion. Free expansion of a gas increases its disorder because the molecules have greater randomness of position after the expansion than before. Figure 20.17 shows another process in which disorder increases.

20.17 When firecrackers explode, disorder increases: The neatly packaged chemicals within each firecracker are dispersed in all directions, and the stored chemical energy is converted to random kinetic energy of the fragments.

Entropy and Disorder

Entropy provides a *quantitative* measure of disorder. To introduce this concept, let's consider an infinitesimal isothermal expansion of an ideal gas. We add heat dQ and let the gas expand just enough to keep the temperature constant. Because the internal energy of an ideal gas depends only on its temperature, the internal energy is also constant; thus from the first law, the work dW done by the gas is equal to the heat dQ added. That is,

$$dQ = dW = p\,dV = \frac{nRT}{V}dV \qquad \text{so} \qquad \frac{dV}{V} = \frac{dQ}{nRT}$$

The gas is in a more disordered state after the expansion than before because the molecules are moving in a larger volume and have more randomness of position. Thus the fractional volume change dV/V is a measure of the increase in disorder, and the above equation shows that it is proportional to the quantity dQ/T. We introduce the symbol S for the entropy of the system, and we define the infinitesimal entropy change dS during an infinitesimal reversible process at absolute temperature T as

$$dS = \frac{dQ}{T} \qquad \text{(infinitesimal reversible process)} \qquad (20.17)$$

If a total amount of heat Q is added during a reversible isothermal process at absolute temperature T, the total entropy change $\Delta S = S_2 - S_1$ is given by

$$\Delta S = S_2 - S_1 = \frac{Q}{T} \qquad \text{(reversible isothermal process)} \qquad (20.18)$$

Entropy has units of energy divided by temperature; the SI unit of entropy is 1 J/K.

We can see how the quotient Q/T is related to the increase in disorder. Higher temperature means greater randomness of motion. If the substance is initially cold, with little molecular motion, adding heat Q causes a substantial fractional increase in molecular motion and randomness. But if the substance is already hot, the same quantity of heat adds relatively little to the greater molecular motion already present. So the quotient Q/T is an appropriate characterization of the increase in randomness or disorder when heat flows into a system.

Example 20.5 Entropy change in melting

One kilogram of ice at 0°C is melted and converted to water at 0°C. Compute its change in entropy, assuming that the melting is done reversibly. The heat of fusion of water is $L_f = 3.34 \times 10^5$ J/kg.

SOLUTION

IDENTIFY: The melting occurs at a constant temperature of 0°C, so this is a reversible isothermal process.

SET UP: We are given the amount of heat added (in terms of the heat of fusion) and the temperature $T = 273$ K. (Note that in entropy calculations we must always use absolute, or Kelvin, temperatures.) We can then calculate the entropy change using Eq. (20.18).

Continued

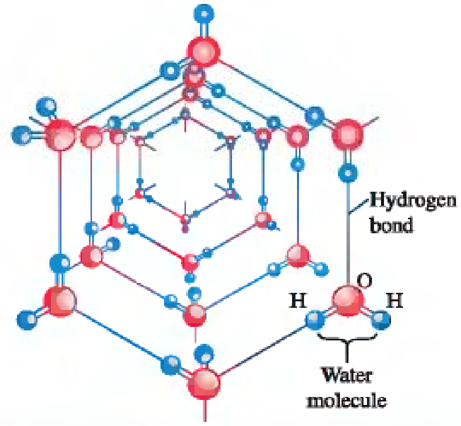
EXECUTE: The heat needed to melt the ice is $Q = mL_f = 3.34 \times 10^5$ J. From Eq. (20.18) the increase in entropy of the system is

$$\Delta S = S_2 - S_1 = \frac{Q}{T} = \frac{3.34 \times 10^5\ \text{J}}{273\ \text{K}} = 1.22 \times 10^3\ \text{J/K}$$

EVALUATE: This increase corresponds to the increase in disorder when the water molecules go from the highly ordered state of a crystalline solid to the much more disordered state of a liquid (Fig. 20.18).

In any *isothermal* reversible process, the entropy change equals the heat transferred divided by the absolute temperature. When we refreeze the water, Q has the opposite sign, and the entropy change of the water is $\Delta S = -1.22 \times 10^3$ J/K. The water molecules rearrange themselves into a crystal to form ice, so disorder and entropy both decrease.

20.18 Water molecules are arranged in a regular, ordered way in an ice crystal. When the ice melts, the hydrogen bonds between molecules are broken, increasing the water's disorder and its entropy.



Entropy in Reversible Processes

We can generalize the definition of entropy change to include *any* reversible process leading from one state to another, whether it is isothermal or not. We represent the process as a series of infinitesimal reversible steps. During a typical step, an infinitesimal quantity of heat dQ is added to the system at absolute temperature T. Then we sum (integrate) the quotients dQ/T for the entire process; that is,

$$\Delta S = \int_1^2 \frac{dQ}{T} \qquad \text{(entropy change in a reversible process)} \qquad (20.19)$$

The limits 1 and 2 refer to the initial and final states.

Because entropy is a measure of the disorder of a system in any specific state, it must depend only on the current state of the system, not on its past history. We will show later that this is indeed the case. When a system proceeds from an initial state with entropy S_1 to a final state with entropy S_2, the change in entropy $\Delta S = S_2 - S_1$ defined by Eq. (20.19) does not depend on the path leading from the initial to the final state but is the same for *all possible* processes leading from state 1 to state 2. Thus the entropy of a system must also have a definite value for any given state of the system. We recall that *internal energy,* introduced in Chapter 19, also has this property, although entropy and internal energy are very different quantities.

Since entropy is a function only of the state of a system, we can also compute entropy changes in *irreversible* (nonequilibrium) processes for which Eqs. (20.17) and (20.19) are not applicable. We simply invent a path connecting the given initial and final states that *does* consist entirely of reversible equilibrium processes and compute the total entropy change for that path. It is not the actual path, but the entropy change must be the same as for the actual path.

As with internal energy, the above discussion does not tell us how to calculate entropy itself, but only the change in entropy in any given process. Just as with internal energy, we may arbitrarily assign a value to the entropy of a system in a specified reference state and then calculate the entropy of any other state with reference to this.

Example 20.6 Entropy change in a temperature change

One kilogram of water at 0°C is heated to 100°C. Compute its change in entropy.

SOLUTION

IDENTIFY: In practice, the process described would be done irreversibly, perhaps by setting a pan of water on an electric range whose cooking surface is maintained at 100°C. But the entropy change of the water depends only on the initial and final states of the system, and is the same whether the process is reversible or irreversible.

SET UP: We can imagine that the temperature of the water is increased reversibly in a series of infinitesimal steps, in each of which the temperature is raised by an infinitesimal amount dT. We then use Eq. (20.19) to integrate over all these steps and calculate the entropy change for the total process.

EXECUTE: From Eq. (17.14) the heat required to carry out each such infinitesimal step is $dQ = mc\,dT$. Substituting this into Eq. (20.19) and integrating, we find

$$\Delta S = S_2 - S_1 = \int_1^2 \frac{dQ}{T} = \int_{T_1}^{T_2} mc\frac{dT}{T} = mc\ln\frac{T_2}{T_1}$$
$$= (1.00\text{ kg})(4190\text{ J/kg}\cdot\text{K})\left(\ln\frac{373\text{ K}}{273\text{ K}}\right)$$
$$= 1.31 \times 10^3\text{ J/K}$$

EVALUATE: The entropy change is positive, as it must be for a process in which the system absorbs heat.

In this calculation we assumed that the specific heat c doesn't depend on temperature. That's a pretty good approximation, since c for water increases by only 1% between 0°C and 100°C.

CAUTION **When $\Delta S = Q/T$ can (and cannot) be used** In solving this problem you might be tempted to avoid doing an integral by using the simpler expression in Eq. (20.18), $\Delta S = Q/T$. This would be incorrect, however, because Eq. (20.18) is applicable only to *isothermal* processes, and the initial and final temperatures in our example are *not* the same. The *only* correct way to find the entropy change in a process with different initial and final temperatures is to use Eq. (20.19).

Conceptual Example 20.7 A reversible adiabatic process

A gas expands adiabatically and reversibly. What is its change in entropy?

SOLUTION

In an adiabatic process, no heat enters or leaves the system. Hence $dQ = 0$ and there is *no* change in entropy in this reversible process: $\Delta S = 0$. Every *reversible* adiabatic process is a constant-entropy process. (For this reason, reversible adiabatic processes are also called *isentropic* processes.) The increase in disorder resulting from the gas occupying a greater volume is exactly balanced by the decrease in disorder associated with the lowered temperature and reduced molecular speeds.

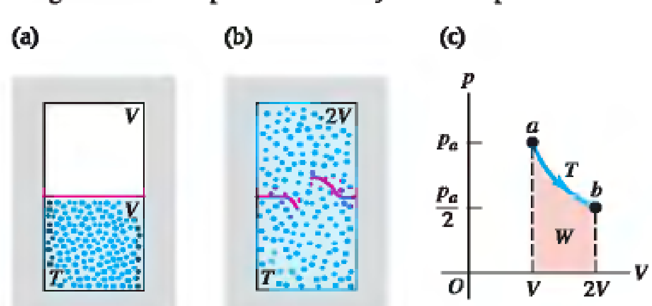
Example 20.8 Entropy change in a free expansion

A thermally insulated box is divided by a partition into two compartments, each having volume V (Fig. 20.19). Initially, one compartment contains n moles of an ideal gas at temperature T, and the other compartment is evacuated. We then break the partition, and the gas expands to fill both compartments. What is the entropy change in this free-expansion process?

SOLUTION

IDENTIFY: For this process, $Q = 0$, $W = 0$, $\Delta U = 0$, and therefore (because the system is an ideal gas) $\Delta T = 0$. We might think that the entropy change is zero because there is no heat exchange. But Eq. (20.19) can be used to calculate entropy changes for *reversible* processes only; this free expansion is *not* reversible, and there *is* an entropy change. The process is adiabatic because $Q = 0$, but it is not isentropic because $\Delta S \neq 0$. As we mentioned

20.19 (a,b) Free expansion of an insulated ideal gas. (c) The free-expansion process doesn't pass through equilibrium states from a to b. However, the entropy change $S_b - S_a$ can be calculated by using the isothermal path shown or *any* reversible path from a to b.



Continued

at the beginning of this section, entropy increases in a free expansion because the positions of the molecules are more random than before the expansion.

SET UP: To calculate ΔS, we recall that the entropy change depends only on the initial and final states. We can devise a *reversible* process having the same endpoints, use Eq. (20.19) to calculate its entropy change, and thus determine the entropy change in the original process. An appropriate reversible process in this case is an isothermal expansion from V to $2V$ at temperature T. The gas does work W during this substitute expansion, so an equal amount of heat Q must be supplied to keep the internal energy constant. We find the entropy change for this reversible isothermal process using Eq. (20.18); the entropy change for the free expansion will be the same.

EXECUTE: We found in Example 19.1 (Section 19.2) that the work done by n moles of ideal gas in an isothermal expansion from V_1 to V_2 is $W = nRT\ln(V_2/V_1)$. Using $V_1 = V$ and $V_2 = 2V$, we have

$$Q = W = nRT\ln\frac{2V}{V} = nRT\ln 2$$

Thus the entropy change is

$$\Delta S = \frac{Q}{T} = nR\ln 2$$

which is also the entropy change for the free expansion with the same initial and final states. For 1 mole,

$$\Delta S = (1\ \text{mol})(8.314\ \text{J/mol}\cdot\text{K})(\ln 2) = 5.76\ \text{J/K}$$

EVALUATE: The entropy change is positive, as we predicted. The factor $(\ln 2)$ in our answer is a result of the volume having increased by a factor of 2. Can you show that if the volume had increased in the free expansion from V to xV, where x is an arbitrary number, the entropy change would have been $\Delta S = nR\ln x$?

Example 20.9 Entropy and the Carnot cycle

For the Carnot engine in Example 20.2 (Section 20.6), find the total entropy change in the engine during one cycle.

SOLUTION

IDENTIFY: All four steps in the Carnot cycle are reversible (see Fig. 20.13), so we can use the expression for the change in entropy in a reversible process.

SET UP: We find the entropy change ΔS for each step and then add the entropy changes to get the total ΔS for the cycle as a whole.

EXECUTE: There is no entropy change during the adiabatic expansion or adiabatic compression. During the isothermal expansion at $T_H = 500$ K the engine takes in 2000 J of heat, and its entropy change, from Eq. (20.18), is

$$\Delta S_H = \frac{Q_H}{T_H} = \frac{2000\ \text{J}}{500\ \text{K}} = 4.0\ \text{J/K}$$

During the isothermal compression at $T_C = 350$ K the engine gives off 1400 J of heat, and its entropy change is

$$\Delta S_C = \frac{Q_C}{T_C} = \frac{-1400\ \text{J}}{350\ \text{K}} = -4.0\ \text{J/K}$$

The total entropy change in the engine during one cycle is $\Delta S_{\text{total}} = \Delta S_H + \Delta S_C = 4.0\ \text{J/K} + (-4.0\ \text{J/K}) = 0$.

EVALUATE: The result $\Delta S_{\text{total}} = 0$ tells us that when the Carnot engine completes a cycle, it has the same entropy as it did at the beginning of the cycle. We'll explore this result in the following subsection.

What is the total entropy change of the engine's *environment* during this cycle? The hot (500 K) reservoir gives off 2000 J of heat during the reversible isothermal expansion, so its entropy change is $(-2000\ \text{J})/(500\ \text{K}) = -4.0\ \text{J/K}$; the cold (350 K) reservoir absorbs 1400 J of heat during the reversible isothermal compression, so its entropy change is $(+1400\ \text{J})/(350\ \text{K}) = +4.0\ \text{J/K}$. Thus each individual reservoir has an entropy change; however, the sum of these changes—that is, the total entropy change of the system's environment—is zero.

These results apply to the special case of the Carnot cycle, for which *all* of the processes are reversible. In this case we find that the total entropy change of the system and the environment together is zero. We will see that if the cycle includes irreversible processes (as is the case for the Otto cycle or Diesel cycle of Section 20.3), the total entropy change of the system and the environment *cannot* be zero, but rather must be positive.

Entropy in Cyclic Processes

Example 20.9 showed that the total entropy change for a cycle of a particular Carnot engine, which uses an ideal gas as its working substance, is zero. This result follows directly from Eq. (20.13), which we can rewrite as

$$\frac{Q_H}{T_H} + \frac{Q_C}{T_C} = 0 \tag{20.20}$$

The quotient Q_H/T_H equals ΔS_H, the entropy change of the engine that occurs at $T = T_H$. Likewise, Q_C/T_C equals ΔS_C, the (negative) entropy change of the engine that occurs at $T = T_C$. Hence Eq. (20.20) says that $\Delta S_H + \Delta S_C = 0$; that is, there is zero net entropy change in one cycle.

What about Carnot engines that use a different working substance? According to the second law, *any* Carnot engine operating between given temperatures T_H and T_C has the same efficiency $e = 1 - T_C/T_H$ [Eq. (20.14)]. Combining this expression for e with Eq. (20.4), $e = 1 + Q_C/Q_H$, just reproduces Eq. (20.20). So Eq. (20.20) is valid for any Carnot engine working between these temperatures, whether its working substance is an ideal gas or not. We conclude that *the total entropy change in one cycle of any Carnot engine is zero.*

This result can be generalized to show that the total entropy change during *any* reversible cyclic process is zero. A reversible cyclic process appears on a *pV*-diagram as a closed path (Fig. 20.20a). We can approximate such as path as closely as we like by a sequence of isothermal and adiabatic processes forming parts of many long, thin Carnot cycles (Fig. 20.20b). The total entropy change for the full cycle is the sum of the entropy changes for each small Carnot cycle, each of which is zero. So **the total entropy change during *any* reversible cycle is zero:**

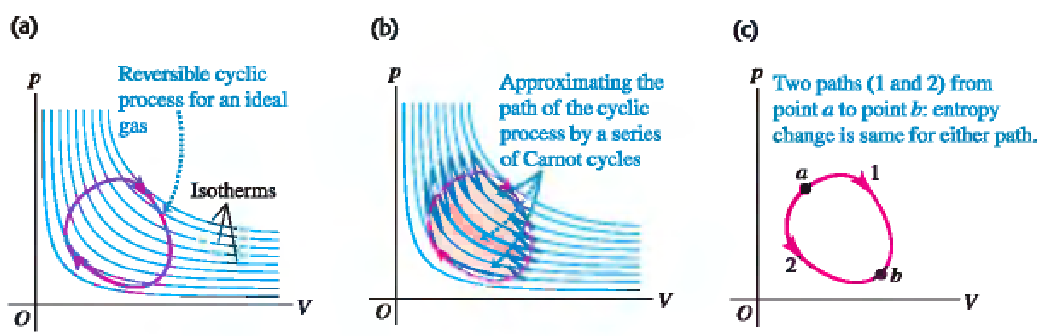
$$\int \frac{dQ}{T} = 0 \qquad \text{(reversible cyclic process)} \qquad (20.21)$$

It follows that when a system undergoes a reversible process leading from any state a to any other state b, *the entropy change of the system is independent of the path* (Fig. 20.20c). If the entropy change for path 1 were different from the change for path 2, the system could be taken along path 1 and then backward along path 2 to the starting point, with a nonzero net change in entropy. This would violate the conclusion that the total entropy change in such a cyclic process must be zero. Because the entropy change in such processes is independent of path, we conclude that in any given state, the system has a definite value of entropy that depends only on the state, not on the processes that led to that state.

Entropy in Irreversible Processes

In an idealized, reversible process involving only equilibrium states, the total entropy change of the system and its surroundings is zero. But all *irreversible* processes involve an increase in entropy. Unlike energy, *entropy is not a conserved quantity.* The entropy of an isolated system *can* change, but as we shall see, it can never decrease. The free expansion of a gas, described in Example 20.8, is an irreversible process in an isolated system in which there is an entropy increase.

20.20 (a) A reversible cyclic process for an ideal gas is shown as a red closed path on a *pV*-diagram. Several ideal-gas isotherms are shown in blue. (b) We can approximate the path in (a) by a series of long, thin Carnot cycles; one of these is highlighted in gold. The total entropy change is zero for each Carnot cycle and for the actual cyclic process. (c) The entropy change between points a and b is independent of the path.



Example 20.10 An irreversible process

Suppose 1.00 kg of water at 100°C is placed in thermal contact with 1.00 kg of water at 0°C. What is the total change in entropy? Assume that the specific heat of water is constant at 4190 J/kg · K over this temperature range.

SOLUTION

IDENTIFY: This process involves irreversible heat flow because of the temperature differences.

SET UP: Since there are equal masses of 0°C water and 100°C water, the final temperature is the average of these two temperatures, or 50°C. Although the processes are irreversible, we can calculate the entropy changes for the (initially) hot water and the (initially) cold water in the same way as in Example 20.6 by assuming that the process occurs reversibly. We must use Eq. (20.19) to calculate ΔS for each substance because the temperatures change in the process.

EXECUTE: The final temperature is 50°C = 323 K. The entropy change of the hot water is

$$\Delta S_{\text{hot}} = mc\int_{T_1}^{T_2}\frac{dT}{T} = (1.00\text{ kg})(4190\text{ J/kg}\cdot\text{K})\int_{373\text{K}}^{323\text{K}}\frac{dT}{T}$$
$$= (4190\text{ J/K})\left(\ln\frac{323\text{ K}}{373\text{ K}}\right) = -603\text{ J/K}$$

The entropy change of the cold water is

$$\Delta S_{\text{cold}} = (4190\text{ J/K})\left(\ln\frac{323\text{ K}}{273\text{ K}}\right) = +705\text{ J/K}$$

The *total* entropy change of the system is

$$\Delta S_{\text{total}} = \Delta S_{\text{hot}} + \Delta S_{\text{cold}} = (-603\text{ J/K}) + 705\text{ J/K} = +102\text{ J/K}$$

EVALUATE: An irreversible heat flow in an isolated system is accompanied by an increase in entropy. We could have reached the same end state by simply mixing the two quantities of water. This, too, is an irreversible process; because the entropy depends only on the state of the system, the total entropy change would be the same, 102 J/K.

It's worth noting that the entropy of the system increases *continuously* as the two quantities of water come to equilibrium. For example, the first 4190 J of heat transferred cools the hot water to 99°C and warms the cold water to 1°C. The net change in entropy for this step is approximately

$$\Delta S = \frac{-4190\text{ J}}{373\text{ K}} + \frac{4190\text{ J}}{273\text{ K}} = +4.1\text{ J/K}$$

Can you show in a similar way that the net entropy change is positive for *any* one-degree temperature change leading to the equilibrium condition?

20.21 The mixing of colored ink and water starts from a state of relative order (low entropy) in which each fluid is separate and distinct from the other. The final state after mixing is more disordered (has greater entropy). Spontaneous unmixing of the ink and water, a process in which there would be a net decrease in entropy, is never observed.

Entropy and the Second Law

The results of Example 20.10 about the flow of heat from a higher to a lower temperature, or the mixing of substances at different temperatures, are characteristic of *all* natural (that is, irreversible) processes. When we include the entropy changes of all the systems taking part in the process, the increases in entropy are always greater than the decreases. In the special case of a *reversible* process, the increases and decreases are equal. Hence we can state the general principle: **When all systems taking part in a process are included, the entropy either remains constant or increases.** In other words: **No process is possible in which the total entropy decreases, when all systems taking part in the process are included.** This is an alternative statement of the second law of thermodynamics in terms of entropy. Thus it is equivalent to the "engine" and "refrigerator" statements discussed earlier. Fig. 20.21 shows a specific example of this general principle.

The increase of entropy in every natural, irreversible process measures the increase of disorder or randomness in the universe associated with that process. Consider again the example of mixing hot and cold water (Example 20.10). We *might* have used the hot and cold water as the high- and low-temperature reservoirs of a heat engine. While removing heat from the hot water and giving heat to the cold water, we could have obtained some mechanical work. But once the hot and cold water have been mixed and have come to a uniform temperature, this opportunity to convert heat to mechanical work is lost irretrievably. The lukewarm water will never *unmix* itself and separate into hotter and colder portions. No decrease in *energy* occurs when the hot and cold water are mixed. What has been lost is not *energy*, but *opportunity*, the opportunity to convert part of the heat from the hot water into mechanical work. Hence when entropy increases, energy becomes less *available*, and the universe becomes more random or "run down."

Test Your Understanding of Section 20.7 Suppose 2.00 kg of water at 50°C spontaneously changes temperature, so that half of the water cools to 0°C while the other half spontaneously warms to 100°C. (All of the water remains liquid, so it doesn't freeze or boil.) What would be the entropy change of the water? Is this process possible? (Hint: See Example 20.10.)

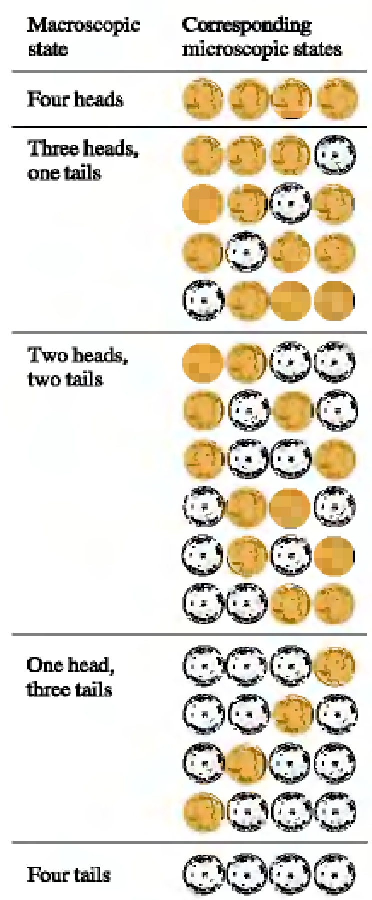
*20.8 Microscopic Interpretation of Entropy

We described in Section 19.4 how the internal energy of a system could be calculated, at least in principle, by adding up all the kinetic energies of its constituent particles and all the potential energies of interaction among the particles. This is called a *microscopic* calculation of the internal energy. We can also make a microscopic calculation of the entropy S of a system. Unlike energy, however, entropy is not something that belongs to each individual particle or pair of particles in the system. Rather, entropy is a measure of the disorder of the system as a whole. To see how to calculate entropy microscopically, we first have to introduce the idea of *macroscopic* and *microscopic states.*

Suppose you toss N identical coins on the floor, and half of them show heads and half show tails. This is a description of the large-scale or **macroscopic state** of the system of N coins. A description of the **microscopic state** of the system includes information about each individual coin: Coin 1 was heads, coin 2 was tails, coin 3 was tails, and so on. There can be many microscopic states that correspond to the same macroscopic description. For instance, with $N = 4$ coins there are six possible states in which half are heads and half are tails (Fig. 20.22). The number of microscopic states grows rapidly with increasing N; for $N = 100$ there are $2^{100} = 1.27 \times 10^{30}$ microscopic states, of which 1.01×10^{29} are half heads and half tails.

The least probable outcomes of the coin toss are the states that are either all heads or all tails. It is certainly possible that you could throw 100 heads in a row, but don't bet on it; the probability of doing this is only 1 in 1.27×10^{30}. The most probable outcome of tossing N coins is that half are heads and half are tails. The reason is that this *macroscopic* state has the greatest number of corresponding *microscopic* states, as shown in Fig. 20.22.

20.22 All possible microscopic states of four coins. There can be several possible microscopic states for each macroscopic state.



To make the connection to the concept of entropy, note that N coins that are all heads constitute a completely ordered macroscopic state; the description "all heads" completely specifies the state of each one of the N coins. The same is true if the coins are all tails. But the macroscopic description "half heads, half tails" by itself tells you very little about the state (heads or tails) of each individual coin. We say that the system is *disordered* because we know so little about its microscopic state. Compared to the state "all heads" or "all tails," the state "half heads, half tails" has a much greater number of possible microscopic states, much greater disorder, and hence much greater entropy (which is a quantitative measure of disorder).

Now instead of N coins, consider a mole of an ideal gas containing Avogadro's number of molecules. The macroscopic state of this gas is given by its pressure p, volume V, and temperature T; a description of the microscopic state involves stating the position and velocity for each molecule in the gas. At a given pressure, volume, and temperature, the gas may be in any one of an astronomically large number of microscopic states, depending on the positions and velocities of its 6.02×10^{23} molecules. If the gas undergoes a free expansion into a greater volume, the range of possible positions increases, as does the number of possible microscopic states. The system becomes more disordered, and the entropy increases as calculated in Example 20.8 (Section 20.7).

We can draw the following general conclusion: **For any system, the most probable macroscopic state is the one with the greatest number of corresponding microscopic states, which is also the macroscopic state with the greatest disorder and the greatest entropy.**

Calculating Entropy: Microscopic States

Let w represent the number of possible microscopic states for a given macroscopic state. (For the four coins shown in Fig. 20.22 the state of four heads has $w = 1$, the state of three heads and one tails has $w = 4$, and so on.) Then the entropy S of a macroscopic state can be shown to be given by

$$S = k\ln w \qquad \text{(microscopic expression for entropy)} \qquad (20.22)$$

where $k = R/N_A$ is the Boltzmann constant (gas constant per molecule) introduced in Section 18.3. As Eq. (20.22) shows, increasing the number of possible microscopic states w increases the entropy S.

What matters in a thermodynamic process is not the absolute entropy S but the *difference* in entropy between the initial and final states. Hence an equally valid and useful definition would be $S = k\ln w + C$, where C is a constant, since C cancels in any calculation of an entropy difference between two states. But it's convenient to set this constant equal to zero and use Eq. (20.22). With this choice, since the smallest possible value of w is unity, the smallest possible value of S for any system is $k\ln 1 = 0$. Entropy can *never* be negative.

In practice, calculating w is a difficult task, so Eq. (20.22) is typically used only to calculate the absolute entropy S of certain special systems. But we can use this relationship to calculate *differences* in entropy between one state and another. Consider a system that undergoes a thermodynamic process that takes it from macroscopic state 1, for which there are w_1 possible microscopic states, to macroscopic state 2, with w_2 associated microscopic states. The change in entropy in this process is

$$\Delta S = S_2 - S_1 = k\ln w_2 - k\ln w_1 = k\ln\frac{w_2}{w_1} \qquad (20.23)$$

The *difference* in entropy between the two macroscopic states depends on the *ratio* of the numbers of possible microscopic states.

As the following example shows, using Eq. (20.23) to calculate a change in entropy from oue macroscopic state to another gives the same results as considering a reversible process connecting those two states and using Eq. (20.19).

Example 20.11 A microscopic calculation of entropy change

Use Eq. (20.23) to calculate the entropy change in the free expansion of n moles of gas at temperature T described in Example 20.8 (Fig. 20.23).

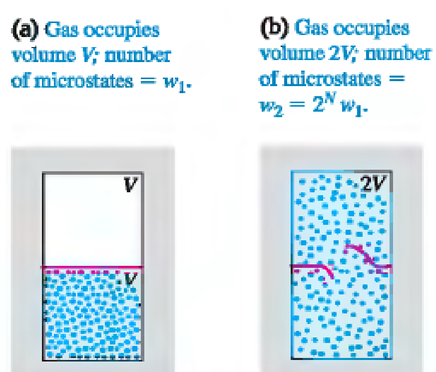
SOLUTION

IDENTIFY: We are asked to calculate the entropy change using the number of microstates in the initial macroscopic state (Fig. 20.23a) and in the final macroscopic state (Fig. 20.23b).

SET UP: When the partition is broken, the velocities of the molecules are unaffected, since no work is done. But each molecule now has twice as much volume in which it can move and hence has twice the number of possible positions. This is all we need to calculate the entropy change using Eq. (20.23).

EXECUTE: Let w_1 be the number of microscopic states of the system as a whole when the gas occupies volume V (Fig. 20.23a). The

20.23 In a free expansion of N molecules in which the volume doubles, the number of possible microscopic states increases by 2^N.



number of molecules is $N = nN_A$, and each molecule has twice as many possible states after the partition is broken. Hence the number w_2 of microscopic states when the gas occupies volume $2V$ (Fig. 20.23b) is greater by a factor of 2^N; that is, $w_2 = 2^N w_1$.

The change in entropy in this process is

$$\Delta S = k\ln\frac{w_2}{w_1} = k\ln\frac{2^N w_1}{w_1} = k\ln 2^N$$
$$= Nk\ln 2$$

Since $N = nN_A$ and $k = R/N_A$, this becomes

$$\Delta S = (nN_A)(R/N_A)\ln 2 = nR\ln 2$$

EVALUATE: We have found the same result as in Example 20.8, but without any reference to the thermodynamic path taken.

Microscopic States and the Second Law

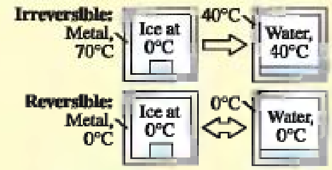
The relationship between entropy and the number of microscopic states gives us new insight into the entropy statement of the second law of thermodynamics, that the entropy of a closed system can never decrease. From Eq. (20.22) this means that a closed system can never spontaneously undergo a process that decreases the number of possible microscopic states.

An example of such a forbidden process would be if all of the air in your room spontaneously moved to one half of the room, leaving a vacuum in the other half. Such a "free compression" would be the reverse of the free expansion of Examples 20.8 and 20.11. This would decrease the number of possible microscopic states by a factor of 2^N. Strictly speaking, this process is not impossible! The probability of finding a given molecule in one half of the room is $\frac{1}{2}$, so the probability of finding all of the molecules in one half of the room at once is $(\frac{1}{2})^N$. (This is exactly the same as the probability of having a tossed coin come up heads N times in a row.) This probability is *not* zero. But lest you worry about suddenly finding yourself gasping for breath in the evacuated half of your room, consider that a typical room might hold 1000 moles of air, and so $N = 1000N_A = 6.02 \times 10^{26}$ molecules. The probability of all the molecules being in the same half of the room is therefore $(\frac{1}{2})^{6.02\times 10^{26}}$. Expressed as a decimal, this number has more than 10^{26} zeros to the right of the decimal point!

Because the probability of such a "free compression" taking place is so vanishingly small, it has almost certainly never occurred anywhere in the universe since the beginning of time. We conclude that for all practical purposes the second law of thermodynamics is never violated.

Test Your Understanding of Section 20.8 A quantity of N molecules of an ideal gas initially occupies volume V. The gas then expands to volume $2V$. The number of microscopic states of the gas increases in this expansion. Under which of the following circumstances will this number increase the most? (i) if the expansion is reversible and isothermal; (ii) if the expansion is reversible and adiabatic; (iii) the number will change by the same amount for both circumstances.

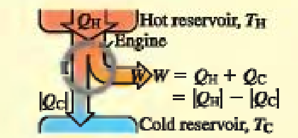
CHAPTER 20 SUMMARY



Reversible and irreversible processes: A reversible process is one whose direction can be reversed by an infinitesimal change in the conditions of the process, and in which the system is always in or very close to thermal equilibrium. All other thermodynamic processes are irreversible.

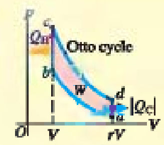
Heat engines: A heat engine takes heat Q_H from a source, converts part of it to work W, and discards the remainder $|Q_C|$ at a lower temperature. The thermal efficiency e of a heat engine measures how much of the absorbed heat is converted to work. (See Example 20.1)

$$e = \frac{W}{Q_H} = 1 + \frac{Q_C}{Q_H} = 1 - \left|\frac{Q_C}{Q_H}\right| \quad (20.4)$$



The Otto cycle: A gasoline engine operating on the Otto cycle has a theoretical maximum thermal efficiency e that depends on the compression ratio r and the ratio of heat capacities γ of the working substance.

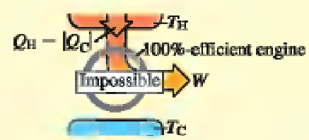
$$e = 1 - \frac{1}{r^{\gamma-1}} \quad (20.6)$$



Refrigerators: A refrigerator takes heat Q_C from a colder place, has a work input $|W|$, and discards heat $|Q_H|$ at a warmer place. The effectiveness of the refrigerator is given by its coefficient of performance K.

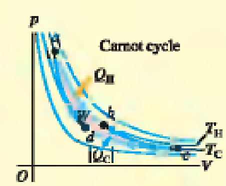
$$K = \frac{|Q_C|}{|W|} = \frac{|Q_C|}{|Q_H| - |Q_C|} \quad (20.9)$$

The second law of thermodynamics: The second law of thermodynamics describes the directionality of natural thermodynamic processes. It can be stated in several equivalent forms. The *engine* statement is that no cyclic process can convert heat completely into work. The *refrigerator* statement is that no cyclic process can transfer heat from a colder place to a hotter place with no input of mechanical work.



The Carnot cycle: The Carnot cycle operates between two heat reservoirs at temperatures T_H and T_C and uses only reversible processes. Its thermal efficiency depends only on T_H and T_C. An additional equivalent statement of the second law is that no engine operating between the same two temperatures can be more efficient than a Carnot engine. (See Examples 20.2 and 20.3.)

$$e_{\text{Carnot}} = 1 - \frac{T_C}{T_H} = \frac{T_H - T_C}{T_H} \quad (20.14)$$



A Carnot engine run backward is a Carnot refrigerator. Its coefficient of performance depends only on T_H and T_C. Another form of the second law states that no refrigerator operating between the same two temperatures can have a larger coefficient of performance than a Carnot refrigerator. (See Example 20.4.)

$$K_{\text{Carnot}} = \frac{T_C}{T_H - T_C} \quad (20.15)$$

Entropy: Entropy is a quantitative measure of the disorder of a system. The entropy change in any reversible process depends on the amount of heat flow and the absolute temperature T. Entropy depends only on the state of the system, and the change in entropy between given initial and final states is the same for all processes leading from one state to the other. This fact can be used to find the entropy change in an irreversible process. (See Examples 20.5–20.10)

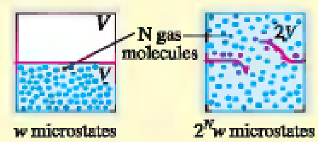
An important statement of the second law of thermodynamics is that the entropy of an isolated system may increase but can never decrease. When a system interacts with its surroundings, the total entropy change of system and surroundings can never decrease. When the interaction involves only reversible processes, the total entropy is constant and $\Delta S = 0$; when there is any irreversible process, the total entropy increases and $\Delta S > 0$.

$$\Delta S = \int_1^2 \frac{dQ}{T} \quad (20.19)$$

(reversible process)

Entropy and microscopic states: When a system is in a particular macroscopic state, the particles that make up the system may be in any of w possible microscopic states. The greater the number w, the greater the entropy. (See Example 20.11.)

$$S = k \ln w \quad (20.22)$$



Key Terms

irreversible process, *673*
reversible process, *674*
equilibrium process, *674*
heat engine, *675*
working substance, *675*
cyclic process, *675*
thermal efficiency, *676*
compression ratio, *678*
Otto cycle, *678*
Diesel cycle, *680*
refrigerator, *680*
coefficient of performance, *680*
energy efficiency rating, *681–682*
heat pump, *682*
second law of thermodynamics, *682*
Carnot cycle, *684*
Kelvin temperature scale, *690*
absolute zero, *690*
entropy, *691*
macroscopic state, *697*
microscopic state, *697*

Answer to Chapter Opening Question ?

Yes. That's what a refrigerator does: It makes heat flow from the cold interior of the refrigerator to the warm outside. The second law of thermodynamics says that heat cannot *spontaneously* flow from a cold body to a hot one. A refrigerator has a motor that does work on the system to *force* the heat to flow in that way.

Answers to Test Your Understanding Questions

20.1 Answer: (ii) Like sliding a book across a table, rubbing your hands together uses friction to convert mechanical energy into heat. The (impossible) reverse process would involve your hands spontaneously getting colder, with the released energy forcing your hands to move rhythmically back and forth!

20.2 Answer: (iii), (i), (ii) From Eq. (20.4) the efficiency is $e = W/Q_H$, and from Eq. (20.2) $W = Q_H + Q_C = |Q_H| - |Q_C|$. For engine (i) $Q_H = 5000\ \text{J}$ and $Q_C = -4500\ \text{J}$, so $W = 5000\ \text{J} + (-4500\ \text{J}) = 500\ \text{J}$ and $e = (500\ \text{J})/(5000\ \text{H}) = 0.100$. For engine (ii) $Q_H = 25{,}000\ \text{J}$ and $W = 2000\ \text{J}$, so $e = (2000\ \text{J})/(25{,}000\ \text{J}) = 0.080$. For engine (iii) $W = 400\ \text{J}$ and $Q_C = -2800\ \text{J}$, so $Q_H = W - Q_C = 400\ \text{J} - (-2800\ \text{J}) = 3200\ \text{J}$ and $e = (400\ \text{J})/(3200\ \text{J}) = 0.125$.

20.3 Answers: (i), (ii) Doubling the amount of fuel burned per cycle means that Q_H is doubled, so the resulting pressure increase from b to c in Fig. 20.6 is greater. The compression ratio and hence the efficiency remain the same, so $|Q_C|$ (the amount of heat rejected to the environment) must increase by the same factor as Q_H. Hence the pressure drop from d to a in Fig. 20.6 is also greater. The volume V and the compression ratio r don't change, so the horizontal dimensions of the pV-diagram don't change.

20.4 Answer: no A refrigerator uses an input of work to transfer heat from one system (the refrigerator's interior) to another system (its exterior, which includes the house in which the refrigerator is installed). If the door is open, these two systems are really the *same* system and will eventually come to the same temperature. By the first law of thermodynamics, all of the work inpnt to the refrigerator motor will be converted into heat and the temperature in your house will actually *increase*. To cool the house you need a system that will transfer heat from it to the outside world, such as an air conditioner or heat pump.

20.5 Answers: no, no Both the 100%-efficient engine of Fig. 20.11a and the workless refrigerator of Fig. 20.11b return to the same state at the end of a cycle as at the beginning, so the net change in internal energy of each system is zero $(\Delta U = 0)$. For the 100%-efficient engine, the net heat flow into the engine equals the net work done, so $Q = W$, $Q - W = 0$, and the first law $(\Delta U = Q - W)$ is obeyed. For the workless refrigerator, no net work is done (so $W = 0$) and as much heat flows into it as out (so $Q = 0$), so again $Q - W = 0$ and $\Delta U = Q - W$ in accordance with the first law. It is the second law of thermodynamics that tells us that both the 100%-efficient engine and the workless refrigerator are impossible.

20.6 Answer: no The efficiency can be no better than that of a Carnot engine running between the same two temperature limits, $e_{\text{Carnot}} = 1 - (T_C/T_H)$ [Eq. (20.14)]. The temperature T_C of the cold reservoir for this air-cooled engine is about 300 K (ambient temperature), and the temperature T_H of the hot reservoir cannot exceed the melting point of copper, 1356 K (see Table 17.4). Hence the maximum possible Carnot efficiency is $e = 1 - (300\text{ K})/(1356\text{ K}) = 0.78$, or 78%. The temperature of any real engine would be less than this, so it would be impossible for the inventor's engine to attain 85% efficiency. You should invest your money elsewhere.

20.7 Answers: -102 J/K, no The process described is exactly the opposite of the process used in Example 20.10. The result violates the second law of thermodynamics, which states that the entropy of an isolated system cannot decrease.

20.8 Answer: (i) For case (i), we saw in Example 20.8 (Section 20.7) that for an ideal gas, the entropy change in a free expansion is the same as in an isothermal expansion. From Eq. (20.23), this implies that the ratio of the number of microscopic states after and before the expansion, w_2/w_1, is also the same for these two cases. From Example 20.11, $w_2/w_1 = 2^N$, so the number of microscopic states increases by a factor 2^N. For case (ii), in a reversible expansion the entropy change is $\Delta S = \int dQ/T = 0$; if the expansion is adiabatic there is no heat flow, so $\Delta S = 0$. From Eq. (20.23), $w_2/w_1 = 1$ and there is *no* change in the number of microscopic states. The difference is that in an adiabatic expansion the temperature drops and the molecules move more slowly, so they have fewer microscopic states available to them than in an isothermal expansion.

PROBLEMS

For instructor-assigned homework, go to www.masteringphysics.com

Discussion Questions

Q20.1. A pot is half-filled with water, and a lid is placed on it, forming a tight seal so that no water vapor can escape. The pot is heated on a stove, forming water vapor inside the pot. The heat is then turned off and the water vapor condenses back to liquid. Is this cycle reversible or irreversible? Why?

Q20.2. Give two examples of reversible processes and two examples of irreversible processes in purely mechanical systems, such as blocks sliding on planes, springs, pulleys, and strings. Explain what makes each process reversible or irreversible.

Q20.3. What irreversible processes occur in a gasoline engine? Why are they irreversible?

Q20.4. Suppose you try to cool the kitchen of your house by leaving the refrigerator door open. What happens? Why? Would the result be the same if you left open a picnic cooler full of ice? Explain the reason for any differences.

Q20.5. A member of the U.S. Congress proposed a scheme to produce energy as follows. Water molecules (H_2O) are to be broken apart to produce hydrogen and oxygen. The hydrogen is then burned (that is, combined with oxygen), releasing energy in the process. The only product of this combustion is water, so there is no pollution. In light of the second law of thermodynamics, what do you think of this energy-producing scheme?

Q20.6. Is it a violation of the second law of thermodynamics to convert mechanical energy completely into heat? To convert heat completely into work? Explain your answers.

Q20.7. Imagine a special air filter placed in a window of a house. The tiny holes in the filter allow only air molecules moving faster than a certain speed to exit the house, and allow only air molecules moving slower than that speed to enter the house from outside. Explain why such an air filter would cool the house, and why the second law of thermodynamics makes building such a filter an impossible task.

Q20.8. An electric motor has its shaft coupled to that of an electric generator. The motor drives the generator, and some current from the generator is used to run the motor. The excess current is used to light a home. What is wrong with this scheme?

Q20.9. When a wet cloth is hung up in a hot wind in the desert, it is cooled by evaporation to a temperature that may be 20 C° or so below that of the air. Discuss this process in light of the second law of thermodynamics.

Q20.10. Compare the *pV*-diagram for the Otto cycle in Fig. 20.6 with the diagram for the Carnot heat engine in Fig. 20.13. Explain some of the important differences between the two cycles.

Q20.11. If no real engine can be as efficient as a Carnot engine operating between the same two temperatures, what is the point of developing and using Eq. (20.14)?

Q20.12. The efficiency of heat engines is high when the temperature difference between the hot and cold reservoirs is large. Refrigerators, on the other hand, work better when the temperature difference is small. Thinking of the mechanical refrigerator cycle shown in Fig. 20.9, explain in physical terms why it takes less work to remove heat from the working substance if the two reservoirs (the inside of the refrigerator and the outside air) are at nearly the same temperature, than if the outside air is much warmer than the interior of the refrigerator.

Q20.13. What would be the efficiency of a Carnot engine operating with $T_H = T_C$? What would be the efficiency if $T_C = 0$ K and T_H were any temperature above 0 K? Interpret your answers.

Q20.14. Real heat engines, like the gasoline engine in a car, always have some friction between their moving parts, although lubricants keep the friction to a minimum. Would a heat engine with completely frictionless parts be 100% efficient? Why or why not? Does the answer depend on whether or not the engine runs on the Carnot cycle? Again, why or why not?

Q20.15. Does a refrigerator full of food consume more power if the room temperature is 20°C than if it is 15°C? Or is the power consumption the same? Explain your reasoning.

Q20.16. In Example 20.4, a Carnot refrigerator requires a work input of only 230 J to extract 346 J of heat from the cold reservoir.

Doesn't this discrepancy imply a violation of the law of conservation of energy? Explain why or why not.

Q20.17. Explain why each of the following processes is an example of increasing disorder or randomness: mixing hot and cold water; free expansion of a gas; irreversible heat flow; developing heat by mechanical friction. Are entropy increases involved in all of these? Why or why not?

Q20.18. The free expansion of a gas is an adiabatic process and so no heat is transferred. No work is done, so the internal energy does not change. Thus, $Q/T = 0$, yet the disorder of the system and thus its entropy have increased after the expansion. Why does Eq. 20.19 not apply to this situation?

Q20.19. Are the earth and sun in thermal equilibrium? Are there entropy changes associated with the transmission of energy from the sun to the earth? Does radiation differ from other modes of heat transfer with respect to entropy changes? Explain your reasoning.

Q20.20. Discuss the entropy changes involved in the preparation and consumption of a hot fudge sundae.

Q20.21. If you run a movie film backwards, it is as if the direction of time were reversed. In the time-reversed movie, would you see processes that violate conservation of energy? Conservation of linear momentum? Would you see processes that violate the second law of thermodynamics? In each case, if law-breaking processes could occur, give some examples.

Q20.22. Some critics of biological evolution claim that it violates the second law of thermodynamics, since evolution involves simple life forms developing into more complex and more highly ordered organisms. Explain why this is not a valid argument against evolution.

Q20.23. A growing plant creates a highly complex and organized structure out of simple materials such as air, water, and trace minerals. Does this violate the second law of thermodynamics? Why or why not? What is the plant's ultimate source of energy? Explain your reasoning.

Exercises

Section 20.2 Heat Engines

20.1. A diesel engine performs 2200 J of mechanical work and discards 4300 J of heat each cycle. (a) How much heat must be supplied to the engine in each cycle? (b) What is the thermal efficiency of the engine?

20.2. An aircraft engine takes in 9000 J of heat and discards 6400 J each cycle. (a) What is the mechanical work output of the engine during one cycle? (b) What is the thermal efficiency of the engine?

20.3. A Gasoline Engine. A gasoline engine takes in 1.61×10^4 J of heat and delivers 3700 J of work per cycle. The heat is obtained by burning gasoline with a heat of combustion of 4.60×10^4 J/g. (a) What is the thermal efficiency? (b) How much heat is discarded in each cycle? (c) What mass of fuel is burned in each cycle? (d) If the engine goes through 60.0 cycles per second, what is its power output in kilowatts? In horsepower?

20.4. A gasoline engine has a power output of 180 kW (about 241 hp). Its thermal efficiency is 28.0%. (a) How much heat must be supplied to the engine per second? (b) How much heat is discarded by the engine per second?

20.5. A certain nuclear-power plant has a mechanical-power output (used to drive an electric generator) of 330 MW. Its rate of heat input from the nuclear reactor is 1300 MW. (a) What is the thermal efficiency of the system? (b) At what rate is heat discarded by the system?

Section 20.3 Internal-Combustion Engines

20.6. (a) Calculate the theoretical efficiency for an Otto cycle engine with $\gamma = 1.40$ and $r = 9.50$. (b) If this engine takes in 10,000 J of heat from burning its fuel, how much heat does it discard to the outside air?

20.7. What compression ratio r must an Otto cycle have to achieve an ideal efficiency of 65.0% if $\gamma = 1.40$?

20.8. The Otto-cycle engine in a Mercedes-Benz SLK230 has a compression ratio of 8.8. (a) What is the ideal efficiency of the engine? Use $\gamma = 1.40$. (b) The engine in a Dodge Viper GT2 has a slightly higher compression ratio of 9.6. How much increase in the ideal efficiency results from this increase in the compression ratio?

Section 20.4 Refrigerators

20.9. A refrigerator has a coefficient of performance of 2.10. In each cycle it absorbs 3.40×10^4 J of heat from the cold reservoir. (a) How much mechanical energy is required each cycle to operate the refrigerator? (b) During each cycle, how much heat is discarded to the high-temperature reservoir?

20.10. A room air conditioner has a coefficient of performance of 2.9 on a hot day, and uses 850 W of electrical power. (a) How many joules of heat does the air conditioner remove from the room in one minute? (b) How many joules of heat does the air conditioner deliver to the hot outside air in one minute? (c) Explain why your answers to parts (a) and (b) are not the same.

20.11. A window air-conditioner unit absorbs 9.80×10^4 J of heat per minute from the room being cooled and in the same time period deposits 1.44×10^5 J of heat into the outside air. (a) What is the power consumption of the unit in watts? (b) What is the energy efficiency rating of the unit?

20.12. A freezer has a coefficient of performance of 2.40. The freezer is to convert 1.80 kg of water at 25.0°C to 1.80 kg of ice at −5.0°C in hour. (a) What amount of heat must be removed from the water at 25.0°C to convert it to ice at −5.0°C? (b) How much electrical energy is consumed by the freezer during this hour? (c) How much wasted heat is delivered to the room in which the freezer sits?

Section 20.6 The Carnot Cycle

20.13. A Carnot engine whose high-temperature reservoir is at 620 K takes in 550 J of heat at this temperature in each cycle and gives up 335 J to the low-temperature reservoir. (a) How much mechanical work does the engine perform during each cycle? (b) What is the temperature of the low-temperature reservoir? (c) What is the thermal efficiency of the cycle?

20.14. A Carnot engine is operated between two heat reservoirs at temperatures of 520 K and 300 K. (a) If the engine receives 6.45 kJ of heat energy from the reservoir at 520 K in each cycle, how many joules per cycle does it discard to the reservoir at 300 K? (b) How much mechanical work is performed by the engine during each cycle? (c) What is the thermal efficiency of the engine?

20.15. A Carnot engine has an efficiency of 59% and performs 2.5×10^4 J of work in each cycle. (a) How much heat does the engine extract from its heat source in each cycle? (b) Suppose the engine exhausts heat at room temperature (20.0°C). What is the temperature of its heat source?

20.16. An ice-making machine operates in a Carnot cycle. It takes heat from water at 0.0°C and rejects heat to a room at 24.0°C. Suppose that 85.0 kg of water at 0.0°C are converted to ice at 0.0°C. (a) How much heat is discharged into the room? (b) How much energy must be supplied to the device?

20.17. A Carnot refrigerator is operated between two heat reservoirs at temperatures of 320 K and 270 K. (a) If in each cycle the refrigerator receives 415 J of heat energy from the reservoir at 270 K, how many joules of heat energy does it deliver to the reservoir at 320 K? (b) If the refrigerator completes 165 cycles each minute, what power input is required to operate it? (c) What is the coefficient of performance of the refrigerator?

20.18. A Carnot device extracts 5.00 kJ of heat from a body at $-10.0°C$. How much work is done if the device exhausts heat into the environment at (a) 25.0°C; (b) 0.0°C; (c) $-25.0°C$; In each case, is the device acting as an engine or as a refrigerator?

20.19. A certain brand of freezer is advertised to use 730 kW · h of energy per year. (a) Assuming the freezer operates for 5 hours each day, how much power does it require while operating? (b) If the freezer keeps its interior at a temperature of $-5.0°C$ in a 20.0°C room, what is its theoretical maximum performance coefficient? (c) What is the theoretical maximum amount of ice this freezer could make in an hour, starting with water at 20.0°C?

20.20. An ideal Carnot engine operates between 500°C and 100°C with a heat input of 250 J per cycle. (a) How much heat is delivered to the cold reservoir in each cycle? (b) What minimum number of cycles is necessary for the engine to lift a 500-kg rock through a height of 100 m?

20.21. A Carnot heat engine has a thermal efficiency of 0.600, and the temperature of its hot reservoir is 800 K. If 3000 J of heat is rejected to the cold reservoir in one cycle, what is the work output of the engine during one cycle?

20.22. A Carnot heat engine uses a hot reservoir consisting of a large amount of boiling water and a cold reservoir consisting of a large tub of ice and water. In 5 minutes of operation, the heat rejected by the engine melts 0.0400 kg of ice. During this time, how much work W is performed by the engine?

20.23. You design an engine that takes in 1.50×10^4 J of heat at 650 K in each cycle and rejects heat at a temperature of 350 K. The engine completes 240 cycles in 1 minute. What is the theoretical maximum power output of your engine, in horsepower?

20.24. (a) Show that the efficiency e of a Carnot engine and the coefficient of performance K of a Carnot refrigerator are related by $K = (1 - e)/e$. The engine and refrigerator operate between the same hot and cold reservoirs. (b) What is K for the limiting values $e \rightarrow 1$ and $e \rightarrow 0$? Explain.

Section 20.7 Entropy

20.25. A sophomore with nothing better to do adds heat to 0.350 kg of ice at 0.0°C until it is all melted. (a) What is the change in entropy of the water? (b) The source of heat is a very massive body at a temperature of 25.0°C. What is the change in entropy of this body? (c) What is the total change in entropy of the water and the heat source?

20.26. You decide to take a nice hot bath but discover that your thoughtless roommate has used up most of the hot water. You fill the tub with 270 kg of 30.0°C water and attempt to warm it further by pouring in 5.00 kg of boiling water from the stove. (a) Is this a reversible or an irreversible process? Use physical reasoning to explain. (b) Calculate the final temperature of the bath water. (c) Calculate the net change in entropy of the system (bath water + boiling water), assuming no heat exchange with the air or the tub itself.

20.27. A 15.0-kg block of ice at 0.0°C melts to liquid water at 0.0°C inside a large room that has a temperature of 20.0°C. Treat the ice and the room as an isolated system, and assume that the room is large enough for its temperature change to be ignored. (a) Is the melting of the ice reversible or irreversible? Explain, using simple physical reasoning without resorting to any equations. (b) Calculate the net entropy change of the system during this process. Explain whether or not this result is consistent with your answer to part (a).

20.28. You make tea with 0.250 kg of 85.0°C water and let it cool to room temperature (20.0°C) before drinking it. (a) Calculate the entropy change of the water while it cools. (b) The cooling process is essentially isothermal for the air in your kitchen. Calculate the change in entropy of the air while the tea cools, assuming that all the heat lost by the water goes into the air. What is the total entropy change of the system tea + air?

20.28. Three moles of an ideal gas undergo a reversible isothermal compression at 20.0°C. During this compression, 1850 J of work is done on the gas. What is the change of entropy of the gas?

20.30. What is the change in entropy of 0.130 kg of helium gas at the normal boiling point of helium when it all condenses isothermally to 1.00 L of liquid helium? (*Hint:* See Table 17.4 in Section 17.6.)

20.31. (a) Calculate the change in entropy when 1.00 kg of water at 100°C is vaporized and converted to steam at 100°C (see Table 17.4). (b) Compare your answer to the change in entropy when 1.00 kg of ice is melted at 0°C, calculated in Example 20.5 (Section 20.7). Is the change in entropy greater for melting or for vaporization? Interpret your answer using the idea that entropy is a measure of the randomness of a system.

20.32. (a) Calculate the change in entropy when 1.00 mol of water (molecular mass 18.0 g/mol) at 100°C evaporates to form water vapor at 100°C. (b) Repeat the calculation of part (a) for 1.00 mol of liquid nitrogen, 1.00 mol of silver, and 1.00 mol of mercury when each is vaporized at its normal boiling point. (See Table 17.4 for the heats of vaporization, and Appendix D for the molar masses. Note that the nitrogen molecule is N_2.) (c) Your results in parts (a) and (b) should be in relatively close agreement. (This is called the *rule of Drepez and Trouton*.) Explain why this should be so, using the idea that entropy is a measure of the randomness of a system.

20.33. If 25.0 g of the metal gallium melts in your hand (see Fig. 17.20), what is the change in entropy of the gallium in this process? What about the change in entropy of your hand? Is it positive or negative? Is its magnitude greater or less than that of the change in entropy of the gallium?

*Section 20.8 Microscopic Interpretation of Entropy

***20.34.** A box is separated by a partition into two parts of equal volume. The left side of the box contains 500 molecules of nitrogen gas; the right side contains 100 molecules of oxygen gas. The two gases are at the same temperature. The partition is punctured, and equilibrium is eventually attained. Assume that the volume of the box is large enough for each gas to undergo a free expansion and not change temperature. (a) On average, how many molecules of each type will there be in either half of the box? (b) What is the change in entropy of the system when the partition is punctured? (c) What is the probability that the molecules will be found in the same distribution as they were before the partition was punctured—that is, 500 nitrogen molecules in the left half and 100 oxygen molecules in the right half?

***20.35.** Two moles of an ideal gas occupy a volume V. The gas expands isothermally and reversibly to a volume $3V$. (a) Is the velocity distribution changed by the isothermal expansion?

Explain. (b) Use Eq. (20.23) to calculate the change in entropy of the gas. (c) Use Eq. (20.18) to calculate the change in entropy of the gas. Compare this result to that obtained in part (b).

***20.36.** A lonely party balloon with a volume of 2.40 L and containing 0.100 mol of air is left behind to drift in the temporarily uninhabited and depressurized International Space Station. Sunlight coming through a porthole heats and explodes the balloon, causing the air in it to undergo a free expansion into the empty station, whose total volume is 425 m^3. Calculate the entropy change of the air during the expansion.

Problems

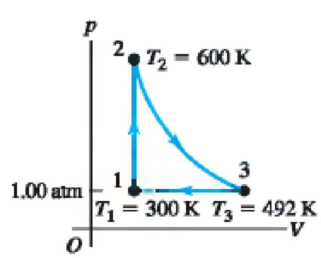
20.37. You design a Carnot engine that operates between temperatures of 500 K and 400 K and produces 2000 J of work in each cycle. (a) Calculate your engine's efficiency. (b) Calculate the amount of heat discarded during the isothermal compression at 400 K. (c) *Sketch* the 500 K and 400 K isotherms on a pV-diagram (no calculations); then sketch the Carnot cycle followed by your engine. (d) On the same diagram, *sketch* the 300 K isotherm; then sketch, in a different color if possible, the Carnot cycle starting at the same point on the 500 K isotherm but operating in a cycle between the 500 K and 300 K isotherms. (e) Compare the areas inside the loops (the net work done) for the two cycles. Notice that the same amount of heat is extracted from the hot reservoir in both cases. Can you explain why less heat is "wasted" during the 300 K isothermal compression than during the 400 K compression?

20.38. You are designing a Carnot engine that has 2 mol of CO_2 as its working substance; the gas may be treated as ideal. The gas is to have a maximum temperature of 527°C and a maximum pressure of 5.00 atm. With a heat input of 400 J per cycle, you want 300 J of useful work. (a) Find the temperature of the cold reservoir. (b) For how many cycles must this engine run to melt completely a 10.0-kg block of ice originally at 0.0°C, using only the heat rejected by the engine?

20.39. A Carnot engine whose low-temperature reservoir is at -90.0°C has an efficiency of 40.0%. An engineer is assigned the problem of increasing this to 45.0%. (a) By how many Celsius degrees must the temperature of the high-temperature reservoir be increased if the temperature of the low-temperature reservoir remains constant? (b) By how many Celsius degrees must the temperature of the low-temperature reservoir be decreased if the temperature of the high-temperature reservoir remains constant?

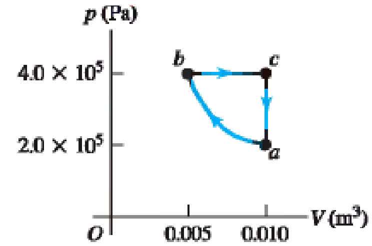
20.40. A heat engine takes 0.350 mol of a diatomic ideal gas around the cycle shown in the pV-diagram of Fig. 20.24. Process $1 \rightarrow 2$ is at constant volume, process $2 \rightarrow 3$ is adiabatic, and process $3 \rightarrow 1$ is at a constant pressure of 1.00 atm. The value of γ for this gas is 1.40. (a) Find the pressure and volume at points 1, 2, and 3. (b) Calculate Q, W, and ΔU for each of the three processes. (c) Find the net work done by the gas in the cycle. (d) Find the net heat flow into the engine in one cycle. (e) What is the thermal efficiency of the engine? How does this compare to the efficiency of a Carnot-cycle engine operating between the same minimum and maximum temperatures T_1 and T_2?

Figure 20.24 Problem 20.40.



20.41. You build a heat engine that takes 1.00 mol of an ideal diatomic gas through the cycle shown in Fig. 20.25. (a) Show that segment ab is an isothermal compression. (b) During which segment(s) of the cycle is heat absorbed by the gas? During which segment(s) is heat rejected? How do you know? (c) Calculate the temperature at points a, b, and c. (d) Calculate the net heat exchanged with the surroundings and the net work done by the engine in one cycle. (e) Calculate the thermal efficiency of the engine.

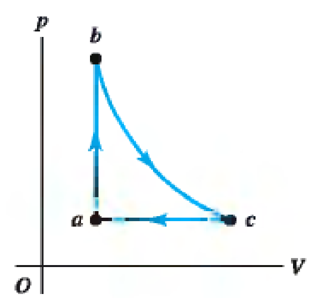
Figure 20.25 Problem 20.41.



20.42. Heat Pump. A heat pump is a heat engine run in reverse. In winter it pumps heat from the cold air outside into the warmer air inside the building, maintaining the building at a comfortable temperature. In summer it pumps heat from the cooler air inside the building to the warmer air outside, acting as an air conditioner. (a) If the outside temperature in winter is -5.0°C and the inside temperature is 17.0°C, how many joules of heat will the heat pump deliver to the inside for each joule of electrical energy used to run the unit, assuming an ideal Carnot cycle? (b) Suppose you have the option of using electrical resistance heating rather than a heat pump. How much electrical energy would you need in order to deliver the same amount of heat to the inside of the house as in part (a)? Consider a Carnot heat pump delivering heat to the inside of a house to maintain it at 68°F. Show that the heat pump delivers less heat for each joule of electrical energy used to operate the unit as the outside temperature decreases. Notice that this behavior is opposite to the dependence of the efficiency of a Carnot heat engine on the difference in the reservoir temperatures. Explain why this is so.

20.43. A heat engine operates using the cycle shown in Fig. 20.26. The working substance is 2.00 mol of helium gas, which reaches a maximum temperature of 327°C. Assume the helium can be treated as an ideal gas. Process bc is isothermal. The pressure in states a and c is 1.00×10^5 Pa, and the pressure in state b is 3.00×10^5 Pa. (a) How much heat enters the gas and how much leaves the gas each cycle? (b) How much work does the engine do each cycle, and what is its efficiency? (c) Compare this engine's efficiency with the maximum possible efficiency attainable with the hot and cold reservoirs used by this cycle.

Figure 20.26 Problem 20.43.

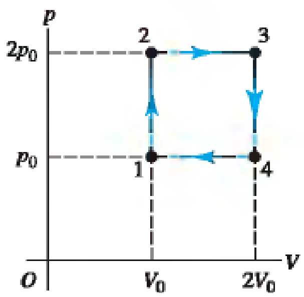


20.44. As a budding mechanical engineer, you are called upon to design a Carnot engine that has 2.00 mol of a monatomic ideal gas as its working substance and operates from a high-temperature reservoir at 500°C. The engine is to lift a 15.0-kg weight 2.00 m per cycle, using 500 J of heat input. The gas in the engine chamber can have a minimum volume of 5.00 L during the cycle. (a) Draw a pV-diagram for this cycle. Show in your diagram where heat enters and leaves the gas. (b) What must be the temperature of the cold reservoir? (c) What is the thermal efficiency of the engine? (d) How much heat energy does this engine waste per cycle? (e) What is the maximum pressure that the gas chamber will have to withstand?

20.45. An experimental power plant at the Natural Energy Laboratory of Hawaii generates electricity from the temperature gradient of the ocean. The surface and deep-water temperatures are 27°C and 6°C, respectively. (a) What is the maximum theoretical efficiency of this power plant? (b) If the power plant is to produce 210 kW of power, at what rate must heat be extracted from the warm water? At what rate must heat be absorbed by the cold water? Assume the maximum theoretical efficiency. (c) The cold water that enters the plant leaves it at a temperature of 10°C. What must be the flow rate of cold water through the system? Give your answer in kg/h and L/h.

20.46. What is the thermal efficiency of an engine that operates by taking n moles of diatomic ideal gas through the cycle $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$ shown in Fig. 20.27?

Figure 20.27 Problem 20.46.



20.47. A cylinder contains oxygen at a pressure of 2.00 atm. The volume is 4.00 L, and the temperature is 300 K. Assume that the oxygen may be treated as an ideal gas. The oxygen is carried through the following processes:

(i) Heated at constant pressure from the initial state (state 1) to state 2, which has $T = 450$ K.
(ii) Cooled at constant volume to 250 K (state 3).
(iii) Compressed at constant temperature to a volume of 4.00 L (state 4).
(iv) Heated at constant volume to 300 K, which takes the system back to state 1.

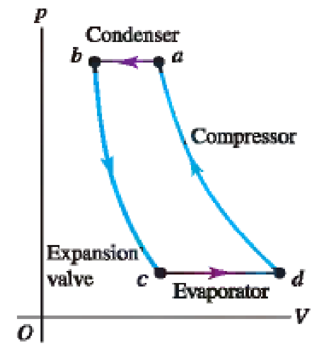
(a) Show these four processes in a pV-diagram, giving the numerical values of p and V in each of the four states. (b) Calculate Q and W for each of the four processes. (c) Calculate the net work done by the oxygen. (d) What is the efficiency of this device as a heat engine? How does this compare to the efficiency of a Carnot-cycle engine operating between the same minimum and maximum temperatures of 250 K and 450 K?

20.48. Thermodynamic Processes for a Refrigerator. A refrigerator operates on the cycle shown in Fig. 20.28. The compression $(d \rightarrow a)$ and expansion $(b \rightarrow c)$ steps are adiabatic. The temperature, pressure, and volume of the coolant in each of the four states a, b, c, and d are given in the table.

State	T (°C)	P (kPa)	V (m^3)	U (kJ)	Percentage That Is Liquid
a	80	2305	0.0682	1969	0
b	80	2305	0.00946	1171	100
c	5	363	0.2202	1005	54
d	5	363	0.4513	1657	5

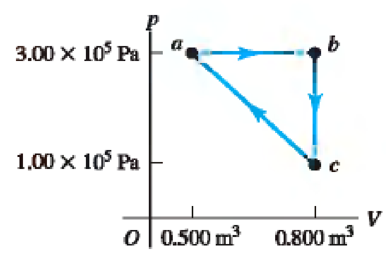
(a) In each cycle, how much heat is taken from inside the refrigerator into the coolant while the coolant is in the evaporator? (b) In each cycle, how much heat is exhausted from the coolant into the air outside the refrigerator while the coolant is in the condenser? (c) In each cycle, how much work is done by the motor that operates the compressor? (d) Calculate the coefficient of performance of the refrigerator.

Figure 20.28 Problem 20.48.



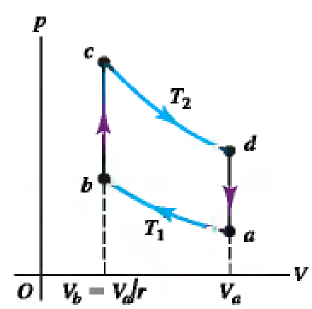
20.46. A monatomic ideal gas is taken around the cycle shown in Fig. 20.29 in the direction shown in the figure. The path for process $c \rightarrow a$ is a straight line in the pV-diagram. (a) Calculate Q, W, and ΔU for each process $a \rightarrow b$, $b \rightarrow c$, and $c \rightarrow a$. (b) What are Q, W, and ΔU for one complete cycle? (c) What is the efficiency of the cycle?

Figure 20.29 Problem 20.49.



20.50. A Stirling-Cycle Engine. The *Stirling cycle* is similar to the Otto cycle, except that the compression and expansion of the gas are done at constant temperature, not adiabatically as in the Otto cycle. The Stirling cycle is used in *external* combustion engines (in fact, burning fuel is not necessary; *any* way of producing a temperature difference will do—solar, geothermal, ocean temperature gradient, etc.), which means that the gas inside the cylinder is not used in the combustion process. Heat is supplied by burning fuel steadily outside the cylinder, instead of explosively inside the cylinder as in the Otto cycle. For this reason Stirling-cycle engines are quieter than Otto-cycle engines, since there are no intake and exhaust valves (a major source of engine noise). While small Stirling engines are used for a variety of purposes, Stirling engines for automobiles have not been successful because they are larger, heavier, and more expensive than conventional automobile engines. In the cycle, the working fluid goes through the following sequence of steps (Fig. 20.30):

Figure 20.30 Problem 20.50.



(i) Compressed isothermally at temperature T_1 from the initial state a to state b, with a compression ratio r.
(ii) Heated at constant volume to state c at temperature T_2.
(iii) Expanded isothermally at T_2 to state d.
(iv) Cooled at constant volume back to the initial state a.

Assume that the working fluid is n moles of an ideal gas (for which C_V is independent of temperature). (a) Calculate Q, W, and ΔU for each of the processes $a \to b$, $b \to c$, $c \to d$, and $d \to a$. (b) In the Stirling cycle, the heat transfers in the processes $b \to c$ and $d \to a$ do not involve external heat sources but rather use *regeneration:* The same substance that transfers heat to the gas inside the cylinder in the process $b \to c$ also absorbs heat back from the gas in the process $d \to a$. Hence the heat transfers $Q_{b \to c}$ and $Q_{d \to a}$ do not play a role in determining the efficiency of the engine. Explain this last statement by comparing the expressions for $Q_{b \to c}$ and $Q_{d \to a}$ calculated in part (a). (c) Calculate the efficiency of a Stirling-cycle engine in terms of the temperatures T_1 and T_2. How does this compare to the efficiency of a Carnot-cycle engine operating between these same two temperatures? (Historically, the Stirling cycle was devised before the Carnot cycle.) Does this result violate the second law of thermodynamics? Explain. Unfortunately, actual Stirling-cycle engines cannot achieve this efficiency due to problems with the heat-transfer processes and pressure losses in the engine.

20.51. A Carnot engine operates between two heat reservoirs at temperatures T_H and T_C. An inventor proposes to increase the efficiency by running one engine between T_H and an intermediate temperature T' and a second engine between T' and T_C, using as input the heat expelled by the first engine. Compute the efficiency of this composite system, and compare it to that of the original engine.

20.52. A typical coal-fired power plant generates 1000 MW of usable power at an overall thermal efficiency of 40%. (a) What is the rate of heat input to the plant? (b) The plant burns anthracite coal, which has a heat of combustion of 2.65×10^7 J/kg. How much coal does the plant use per day, if it operates continuously? (c) At what rate is heat ejected into the cool reservoir, which is the nearby river? (d) The river's temperature is 18.0°C before it reaches the power plant and 18.5°C after it has received the plant's waste heat. Calculate the river's flow rate, in cubic meters per second. (e) By how much does the river's entropy increase each second?

20.53. Automotive Thermodynamics. A Volkswagen Passat has a six-cylinder Otto-cycle engine with compression ratio $r = 10.6$. The diameter of each cylinder, called the *bore* of the engine, is 82.5 mm. The distance that the piston moves during the compression in Fig. 20.5, called the *stroke* of the engine, is 86.4 mm. The initial pressure of the air–fuel mixture (at point a in Fig. 20.6) is 8.50×10^4 Pa, and the initial temperature is 300 K (the same as the outside air). Assume that 200 J of heat is added to each cylinder in each cycle by the burning gasoline, and that the gas has $C_V = 20.5$ J/mol · K and $\gamma = 1.40$. (a) Calculate the total work done in one cycle in each cylinder of the engine, and the heat released when the gas is cooled to the temperature of the outside air. (b) Calculate the volume of the air–fuel mixture at point a in the cycle. (c) Calculate the pressure, volume, and temperature of the gas at points b, c, and d in the cycle. In a pV-diagram, show the numerical values of p, V, and T for each of the four states. (d) Compare the efficiency of this engine with the efficiency of a Carnot-cycle engine operating between the same maximum and minimum temperatures.

20.54. An air conditioner operates on 800 W of power and has a performance coefficient of 2.80 with a room temperature of 21.0°C and an outside temperature of 35.0°C. (a) Calculate the rate of heat removal for this unit. (b) Calculate the rate at which heat is discharged to the outside air. (c) Calculate the total entropy change in the room if the air conditioner runs for 1 hour. Calculate the total entropy change in the outside air for the same time period. (d) What is the net change in entropy for the system (room + outside air)?

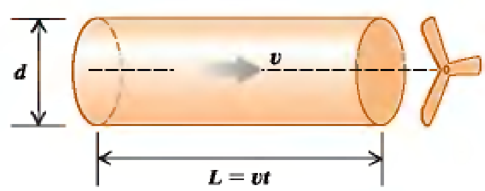
20.55. Unavailable Energy. The discussion of entropy and the second law that follows Example 20.10 (Section 20.7) says that the increase in entropy in an irreversible process is associated with energy becoming less available. Consider a Carnot cycle that uses a low-temperature reservoir with Kelvin temperature T_c. This is a true reservoir—that is, large enough not to change temperature when it accepts heat from the engine. Let the engine accept heat from an object of temperature T', where $T' > T_c$. The object is of finite size, so it cools as heat is extracted from it. The engine continues to operate until $T' = T_c$. (a) Show that the total magnitude of heat rejected to the low-temperature reservoir is $T_c|\Delta S_h|$, where ΔS_h is the change in entropy of the high-temperature reservoir. (b) Apply the result of part (a) to 1.00 kg of water initially at a temperature of 373 K as the heat source for the engine and $T_c = 273$ K. How much total mechanical work can be performed by the engine until it stops? (c) Repeat part (b) for 2.00 kg of water at 323 K. (d) Compare the amount of work that can be obtained from the energy in the water of Example 20.10 before and after it is mixed. Discuss whether your result shows that energy has become less available.

20.56. The maximum power that can be extracted by a wind turbine from an air stream is approximately

$$P = kd^2v^3$$

where d is the blade diameter, v is the wind speed, and the constant $k = 0.5$ W · s³/m⁵. (a) Explain the dependence of P on d and on v by considering a cylinder of air that passes over the turbine blades in time t (Fig. 20.31). This cylinder has diameter d, length $L = vt$, and density ρ. (b) The Mod-5B wind turbine at Kahaku on the Hawaiian island of Oahu has a blade diameter of 97 m (slightly longer than a football field) and sits atop a 58-m tower. It can produce 3.2 MW of electric power. Assuming 25% efficiency, what wind speed is required to produce this amount of power? Give your answer in m/s and in km/h. (c) Commercial wind turbines are commonly located in or downwind of mountain passes. Why?

Figure 20.31 Problem 20.56.



20.57. (a) How much work must a Carnot refrigerator do on a hot day to transfer 1000 J of heat from its interior at 10°C to the outside air at 35.0°C? (b) How much work must the same refrigerator do to transfer the same amount of heat if the interior temperature is the same, but the outside air is at only 15.0°C? (c) Sketch pV-diagrams for these two situations. Can you explain in physical terms why more work must be done when the temperature difference between the two isothermal stages is greater?

20.56. A 0.0500-kg cube of ice at an initial temperature of −15.0°C is placed in 0.600 kg of water at $T = 45.0$°C in an insulated container of negligible mass. (a) Calculate the final temperature of the water once the ice has melted. (b) Calculate the change in entropy of the system.

20.59. (a) For the Otto cycle shown in Fig. 20.6, calculate the changes in entropy of the gas in each of the constant-volume processes $b \to c$ and $d \to a$ in terms of the temperatures T_a, T_b, T_c, and T_d and the number of moles n and the heat capacity C_V of the gas. (b) What is the total entropy change in the engine during one cycle? (*Hint:* Use the relationships between T_a and T_b and between T_d and T_c.) (c) The processes $b \rightarrow c$ and $d \rightarrow a$ occur irreversibly in a real Otto engine. Explain how can this be reconciled with your result in part (b).

20.60. A *TS*-Diagram. (a) Graph a Carnot cycle, plotting Kelvin temperature vertically and entropy horizontally. This is called a temperature–entropy diagram, or *TS*-diagram. (b) Show that the area under any curve representing a reversible path in a temperature–entropy diagram represents the heat absorbed by the system. (c) Derive from your diagram the expression for the thermal efficiency of a Carnot cycle. (d) Draw a temperature–entropy diagram for the Stirling cycle, described in Problem 20.50. Use this diagram to relate the efficiency of the Carnot and Stirling cycles.

20.61. A physics student immerses one end of a copper rod in boiling water at 100°C and the other end in an ice–water mixture at 0°C. The sides of the rod are insulated. After steady-state conditions have been achieved in the rod, 0.160 kg of ice melts in a certain time interval. For this time interval, find (a) the entropy change of the boiling water; (b) the entropy change of the ice–water mixture; (c) the entropy change of the copper rod; (d) the total entropy change of the entire system.

20.62. To heat 1 cup of water $(250\,\text{cm}^3)$ to make coffee, you place an electric heating element in the cup. As the water temperature increases from 20°C to 65°C, the temperature of the heating element remains at a constant 120°C. Calculate the change in entropy of (a) the water; (b) the heating element; (c) the system of water and heating element. (Make the same assumption about the specific heat of water as in Example 20.10 in Section 20.7, and ignore the heat that flows into the ceramic coffee cup itself.) (d) Is this process reversible or irreversible? Explain.

20.63. An object of mass m_1, specific heat capacity c_1, and temperature T_1 is placed in contact with a second object of mass m_2, specific heat capacity c_2, and temperature $T_2 > T_1$. As a result, the temperature of the first object increases to T and the temperature of the second object decreases to T'. (a) Show that the entropy increase of the system is

$$\Delta S = m_1 c_1 \ln\frac{T}{T_1} + m_2 c_2 \ln\frac{T'}{T_2}$$

and show that energy conservation requires that

$$m_1 c_1 (T - T_1) = m_2 c_2 (T_2 - T')$$

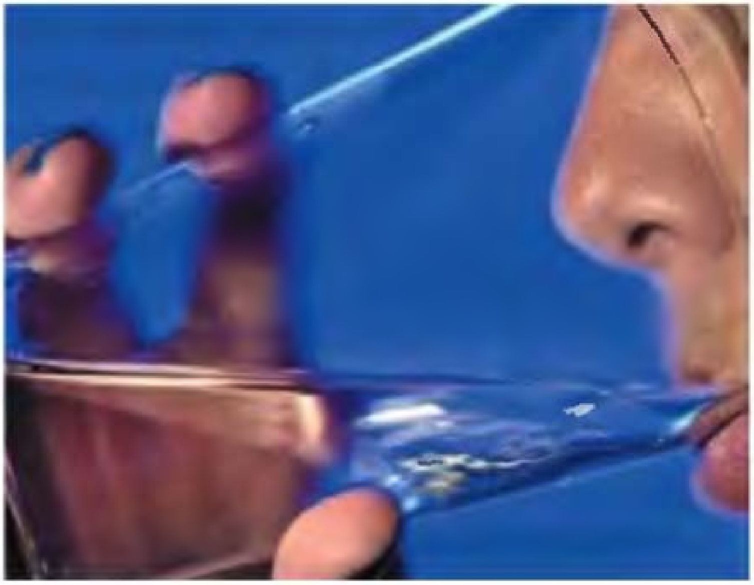
(b) Show that the entropy change ΔS, considered as a function of T, is a *maximum* if $T = T'$, which is just the condition of thermodynamic equilibrium. (c) Discuss the result of part (b) in terms of the idea of entropy as a measure of disorder.

Challenge Problem

20.64. Consider a Diesel cycle that starts (at point a in Fig. 20.7) with air at temperature T_a. The air may be treated as an ideal gas. (a) If the temperature at point c is T_c, derive an expression for the efficiency of the cycle in terms of the compression ratio r. (b) What is the efficiency if $T_a = 300$ K, $T_c = 950$ K, $\gamma = 1.40$, and $r = 21.0$?

ELECTRIC CHARGE AND ELECTRIC FIELD

21



? Water makes life possible: The cells of your body could not function without water in which to dissolve essential biological molecules. What electrical properties of water make it such a good solvent?

LEARNING GOALS

By studying this chapter, you will learn:

- The nature of electric charge, and how we know that electric charge is conserved.
- How objects become electrically charged.
- How to use Coulomb's law to calculate the electric force between charges.
- The distinction between electric force and electric field.
- How to calculate the electric field due to a collection of charges.
- How to use the idea of electric field lines to visualize and interpret electric fields.
- How to calculate the properties of electric dipoles.

Back in Chapter 5, we briefly mentioned the four kinds of fundamental forces. To this point the only one of these forces that we have examined in any detail is gravity. Now we are ready to examine the force of *electromagnetism,* which encompasses both electricity and magnetism. Our exploration of electromagnetic phenomena will occupy our attention for most of the remainder of this book.

Electromagnetic interactions involve particles that have a property called *electric charge,* an attribute that is as fundamental as mass. Just as objects with mass are accelerated by gravitational forces, so electrically charged objects are accelerated by electric forces. The annoying electric spark you feel when you scuff your shoes across a carpet and then reach for a metal doorknob is due to charged particles leaping between your finger and the doorknob. Electric currents, such as those in a flashlight or a television, are simply streams of charged particles flowing within wires in response to electric forces. Even the forces that hold atoms together to form solid matter, and that keep the atoms of solid objects from passing through each other, are fundamentally due to electric interactions between the charged particles within atoms.

We begin our study of electromagnetism in this chapter by examining the nature of electric charge. We'll find that electric charge is quantized and that it obeys a conservation principle. We then turn to a discussion of the interactions of electric charges that are at rest in our frame of reference, called *electrostatic* interactions. Such interactions are of tremendous importance in chemistry and biology and have many technological applications. Electrostatic interactions are governed by a simple relationship known as *Coulomb's law* and are most conveniently described by using the concept of *electric field.* In later chapters we'll expand our discussion to include electric charges in motion. This will lead us to an understanding of magnetism and, remarkably, of the nature of light.

While the key ideas of electromagnetism are conceptually simple, applying them to practical problems will make use of many of your mathematical skills, especially your knowledge of geometry and integral calculus. For this reason you may find this chapter and those that follow to be more mathematically demanding than earlier chapters. The reward for your extra effort will be a deeper understanding of principles that are at the heart of modern physics and technology.

21.1 Electric Charge

The ancient Greeks discovered as early as 600 B.C. that after they rubbed amber with wool, the amber could attract other objects. Today we say that the amber has acquired a net **electric charge,** or has become *charged.* The word "electric" is derived from the Greek word *elektron,* meaning amber. When you scuff your shoes across a nylon carpet, you become electrically charged, and you can charge a comb by passing it through dry hair.

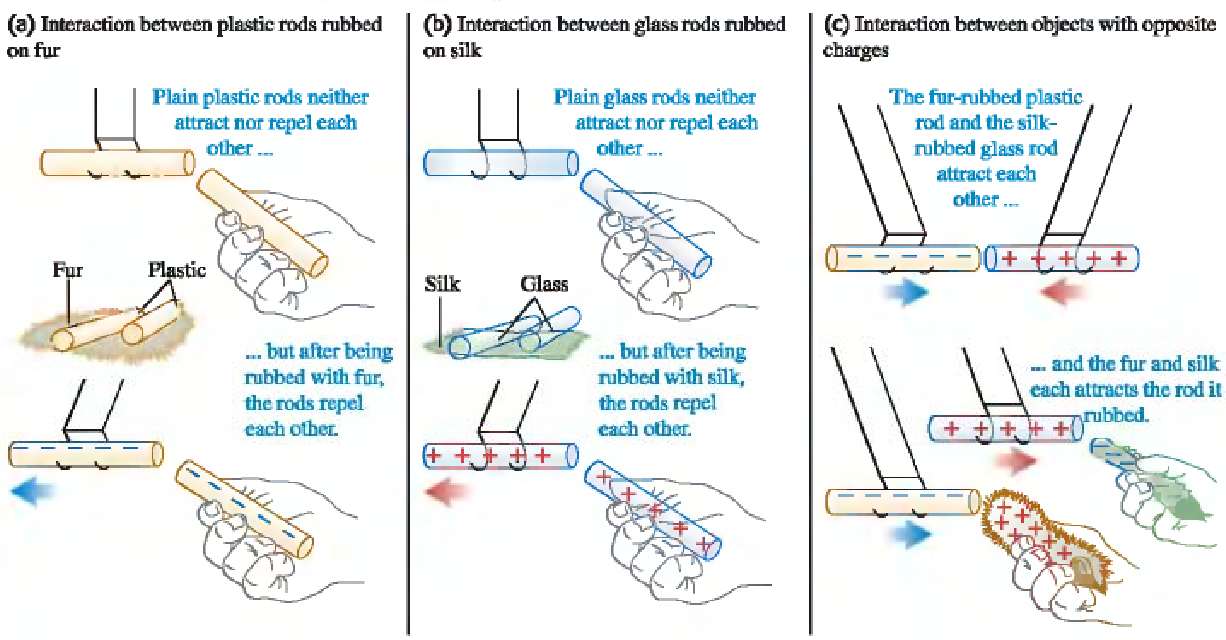
Plastic rods and fur (real or fake) are particularly good for demonstrating **electrostatics,** the interactions between electric charges that are at rest (or nearly so). Figure 21.1a shows two plastic rods and a piece of fur. After we charge each rod by rubbing it with the piece of fur, we find that the rods repel each other.

When we rub glass rods with silk, the glass rods also become charged and repel each other (Fig. 21.1b). But a charged plastic rod *attracts* a charged glass rod; furthermore, the plastic rod and the fur attract each other, and the glass rod and the silk attract each other (Fig. 21.1c).

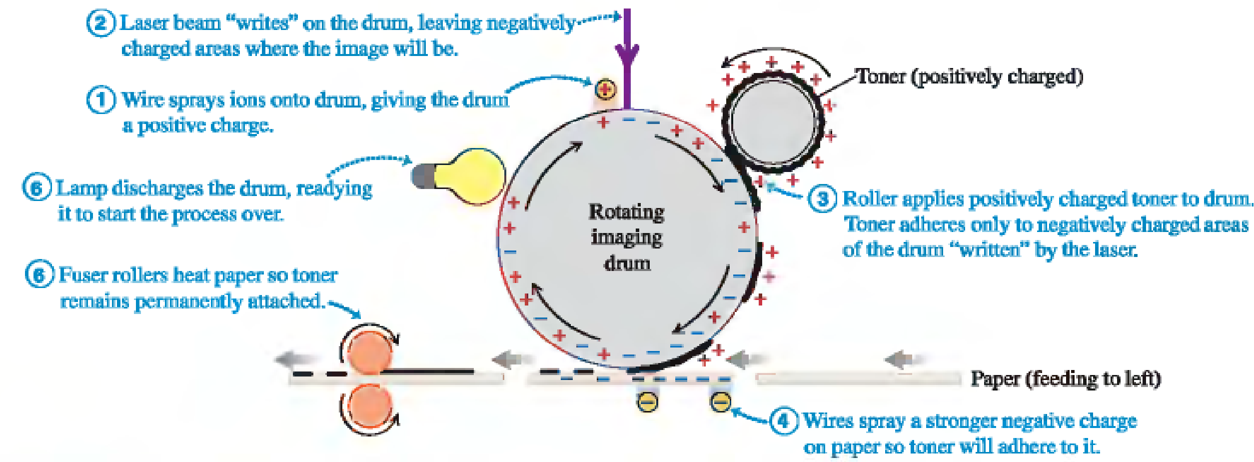
These experiments and many others like them have shown that there are exactly two kinds of electric charge: the kind on the plastic rod rubbed with fur and the kind on the glass rod rubbed with silk. Benjamin Franklin (1706–1790) suggested calling these two kinds of charge *negative* and *positive,* respectively, and these names are still used. The plastic rod and the silk have negative charge; the glass rod and the fur have positive charge.

Two positive charges or two negative charges repel each other. A positive charge and a negative charge attract each other.

21.1 Experiments in electrostatics. (a) Negatively charged objects repel each other. (b) Positively charged objects repel each other. (c) Positvely charged objects and negatively charged objects attract each other.



21.2 Schematic diagram of the operation of a laser printer.



CAUTION **Electric attraction and repulsion** The attraction and repulsion of two charged objects are sometimes summarized as "Like charges repel, and opposite charges attract." But keep in mind that the phrase "like charges" does *not* mean that the two charges are exactly identical, only that both charges have the same algebraic *sign* (both positive or both negative). "Opposite charges" means that both objects have an electric charge, and those charges have different signs (one positive and the other negative).

One technological application of forces between charged bodies is in a laser printer (Fig. 21.2). Initially the printer's light-sensitive imaging drum is given a positive charge. As the drum rotates, a laser beam shines on selected areas of the drum, leaving those areas with a *negative* charge. Positively charged particles of toner adhere only to the areas of the drum "written" by the laser. When a piece of paper is placed in contact with the drum, the toner particles stick to the paper and form an image.

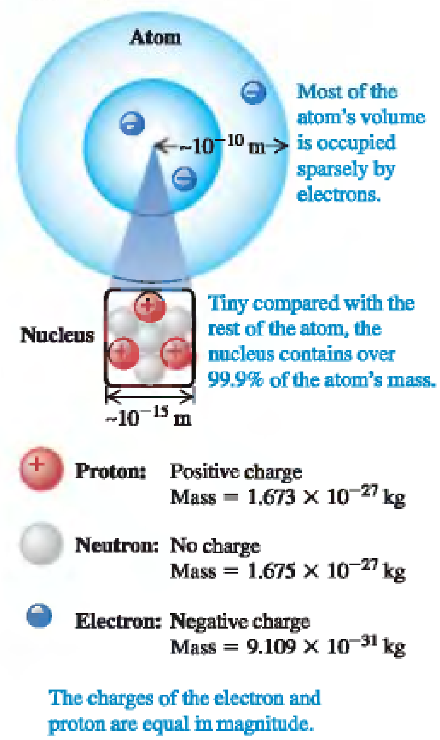
Electric Charge and the Structure of Matter

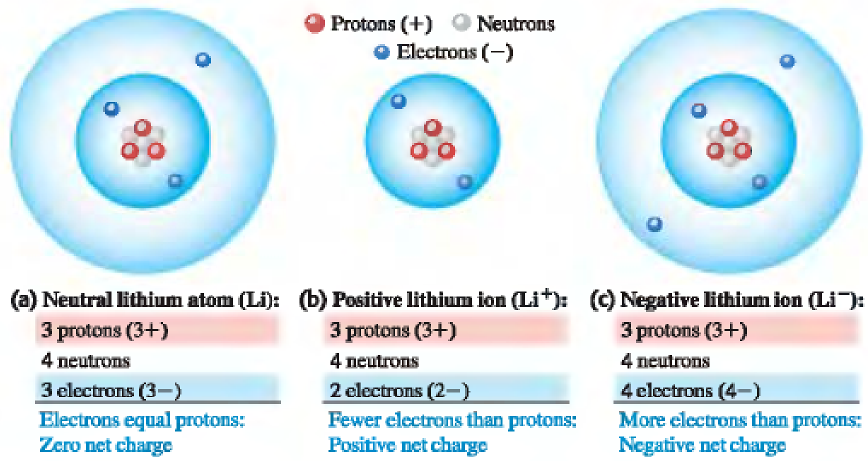
When you charge a rod by rubbing it with fur or silk as in Fig. 21.1, there is no visible change in the appearance of the rod. What, then, actually happens to the rod when you charge it? To answer this question, we must look more closely at the structure and electric properties of atoms, the building blocks of ordinary matter of all kinds.

The structure of atoms can be described in terms of three particles: the negatively charged **electron**, the positively charged **proton**, and the uncharged **neutron** (Fig. 21.3). The proton and neutron are combinations of other entities called *quarks*, which have charges of $\pm\frac{1}{3}$ and $\pm\frac{2}{3}$ times the electron charge. Isolated quarks have not been observed, and there are theoretical reasons to believe that it is impossible in principle to observe a quark in isolation.

The protons and neutrons in an atom make up a small, very dense core called the **nucleus**, with dimensions of the order of 10^{-15} m. Surrounding the nucleus are the electrons, extending out to distances of the order of 10^{-10} m from the nucleus. If an atom were a few kilometers across, its nucleus would be the size of a tennis ball. The negatively charged electrons are held within the atom by the attractive electric forces exerted on them by the positively charged nucleus. (The protons and neutrons are held within the stable atomic nuclei by an attractive interaction, called the *strong nuclear force*, that overcomes the electric repulsion of the protons. The strong nuclear force has a short range, and its effects do not extend far beyond the nucleus.)

21.3 The structure of an atom. The particular atom depicted here is lithium (see Fig. 21.4a).





21.4 (a) A neutral atom has as many electrons as it does protons. (b) A positive ion has a deficit of electrons. (c) A negative ion has an excess of electrons. (The electron "shells" are a schematic representation of the actual electron distribution, a diffuse cloud many times larger than the nucleus.)

The masses of the individual particles, to the precision that they are presently known, are

$$\text{Mass of electron} = m_e = 9.1093826(16) \times 10^{-31}\ \text{kg}$$
$$\text{Mass of proton} = m_p = 1.67262171(29) \times 10^{-27}\ \text{kg}$$
$$\text{Mass of neutron} = m_n = 1.67492728(29) \times 10^{-27}\ \text{kg}$$

The numbers in parentheses are the uncertainties in the last two digits. Note that the masses of the proton and neutron are nearly equal and are roughly 2000 times the mass of the electron. Over 99.9% of the mass of any atom is concentrated in its nucleus.

The negative charge of the electron has (within experimental error) *exactly* the same magnitude as the positive charge of the proton. In a neutral atom the number of electrons equals the number of protons in the nucleus, and the net electric charge (the algebraic sum of all the charges) is exactly zero (Fig. 21.4a). The number of protons or electrons in a neutral atom of an element is called the **atomic number** of the element. If one or more electrons are removed, the remaining positively charged structure is called a **positive ion** (Fig. 21.4b). A **negative ion** is an atom that has *gained* one or more electrons (Fig. 21.4c). This gaining or losing of electrons is called **ionization.**

When the total number of protons in a macroscopic body equals the total number of electrons, the total charge is zero and the body as a whole is electrically neutral. To give a body an excess negative charge, we may either *add negative* charges to a neutral body or *remove positive* charges from that body. Similarly, we can create an excess positive charge by either *adding positive* charge or *removing negative* charge. In most cases, negatively charged (and highly mobile) electrons are added or removed, and a "positively charged body" is one that has lost some of its normal complement of electrons. When we speak of the charge of a body, we always mean its *net* charge. The net charge is always a very small fraction (typically no more than 10^{-12}) of the total positive charge or negative charge in the body.

Electric Charge Is Conserved

Implicit in the foregoing discussion are two very important principles. First is the **principle of conservation of charge:**

The algebraic sum of all the electric charges in any closed system is constant.

If we rub together a plastic rod and a piece of fur, both initially uncharged, the rod acquires a negative charge (since it takes electrons from the fur) and the fur acquires a positive charge of the *same* magnitude (since it has lost as many electrons as the rod has gained). Hence the total electric charge on the two bodies together does not change. In any charging process, charge is not created or destroyed; it is merely *transferred* from one body to another.

Conservation of charge is thought to be a *universal* conservation law. No experimental evidence for any violation of this principle has ever been observed. Even in high-energy interactions in which particles are created and destroyed, such as the creation of electron–positron pairs, the total charge of any closed system is exactly constant.

The second important principle is:

The magnitude of charge of the electron or proton is a natural unit of charge.

Every observable amount of electric charge is always an integer multiple of this basic unit. We say that charge is *quantized.* A familiar example of quantization is money. When you pay cash for an item in a store, you have to do it in one-cent increments. Cash can't be divided into amounts smaller than one cent, and electric charge can't be divided into amounts smaller than the charge of one electron or proton. (The quark charges, $\pm\frac{1}{3}$ and $\pm\frac{2}{3}$ of the electron charge, are probably not observable as isolated charges.) Thus the charge on any macroscopic body is always either zero or an integer multiple (negative or positive) of the electron charge.

Understanding the electric nature of matter gives us insight into many aspects of the physical world (Fig. 21.5). The chemical bonds that hold atoms together to form molecules are due to electric interactions between the atoms. They include the strong ionic bonds that hold sodium and chlorine atoms together to make table salt and the relatively weak bonds between the strands of DNA that record your body's genetic code. The normal force exerted on you by the chair in which you're sitting arises from electric forces between charged particles in the atoms of your seat and in the atoms of your chair. The tension force in a stretched string and the adhesive force of glue are likewise due to the electric interactions of atoms.

21.5 Most of the forces on this water skier are electric. Electric interactions between adjacent molecules give rise to the force of the water on the ski, the tension in the tow rope, and the resistance of the air on the skier's body. Electric interactions also hold the atoms of the skier's body together. Only one wholly nonelectric force acts on the skier: the force of gravity.

Test Your Understanding of Section 21.1 (a) Strictly speaking, does the plastic rod in Fig. 21.1 weigh more, less, or the same after rubbing it with fur? (b) What about the glass rod after rubbing it with silk? What about (c) the fur and (d) the silk?

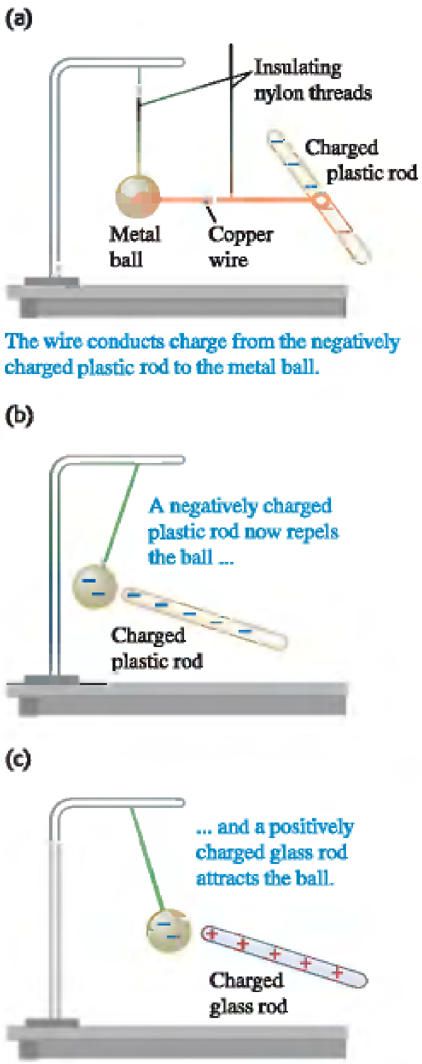
21.2 Conductors, Insulators, and Induced Charges

Some materials permit electric charge to move easily from one region of the material to another, while others do not. For example, Fig. 21.6a shows a copper wire supported by a nylon thread. Suppose you touch one end of the wire to a charged plastic rod and attach the other end to a metal ball that is initially uncharged; you then remove the charged rod and the wire. When you bring another charged body up close to the ball (Figs. 21.6b and 21.6c), the ball is attracted or repelled, showing that the ball has become electrically charged. Electric charge has been transferred through the copper wire between the ball and the surface of the plastic rod.

The copper wire is called a **conductor** of electricity. If you repeat the experiment using a rubber band or nylon thread in place of the wire, you find that *no* charge is transferred to the ball. These materials are called **insulators.** Conductors permit the easy movement of charge through them, while insulators do not. (The supporting nylon threads shown in Fig. 21.6 are insulators, which prevents charge from leaving the metal ball and copper wire.)

As an example, carpet fibers on a dry day are good insulators. As you walk across a carpet, the rubbing of your shoes against the fibers causes charge to build

21.6 Copper is a good conductor of electricity; nylon is a good insulator. **(a)** The copper wire conducts charge between the metal ball and the charged plastic rod to charge the ball negatively. Afterward, the metal ball is **(b)** repelled by a negatively charged plastic rod and **(c)** attracted to a positively charged glass rod.



up on you, and this charge remains on you because it can't flow through the insulating fibers. If you then touch a conducting object such as a doorknob, a rapid charge transfer takes place between your finger and the doorknob, and you feel a shock. One way to prevent this is to wind some of the carpet fibers around conducting cores so that any charge that builds up on you can be transferred harmlessly to the carpet. Another solution is to coat the carpet fibers with an antistatic layer that does not easily transfer electrons to or from your shoes; this prevents any charge from building up on you in the first place.

Most metals are good conductors, while most nonmetals are insulators. Within a solid metal such as copper, one or more outer electrons in each atom become detached and can move freely throughout the material, just as the molecules of a gas can move through the spaces between the grains in a bucket of sand. The motion of these negatively charged electrons carries charge through the metal. The other electrons remain bound to the positively charged nuclei, which themselves are bound in nearly fixed positions within the material. In an insulator there are no, or very few, free electrons, and electric charge cannot move freely through the material. Some materials called *semiconductors* are intermediate in their properties between good conductors and good insulators.

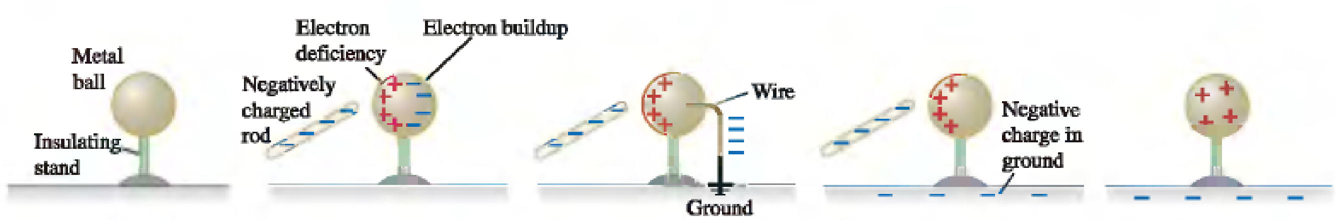
Charging by Induction

We can charge a metal ball using a copper wire and an electrically charged plastic rod, as in Fig. 21.6a. In this process, some of the excess electrons on the rod are transferred from it to the ball, leaving the rod with a smaller negative charge. There is a different technique in which the plastic rod can give another body a charge of *opposite* sign without losing any of its own charge. This process is called charging by **induction.**

Figure 21.7 shows an example of charging by induction. An uncharged metal ball is supported on an insulating stand (Fig. 21.7a). When you bring a negatively charged rod near it, without actually touching it (Fig. 21.7b), the free electrons in the metal ball are repelled by the excess electrons on the rod, and they shift toward the right, away from the rod. They cannot escape from the ball because the supporting stand and the surrounding air are insulators. So we get excess negative charge at the right surface of the ball and a deficiency of negative charge (that is, a net positive charge) at the left surface. These excess charges are called **induced charges.**

Not all of the free electrons move to the right surface of the ball. As soon as any induced charge develops, it exerts forces toward the *left* on the other free electrons. These electrons are repelled by the negative induced charge on the right and attracted toward the positive induced charge on the left. The system reaches an equilibrium state in which the force toward the right on an electron, due to the charged rod, is just balanced by the force toward the left due to the induced charge. If we remove the charged rod, the free electrons shift back to the left, and the original neutral condition is restored.

21.7 Charging a metal ball by induction.



(a) Uncharged metal ball

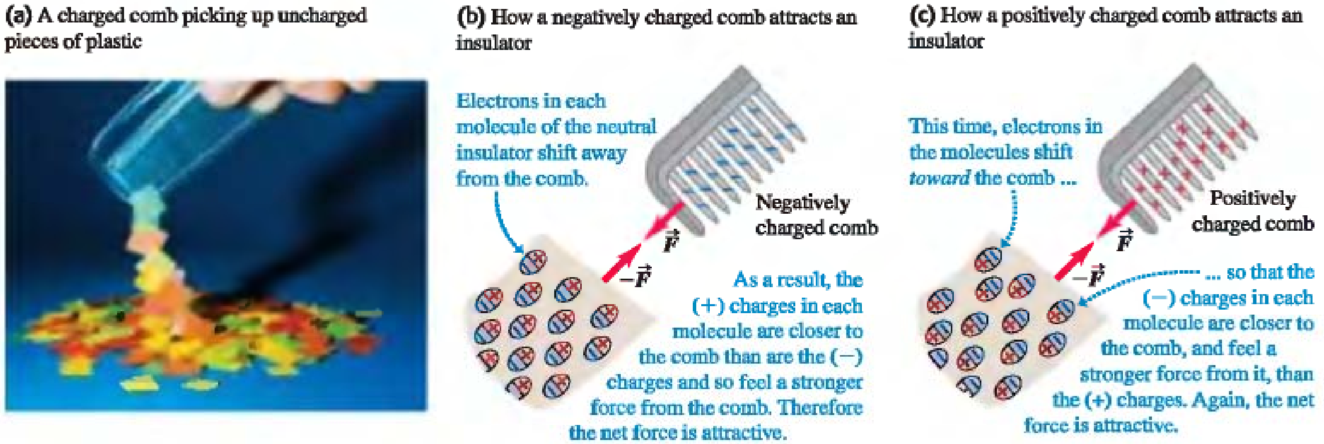
(b) Negative charge on rod repels electrons, creating zones of negative and positive **induced charge.**

(c) Wire lets electron buildup (induced negative charge) flow into ground.

(d) Wire removed; ball now has only an electron-deficient region of positive charge.

(e) Rod removed; electrons rearrange themselves, ball has overall electron deficiency (net positive charge).

21.8 The charges within the molecules of an insulating material can shift slightly. As a result, a comb with either sign of charge attracts a neutral insulator. By Newton's third law the neutral insulator exerts an equal-magnitude attractive force on the comb.



What happens if, while the plastic rod is nearby, you touch one end of a conducting wire to the right surface of the ball and the other end to the earth (Fig. 21.7c)? The earth is a conductor, and it is so large that it can act as a practically infinite source of extra electrons or sink of unwanted electrons. Some of the negative charge flows through the wire to the earth. Now suppose you disconnect the wire (Fig. 21.7d) and then remove the rod (Fig. 21.7e); a net positive charge is left on the ball. The charge on the negatively charged rod has not changed during this process. The earth acquires a negative charge that is equal in magnitude to the induced positive charge remaining on the ball.

Charging by induction would work just as well if the mobile charges in the ball were positive charges instead of negatively charged electrons, or even if both positive and negative mobile charges were present. In a metallic conductor the mobile charges are always negative electrons, but it is often convenient to describe a process *as though* the moving charges were positive. In ionic solutions and ionized gases, both positive and negative charges are mobile.

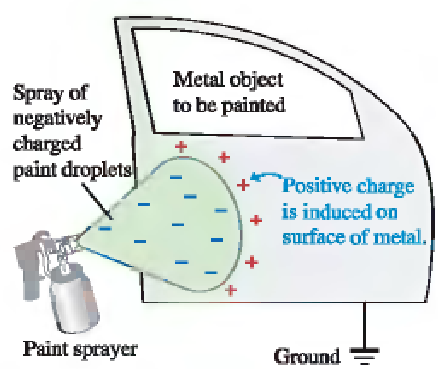
Electric Forces on Uncharged Objects

Finally, we note that a charged body can exert forces even on objects that are *not* charged themselves. If you rub a balloon on the rug and then hold the balloon against the ceiling, it sticks, even though the ceiling has no net electric charge. After you electrify a comb by running it through your hair, you can pick up uncharged bits of paper or plastic with the comb (Fig. 21.8a). How is this possible?

This interaction is an induced-charge effect. Even in an insulator, electric charges can shift back and forth a little when there is charge nearby. This is shown in Fig. 21.8b; the negatively charged plastic comb causes a slight shifting of charge within the molecules of the neutral insulator, an effect called *polarization*. The positive and negative charges in the material are present in equal amounts, but the positive charges are closer to the plastic comb and so feel an attraction that is stronger than the repulsion felt by the negative charges, giving a net attractive force. (In Section 21.3 we will study how electric forces depend on distance.) Note that a neutral insulator is also attracted to a *positively* charged comb (Fig. 21.8c). Now the charges in the insulator shift in the opposite direction; the negative charges in the insulator are closer to the comb and feel an attractive force that is stronger than the repulsion felt by the positive charges in the insulator. Hence a charged object of *either* sign exerts an attractive force on an uncharged insulator.

The attraction between a charged object and an uncharged one has many important practical applications, including the electrostatic painting process used in the automobile industry (Fig. 21.9). A metal object to be painted is connected to the earth ("ground" in Fig. 21.9), and the paint droplets are given an electric charge as they exit the sprayer nozzle. Induced charges of the opposite sign

21.9 The electrostatic painting process (compare Figs. 21.7b and 21.7c).



appear in the object as the droplets approach, just as in Fig. 21.7b, and they attract the droplets to the surface. This process minimizes overspray from clouds of stray paint particles and gives a particularly smooth finish.

Test Your Understanding of Section 21.2 You have two lightweight metal spheres, each hanging from an insulating nylon thread. One of the spheres has a net negative charge, while the other sphere has no net charge. (a) If the spheres are close together but do not touch, will they (i) attract each other, (ii) repel each other, or (iii) exert no force on each other? (b) You now allow the two spheres to touch. Once they have touched, will the two spheres (i) attract each other, (ii) repel each other, or (iii) exert no force on each other?

21.3 Coulomb's Law

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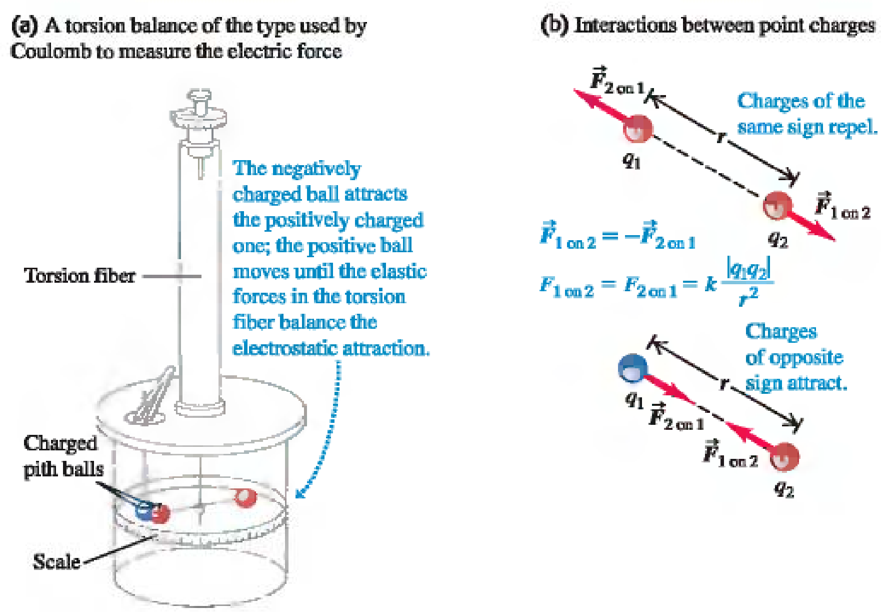
11.1 Electric Force: Coulomb's Law
11.2 Electric Force: Superposition Principle
11.3 Electric Force: Superposition (Quantitative)

Charles Augustin de Coulomb (1736–1806) studied the interaction forces of charged particles in detail in 1784. He used a torsion balance (Fig. 21.10a) similar to the one used 13 years later by Cavendish to study the much weaker gravitational interaction, as we discussed in Section 12.1. For **point charges,** charged bodies that are very small in comparison with the distance r between them, Coulomb found that the electric force is proportional to $1/r^2$. That is, when the distance r doubles, the force decreases to $\frac{1}{4}$ of its initial value; when the distance is halved, the force increases to four times its initial value.

The electric force between two point charges also depends on the quantity of charge on each body, which we will denote by q or Q. To explore this dependence, Coulomb divided a charge into two equal parts by placing a small charged spherical conductor into contact with an identical but uncharged sphere; by symmetry, the charge is shared equally between the two spheres. (Note the essential role of the principle of conservation of charge in this procedure.) Thus he could obtain one-half, one-quarter, and so on, of any initial charge. He found that the forces that two point charges q_1 and q_2 exert on each other are proportional to each charge and therefore are proportional to the *product* q_1q_2 of the two charges.

Thus Coulomb established what we now call **Coulomb's law:**

The magnitude of the electric force between two point charges is directly proportional to the product of the charges and inversely proportional to the square of the distance between them.



21.10 (a) Measuring the electric force between point charges. (b) The electric forces between point charges obey Newton's third law: $\vec{F}_{1\text{ on }2} = -\vec{F}_{2\text{ on }1}$.

In mathematical terms, the magnitude F of the force that each of two point charges q_1 and q_2 a distance r apart exerts on the other can be expressed as

$$F = k\frac{|q_1q_2|}{r^2} \tag{21.1}$$

where k is a proportionality constant whose numerical value depends on the system of units used. The absolute value bars are used in Eq. (21.1) because the charges q_1 and q_2 can be either positive or negative, while the force magnitude F is always positive.

The directions of the forces the two charges exert on each other are always along the line joining them. When the charges q_1 and q_2 have the same sign, either both positive or both negative, the forces are repulsive; when the charges have opposite signs, the forces are attractive (Fig. 21.10b). The two forces obey Newton's third law; they are always equal in magnitude and opposite in direction, even when the charges are not equal in magnitude.

The proportionality of the electric force to $1/r^2$ has been verified with great precision. There is no reason to suspect that the exponent is different from precisely 2. Thus the form of Eq. (21.1) is the same as that of the law of gravitation. But electric and gravitational interactions are two distinct classes of phenomena. Electric interactions depend on electric charges and can be either attractive or repulsive, while gravitational interactions depend on mass and are always attractive (because there is no such thing as negative mass).

Fundamental Electric Constants

The value of the proportionality constant k in Coulomb's law depends on the system of units used. In our study of electricity and magnetism we will use SI units exclusively. The SI electric units include most of the familiar units such as the volt, the ampere, the ohm, and the watt. (There is *no* British system of electric units.) The SI unit of electric charge is called one **coulomb** (1 C). In SI units the constant k in Eq. (21.1) is

$$k = 8.987551787 \times 10^9\ \mathrm{N \cdot m^2/C^2} \cong 8.988 \times 10^9\ \mathrm{N \cdot m^2/C^2}$$

The value of k is known to such a large number of significant figures because this value is closely related to the speed of light in vacuum. (We will show this in Chapter 32 when we study electromagnetic radiation.) As we discussed in Section 1.3, this speed is *defined* to be exactly $c = 2.99792458 \times 10^8$ m/s. The numerical value of k is defined in terms of c to be precisely

$$k = (10^{-7}\ \mathrm{N \cdot s^2/C^2})c^2$$

You should check this expression to confirm that k has the right units.

In principle we can measure the electric force F between two equal charges q at a measured distance r and use Coulomb's law to determine the charge. Thus we could regard the value of k as an operational definition of the coulomb. For reasons of experimental precision it is better to define the coulomb instead in terms of a unit of electric *current* (charge per unit time), the *ampere,* equal to 1 coulomb per second. We will return to this definition in Chapter 28.

In SI units we usually write the constant k in Eq. (21.1) as $1/4\pi\epsilon_0$, where ϵ_0 ("epsilon-nought" or "epsilon-zero") is another constant. This appears to complicate matters, but it actually simplifies many formulas that we will encounter in later chapters. From now on, we will usually write Coulomb's law as

$$F = \frac{1}{4\pi\epsilon_0}\frac{|q_1q_2|}{r^2} \quad \text{(Coulomb's law: force between two point charges)} \tag{21.2}$$

The constants in Eq. (21.2) are approximately

$$\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{C^2/N\cdot m^2} \quad \text{and} \quad \frac{1}{4\pi\epsilon_0} = k = 8.988 \times 10^9\ \mathrm{N\cdot m^2/C^2}$$

In examples and problems we will often use the approximate value

$$\frac{1}{4\pi\epsilon_0} = 9.0 \times 10^9\ \mathrm{N\cdot m^2/C^2}$$

which is within about 0.1% of the correct value.

As we mentioned in Section 21.1, the most fundamental unit of charge is the magnitude of the charge of an electron or a proton, which is denoted by e. The most precise value available as of the writing of this book is

$$e = 1.60217653(14) \times 10^{-19}\ \mathrm{C}$$

One coulomb represents the negative of the total charge of about 6×10^{18} electrons. For comparison, a copper cube 1 cm on a side contains about 2.4×10^{24} electrons. About 10^{19} electrons pass through the glowing filament of a flashlight bulb every second.

In electrostatics problems (that is, problems that involve charges at rest), it's very unusual to encounter charges as large as 1 coulomb. Two 1-C charges separated by 1 m would exert forces on each other of magnitude 9×10^9 N (about 1 million tons)! The total charge of all the electrons in a copper one-cent coin is even greater, about 1.4×10^5 C, which shows that we can't disturb electric neutrality very much without using enormous forces. More typical values of charge range from about 10^{-9} to about 10^{-6} C. The microcoulomb $(1\ \mu\mathrm{C} = 10^{-6}\ \mathrm{C})$ and the nanocoulomb $(1\ \mathrm{nC} = 10^{-9}\ \mathrm{C})$ are often used as practical units of charge.

Example 21.1 Electric force versus gravitational force

An α particle ("alpha") is the nucleus of a helium atom. It has mass $m = 6.64 \times 10^{-27}$ kg and charge $q = +2e = 3.2 \times 10^{-19}$ C. Compare the force of the electric repulsion between two α particles with the force of gravitational attraction between them.

SOLUTION

IDENTIFY: This problem involves Newton's law for the gravitational force F_g between particles (see Section 12.1) and Coulomb's law for the electric force F_e between point charges. We are asked to compare these forces, so our target variable is the *ratio* of these two forces, F_e/F_g.

SET UP: Figure 21.11 shows our sketch. The magnitude of the repulsive electric force is given by Eq. (21.2):

$$F_e = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2}$$

The magnitude F_g of the attractive gravitational force is given by Eq. (12.1):

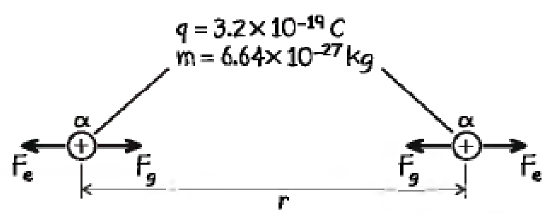
$$F_g = G\frac{m^2}{r^2}$$

EXECUTE: The ratio of the electric force to the gravitational force is

$$\frac{F_e}{F_g} = \frac{1}{4\pi\epsilon_0 G}\frac{q^2}{m^2} = \frac{9.0 \times 10^9\ \mathrm{N\cdot m^2/C^2}}{6.67 \times 10^{-11}\ \mathrm{N\cdot m^2/kg^2}}\frac{(3.2 \times 10^{-19}\ \mathrm{C})^2}{(6.64 \times 10^{-27}\ \mathrm{kg})^2}$$
$$= 3.1 \times 10^{35}$$

EVALUATE: This astonishingly large number shows that the gravitational force in this situation is completely negligible in comparison to the electric force. This is always true for interactions of atomic and subatomic particles. (Notice that this result doesn't depend on the distance r between the two α particles.) But within objects the size of a person or a planet, the positive and negative charges are nearly equal in magnitude, and the net electric force is usually much *smaller* than the gravitational force.

21.11 Our sketch for this problem.



Superposition of Forces

Coulomb's law as we have stated it describes only the interaction of two *point* charges. Experiments show that when two charges exert forces simultaneously on a third charge, the total force acting on that charge is the *vector sum* of the forces that the two charges would exert individually. This important property, called the **principle of superposition of forces,** holds for any number of charges. By using this principle, we can apply Coulomb's law to *any* collection of charges. Several of the examples at the end of this section show applications of the superposition principle.

Strictly speaking, Coulomb's law as we have stated it should be used only for point charges *in a vacuum.* If matter is present in the space between the charges, the net force acting on each charge is altered because charges are induced in the molecules of the intervening material. We will describe this effect later. As a practical matter, though, we can use Coulomb's law unaltered for point charges in air. At normal atmospheric pressure, the presence of air changes the electric force from its vacuum value by only about one part in 2000.

Problem-Solving Strategy 21.1 Coulomb's Law

MP

IDENTIFY *the relevant concepts:* Coulomb's law comes into play whenever you need to know the electric force acting between charged particles.

SET UP *the problem* using the following steps:

1. Make a drawing showing the locations of the charged particles, and label each particle with its charge. This step is particularly important if more than two charged particles are present.
2. If three or more charges are present and they do not all lie on the same line, set up an *xy*-coordinate system.
3. Often you will need to find the electric force on just one particle. If so, identify that particle.

EXECUTE *the solution* as follows:

1. For each particle that exerts a force on the particle of interest, calculate the magnitude of that force using Eq.(21.2).
2. Sketch the electric force vectors acting on the particle(s) of interest due to each of the other particles (that is, make a free-body diagram). Remember that the force exerted by particle 1 on particle 2 points from particle 2 toward particle 1 if the two charges have opposite signs, but points from particle 2 directly away from particle 1 if the charges have the same sign.
3. Calculate the total electric force on the particle(s) of interest. Remember that the electric force, like any force, is a *vector.* When the forces acting on a charge are caused by two or more other charges, the total force on the charge is the *vector sum* of the individual forces. You may want to go back and review the vector algebra in Sections 1.7 through 1.9. It's often helpful to use components in an *xy*-coordinate system. Be sure to use correct vector notation; if a symbol represents a vector quantity, put an arrow over it. If you get sloppy with your notation, you will also get sloppy with your thinking.
4. As always, using consistent units is essential. With the value of $k = 1/4\pi\epsilon_0$ given above, distances *must* be in meters, charge in coulombs, and force in newtons. If you are given distances in centimeters, inches, or furlongs, don't forget to convert! When a charge is given in microcoulombs (μC) or nanocoulombs (nC), remember that $1\ \mu\text{C} = 10^{-6}\ \text{C}$ and $1\ \text{nC} = 10^{-9}\ \text{C}$.
5. Some examples and problems in this and later chapters involve a continuous distribution of charge along a line or over a surface. In these cases the vector sum described in step 3 becomes a vector integral, usually carried out by use of components. We divide the total charge distribution into infinitesimal pieces, use Coulomb's law for each piece, and then integrate to find the vector sum. Sometimes this process can be done without explicit use of integration.
6. In many situations the charge distribution will be *symmetrical.* For example, you might be asked to find the force on a charge Q in the presence of two other identical charges q, one above and to the left of Q and the other below and to the left of Q. If the distances from Q to each of the other charges are the same, the force on Q from each charge has the same magnitude; if each force vector makes the same angle with the horizontal axis, adding these vectors to find the net force is particularly easy. Whenever possible, exploit any symmetries to simplify the problem-solving process.

EVALUATE *your answer:* Check whether your numerical results are reasonable, and confirm that the direction of the net electric force agrees with the principle that like charges repel and opposite charges attract.

Example 21.2 Force between two point charges

Two point charges, $q_1 = +25\text{ nC}$ and $q_2 = -75\text{ nC}$, are separated by a distance of 3.0 cm (Fig. 21.12a). Find the magnitude and direction of (a) the electric force that q_1 exerts on q_2; and (b) the electric force that q_2 exerts on q_1.

SOLUTION

IDENTIFY: This problem asks for the electric forces that two charges exert on each other, so we will need to use Coulomb's law.

SET UP: We use Eq. (21.2) to calculate the magnitude of the force that each particle exerts on the other. We use Newton's third law to relate the forces that the two particles exert on each other.

EXECUTE: (a) After we convert charge to coulombs and distance to meters, the magnitude of the force that q_1 exerts on q_2 is

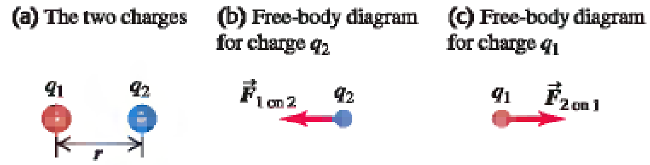
$$F_{1\text{ on }2} = \frac{1}{4\pi\epsilon_0}\frac{|q_1q_2|}{r^2}$$

$$= (9.0\times10^9\text{ N}\cdot\text{m}^2/\text{C}^2)\frac{|(+25\times10^{-9}\text{ C})(-75\times10^{-9}\text{ C})|}{(0.030\text{ m})^2}$$

$$= 0.019\text{ N}$$

Since the two charges have opposite signs, the force is attractive; that is, the force that acts on q_2 is directed toward q_1 along the line joining the two charges, as shown in Fig. 21.12b.

21.12 What force does q_1 exert on q_2, and what force does q_2 exert on q_1? Gravitational forces are negligible.



(b) Newton's third law applies to the electric force. Even though the charges have different magnitudes, the magnitude of the force that q_2 exerts on q_1 is the *same* as the magnitude of the force that q_1 exerts on q_2:

$$F_{2\text{ on }1} = 0.019\text{ N}$$

Newton's third law also states that the direction of the force that q_2 exerts on q_1 is exactly opposite the direction of the force that q_1 exerts on q_2; this is shown in Fig. 21.12c.

EVALUATE: Note that the force on q_1 is directed toward q_2, as it must be, since charges of opposite sign attract each other.

Example 21.3 Vector addition of electric forces on a line

Two point charges are located on the positive x-axis of a coordinate system. Charge $q_1 = 1.0\text{ nC}$ is 2.0 cm from the origin, and charge $q_2 = -3.0\text{ nC}$ is 4.0 cm from the origin. What is the total force exerted by these two charges on a charge $q_3 = 5.0\text{ nC}$ located at the origin? Gravitational forces are negligible.

SOLUTION

IDENTIFY: Here there are *two* electric forces acting on the charge q_3, and we must add these forces to find the total force.

SET UP: Figure 21.13a shows the coordinate system. Our target variable is the net electric force exerted *on* charge q_3 by the other two charges. This is the vector sum of the forces due to q_1 and q_2 individually.

EXECUTE: Figure 21.13b is a free-body diagram for charge q_3. Note that q_3 is repelled by q_1 (which has the same sign) and attracted to q_2 (which has the opposite sign). Converting charge to coulombs and distance to meters, we use Eq. (21.2) to find the magnitude $F_{1\text{ on }3}$ of the force of q_1 on q_3:

$$F_{1\text{ on }3} = \frac{1}{4\pi\epsilon_0}\frac{|q_1q_3|}{r^2}$$

$$= (9.0\times10^9\text{ N}\cdot\text{m}^2/\text{C}^2)\frac{(1.0\times10^{-9}\text{ C})(5.0\times10^{-9}\text{ C})}{(0.020\text{ m})^2}$$

$$= 1.12\times10^{-4}\text{ N} = 112\ \mu\text{N}$$

This force has a negative x-component because q_3 is repelled (that is, pushed in the negative x-direction) by q_1.

The magnitude $F_{2\text{ on }3}$ of the force of q_2 on q_3 is

$$F_{2\text{ on }3} = \frac{1}{4\pi\epsilon_0}\frac{|q_2q_3|}{r^2}$$

$$= (9.0\times10^9\text{ N}\cdot\text{m}^2/\text{C}^2)\frac{(3.0\times10^{-9}\text{ C})(5.0\times10^{-9}\text{ C})}{(0.040\text{ m})^2}$$

$$= 8.4\times10^{-5}\text{ N} = 84\ \mu\text{N}$$

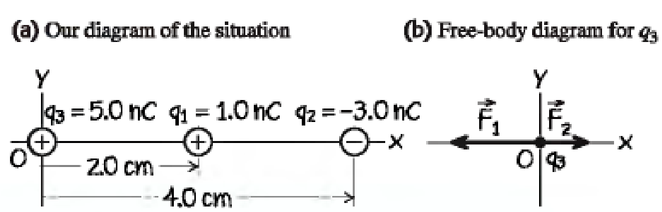
This force has a positive x-component because q_3 is attracted (that is, pulled in the positive x-direction) by q_2. The sum of the x-components is

$$F_x = -112\ \mu\text{N} + 84\ \mu\text{N} = -28\ \mu\text{N}$$

There are no y- or z-components. Thus the total force on q_3 is directed to the left, with magnitude $28\ \mu\text{N} = 2.8\times10^{-5}\text{ N}$.

EVALUATE: To check the magnitudes of the individual forces, note that q_2 has three times as much charge (in magnitude) as q_1 but is twice as far from q_3. From Eq. (21.2) this means that $F_{2\text{ on }3}$ must be $3/2^2 = \frac{3}{4}$ as large as $F_{1\text{ on }3}$. Indeed, our results show that this ratio is $(84\ \mu\text{N})/(112\ \mu\text{N}) = 0.75$. The direction of the net force also makes sense: $\vec{F}_{1\text{ on }3}$ is opposite to and has a larger magnitude than $\vec{F}_{2\text{ on }3}$, so the net force is in the direction of $\vec{F}_{1\text{ on }3}$.

21.13 Our sketches for this problem.



Example 21.4 Vector addition of electric forces in a plane

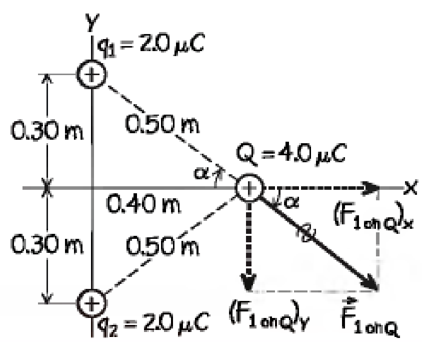
Two equal positive point charges $q_1 = q_2 = 2.0\ \mu\text{C}$ are located at $x = 0, y = 0.30\ \text{m}$ and $x = 0, y = -0.30\ \text{m}$, respectively. What are the magnitude and direction of the total (net) electric force that these charges exert on a third point charge $Q = 4.0\ \mu\text{C}$ at $x = 0.40\ \text{m}, y = 0$?

SOLUTION

IDENTIFY: As in Example 21.3, we have to compute the force that each charge exerts on Q and then find the vector sum of the forces.

SET UP: Figure 21.14 shows the situation. Since the three charges do not all lie on a line, the best way to calculate the forces that q_1 and q_2 exert on Q is to use components.

21.14 Our sketch for this problem.



EXECUTE: Figure 21.14 shows the force on Q due to the upper charge q_1. From Coulomb's law the magnitude F of this force is

$$F_{1\,\text{on}\,Q} = (9.0 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\frac{(4.0 \times 10^{-6}\ \text{C})(2.0 \times 10^{-6}\ \text{C})}{(0.50\ \text{m})^2}$$
$$= 0.29\ \text{N}$$

The angle α is below the x-axis, so the components of this force are given by

$$(F_{1\,\text{on}\,Q})_x = (F_{1\,\text{on}\,Q})\cos\alpha = (0.29\ \text{N})\frac{0.40\ \text{m}}{0.50\ \text{m}} = 0.23\ \text{N}$$
$$(F_{1\,\text{on}\,Q})_y = -(F_{1\,\text{on}\,Q})\sin\alpha = -(0.29\ \text{N})\frac{0.30\ \text{m}}{0.50\ \text{m}} = -0.17\ \text{N}$$

The lower charge q_2 exerts a force with the same magnitude but at an angle α *above* the x-axis. From symmetry we see that its x-component is the same as that due to the upper charge, but its y-component has the opposite sign. So the components of the total force $\vec{F}$ on Q are

$$F_x = 0.23\ \text{N} + 0.23\ \text{N} = 0.46\ \text{N}$$
$$F_y = -0.17\ \text{N} + 0.17\ \text{N} = 0$$

The total force on Q is in the $+x$-direction, with magnitude 0.46 N.

EVALUATE: The total force on Q is in a direction that points neither directly away from q_1 nor directly away from q_2. Rather, this direction is a compromise that points away from the *system* of charges q_1 and q_2. Can you see that the total force would *not* be in the $+x$-direction if q_1 and q_2 were not equal or if the geometrical arrangement of the changes were not so symmetrical?

Test Your Understanding of Section 21.3 Suppose that charge q_2 in Example 21.4 were $-2.0\ \mu\text{C}$. In this case, the total electric force on Q would be (i) in the positive x-direction; (ii) in the negative x-direction; (iii) in the positive y-direction; (iv) in the negative y-direction; (v) zero; (vi) none of these.

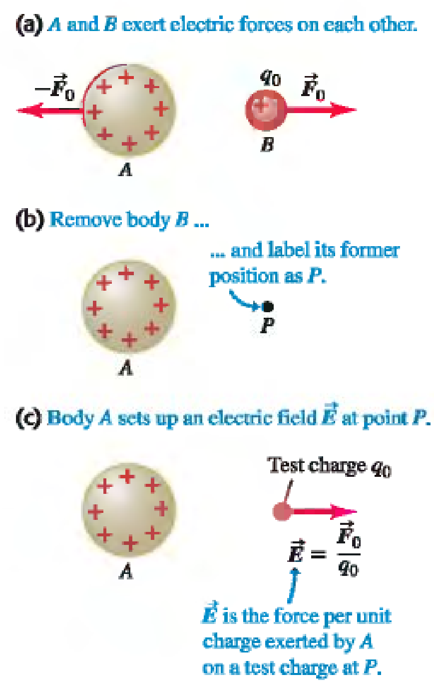
21.4 Electric Field and Electric Forces

When two electrically charged particles in empty space interact, how does each one know the other is there? What goes on in the space between them to communicate the effect of each one to the other? We can begin to answer these questions, and at the same time reformulate Coulomb's law in a very useful way, by using the concept of *electric field.*

Electric Field

To introduce this concept, let's look at the mutual repulsion of two positively charged bodies A and B (Fig. 21.15a). Suppose B has charge q_0, and let $\vec{F}_0$ be the electric force of A on B. One way to think about this force is as an "action-at-a-distance" force—that is, as a force that acts across empty space without needing any matter (such as a push rod or a rope) to transmit it through the intervening space. (Gravity can also be thought of as an "action-at-a-distance" force.) But a more fruitful way to visualize the repulsion between A and B is as a two-stage process. We first envision that body A, as a result of the charge that it carries, somehow *modifies the properties of the space around it.* Then body B, as

21.15 A charged body creates an electric field in the space around it.



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11.4 Electric Field: Point Change

11.9 Motion of a Charge in an Electric Field: Introduction

11.10 Motion in an Electric Field: Problems

a result of the charge that *it* carries, senses how space has been modified at its position. The response of body B is to experience the force $\vec{F}_0$.

To elaborate how this two-stage process occurs, we first consider body A by itself: We remove body B and label its former position as point P (Fig. 21.15b). We say that the charged body A produces or causes an **electric field** at point P (and at all other points in the neighborhood). This electric field is present at P even if there is no charge at P; it is a consequence of the charge on body A only. If a point charge q_0 is then placed at point P, it experiences the force $\vec{F}_0$. We take the point of view that this force is exerted on q_0 *by the field* at P (Fig. 21.15c). Thus the electric field is the intermediary through which A communicates its presence to q_0. Because the point charge q_0 would experience a force at *any* point in the neighborhood of A, the electric field that A produces exists at all points in the region around A.

We can likewise say that the point charge q_0 produces an electric field in the space around it and that this electric field exerts the force $-\vec{F}_0$ on body A. For each force (the force of A on q_0 and the force of q_0 on A), one charge sets up an electric field that exerts a force on the second charge. We emphasize that this is an *interaction* between *two* charged bodies. A single charge produces an electric field in the surrounding space, but this electric field cannot exert a net force on the charge that created it; this is an example of the general principle that a body cannot exert a net force on itself, as discussed in Section 4.3. (If this principle wasn't valid, you would be able to lift yourself to the ceiling by pulling up on your belt!)

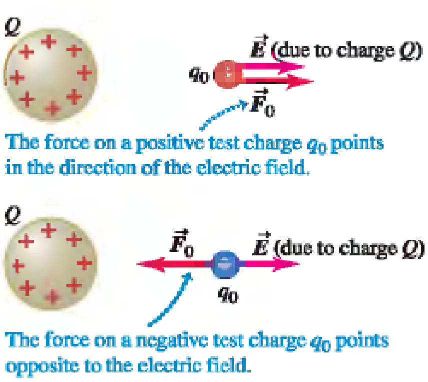
The electric force on a charged body is exerted by the electric field created by *other* charged bodies.

To find out experimentally whether there is an electric field at a particular point, we place a small charged body, which we call a **test charge**, at the point (Fig. 21.15c). If the test charge experiences an electric force, then there is an electric field at that point. This field is produced by charges other than q_0.

Force is a vector quantity, so electric field is also a vector quantity. (Note the use of vector signs as well as boldface letters and plus, minus, and equals signs in the following discussion.) We define the *electric field* $\vec{E}$ at a point as the electric force $\vec{F}_0$ experienced by a test charge q_0 at the point, divided by the charge q_0. That is, the electric field at a certain point is equal to the *electric force per unit charge* experienced by a charge at that point:

$$\vec{E} = \frac{\vec{F}_0}{q_0} \qquad \text{(definition of electric field as electric force per unit charge)} \tag{21.3}$$

21.16 The force $\vec{F}_0 = q_0\vec{E}$ exerted on a point charge q_0 placed in an electric field $\vec{E}$.



In SI units, in which the unit of force is 1 N and the unit of charge is 1 C, the unit of electric field magnitude is 1 newton per coulomb (1 N/C).

If the field $\vec{E}$ at a certain point is known, rearranging Eq. (21.3) gives the force $\vec{F}_0$ experienced by a point charge q_0 placed at that point. This force is just equal to the electric field $\vec{E}$ produced at that point by charges other than q_0, multiplied by the charge q_0:

$$\vec{F}_0 = q_0\vec{E} \qquad \text{(force exerted on a point charge } q_0 \text{ by an electric field } \vec{E}\text{)} \tag{21.4}$$

The charge q_0 can be either positive or negative. If q_0 is *positive*, the force $\vec{F}_0$ experienced by the charge is the same direction as $\vec{E}$; if q_0 is *negative*, $\vec{F}_0$ and $\vec{E}$ are in opposite directions (Fig. 21.16).

While the electric field concept may be new to you, the basic idea—that one body sets up a field in the space around it and a second body responds to that field—is one that you've actually used before. Compare Eq. (21.4) to the familiar expression for the gravitational force $\vec{F}_g$ that the earth exerts on a mass m_0:

$$\vec{F}_g = m_0\vec{g} \tag{21.5}$$

In this expression, $\vec{g}$ is the acceleration due to gravity. If we divide both sides of Eq. (21.5) by the mass m_0, we obtain

$$\vec{g} = \frac{\vec{F}_g}{m_0}$$

Thus $\vec{g}$ can be regarded as the gravitational force per unit mass. By analogy to Eq. (21.3), we can interpret $\vec{g}$ as the *gravitational field*. Thus we treat the gravitational interaction between the earth and the mass m_0 as a two-stage process: The earth sets up a gravitational field $\vec{g}$ in the space around it, and this gravitational field exerts a force given by Eq. (21.5) on the mass m_0 (which we can regard as a *test mass*). In this sense, you've made use of the field concept every time you've used Eq. (21.5) for the force of gravity. The gravitational field $\vec{g}$, or gravitational force per unit mass, is a useful concept because it does not depend on the mass of the body on which the gravitational force is exerted; likewise, the electric field $\vec{E}$, or electric force per unit charge, is useful because it does not depend on the charge of the body on which the electric force is exerted.

CAUTION **$\vec{F}_0 = q_0\vec{E}_0$ is for *point* test charges only** The electric force experienced by a test charge q_0 can vary from point to point, so the electric field can also be different at different points. For this reason, Eq. (21.4) can be used only to find the electric force on a *point* charge. If a charged body is large enough in size, the electric field $\vec{E}$ may be noticeably different in magnitude and direction at different points on the body, and calculating the net electric force on the body can become rather complicated.

We have so far ignored a subtle but important difficulty with our definition of electric field: In Fig. 21.15 the force exerted by the test charge q_0 on the charge distribution on body A may cause this distribution to shift around. This is especially true if body A is a conductor, on which charge is free to move. So the electric field around A when q_0 is present may not be the same as when q_0 is absent. But if q_0 is very small, the redistribution of charge on body A is also very small. So to make a completely correct definition of electric field, we take the *limit* of Eq. (21.3) as the test charge q_0 approaches zero and as the disturbing effect of q_0 on the charge distribution becomes negligible:

$$\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}_0}{q_0}$$

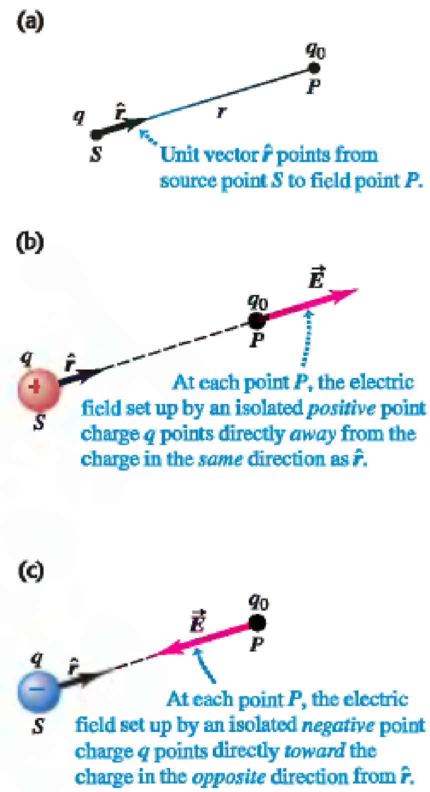
In practical calculations of the electric field $\vec{E}$ produced by a charge distribution, we will consider the charge distribution to be fixed, and so we will not need this limiting process.

Electric Field of a Point Charge

If the source distribution is a point charge q, it is easy to find the electric field that it produces. We call the location of the charge the **source point**, and we call the point P where we are determining the field the **field point.** It is also useful to introduce a *unit vector* $\hat{r}$ that points along the line from source point to field point (Fig. 21.17a). This unit vector is equal to the displacement vector $\vec{r}$ from the source point to the field point, divided by the distance $\hat{r} = |\vec{r}|$ between these two points; that is, $\hat{r} = \vec{r}/r$. If we place a small test charge q_0 at the field point P, at a distance r from the source point, the magnitude F_0 of the force is given by Coulomb's law, Eq. (21.2):

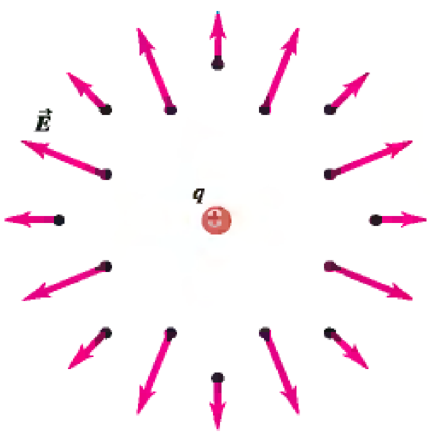
$$F_0 = \frac{1}{4\pi\epsilon_0}\frac{|qq_0|}{r^2}$$

21.17 The electric field $\vec{E}$ produced at point P by an isolated point charge q at S. Note that in both (b) and (c), $\vec{E}$ is *produced* by q [see Eq. (21.7)] but *acts* on the charge q_0 at point P [see Eq. (21.4)].

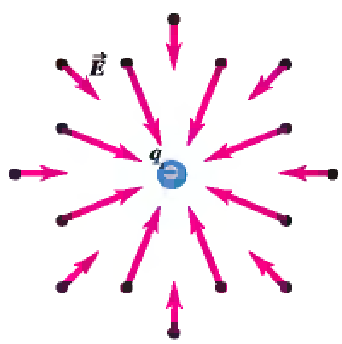


21.18 A point charge q produces an electric field $\vec{E}$ at *all* points in space. The field strength decreases with increasing distance.

(a) The field produced by a positive point charge points *away from* the charge.



(b) The field produced by a negative point charge points *toward* the charge.



From Eq. (21.3) the magnitude E of the electric field at P is

$$E = \frac{1}{4\pi\epsilon_0}\frac{|q|}{r^2} \quad \text{(magnitude of electric field of a point charge)} \tag{21.6}$$

Using the unit vector $\hat{r}$, we can write a *vector* equation that gives both the magnitude and direction of the electric field $\vec{E}$:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \quad \text{(electric field of a point charge)} \tag{21.7}$$

By definition, the electric field of a point charge always points *away from* a positive charge (that is, in the same direction as $\hat{r}$; see Fig. 21.17b) but *toward* a negative charge (that is, in the direction opposite $\hat{r}$; see Fig. 21.17c).

We have emphasized calculating the electric field $\vec{E}$ at a certain point. But since $\vec{E}$ can vary from point to point, it is not a single vector quantity but rather an *infinite* set of vector quantities, one associated with each point in space. This is an example of a **vector field.** Figure 21.18 shows a number of the field vectors produced by a positive or negative point charge. If we use a rectangular (x, y, z) coordinate system, each component of $\vec{E}$ at any point is in general a function of the coordinates (x, y, z) of the point. We can represent the functions as $E_x(x, y, z)$, $E_y(x, y, z)$, and $E_z(x, y, z)$. Vector fields are an important part of the language of physics, not just in electricity and magnetism. One everyday example of a vector field is the velocity $\vec{v}$ of wind currents; the magnitude and direction of $\vec{v}$, and hence its vector components, vary from point to point in the atmosphere.

In some situations the magnitude and direction of the field (and hence its vector components) have the same values everywhere throughout a certain region; we then say that the field is *uniform* in this region. An important example of this is the electric field inside a *conductor.* If there is an electric field within a conductor, the field exerts a force on every charge in the conductor, giving the free charges a net motion. By definition an electrostatic situation is one in which the charges have *no* net motion. We conclude that *in electrostatics the electric field at every point within the material of a conductor must be zero.* (Note that we are not saying that the field is necessarily zero in a *hole* inside a conductor.)

With the concept of electric field, our description of electric interactions has two parts. First, a given charge distribution acts as a source of electric field. Second, the electric field exerts a force on any charge that is present in the field. Our analysis often has two corresponding steps: first, calculating the field caused by a source charge distribution; second, looking at the effect of the field in terms of force and motion. The second step often involves Newton's laws as well as the principles of electric interactions. In the next section we show how to calculate fields caused by various source distributions, but first here are some examples of calculating the field due to a point charge and of finding the force on a charge due to a given field $\vec{E}$.

Example 21.5 Electric-field magnitude for a point charge

What is the magnitude of the electric field at a field point 2.0 m from a point charge $q = 4.0$ nC? (The point charge could represent any small charged object with this value of q, provided the dimensions of the object are much less than the distance from the object to the field point.)

SOLUTION

IDENTIFY: This problem uses the expression for the electric field due to a point charge.

SET UP: We are given the magnitude of the charge and the distance from the object to the field point, so we use Eq. (21.6) to calculate the field magnitude E.

EXECUTE: From Eq. (21.6),

$$E = \frac{1}{4\pi\epsilon_0}\frac{|q|}{r^2} = (9.0 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\frac{4.0 \times 10^{-9}\ \text{C}}{(2.0\ \text{m})^2}$$
$$= 9.0\ \text{N/C}$$

EVALUATE: To check our result, we use the definition of electric field as the electric force per unit charge. We can first use Coulomb's law, Eq. (21.2), to find the magnitude F_0 of the force on a test charge q_0 placed 2.0 m from q:

$$F_0 = \frac{1}{4\pi\epsilon_0}\frac{|qq_0|}{r^2} = (9.0 \times 10^9\,\mathrm{N\cdot m^2/C^2})\frac{4.0 \times 10^{-9}\,\mathrm{C}|q_0|}{(2.0\,\mathrm{m})^2}$$
$$= (9.0\,\mathrm{N/C})|q_0|$$

Then, from Eq. (21.3), the magnitude of $\vec{E}$ is

$$E = \frac{F_0}{|q_0|} = 9.0\,\mathrm{N/C}$$

Because q is positive, the *direction* of $\vec{E}$ at this point is along the line from q toward q_0, as shown in Fig. 21.17b. However, the magnitude and direction of $\vec{E}$ do not depend on the sign of q_0. Do you see why not?

Example 21.6 Electric-field vector for a point charge

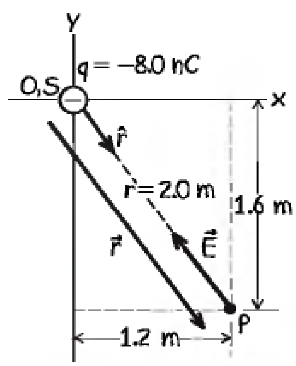
A point charge $q = -8.0\,\mathrm{nC}$ is located at the origin. Find the electric-field vector at the field point $x = 1.2\,\mathrm{m}$, $y = -1.6\,\mathrm{m}$.

SOLUTION

IDENTIFY: In this problem we are asked to find the electric-field *vector* $\vec{E}$ due to a point charge. Hence we need to find either the components of $\vec{E}$ or its magnitude and direction.

SET UP: Figure 21.19 shows the situation. The electric field is given in vector form by Eq. (21.7). To use this equation, we first find the distance r from the source point S (the position of the charge q) to the field point P, as well as the unit vector $\hat{r}$ that points in the direction from S to P.

21.19 Our sketch for this problem.



EXECUTE: The distance from the charge at the source point S (which in this example is at the origin O) to the field point P is

$$r = \sqrt{x^2 + y^2} = \sqrt{(1.2\,\mathrm{m})^2 + (-1.6\,\mathrm{m})^2} = 2.0\,\mathrm{m}$$

The unit vector $\hat{r}$ is directed from the source point to the field point. This is equal to the displacement vector $\vec{r}$ from the source point to the field point (shown shifted to one side in Fig. 21.19 so as not to obscure the other vectors), divided by its magnitude r:

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{\imath} + y\hat{\jmath}}{r}$$
$$= \frac{(1.2\,\mathrm{m})\hat{\imath} + (-1.6\,\mathrm{m})\hat{\imath}}{2.0\,\mathrm{m}} = 0.60\hat{\imath} - 0.80\hat{\jmath}$$

Hence the electric-field vector is

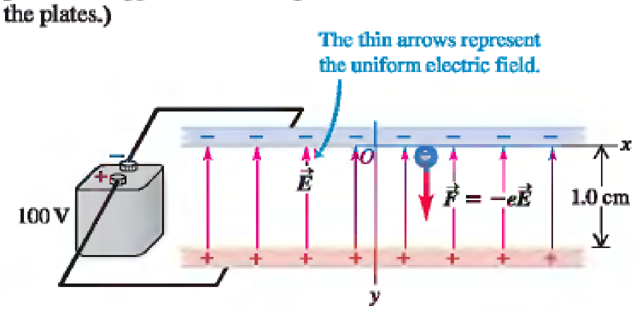
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$
$$= (9.0 \times 10^9\,\mathrm{N\cdot m^2/C^2})\frac{(-8.0 \times 10^{-9}\,\mathrm{C})}{(2.0\,\mathrm{m})^2}(0.60\hat{\imath} - 0.80\hat{\jmath})$$
$$= (-11\,\mathrm{N/C})\hat{\imath} + (14\,\mathrm{N/C})\hat{\jmath}$$

EVALUATE: Since q is negative, $\vec{E}$ points from the field point to the charge (the source point), in the direction opposite to $\hat{r}$ (compare Fig. 21.17c). We leave the calculation of the magnitude and direction of $\vec{E}$ to you (see Exercise 21.36).

Example 21.7 Electron in a uniform field

When the terminals of a battery are connected to two large parallel conducting plates, the resulting charges on the plates cause an electric field $\vec{E}$ in the region between the plates that is very nearly uniform. (We will see the reason for this uniformity in the next section. Charged plates of this kind are used in common electrical devices called *capacitors*, to be discussed in Chapter 24.) If the plates are horizontal and separated by 1.0 cm and the plates are connected to a 100-volt battery, the magnitude of the field is $E = 1.00 \times 10^4\,\mathrm{N/C}$. Suppose the direction of $\vec{E}$ is vertically upward, as shown by the vectors in Fig. 21.20. (a) If an electron is released from rest at the upper plate, what is its acceleration? (b) What speed and kinetic energy does the electron acquire while traveling 1.0 cm to the lower plate? (c) How much time is

21.20 A uniform electric field between two parallel conducting plates connected to a 100-volt battery. (The separation of the plates is exaggerated in this figure relative to the dimensions of the plates.)



Continued

required for it to travel this distance? An electron has charge $-e = -1.60 \times 10^{-19}$ C and mass $m = 9.11 \times 10^{-31}$ kg.

SOLUTION

IDENTIFY: This example involves several concepts: the relationship between electric field and electric force, the relationship between force and acceleration, the definition of kinetic energy, and the kinematic relationships among acceleration, distance, velocity, and time.

SET UP: Figure 21.20 shows our coordinate system. We are given the electric field, so we use Eq. (21.4) to find the force on the electron and Newton's second law to find its acceleration. Because the field is uniform between the plates, the force and acceleration are constant and we can use the constant-acceleration formulas from Chapter 3 to find the electron's velocity and travel time. We find the kinetic energy using the definition $K = \frac{1}{2}mv^2$.

EXECUTE: (a) Note that $\vec{E}$ is upward (in the $+y$-direction) but $\vec{F}$ is downward because the charge of the electron is negative. Thus F_y is negative. Because F_y is constant, the electron moves with constant acceleration a_y given by

$$a_y = \frac{F_y}{m} = \frac{-eE}{m} = \frac{(-1.60 \times 10^{-19}\,\text{C})(1.00 \times 10^4\,\text{N/C})}{9.11 \times 10^{-31}\,\text{kg}}$$
$$= -1.76 \times 10^{15}\,\text{m/s}^2$$

This is an enormous acceleration! To give a 1000-kg car this acceleration, we would need a force of about 2×10^{18} N (about 2×10^{14} tons). The gravitational force on the electron is completely negligible compared to the electric force.

(b) The electron starts from rest, so its motion is in the y-direction only (the direction of the acceleration). We can find the electron's speed at any position using the constant-acceleration formula $v_y^2 = v_{0y}^2 + 2a_y(y - y_0)$. We have $v_{0y} = 0$ and $y_0 = 0$, so the speed $|v_y|$ when $y = -1.0\,\text{cm} = -1.0 \times 10^{-2}$ m is

$$|v_y| = \sqrt{2a_y y} = \sqrt{2(-1.76 \times 10^{15}\,\text{m/s}^2)(-1.0 \times 10^{-2}\,\text{m})}$$
$$= 5.9 \times 10^6\,\text{m/s}$$

The velocity is downward, so its y-component is $v_y = -5.9 \times 10^6$ m/s. The electron's kinetic energy is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}(9.11 \times 10^{-31}\,\text{kg})(5.9 \times 10^6\,\text{m/s})^2$$
$$= 1.6 \times 10^{-17}\,\text{J}$$

(c) From the constant-acceleration formula $v_y = v_{0y} + a_y t$, we find that the time required is very brief:

$$t = \frac{v_y - v_{0y}}{a_y} = \frac{(-5.9 \times 10^6\,\text{m/s}) - (0\,\text{m/s})}{-1.76 \times 10^{15}\,\text{m/s}^2}$$
$$= 3.4 \times 10^{-9}\,\text{s}$$

(We could also have found the time by solving the equation $y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2$ for t.)

EVALUATE: This example shows that in problems about subatomic particles such as electrons, many quantities—including acceleration, speed, kinetic energy, and time—will have *very* different values from what we have seen for ordinary objects such as baseballs and automobiles.

Example 21.8 An electron trajectory

If we launch an electron into the electric field of Example 21.7 with an initial horizontal velocity v_0 (Fig. 21.21), what is the equation of its trajectory?

SOLUTION

IDENTIFY: We found the electron's acceleration in Example 21.7. Our goal is to find the trajectory that corresponds to that acceleration.

SET UP: The acceleration is constant and in the negative y-direction (there is no acceleration in the x-direction). Hence we can use the kinematic equations from Chapter 3 for two-dimensional motion with constant acceleration.

EXECUTE: We have $a_x = 0$ and $a_y = (-e)E/m$. At $t = 0$, $x_0 = y_0 = 0$, $v_{0x} = v_0$, and $v_{0y} = 0$; hence at time t,

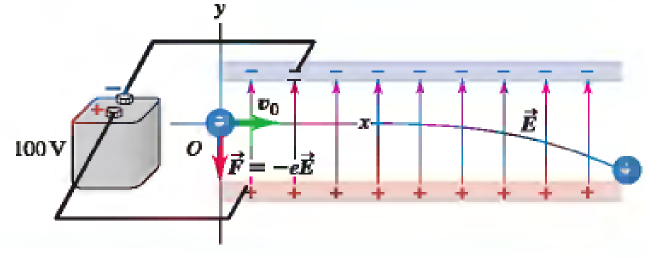
$$x = v_0 t \quad \text{and} \quad y = \frac{1}{2}a_y t^2 = -\frac{1}{2}\frac{eE}{m}t^2$$

Eliminating t between these equations, we get

$$y = -\frac{1}{2}\frac{eE}{mv_0^2}x^2$$

EVALUATE: This is the equation of a parabola, just like the trajectory of a projectile launched horizontally in the earth's gravitational field (discussed in Section 3.3). For a given initial velocity of the electron, the curvature of the trajectory depends on the field magnitude E. If we reverse the signs of the charges on the two plates in Fig. 21.21, the direction of $\vec{E}$ reverses, and the electron trajectory will curve up, not down. Hence we can "steer" the electron by varying the charges on the plates. The electric field between charged conducting plates can be used in this way to control the trajectory of electron beams in oscilloscopes.

21.21 The parabolic trajectory of an electron in a uniform electric field.



Test Your Understanding of Section 21.4 (a) A negative point charge moves along a straight-line path directly toward a stationary positive point charge. Which aspect(s) of the electric force on the negative point charge will remain constant as it moves? (i) magnitude; (ii) direction; (iii) both magnitude and direction; (iv) neither magnitude nor direction. (b) A negative point charge moves along a circular orbit around a positive point charge. Which aspect(s) of the electric force on the negative point charge will remain constant as it moves? (i) magnitude; (ii) direction; (iii) both magnitude and direction; (iv) neither magnitude nor direction.

21.5 Electric-Field Calculations

ActivPhysics ONLINE
11.5 Electric Field Due to a Dipole
11.6 Electric Field: Problems

Equation (21.7) gives the electric field caused by a single point charge. But in most realistic situations that involve electric fields and forces, we encounter charge that is *distributed* over space. The charged plastic and glass rods in Fig. 21.1 have electric charge distributed over their surfaces, as does the imaging drum of a laser printer (Fig. 21.2). In this section we'll learn to calculate electric fields caused by various distributions of electric charge. Calculations of this kind are of tremendous importance for technological applications of electric forces. To determine the trajectories of electrons in a TV tube, of atomic nuclei in an accelerator for cancer radiotherapy, or of charged particles in a semiconductor electronic device, you have to know the detailed nature of the electric field acting on the charges.

The Superposition of Electric Fields

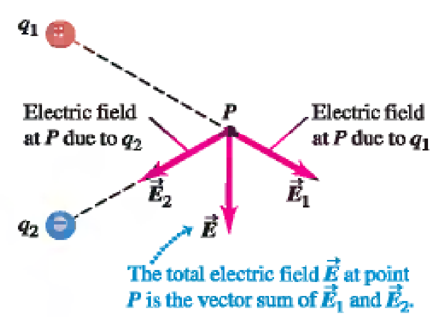
To find the field caused by a charge distribution, we imagine the distribution to be made up of many point charges $q_1, q_2, q_3, \ldots$. (This is actually quite a realistic description, since we have seen that charge is carried by electrons and protons that are so small as to be almost pointlike.) At any given point P, each point charge produces its own electric field $\vec{E}_1, \vec{E}_2, \vec{E}_3, \ldots$, so a test charge q_0 placed at P experiences a force $\vec{F}_1 = q_0\vec{E}_1$ from charge q_1, a force $\vec{F}_2 = q_0\vec{E}_2$ from charge q_2, and so on. From the principle of superposition of forces discussed in Section 21.3, the *total* force $\vec{F}_0$ that the charge distribution exerts on q_0 is the vector sum of these individual forces:

$$\vec{F}_0 = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots = q_0\vec{E}_1 + q_0\vec{E}_2 + q_0\vec{E}_3 + \cdots$$

The combined effect of all the charges in the distribution is described by the *total* electric field $\vec{E}$ at point P. From the definition of electric field, Eq. (21.3), this is

$$\vec{E} = \frac{\vec{F}_0}{q_0} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \cdots$$

The total electric field at P is the vector sum of the fields at P due to each point charge in the charge distribution (Fig. 21.22). This is the **principle of superposition of electric fields.**



21.22 Illustrating the principle of superposition of electric fields.

When charge is distributed along a line, over a surface, or through a volume, a few additional terms are useful. For a line charge distribution (such as a long, thin, charged plastic rod), we use λ (the Greek letter lambda) to represent the **linear charge density** (charge per unit length, measured in C/m). When charge is distributed over a surface (such as the surface of the imaging drum of a laser printer), we use σ (sigma) to represent the **surface charge density** (charge per unit area, measured in C/m^2). And when charge is distributed through a volume, we use ρ (rho) to represent the **volume charge density** (charge per unit volume, C/m^3).

Some of the calculations in the following examples may look fairly intricate; in electric-field calculations a certain amount of mathematical complexity is in the nature of things. After you've worked through the examples one step at a time, the process will seem less formidable. We will use many of the calculational techniques in these examples in Chapter 28 to calculate the *magnetic* fields caused by charges in motion.

Problem-Solving Strategy 21.2 Electric-Field Calculations

MP

IDENTIFY *the relevant concepts:* Use the principle of superposition whenever you need to calculate the electric field due to a charge distribution (two or more point charges, a distribution over a line, surface, or volume, or a combination of these).

SET UP *the problem* using the following steps:

1. Make a drawing that clearly shows the locations of the charges and your choice of coordinate axes.
2. On your drawing, indicate the position of the *field point* (the point at which you want to calculate the electric field $\vec{E}$). Sometimes the field point will be at some arbitrary position along a line. For example, you may be asked to find $\vec{E}$ at any point on the x-axis.

EXECUTE *the solution* as follows:

1. Be sure to use a consistent set of units. Distances must be in meters and charge must be in coulombs. If you are given centimeters or nanocoulombs, don't forget to convert.
2. When adding up the electric fields caused by different parts of the charge distribution, remember that electric field is a vector, so you *must* use vector addition. Don't simply add together the magnitudes of the individual fields; the directions are important, too.
3. Take advantage of any symmetries in the charge distribution. For example, if a positive charge and a negative charge of equal magnitude are placed symmetrically with respect to the field point, they produce electric fields of the same magnitude but with mirror-image directions. Exploiting these symmetries will simplify your calculations.
4. Most often you will use components to compute vector sums. Use the methods you learned in Chapter 1; review them if necessary. Use proper vector notation; distinguish carefully between scalars, vectors, and components of vectors. Be certain the components are consistent with your choice of coordinate axes.
5. In working out the directions of $\vec{E}$ vectors, be careful to distinguish between the *source point* and the *field point.* The field produced by a point charge always points from source point to field point if the charge is positive; it points in the opposite direction if the charge is negative.
6. In some situations you will have a continuous distribution of charge along a line, over a surface, or through a volume. Then you must define a small element of charge that can be considered as a point, find its electric field at point P, and find a way to add the fields of all the charge elements. Usually it is easiest to do this for each component of $\vec{E}$ separately, and often you will need to evaluate one or more integrals. Make certain the limits on your integrals are correct; especially when the situation has symmetry, make sure you don't count the charge twice.

EVALUATE *your answer:* Check that the direction of $\vec{E}$ is reasonable. If your result for the electric-field magnitude E is a function of position (say, the coordinate x), check your result in any limits for which you know what the magnitude should be. When possible, check your answer by calculating it in a different way.

Example 21.9 Field of an electric dipole

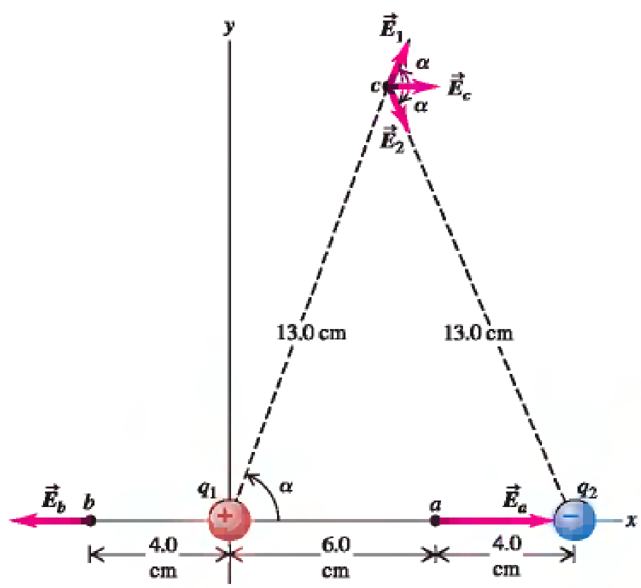
Point charges q_1 and q_2 of $+12$ nC and -12 nC, respectively, are placed 0.10 m apart (Fig. 21.23). This combination of two charges with equal magnitude and opposite sign is called an *electric dipole.* (Such combinations occur frequently in nature. For example, in Figs. 21.8b and 21.8c, each molecule in the neutral insulator is an electric dipole. We'll study dipoles in more detail in Section 21.7.) Compute the electric field caused by q_1, the field caused by q_2, and the total field (a) at point a; (b) at point b; and (c) at point c.

SOLUTION

IDENTIFY: We need to find the total electric field at three different points due to two point charges. We will use the principle of superposition: $\vec{E} = \vec{E}_1 + \vec{E}_2$.

SET UP: Figure 21.23 shows the coordinate system and the locations of the three field points a, b, and c.

21.23 Electric field at three points, a, b, and c, set up by charges q_1 and q_2, which form an electric dipole.



EXECUTE: (a) At point a the field $\vec{E}_1$ caused by the positive charge q_1 and the field $\vec{E}_2$ caused by the negative charge q_2 are both directed toward the right. The magnitudes of $\vec{E}_1$ and $\vec{E}_2$ are

$$E_1 = \frac{1}{4\pi\epsilon_0}\frac{|q_1|}{r^2} = (9.0 \times 10^9\,\mathrm{N \cdot m^2/C^2})\frac{12 \times 10^{-9}\,\mathrm{C}}{(0.060\,\mathrm{m})^2}$$
$$= 3.0 \times 10^4\,\mathrm{N/C}$$
$$E_2 = \frac{1}{4\pi\epsilon_0}\frac{|q_2|}{r^2} = (9.0 \times 10^9\,\mathrm{N \cdot m^2/C^2})\frac{12 \times 10^{-9}\,\mathrm{C}}{(0.040\,\mathrm{m})^2}$$
$$= 6.8 \times 10^4\,\mathrm{N/C}$$

The components of $\vec{E}_1$ and $\vec{E}_2$ are

$$E_{1x} = 3.0 \times 10^4\,\mathrm{N/C} \qquad E_{1y} = 0$$
$$E_{2x} = 6.8 \times 10^4\,\mathrm{N/C} \qquad E_{2y} = 0$$

Hence at point a the total electric field $\vec{E}_a = \vec{E}_1 + \vec{E}_2$ has components

$$(E_a)_x = E_{1x} + E_{2x} = (3.0 + 6.8) \times 10^4\,\mathrm{N/C}$$
$$(E_a)_y = E_{1y} + E_{2y} = 0$$

At point a the total field has magnitude $9.8 \times 10^4\,\mathrm{N/C}$ and is directed toward the right, so

$$\vec{E}_a = (9.8 \times 10^4\,\mathrm{N/C})\hat{\imath}$$

(b) At point b the field $\vec{E}_1$ due to q_1 is directed toward the left, while the field $\vec{E}_2$ due to q_2 is directed toward the right. The magnitudes of $\vec{E}_1$ and $\vec{E}_2$ are

$$E_1 = \frac{1}{4\pi\epsilon_0}\frac{|q_1|}{r^2} = (9.0 \times 10^9\,\mathrm{N \cdot m^2/C^2})\frac{12 \times 10^{-9}\,\mathrm{C}}{(0.040\,\mathrm{m})^2}$$
$$= 6.8 \times 10^4\,\mathrm{N/C}$$
$$E_2 = \frac{1}{4\pi\epsilon_0}\frac{|q_2|}{r^2} = (9.0 \times 10^9\,\mathrm{N \cdot m^2/C^2})\frac{12 \times 10^{-9}\,\mathrm{C}}{(0.140\,\mathrm{m})^2}$$
$$= 0.55 \times 10^4\,\mathrm{N/C}$$

The components of $\vec{E}_1$, $\vec{E}_2$, and the total field $\vec{E}_b$ at point b are

$$E_{1x} = -6.8 \times 10^4\,\mathrm{N/C} \qquad E_{1y} = 0$$
$$E_{2x} = 0.55 \times 10^4\,\mathrm{N/C} \qquad E_{2y} = 0$$
$$(E_b)_x = E_{1x} + E_{2x} = (-6.8 + 0.55) \times 10^4\,\mathrm{N/C}$$
$$(E_b)_y = E_{1y} + E_{2y} = 0$$

That is, the electric field at b has magnitude $6.2 \times 10^4\,\mathrm{N/C}$ and is directed toward the left, so

$$\vec{E}_b = (-6.2 \times 10^4\,\mathrm{N/C})\hat{\imath}$$

(c) At point c, both $\vec{E}_1$ and $\vec{E}_2$ have the same magnitude, since this point is equidistant from both charges and the charge magnitudes are the same:

$$E_1 = E_2 = \frac{1}{4\pi\epsilon_0}\frac{|q|}{r^2} = (9.0 \times 10^9\,\mathrm{N \cdot m^2/C^2})\frac{12 \times 10^{-9}\,\mathrm{C}}{(0.130\,\mathrm{m})^2}$$
$$= 6.39 \times 10^3\,\mathrm{N/C}$$

The directions of $\vec{E}_1$ and $\vec{E}_2$ are shown in Fig 21.23. The x-components of both vectors are the same:

$$E_{1x} = E_{2x} = E_1\cos\alpha = (6.39 \times 10^3\,\mathrm{N/C})\left(\frac{5}{13}\right)$$
$$= 2.46 \times 10^3\,\mathrm{N/C}$$

From symmetry the y-components E_{1y} and E_{2y} are equal and opposite and so add to zero. Hence the components of the total field $\vec{E}_c$ are

$$(E_c)_x = E_{1x} + E_{2x} = 2(2.46 \times 10^3\,\mathrm{N/C}) = 4.9 \times 10^3\,\mathrm{N/C}$$
$$(E_c)_y = E_{1y} + E_{2y} = 0$$

So at point c the total electric field has magnitude $4.9 \times 10^3\,\mathrm{N/C}$ and is directed toward the right, so

$$\vec{E}_c = (4.9 \times 10^3\,\mathrm{N/C})\hat{\imath}$$

Does it surprise you that the field at point c is parallel to the line between the two charges?

EVALUATE: An alternative way to find the electric field at c is to use the vector expression for the field of a point charge, Eq. (21.7). The displacement vector $\vec{r}_1$ from q_1 to point c, a distance $r = 13.0$ cm away, is

$$\vec{r}_1 = r\cos\alpha\hat{\imath} + r\sin\alpha\hat{\jmath}$$

Hence the unit vector that points from q_1 to c is

$$\hat{r}_1 = \frac{\vec{r}_1}{r} = \cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath}$$

and the field due to q_1 at point c is

$$\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r}_1 = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}(\cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath})$$

By symmetry the unit vector $\hat{r}_2$ that points from q_2 to point c has the opposite x-component but the same y-component, so the field at c due to q_2 is

$$\vec{E}_2 = \frac{1}{4\pi\epsilon_0}\frac{q_2}{r^2}\hat{r}_2 = \frac{1}{4\pi\epsilon_0}\frac{q_2}{r^2}(-\cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath})$$

Since $q_2 = -q_1$, the total field at c is

$$\vec{E}_c = \vec{E}_1 + \vec{E}_2$$
$$= \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}(\cos\alpha\hat{\imath} + \mathrm{in}\alpha\hat{\jmath}) + \frac{1}{4\pi\epsilon_0}\frac{(-q_1)}{r^2}(-\cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath})$$
$$= \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}(2\cos\alpha\,\hat{\imath})$$
$$= (9.0 \times 10^9\,\mathrm{N \cdot m^2/C^2})\frac{12 \times 10^{-9}\,\mathrm{C}}{(0.13\,\mathrm{m})^2}\left[2\left(\frac{5}{13}\right)\right]\hat{\imath}$$
$$= (4.9 \times 10^3\,\mathrm{N/C})\hat{\imath}$$

as before.

Example 21.10 Field of a ring of charge

A ring-shaped conductor with radius a carries a total charge Q uniformly distributed around it (Fig. 21.24). Find the electric field at a point P that lies on the axis of the ring at a distance x from its center.

SOLUTION

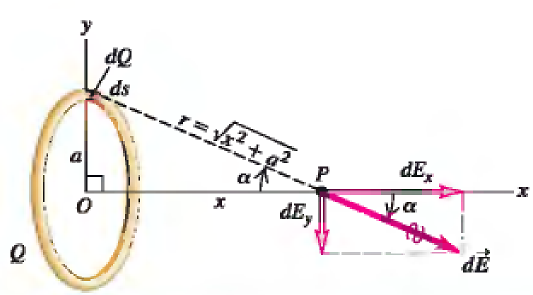
IDENTIFY: This is a problem in the superposition of electric fields. The new wrinkle is that the charge is distributed continuously around the ring rather than in a number of point charges.

SET UP: The field point is an arbitrary point on the x-axis in Fig. 21.24. Our target variable is the electric field at such a point as a function of the coordinate x.

EXECUTE: As shown in Fig. 21.24, we imagine the ring divided into infinitesimal segments of length ds. Each segment has charge dQ and acts as a point-charge source of electric field. Let $d\vec{E}$ be the electric field from one such segment; the net electric field at P is then the sum of all contributions $d\vec{E}$ from all the segments that make up the ring. (This same technique works for any situation in which charge is distributed along a line or a curve.)

The calculation of $\vec{E}$ is greatly simplified because the field point P is on the symmetry axis of the ring. Consider two segments at the top and bottom of the ring: The contributions $d\vec{E}$ to the field at P from these segments have the same x-component but opposite y-components. Hence the total y-component of field due to this pair of segments is zero. When we add up the contributions from all such pairs of segments, the total field $\vec{E}$ will have only a component along the ring's symmetry axis (the x-axis), with no component perpendicular to that axis (that is, no y-component or z-component). So the field at P is described completely by its x-component E_x.

21.24 Calculating the electric field on the axis of a ring of charge. In this figure, the charge is assumed to be positive.



To calculate E_x, note that the square of the distance r from a ring segment to the point P is $r^2 = x^2 + a^2$. Hence the magnitude of this segment's contribution $d\vec{E}$ to the electric field at P is

$$dE = \frac{1}{4\pi\epsilon_0}\frac{dQ}{x^2 + a^2}$$

Using $\cos\alpha = x/r = x/(x^2 + a^2)^{1/2}$, the x-component dE_x of this field is

$$dE_x = dE\cos\alpha = \frac{1}{4\pi\epsilon_0}\frac{dQ}{x^2 + a^2}\frac{x}{\sqrt{x^2 + a^2}} = \frac{1}{4\pi\epsilon_0}\frac{x\,dQ}{(x^2 + a^2)^{3/2}}$$

To find the *total* x-component E_x of the field at P, we integrate this expression over all segments of the ring:

$$E_x = \int \frac{1}{4\pi\epsilon_0}\frac{x\,dQ}{(x^2 + a^2)^{3/2}}$$

Since x does not vary as we move from point to point around the ring, all the factors on the right side except dQ are constant and can be taken outside the integral. The integral of dQ is just the total charge Q, and we finally get

$$\vec{E} = E_x\hat{\imath} = \frac{1}{4\pi\epsilon_0}\frac{Qx}{(x^2 + a^2)^{3/2}}\hat{\imath} \qquad (21.8)$$

EVALUATE: Our result for $\vec{E}$ shows that at the center of the ring $(x = 0)$ the field is zero. We should expect this; charges on opposite sides of the ring would push in opposite directions on a test charge at the center, and the forces would add to zero. When the field point P is much farther from the ring than its size (that is, $x \gg a$), the denominator in Eq. (21.8) becomes approximately equal to x^3, and the expression becomes approximately

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{x^2}\hat{\imath}$$

In other words, when we are so far from the ring that its size a is negligible in comparison to the distance x, its field is the same as that of a point charge. To an observer far from the ring, the ring would appear like a point, and the electric field reflects this.

In this example we used a *symmetry argument* to conclude that $\vec{E}$ had only an x-component at a point on the ring's axis of symmetry. We'll use symmetry arguments many times in this and subsequent chapters. Keep in mind, however, that such arguments can be used only in special cases. At a point in the xy-plane that is not on the x-axis in Fig. 21.24, the symmetry argument doesn't apply, and the field has in general both x- and y-components.

Example 21.11 Field of a line of charge

Positive electric charge Q is distributed uniformly along a line with length $2a$, lying along the y-axis between $y = -a$ and $y = +a$. (This might represent one of the charged rods in Fig. 21.1.) Find the electric field at point P on the x-axis at a distance x from the origin.

SOLUTION

IDENTIFY: As in Example 21.10, our target variable is the electric field due to a continuous distribution of charge.

SET UP: Figure 21.25 shows the situation. We need to find the electric field at P as a function of the coordinate x. The x-axis is the perpendicular bisector of the charged line, so as in Example 21.10 we will be able to make use of a symmetry argument.

EXECUTE: We divide the line charge into infinitesimal segments, each of which acts as a point charge; let the length of a typical segment at height y be dy. If the charge is distributed uniformly, the linear charge density λ at any point on the line is equal to $Q/2a$ (the total charge divided by the total length). Hence the charge dQ in a segment of length dy is

$$dQ = \lambda dy = \frac{Q dy}{2a}$$

The distance r from this segment to P is $(x^2 + y^2)^{1/2}$, so the magnitude of field dE at P due to this segment is

$$dE = \frac{1}{4\pi\epsilon_0}\frac{dQ}{r^2} = \frac{Q}{4\pi\epsilon_0}\frac{dy}{2a(x^2 + y^2)}$$

We represent this field in terms of its x- and y-components:

$$dE_x = dE\cos\alpha \qquad dE_y = -dE\sin\alpha$$

We note that $\sin\alpha = y/(x^2 + y^2)^{1/2}$ and $\cos\alpha = x/(x^2 + y^2)^{1/2}$; combining these with the expression for dE, we find

$$dE_x = \frac{Q}{4\pi\epsilon_0}\frac{x\,dy}{2a(x^2 + y^2)^{3/2}}$$

$$dE_y = -\frac{Q}{4\pi\epsilon_0}\frac{y\,dy}{2a(x^2 + y^2)^{3/2}}$$

To find the total field components E_x and E_y, we integrate these expressions, noting that to include all of Q, we must integrate from $y = -a$ to $y = +a$. We invite you to work out the details of the integration; an integral table is helpful. The final results are

$$E_x = \frac{1}{4\pi\epsilon_0}\frac{Qx}{2a}\int_{-a}^{a}\frac{dy}{(x^2 + y^2)^{3/2}} = \frac{Q}{4\pi\epsilon_0}\frac{1}{x\sqrt{x^2 + a^2}}$$

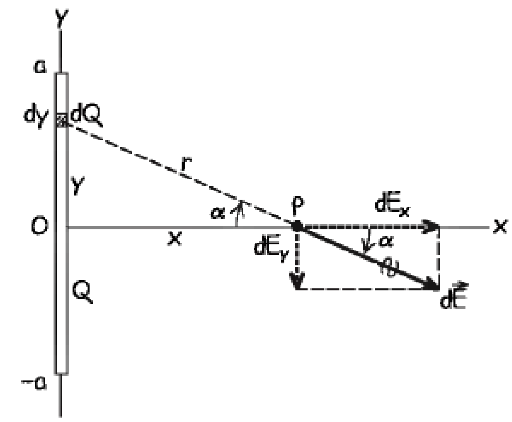
$$E_y = -\frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\int_{-a}^{a}\frac{y\,dy}{(x^2 + y^2)^{3/2}} = 0$$

or, in vector form,

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{x\sqrt{x^2 + a^2}}\hat{\imath} \qquad (21.9)$$

EVALUATE: Using a symmetry argument as in Example 21.10, we could have guessed that E_y would be zero; if we place a positive test charge at P, the upper half of the line of charge pushes downward on it, and the lower half pushes up with equal magnitude.

21.25 Our sketch for this problem.



To explore our result, let's first see what happens in the limit that x is much larger than a. Then we can neglect a in the denominator of Eq. (21.9), and our result becomes

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{x^2}\hat{\imath}$$

This means that if point P is very far from the line charge in comparison to the length of the line, the field at P is the same as that of a point charge. We found a similar result for the charged ring in Example 21.10.

To further explore our exact result for $\vec{E}$, Eq. (21.9), let's express it in terms of the linear charge density $\lambda = Q/2a$. Substituting $Q = 2a\lambda$ into Eq. (21.9) and simplifying, we get

$$\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{x\sqrt{(x^2/a^2) + 1}}\hat{\imath} \qquad (21.10)$$

Now we can answer the question: What is $\vec{E}$ at a distance x from a *very* long line of charge? To find the answer we take the *limit* of Eq. (21.10) as a becomes very large. In this limit, the term x^2/a^2 in the denominator becomes much smaller than unity and can be thrown away. We are left with

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 x}\hat{\imath}$$

The field magnitude depends only on the distance of point P from the line of charge. So at any point P at a perpendicular distance r from the line in any direction, $\vec{E}$ has magnitude

$$E = \frac{\lambda}{2\pi\epsilon_0 r} \qquad \text{(infinite line of charge)}$$

Thus the electric field due to an infinitely long line of charge is proportional to $1/r$ rather than to $1/r^2$ as for a point charge. The direction of $\vec{E}$ is radially outward from the line if λ is positive and radially inward if λ is negative.

There's really no such thing in nature as an infinite line of charge. But when the field point is close enough to the line, there's very little difference between the result for an infinite line and the real-life finite case. For example, if the distance r of the field point from the center of the line is 1% of the length of the line, the value of E differs from the infinite-length value by less than 0.02%.

Example 21.12 Field of a uniformly charged disk

Find the electric field caused by a disk of radius R with a uniform positive surface charge density (charge per unit area) σ, at a point along the axis of the disk a distance x from its center. Assume that x is positive.

SOLUTION

IDENTIFY: This example is similar to Examples 21.10 and 21.11 in that our target variable is the electric field along a symmetry axis of a continuous charge distribution.

SET UP: Figure 21.26 shows the situation. We can represent the charge distribution as a collection of concentric rings of charge dQ, as shown in Fig. 21.26. From Example 21.10 we know the field of a single ring on its axis of symmetry, so all we have to do is add the contributions of the rings.

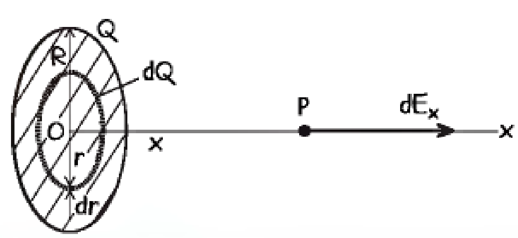
EXECUTE: A typical ring has charge dQ, inner radius r, and outer radius $r + dr$ (Fig. 21.26). Its area dA is approximately equal to its width dr times its circumference $2\pi r$, or $dA = 2\pi r\,dr$. The charge per unit area is $\sigma = dQ/dA$, so the charge of the ring is $dQ = \sigma\,dA = \sigma\,(2\pi r\,dr)$, or

$$dQ = 2\pi\sigma r\,dr$$

We use this in place of Q in the expression for the field due to a ring found in Example 21.10, Eq. (21.8), and also replace the ring radius a with r. The field component dE_x at point P due to charge dQ is

$$dE_x = \frac{1}{4\pi\epsilon_0}\frac{dQ}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{(2\pi\sigma r\,dr)x}{(x^2 + r^2)^{3/2}}$$

21.26 Our sketch for this problem.



To find the total field due to all the rings, we integrate dE_x over r from $r = 0$ to $r = R$ (*not* from $-R$ to R):

$$E_x = \int_0^R \frac{1}{4\pi\epsilon_0}\frac{(2\pi\sigma r\,dr)x}{(x^2 + r^2)^{3/2}} = \frac{\sigma x}{2\epsilon_0}\int_0^R \frac{r\,dr}{(x^2 + r^2)^{3/2}}$$

Remember that x is a constant during the integration and that the integration variable is r. The integral can be evaluated by use of the substitution $z = x^2 + r^2$. We'll let you work out the details; the result is

$$\begin{aligned} E_x &= \frac{\sigma x}{2\epsilon_0}\left[-\frac{1}{\sqrt{x^2 + R^2}} + \frac{1}{x}\right] \\ &= \frac{\sigma}{2\epsilon_0}\left[1 - \frac{1}{\sqrt{(R^2/x^2) + 1}}\right] \end{aligned} \tag{21.11}$$

The electric field due to the ring has no components perpendicular to the axis. Hence at point P in Fig. 21.26, $dE_y = dE_z = 0$ for each ring, and the total field has $E_y = E_z = 0$.

EVALUATE Suppose we keep increasing the radius R of the disk, simultaneously adding charge so that the surface charge density σ (charge per unit area) is constant. In the limit that R is much larger than the distance x of the field point from the disk, the term $1/\sqrt{(R^2/x^2) + 1}$ in Eq. (21.11) becomes negligibly small, and we get

$$E = \frac{\sigma}{2\epsilon_0} \tag{21.12}$$

Our final result does not contain the distance x from the plane. Hence the electric field produced by an *infinite* plane sheet of charge is *independent of the distance from the sheet.* The field direction is everywhere perpendicular to the sheet, away from it. There is no such thing as an infinite sheet of charge, but if the dimensions of the sheet are much larger than the distance x of the field point P from the sheet, the field is very nearly given by Eq. (21.11).

If P is to the *left* of the plane $(x < 0)$, the result is the same except that the direction of $\vec{E}$ is to the left instead of the right. If the surface charge density is negative, the directions of the fields on both sides of the plane are toward it rather than away from it.

Example 21.13 Field of two oppositely charged infinite sheets

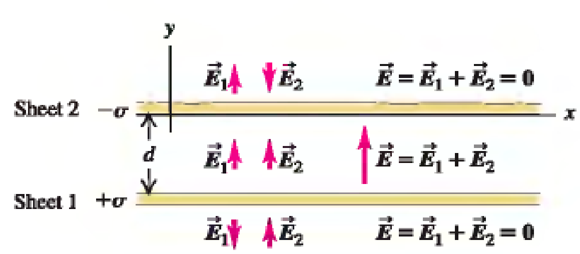
Two infinite plane sheets are placed parallel to each other, separated by a distance d (Fig. 21.27). The lower sheet has a uniform positive surface charge density σ, and the upper sheet has a uniform negative surface charge density $-\sigma$ with the same magnitude. Find the electric field between the two sheets, above the upper sheet, and below the lower sheet.

SOLUTION

IDENTIFY: From Example 21.12 we know the electric field due to a single infinite plane sheet of charge. Our goal is to find the electric field due to *two* such sheets.

SET UP: We use the principle of superposition to combine the electric fields produced by the two sheets, as shown in Fig. 21.27.

21.27 Finding the electric field due to two oppositely charged infinite sheets. The sheets are seen edge-on; only a portion of the infinite sheets can be shown!



EXECUTE: Let sheet 1 be the lower sheet of positive charge, and let sheet 2 be the upper sheet of negative charge; the fields due to each sheet are $\vec{E}_1$ and $\vec{E}_2$, respectively. From Eq. (21.12) of Example 21.12, both $\vec{E}_1$ and $\vec{E}_2$ have the same magnitude at all points, no matter how far from either sheet:

$$E_1 = E_2 = \frac{\sigma}{2\epsilon_0}$$

At all points, the direction of $\vec{E}_1$ is away from the positive charge of sheet 1, and the direction of $\vec{E}_2$ is toward the negative charge of sheet 2. These fields and the x- and y-axes are shown in Fig. 21.27.

CAUTION **Electric fields are not "flows"** You may be surprised that $\vec{E}_1$ is unaffected by the presence of sheet 2 and that $\vec{E}_2$ is unaffected by the presence of sheet 1. Indeed, you may have thought that the field of one sheet would be unable to "penetrate" the other sheet. You might conclude this if you think of the electric field as some kind of physical substance that "flows" into or out of charges. But in fact there is no such substance, and the electric fields $\vec{E}_1$ and $\vec{E}_2$ depend only on the individual charge distributions that create them. The *total* field is just the vector sum of $\vec{E}_1$ and $\vec{E}_2$.

At points between the sheets, $\vec{E}_1$ and $\vec{E}_2$ reinforce each other; at points above the upper sheet or below the lower sheet, $\vec{E}_1$ and $\vec{E}_2$ cancel each other. Thus the total field is

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \begin{cases} 0 & \text{above the upper sheet} \\ \dfrac{\sigma}{\epsilon_0}\hat{j} & \text{between the sheets} \\ 0 & \text{below the lower sheet} \end{cases}$$

Because we considered the sheets to be infinite, our result does not depend on the separation d.

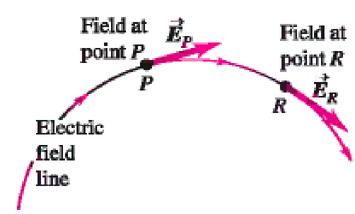
EVALUATE: Note that the field between the oppositely charged sheets is uniform. We used this in Examples 21.7 and 21.8, in which two large parallel conducting plates were connected to the terminals of a battery. The battery causes the two plates to become oppositely charged, giving a field between the plates that is essentially uniform if the plate separation is much smaller than the dimensions of the plates. In Chapter 23 we will examine how a battery can produce such separation of positive and negative charge. An arrangement of two oppositely charged conducting plates is called a *capacitor;* these devices prove to be of tremendous practical utility and are the principal subject of Chapter 24.

Test Your Understanding of Section 21.5 Suppose that the line of charge in Fig. 21.25 (Example 21.11) had charge $+Q$ distributed uniformly between $y = 0$ and $y = +a$ and had charge $-Q$ distributed uniformly between $y = 0$ and $y = -a$. In this situation, the electric field at P would be (i) in the positive x-direction; (ii) in the negative x-direction; (iii) in the positive y-direction; (iv) in the negative y-direction; (v) zero; (vi) none of these.

21.6 Electric Field Lines

The concept of an electric field can be a little elusive because you can't see an electric field directly. Electric field *lines* can be a big help for visualizing electric fields and making them seem more real. An **electric field line** is an imaginary line or curve drawn through a region of space so that its tangent at any point is in the direction of the electric-field vector at that point. Figure 21.28 shows the basic idea. (We used a similar concept in our discussion of fluid flow in Section 14.5. A *streamline* is a line or curve whose tangent at any point is in the direction of the velocity of the fluid at that point. However, the similarity between electric field lines and fluid streamlines is a mathematical one only; there is nothing "flowing" in an electric field.) The English scientist Michael Faraday (1791–1867) first introduced the concept of field lines. He called them "lines of force," but the term "field lines" is preferable.

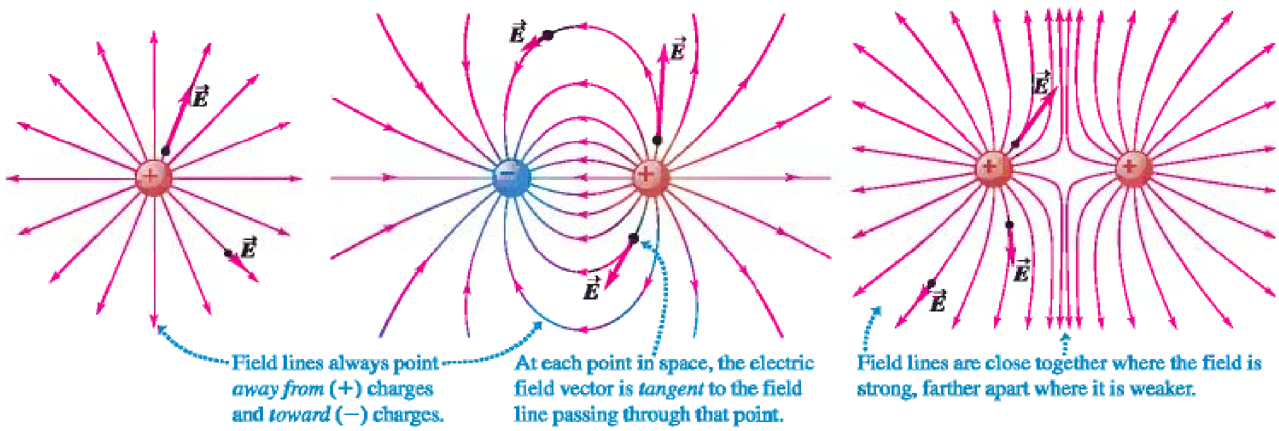
21.28 The direction of the electric field at any point is tangent to the field line through that point.



Electric field lines show the direction of $\vec{E}$ at each point, and their spacing gives a general idea of the *magnitude* of $\vec{E}$ at each point. Where $\vec{E}$ is strong, we draw lines bunched closely together; where $\vec{E}$ is weaker, they are farther apart. At any particular point, the electric field has a unique direction, so only one field line can pass through each point of the field. In other words, *field lines never intersect.*

Figure 21.29 shows some of the electric field lines in a plane containing (a) a single positive charge; (b) two equal-magnitude charges, one positive and one negative (a dipole); and (c) two equal positive charges. Diagrams such as these are sometimes called *field maps;* they are cross sections of the actual three-dimensional patterns. The direction of the total electric field at every point in each diagram is along the tangent to the electric field line passing through the point. Arrowheads indicate the direction of the $\vec{E}$-field vector along each field

21.29 Electric field lines for three different charge distributions. In general, the magnitude of $\vec{E}$ is different at different points along a given field line.



line. The actual field vectors have been drawn at several points in each pattern. Notice that in general, the magnitude of the electric field is different at different points on a given field line; a field line is *not* a curve of constant electric-field magnitude!

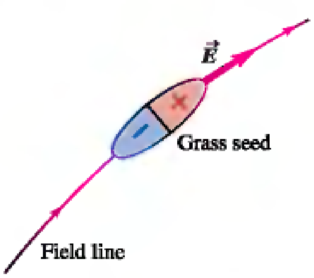
Figure 21.29 shows that field lines are directed *away* from positive charges (since close to a positive point charge, $\vec{E}$ points away from the charge) and *toward* negative charges (since close to a negative point charge, $\vec{E}$ points toward the charge). In regions where the field magnitude is large, such as between the positive and negative charges in Fig. 21.29b, the field lines are drawn close together. In regions where the field magnitude is small, such as between the two positive charges in Fig. 21.29c, the lines are widely separated. In a *uniform* field, the field lines are straight, parallel, and uniformly spaced, as in Fig. 21.20.

Figure 21.30 is a view from above of a demonstration setup for visualizing electric field lines. In the arrangement shown here, the tips of two positively charged wires are inserted in a container of insulating liquid, and some grass seeds are floated on the liquid. The grass seeds are electrically neutral insulators, but the electric field of the two charged wires causes *polarization* of the grass seeds; there is a slight shifting of the positive and negative charges within the molecules of each seed, like that shown in Fig. 21.8. The positively charged end of each grass seed is pulled in the direction of $\vec{E}$ and the negatively charged end is pulled opposite $\vec{E}$. Hence the long axis of each grass seed tends to orient parallel to the electric field, in the direction of the field line that passes through the position of the seed (Fig. 21.30b).

21.30 (a) Electric field lines produced by two equal point charges. The pattern is formed by grass seeds floating on a liquid above two charged wires. Compare this pattern with Fig. 21.29c. (b) The electric field causes polarization of the grass seeds, which in turn causes the seeds to align with the field.

(a)

(b)



CAUTION **Electric field lines are not the same as trajectories** It's a common misconception that if a charged particle of charge q is in motion where there is an electric field, the particle must move along an electric field line. Because $\vec{E}$ at any point is tangent to the field line that passes through that point, it is indeed true that the *force* $\vec{F} = q\vec{E}$ on the particle, and hence the particle's acceleration, are tangent to the field line. But we learned in Chapter 3 that when a particle moves on a curved path, its acceleration *cannot* be tangent to the path. So in general, the trajectory of a charged particle is *not* the same as a field line.

Test Your Understanding of Section 21.6 Suppose the electric field lines in a region of space are straight lines. If a charged particle is released from rest in that region, will the trajectory of the particle be along a field line?

21.7 Electric Dipoles

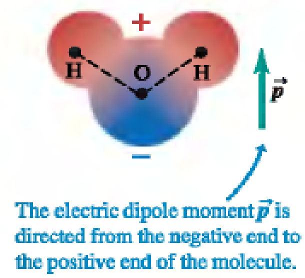
An **electric dipole** is a pair of point charges with equal magnitude and opposite sign (a positive charge q and a negative charge $-q$) separated by a distance d. We introduced electric dipoles in Example 21.9 (Section 21.5); the concept is worth exploring further because many physical systems, from molecules to TV antennas, can be described as electric dipoles. We will also use this concept extensively in our discussion of dielectrics in Chapter 24.

Figure 21.31 a shows a molecule of water (H_2O), which in many ways behaves like an electric dipole. The water molecule as a whole is electrically neutral, but the chemical bonds within the molecule cause a displacement of charge; the result is a net negative charge on the oxygen end of the molecule and a net positive charge on the hydrogen end, forming an electric dipole. The effect is equivalent to shifting one electron only about 4×10^{-11} m (about the radius of a hydrogen atom), but the consequences of this shift are profound. Water is an excellent solvent for ionic substances such as table salt (sodium chloride, NaCl) precisely because the water molecule is an electric dipole (Fig. 21.31b). When dissolved in water, salt dissociates into a positive sodium ion (Na^+) and a negative chlorine ion (Cl^-), which tend to be attracted to the negative and positive ends, respectively, of water molecules; this holds the ions in solution. If water molecules were not electric dipoles, water would be a poor solvent, and almost all of the chemistry that occurs in aqueous solutions would be impossible. This includes all of the biochemical reactions that occur in all of the life on earth. In a very real sense, your existence as a living being depends on electric dipoles!

We examine two questions about electric dipoles. First, what forces and torques does an electric dipole experience when placed in an external electric field (that is, a field set up by charges outside the dipole)? Second, what electric field does an electric dipole itself produce?

21.31 (a) A water molecule is an example of an electric dipole. (b) Each test tube contains a solution of a different substance in water. The large electric dipole moment of water makes it an excellent solvent.

(a) A water molecule, showing positive charge as red and negative charge as blue



(b) Various substances dissolved in water

Force and Torque on an Electric Dipole

To start with the first question, let's place an electric dipole in a *uniform* external electric field $\vec{E}$, as shown in Fig. 21.32. The forces $\vec{F}_+$ and $\vec{F}_-$ on the two charges both have magnitude qE, but their directions are opposite, and they add to zero. *The net force on an electric dipole in a uniform external electric field is zero.*

However, the two forces don't act along the same line, so their *torques* don't add to zero. We calculate torques with respect to the center of the dipole. Let the angle between the electric field $\vec{E}$ and the dipole axis be ϕ; then the lever arm for both $\vec{F}_+$ and $\vec{F}_-$ is $(d/2)\sin\phi$. The torque of $\vec{F}_+$ and the torque of $\vec{F}_-$ both have the same magnitude of $(qE)(d/2)\sin\phi$, and both torques tend to rotate the dipole clockwise (that is, $\vec{\tau}$ is directed into the page in Fig. 21.32). Hence the magnitude of the net torque is twice the magnitude of either individual torque:

$$\tau = (qE)(d\sin\phi) \tag{21.13}$$

where $d \sin \phi$ is the perpendicular distance between the lines of action of the two forces.

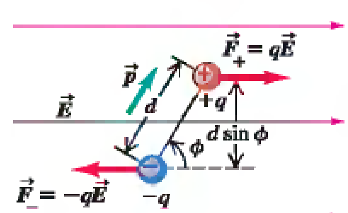
The product of the charge q and the separation d is the magnitude of a quantity called the **electric dipole moment,** denoted by p:

$$p = qd \qquad \text{(magnitude of electric dipole moment)} \tag{21.14}$$

The units of p are charge times distance $(\mathrm{C \cdot m})$. For example, the magnitude of the electric dipole moment of a water molecule is $p = 6.13 \times 10^{-30}\ \mathrm{C \cdot m}$.

CAUTION The symbol p has multiple meanings Be careful not to confuse dipole moment with momentum or pressure. There aren't as many letters in the alphabet as there are physical quantities, so some letters are used several times. The context usually makes it clear what we mean, but be careful.

21.32 The net force on this electric dipole is zero, but there is a torque directed into the page that tends to rotate the dipole clockwise.



We further define the electric dipole moment to be a *vector* quantity $\vec{p}$. The magnitude of $\vec{p}$ is given by Eq. (21.14), aud its direction is along the dipole axis from the negative charge to the positive charge as shown in Fig. 21.32.

In terms of p, Eq. (21.13) for the magnitude τ of the torque exerted by the field becomes

$$\tau = pE\sin\phi \qquad \text{(magnitude of the torque on an electric dipole)} \qquad (21.15)$$

Since the angle ϕ in Fig. 21.32 is the angle between the directions of the vectors $\vec{p}$ and $\vec{E}$, this is reminiscent of the expression for the magnitude of the *vector product* discussed in Section 1.10. (You may want to review that discussion.) Hence we can write the torque on the dipole in vector form as

$$\vec{\tau} = \vec{p} \times \vec{E} \qquad \text{(torque on an electric dipole, in vector form)} \qquad (21.16)$$

You can use the right-hand rule for the vector product to verify that in the situation shown in Fig. 21.32, $\vec{\tau}$ is directed into the page. The torque is greatest when $\vec{p}$ and $\vec{E}$ are perpendicular and is zero when they are parallel or antiparallel. The torque always tends to turn $\vec{p}$ to line it up with $\vec{E}$. The position $\phi = 0$, with $\vec{p}$ parallel to $\vec{E}$, is a position of stable equilibrium, and the position $\phi = \pi$, with $\vec{p}$ and $\vec{E}$ antiparallel, is a position of unstable equilibrium. The polarization of a grass seed in the apparatus of Fig. 21.30b gives it an electric dipole moment; the torque exerted by $\vec{E}$ then causes the seed to align with $\vec{E}$ and hence with the field lines.

Potential Energy of an Electric Dipole

When a dipole changes direction in an electric field, the electric-field torque does *work* on it, with a corresponding change in potential energy. The work dW done by a torque τ during an infinitesimal displacement $d\phi$ is given by Eq. (10.19): $dW = \tau\, d\phi$. Because the torque is in the direction of decreasing ϕ, we must write the torque as $\tau = -pE\sin\phi$, and

$$dW = \tau\, d\phi = -pE\sin\phi\, d\phi$$

In a finite displacement from ϕ_1 to ϕ_2 the total work done on the dipole is

$$\begin{aligned} W &= \int_{\phi_1}^{\phi_2} (-pE\sin\phi)\, d\phi \\ &= pE\cos\phi_2 - pE\cos\phi_1 \end{aligned}$$

The work is the negative of the change of potential energy, just as in Chapter 7: $W = U_1 - U_2$. So we see that a suitable definition of potential energy U for this system is

$$U(\phi) = -pE\cos\phi \qquad (21.17)$$

In this expression we recognize the *scalar product* $\vec{p}\cdot\vec{E} = pE\cos\phi$, so we can also write

$$U = -\vec{p}\cdot\vec{E} \qquad \text{(potential energy for a dipole in an electric field)} \qquad (21.18)$$

The potential energy has its minimum value $U = -pE$ (i.e., its most negative value) at the stable equilibrium position, where $\phi = 0$ and $\vec{p}$ is parallel to $\vec{E}$. The potential energy is maximum when $\phi = \pi$ and $\vec{p}$ is antiparallel to $\vec{E}$; then $U = +pE$. At $\phi = \pi/2$, where $\vec{p}$ is perpendicular to $\vec{E}$, U is zero. We could of course define U differently so that it is zero at some other orientation of $\vec{p}$, but our definition is simplest.

Equation (21.18) gives us another way to look at the effect shown in Fig. 21.30. The electric field $\vec{E}$ gives each grass seed an electric dipole moment, and the grass seed then aligns itself with $\vec{E}$ to minimize the potential energy.

Example 21.14 Force and torque on an electric dipole

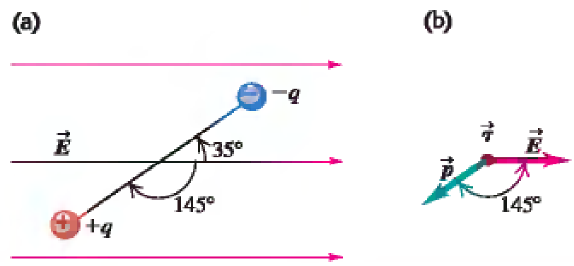
Figure 21.33a shows an electric dipole in a uniform electric field with magnitude 5.0×10^5 N/C directed parallel to the plane of the figure. The charges are $\pm 1.6 \times 10^{-19}$ C; both lie in the plane and are separated by 0.125 nm $= 0.125 \times 10^{-9}$ m. (Both the charge magnitude and the distance are typical of molecular quantities.) Find (a) the net force exerted by the field on the dipole; (b) the magnitude and direction of the electric dipole moment; (c) the magnitude and direction of the torque; (d) the potential energy of the system in the position shown.

SOLUTION

IDENTIFY: This problem uses the ideas of this section about an electric dipole placed in an electric field.

SET UP: We use the relationship $\vec{F} = q\vec{E}$ for each point charge to find the force on the dipole as a whole. Equation (21.14) tells us the dipole moment, Eq. (21.16) tells us the torque on the dipole, and Eq. (21.18) tells us the potential energy of the system.

21.33 (a) An electric dipole. (b) Directions of the electric dipole moment, electric field, and torque.



EXECUTE: (a) Since the field is uniform, the forces on the two charges are equal and opposite, and the total force is zero.

(b) The magnitude p of the electric dipole moment $\vec{p}$ is

$$p = qd = (1.6 \times 10^{-19}\,\text{C})(0.125 \times 10^{-9}\,\text{m}) = 2.0 \times 10^{-29}\,\text{C} \cdot \text{m}$$

The direction of $\vec{p}$ is from the negative to the positive charge, 145° clockwise from the electric-field direction (Fig. 21.33b).

(c) The magnitude of the torque is

$$\tau = pE\sin\phi = (2.0 \times 10^{-29}\,\text{C})(5.0 \times 10^5\,\text{N/C})(\sin 145°) = 5.7 \times 10^{-24}\,\text{N} \cdot \text{m}$$

From the right-hand rule for vector products (see Section 1.10), the direction of the torque $\vec{\tau} = \vec{p} \times \vec{E}$ is out of the page. This corresponds to a counterclockwise torque that tends to align $\vec{p}$ with $\vec{E}$.

(d) The potential energy is

$$U = -pE\cos\phi = -(2.0 \times 10^{-29}\,\text{C} \cdot \text{m})(5.0 \times 10^5\,\text{N/C})(\cos 145°) = 8.2 \times 10^{-24}\,\text{J}$$

EVALUATE: The dipole moment, torque, and potential energy are all exceedingly small. Don't be surprised by this result: Remember that we are looking at a single molecule, which is a very small object indeed!

In this discussion we have assumed that $\vec{E}$ is uniform, so there is no net force on the dipole. If $\vec{E}$ is not uniform, the forces at the ends may not cancel completely, and the net force may not be zero. Thus a body with zero net charge but an electric dipole moment can experience a net force in a nonuniform electric field. As we mentioned in Section 21.1, an uncharged body can be polarized by an electric field, giving rise to a separation of charge and an electric dipole moment. This is how uncharged bodies can experience electrostatic forces (see Fig. 21.8).

Field of an Electric Dipole

Now let's think of an electric dipole as a *source* of electric field. What does the field look like? The general shape of things is shown by the field map of Fig. 21.29b. At each point in the pattern the total $\vec{E}$ field is the vector sum of the fields from the two individual charges, as in Example 21.9 (Section 21.5). Try drawing diagrams showing this vector sum for several points.

To get quantitative information about the field of an electric dipole, we have to do some calculating, as illustrated in the next example. Notice the use of the principle of superposition of electric fields to add up the contributions to the field of the individual charges. Also notice that we need to use approximation techniques even for the relatively simple case of a field due to two charges. Field calculations often become very complicated, and computer analysis is typically used to determine the field due to an arbitrary charge distribution.

Example 21.15 Field of an electric dipole, revisited

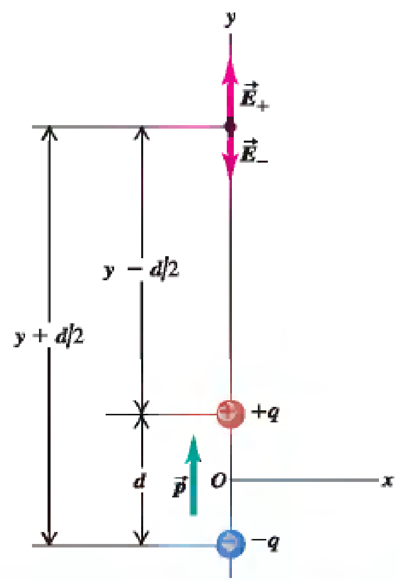
In Fig. 21.34 an electric dipole is centered at the origin, with $\vec{p}$ in the direction of the $+y$-axis. Derive an approximate expression for the electric field at a point on the y-axis for which y is much larger than d. Use the binomial expansion of $(1 + x)^n$—that is, $(1 + x)^n \cong 1 + nx + n(n - 1)x^2/2 + \cdots$ —for the case $|x| < 1$. (This problem illustrates a useful calculational technique.)

SOLUTION

IDENTIFY: We use the principle of superposition: The total electric field is the vector sum of the field produced by the positive charge and the field produced by the negative charge.

SET UP: At the field point shown in Fig. 21.34, the field of the positive charge has a positive (upward) y-component and the field of the negative charge has a negative (downward) y-component. We add these components to find the total field and then apply the approximation that y is much greater than d.

21.34 Finding the electric field of an electric dipole at a point on its axis.



EXECUTE: The total y-component E_y of electric field from the two charges is

$$E_y = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{(y - d/2)^2} - \frac{1}{(y + d/2)^2}\right]$$
$$= \frac{q}{4\pi\epsilon_0 y^2}\left[\left(1 - \frac{d}{2y}\right)^{-2} - \left(1 + \frac{d}{2y}\right)^{-2}\right]$$

We used this same approach in Example 21.9 (Section 21.5). Now comes the approximation. When y is much greater than d—that is, when we are far away from the dipole compared to its size—the quantity $d/2y$ is much smaller than 1. With $n = -2$ and $d/2y$ playing the role of x in the binomial expansion, we keep only the first two terms. The terms we discard are much smaller than those we keep, and we have

$$\left(1 - \frac{d}{2y}\right)^{-2} \cong 1 + \frac{d}{y} \quad \text{and} \quad \left(1 + \frac{d}{2y}\right)^{-2} \cong 1 - \frac{d}{y}$$

Hence E_y is given approximately by

$$E \cong \frac{q}{4\pi\epsilon_0 y^2}\left[1 + \frac{d}{y} - \left(1 - \frac{d}{y}\right)\right]$$
$$= \frac{qd}{2\pi\epsilon_0 y^3}$$
$$= \frac{p}{2\pi\epsilon_0 y^3}$$

EVALUATE: An alternative route to this expression is to put the fractions in the E_y expression over a common denominator and combine, then approximate the denominator $(y - d/2)^2(y + d/2)^2$ as y^4. We leave the details to you (see Exercise 21.65).

For points P off the coordinate axes, the expressions are more complicated, but at *all* points far away from the dipole (in any direction) the field drops off as $1/r^3$. We can compare this with the $1/r^2$ behavior of a point charge, the $1/r$ behavior of a long line charge, and the independence of r for a large sheet of charge. There are charge distributions for which the field drops off even more quickly. An *electric quadrupole* consists of two equal dipoles with opposite orientation, separated by a small distance. The field of a quadrupole at large distances drops off as $1/r^4$.

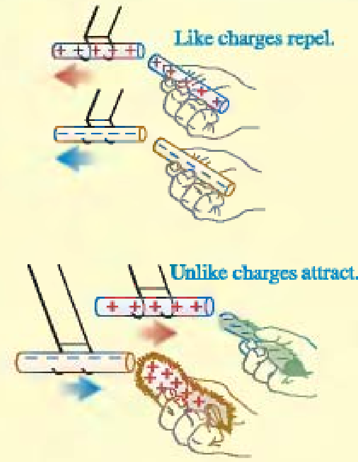
Test Your Understanding of Section 21.7 An electric dipole is placed in a region of uniform electric field $\vec{E}$, with the electric dipole moment $\vec{p}$, pointing in the direction opposite to $\vec{E}$. Is the dipole (i) in stable equilibrium, (ii) in unstable equilibrium, or (iii) neither? (*Hint:* You many want to review Section 7.5.)

CHAPTER 21 SUMMARY

Electric charge, conductors, and insulators: The fundamental quantity in electrostatics is electric charge. There are two kinds of charge, positive and negative. Charges of the same sign repel each other; charges of opposite sign attract. Charge is conserved; the total charge in an isolated system is constant.

All ordinary matter is made of protons, neutrons, and electrons. The positive protons and electrically neutral neutrons in the nucleus of an atom are bound together by the nuclear force; the negative electrons surround the nucleus at distances much greater than the nuclear size. Electric interactions are chiefly responsible for the structure of atoms, molecules, and solids.

Conductors are materials that permit electric charge to move easily within them. Insulators permit charge to move much less readily. Most metals are good conductors; most nonmetals are insulators.

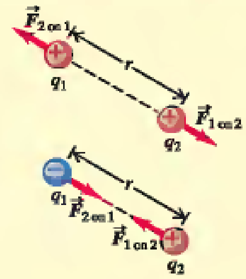


Coulomb's law: Coulomb's law is the basic law of interaction for point electric charges. For charges q_1 and q_2 separated by a distance r, the magnitude of the force on either charge is proportional to the product q_1q_2 and inversely proportional to r^2. The force on each charge is along the line joining the two charges—repulsive if q_1 and q_2 have the same sign, attractive if they have opposite signs. The forces form an action–reaction pair and obey Newton's third law. In SI units the unit of electric charge is the coulomb, abbreviated C. (See Examples 21.1 and 21.2.)

The principle of superposition of forces states that when two or more charges each exert a force on a charge, the total force on that charge is the vector sum of the forces exerted by the individual charges. (See Examples 21.3 and 21.4.)

$$F = \frac{1}{4\pi\epsilon_0}\frac{|q_1q_2|}{r^2} \qquad (21.2)$$

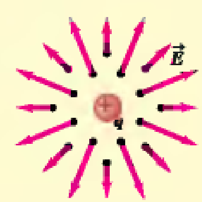
$$\frac{1}{4\pi\epsilon_0} = 8.988 \times 10^9\ \mathrm{N \cdot m^2/C^2}$$



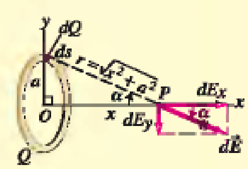
Electric field: Electric field $\vec{E}$, a vector quantity, is the force per unit charge exerted on a test charge at any point, provided the test charge is small enough that it does not disturb the charges that cause the field. The electric field produced by a point charge is directed radially away from or toward the charge. (See Examples 21.5–21.8.)

$$\vec{E} = \frac{\vec{F}_0}{q_0} \qquad (21.3)$$

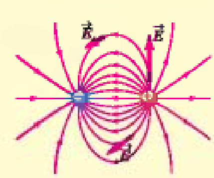
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \qquad (21.7)$$



Superposition of electric fields: The principle of superposition of electric fields states that the electric field $\vec{E}$ of any combination of charges is the vector sum of the fields caused by the individual charges. To calculate the electric field caused by a continuous distribution of charge, divide the distribution into small elements, calculate the field caused by each element, and then carry out the vector sum or each component sum, usually by integrating. Charge distributions are described by linear charge density λ, surface charge density σ, and volume charge density ρ. (See Examples 21.9–21.13.)



Electric field lines: Field lines provide a graphical representation of electric fields. At any point on a field line, the tangent to the line is in the direction of $\vec{E}$ at that point. The number of lines per unit area (perpendicular to their direction) is proportional to the magnitude of $\vec{E}$ at the point.

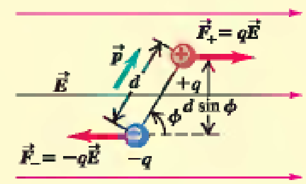


Electric dipoles: An electric dipole is a pair of electric charges of equal magnitode q but opposite sign, separated by a distance d. The electric dipole moment $\vec{p}$ is defined to have magnitude $p = qd$. The direction of $\vec{p}$ is from negative toward positive charge. An electric dipole in an electric field $\vec{E}$ experiences a torque $\vec{\tau}$ equal to the vector product of $\vec{p}$ and $\vec{E}$. The magnitude of the torque depends on the angle ϕ between $\vec{p}$ and $\vec{E}$. The potential energy U for an electric dipole in an electric field also depends on the relative orientation of $\vec{p}$ and $\vec{E}$. (See Examples 21.14 and 21.15.)

$$\tau = pE\sin\phi \tag{21.15}$$

$$\vec{\tau} = \vec{p} \times \vec{E} \tag{21.16}$$

$$U = -\vec{p} \cdot \vec{E} \tag{21.18}$$



Key Terms

electric charge, *710*
electrostatics, *710*
electron, *711*
proton, *711*
neutron, *711*
nucleus, *711*
atomic number, *712*
positive ion, *712*
negative ion, *712*
ionization, *712*
principle of conservation of charge, *712*
conductor, *713*
insulator, *713*
induction, *714*
induced charge, *714*
point charge, *716*
Coulomb's law, *716*
coulomb, *717*
principle of superposition of forces, *719*
electric field, *722*
test charge, *722*
source point, *723*
field point, *723*
vector field, *724*
principle of superposition of electric fields, *727*
linear charge density, *727*
surface charge density, *727*
volume charge density, *727*
electric field line, *733*
electric dipole, *735*
electric dipole moment, *735*

Answer to Chapter Opening Question ?

Water molecules have a permanent electric dipole moment: One end of the molecule has a positive charge and the other end has a negative charge. These ends attract negative and positive ions, respectively, holding the ions apart in solution. Water is less effective as a solvent for materials whose molecules do not ionize (called *nonionic* substances), such as oils.

Answers to Test Your Understanding Questions

21.1 Answers: (a) the plastic rod weighs more, (b) the glass rod weighs less, (c) the fur weighs a little less, (d) the silk weighs a little less The plastic rod gets a negative charge by taking electrons from the fur, so the rod weighs a little more and the fur weighs a little less after the rubbing. By contrast, the glass rod gets a positive charge by giving electrons to the silk. Hence, after they are rubbed together, the glass rod weighs a little less and the silk weighs a little more. The weight change is *very* small: The number of electrons transferred is a small fraction of a mole, and a mole of electrons has a mass of only $(6.02 \times 10^{23}\ \text{electrons})(9.11 \times 10^{-31}\ \text{kg/electron}) = 5.48 \times 10^{-7}\ \text{kg} = 0.548$ milligram!

21.2 Answers: (a) (i), (b) (ii) Before the two spheres touch, the negatively charged sphere exerts a repulsive force on the electrons in the other sphere, causing zones of positive and negative induced charge (see Fig. 21.7b). The positive zone is closer to the negatively charged sphere than the negative zone, so there is a net force of attraction that pulls the spheres together, like the comb and insulator in Fig. 21.8b. Once the two metal spheres touch, some of the excess electrons on the negatively charged sphere will flow onto the other sphere (because metals are conductors). Then both spheres will have a net negative charge and will repel each other.

21.3 Answer: (iv) The force exerted by q_1 on Q is still as in Example 21.4. The magnitude of the force exerted by q_2 on Q is still equal to $F_{1\text{ on }Q}$, but the direction of the force is now *toward* q_2 at an angle α below the x-axis. Hence the x-components of the two forces cancel while the (negative) y-components add together, and the total electric force is in the negative y-direction.

21.4 Answers: (a) (ii), (b) (i) The electric field $\vec{E}$ produced by a positive point charge points directly away from the charge (see Fig. 21.18a) and has a magnitude that depends on the distance r from the charge to the field point. Hence a second, negative point charge $q < 0$ will feel a force $\vec{F} = q\vec{E}$ that points directly toward the positive charge and has a magnitude that depends on the distance r between the two charges. If the negative charge moves directly toward the positive charge, the direction of the force remains the same (along the line of the negative charge's motion) but the force magnitode increases as the distance r decreases. If the negative charge moves in a circle around the positive charge, the force magnitude stays the same (because the distance r is constant) but the force direction changes (when the negative charge is on the right side of the positive charge, the force is to the left; when the negative charge is on the left side of the positive charge, the force is to the right).

21.5 Answer: (iv) Think of a pair of segments of length dy, one at coordinate $y > 0$ and the other at coordinate $-y < 0$. The upper segment has a positive charge and produces an electric field $d\vec{E}$ at P that points away from the segment, so this $d\vec{E}$ has a positive x-component and a negative y-component, like the vector $d\vec{E}$ in Fig. 21.25. The lower segment has the same amount of negative charge. It produces a $d\vec{E}$ that has the same magnitude but points *toward* the lower segment, so it has a negative x-component and a negative y-component. By symmetry, the two x-components are equal but opposite, so they cancel. Thus the total electric field has only a negative y-component.

21.6 Answer: yes If the field lines are straight, $\vec{E}$ must point in the same direction throughout the region. Hence the force $\vec{F} = q\vec{E}$ on a particle of charge q is always in the same direction. A particle released from rest accelerates in a straight line the direction of $\vec{F}$, and so its trajectory is a straight line that will be along a field line.

21.7 Answer: (ii) Equations (21.17) and (21.18) tell is that the potential energy for a dipole in an electric field is $U = -\vec{p} \cdot \vec{E} = -pE \cos\phi$, where ϕ is the angle between the directions of $\vec{p}$ and $\vec{E}$. If $\vec{p}$ and $\vec{E}$ point in opposite directions, so that $\phi = 180°$, we have $\cos\phi = -1$ and $U = +pE$. This is the maximum value that U can have. From our discussion of energy diagrams in Section 7.5, it follows that this is a situation of unstable equilibrium.

Another way to see this is from Eq. (21.15), which tells us that the magnitude of the torque on an electric dipole is $\tau = pE \sin\phi$. This is zero if $\phi = 180°$, so there is no torque, and if left undisturbed the dipole will not rotate. However, if the dipole is disturbed slightly so that ϕ is a little less than 180°, there will be a nonzero torque that tries to rotate the dipole toward $\phi = 0$ so that $\vec{p}$ and $\vec{E}$ point in the same direction. Hence if the dipole is disturbed from the equilibrium orientation at $\phi = 180°$, it moves farther away from that orientation—which is the hallmark of unstable equilibrium.

You can show that the situation in which $\vec{p}$ and $\vec{E}$ point in the same direction ($\phi = 0$) is a case of *stable* equilibrium: The potential energy is minimum, and if the dipole is displaced slightly there is a torque that tries to return it to the original orientation (a *restoring* torque).

PROBLEMS

For instructor-assigned homework, go to www.masteringphysics.com

Discussion Questions

Q21.1. If you peel two strips of transparent tape off the same roll and immediately let them hang near each other, they will repel each other. If you then stick the sticky side of one to the shiny side of the other and rip them apart, they will attract each other. Give a plausible explanation, involving transfer of electrons between the strips of tape, for this sequence of events.

Q21.2. Two metal spheres are hanging from nylon threads. When you bring the spheres close to each other, they tend to attract. Based on this information alone, discuss all the possible ways that the spheres could be charged. Is it possible that after the spheres touch, they will cling together? Explain.

Q21.3. The electric force between two charged particles becomes weaker with increasing distance. Suppose instead that the electric force were *independent* of distance. In this case, would a charged comb still cause a neutral insulator to become polarized as in Fig. 21.8? Why or why not? Would the neutral insulator still be attracted to the comb? Again, why or why not?

Q21.4. Your clothing tends to cling together after going through the dryer. Why? Would you expect more or less clinging if all your clothing were made of the same material (say, cotton) than if you dried different kinds of clothing together? Again, why? (You may want to experiment with your next load of laundry.)

Q21.5. An uncharged metal sphere hangs from a nylon thread. When a positively charged glass rod is brought close to the metal sphere, the sphere is drawn toward the rod. But if the sphere touches the rod, it suddenly flies away from the rod. Explain why the sphere is first attracted and then repelled.

Q21.6. The free electrons in a metal are gravitationally attracted toward the earth. Why, then, don't they all settle to the bottom of the conductor, like sediment settling to the bottom of a river?

Q21.7. Some of the free electrons in a good conductor (such as a piece of copper) move at speeds of 10^6 m/s or faster. Why don't these electrons fly out of the conductor completely?

Q21.8. Good electrical conductors, such as metals, are typically good conductors of heat; electrical insulators, such as wood, are typically poor conductors of heat. Explain why there should be a relationship between electrical conduction and heat conduction in these materials.

Q21.9. Defend this statement: "If there were only one electrically charged particle in the entire universe, the concept of electric charge would be meaningless."

Q21.10. Two identical metal objects are mounted on insulating stands. Describe how you could place charges of opposite sign but exactly equal magnitude on the two objects.

Q21.11. You can use plastic food wrap to cover a container by stretching the material across the top and pressing the overhanging material against the sides. What makes it stick? (*Hint:* The answer involves the electric force.) Does the food wrap stick to itself with equal tenacity? Why or why not? Does it work with metallic containers? Again, why or why not?

Q21.12. If you walk across a nylon rug and then touch a large metal object such as a doorknob, you may get a spark and a shock. Why does this tend to happen more on dry days than on humid days? (*Hint:* See Fig. 21.31.) Why are you less likely to get a shock if you touch a *small* metal object, such as a paper clip?

Q21.13. You have a negatively charged object. How can you use it to place a net negative charge on an insulated metal sphere? To place a net positive charge on the sphere?

Q21.14. When two point charges of equal mass and charge are released on a frictionless table, each has an initial acceleration a_0. If instead you keep one fixed and release the other one, what will be its initial acceleration: a_0, $2a_0$, or $a_0/2$? Explain.

Q21.15. A point charge of mass m and charge Q and another point charge of mass m but charge $2Q$ are released on a frictionless table. If the charge Q has an initial acceleration a_0, what will be the acceleration of $2Q$: a_0, $2a_0$, $4a_0$, $a_0/2$, or $a_0/4$? Explain.

Q21.16. A proton is placed in a uniform electric field and then released. Then an electron is placed at this same point and released. Do these two particles experience the same force? The same acceleration? Do they move in the same direction when released?

Q21.17. In Example 21.1 (Section 21.3) we saw that the electric force between two α particles is of the order of 10^{35} times as strong as the gravitational force. So why do we readily feel the gravity of the earth but no electrical force from it?

Q21.18. What similarities do electrical forces have with gravitational forces? What are the most significant differences?

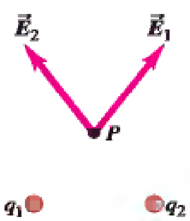
Q21.19. At a distance R from a point charge its electric field is E_0. At what distance (in terms of R) from the point charge would the electric field be $\frac{1}{3}E_0$

Q21.20. Atomic nuclei are made of protons and neutrons. This shows that there must be another kind of interaction in addition to gravitational and electric forces. Explain.

Q21.21. Sufficiently strong electric fields can cause atoms to become positively ionized—that is, to lose one or more electrons. Explain how this can happen. What determines how strong the field must be to make this happen?

Q21.22. The electric fields at point P due to the positive charges q_1 and q_2 are shown in Fig. 21.35. Does the fact that they cross each other violate the statement in Section 21.6 that electric field lines never cross? Explain.

Figure 21.35 Question Q21.22.



Q21.23. The air temperature and the velocity of the air have different values at different places in the earth's atmosphere. Is the air velocity a vector field? Why or why not? Is the air temperature a vector field? Again, why or why not?

Exercises

Section 21.3 Coulomb's Law

21.1. Excess electrons are placed on a small lead sphere with mass 8.00 g so that its net charge is -3.20×10^{-9} C. (a) Find the number of excess electrons on the sphere. (b) How many excess electrons are there per lead atom? The atomic number of lead is 82, and its atomic mass is 207 g/mol.

21.2. Lightning occurs when there is a flow of electric charge (principally electrons) between the ground and a thundercloud. The maximum rate of charge flow in a lightning bolt is about 20,000 C/s; this lasts for 100 μs or less. How much charge flows between the ground and the cloud in this time? How many electrons flow during this time?

21.3. Estimate how many electrons there are in your body. Make any assumptions you feel are necessary, but clearly state what they are. (*Hint:* Most of the atoms in your body have equal numbers of electrons, protons, and neutrons.) What is the combined charge of all these electrons?

21.4. Particles in a Gold Ring. You have a pure (24-karat) gold ring with mass 17.7 g. Gold has an atomic mass of 197 g/mol and an atomic number of 79. (a) How many protons are in the ring, and what is their total positive charge? (b) If the ring carries no net charge, how many electrons are in it?

21.5. An average human weighs about 650 N. If two such generic humans each carried 1.0 coulomb of excess charge, one positive and one negative, how far apart would they have to be for the electric attraction between them to equal their 650-N weight?

21.6. Two small spheres spaced 20.0 cm apart have equal charge. How many excess electrons must be present on each sphere if the magnitude of the force of repulsion between them is 4.57×10^{-21} N?

21.7. Two small plastic spheres are given positive electrical charges. When they are 15.0 cm apart, the repulsive force between them has magnitude 0.220 N. What is the charge on each sphere (a) if the two charges are equal and (b) if one sphere has four times the charge of the other?

21.8. Two small aluminum spheres, each having mass 0.0250 kg, are separated by 80.0 cm. (a) How many electrons does each sphere contain? (The atomic mass of aluminum is 26.982 g/mol, and its atomic number is 13.) (b) How many electrons would have to be removed from one sphere and added to the other to cause an attractive force between the spheres of magnitude 1.00×10^4 N (roughly 1 ton)? Assume that the spheres may be treated as point charges. (c) What fraction of all the electrons in each sphere does this represent?

21.9. Two very small 8.55-g spheres, 15.0 cm apart from center to center, are charged by adding equal numbers of electrons to each of them. Disregarding all other forces, how many electrons would you have to add to each sphere so that the two spheres will accelerate at 25.0g when released? Which way will they accelerate?

21.10. (a) Assuming that only gravity is acting on it, how far does an electron have to be from a proton so that its acceleration is the same as that of a freely falling object at the earth's surface? (b) Suppose the earth were made only of protons but had the same size and mass it presently has. What would be the acceleration of an electron released at the surface? Is it necessary to consider the gravitational attraction as well as the electrical force? Why or why not?

21.11. In an experiment in space, one proton is held fixed and another proton is released from rest a distance of 2.50 mm away. (a) What is the initial acceleration of the proton after it is released? (b) Sketch qualitative (no numbers!) acceleration–time and velocity–time graphs of the released proton's motion.

21.12. A negative charge $-0.550\ \mu$C exerts an upward 0.200-N force on an unknown charge 0.300 m directly below it. (a) What is the unknown charge (magnitude and sign)? (b) What are the magnitude and direction of the force that the unknown charge exerts on the -0.550-μC charge?

21.13. Three point charges are arranged on a line. Charge $q_3 = +5.00$ nC and is at the origin. Charge $q_2 = -3.00$ nC and is at $x = +4.00$ cm. Charge q_1 is at $x = +2.00$ cm. What is q_1 (magnitude and sign) if the net force on q_3 is zero?

21.14. In Example 21.4, suppose the point charge on the y-axis at $y = -0.30$ m has negative charge $-2.0\ \mu$C, and the other charges remain the same. Find the magnitude and direction of the net force on Q. How does your answer differ from that in Example 21.3? Explain the differences.

21.15. In Example 21.3, calculate the net force on charge q_1.

21.16. In Example 21.4, what is the net force (magnitude and direction) on charge q_1 exerted by the other two charges?

21.17. Three point charges are arranged along the x-axis. Charge $q_1 = +3.00\ \mu$C is at the origin, and charge $q_2 = -5.00\ \mu$C is at $x = 0.200$ m. Charge $q_3 = -8.00\ \mu$C. Where is q_3 located if the net force on q_1 is 7.00 N in the $-x$-direction?

21.18. Repeat Exercise 21.17, for $q_3 = +8.00\ \mu$C.

21.19. Two point charges are located on the y-axis as follows: charge $q_1 = -1.50$ nC at $y = -0.600$ m, and charge $q_2 = +3.20$ nC at the origin $(y = 0)$. What is the total force (magnitude and direction) exerted by these two charges on a third charge $q_3 = +5.00$ nC located at $y = -0.400$ m?

21.20. Two point charges are placed on the x-axis as follows: Charge $q_1 = +4.00$ nC is located at $x = 0.200$ m, and charge $q_2 = +5.00$ nC is at $x = -0.300$ m. What are the magnitude and direction of the total force exerted by these two charges on a negative point charge $q_3 = -6.00$ nC that is placed at the origin?

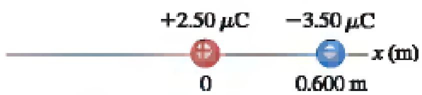
21.21. A positive point charge q is placed on the $+y$-axis at $y = a$, and a negative point charge $-q$ is placed on the $-y$-axis at $y = -a$. A negative point charge $-Q$ is located at some point on the $+x$-axis. (a) In a free-body diagram, show the forces that act on the charge $-Q$. (b) Find the x- and y-components of the net force that the two charges q and $-q$ exert on $-Q$. (Your answer should involve only k, q, Q, a and the coordinate x of the third charge.) (c) What is the net force on the charge $-Q$ when it is at the origin $(x = 0)$? (d) Graph the y-component of the net force on the charge $-Q$ as a function of x for values of x between $-4a$ and $+4a$.

21.22. Two positive point charges q are placed on the y-axis at $y = a$ and $y = -a$. A negative point charge $-Q$ is located at some point on the $+x$-axis. (a) In a free-body diagram, show the forces that act on the charge $-Q$. (b) Find the x- and y-components of the net force that the two positive charges exert on $-Q$. (Your answer should involve only k, q, Q, a and the coordinate x of the third charge.) (c) What is the net force on the charge $-Q$ when it is at the origin $(x = 0)$? (d) Graph the x-component of the net force on the charge $-Q$ as a function of x for values of x between $-4a$ and $+4a$.

21.23. Four identical charges Q are placed at the corners of a square of side L. (a) In a free-body diagram, show all of the forces that act on one of the charges. (b) Find the magnitude and direction of the total force exerted on one charge by the other three charges.

21.24. Two charges, one of 2.50 μC and the other of -3.50 μC, are placed on the x-axis, one at the origin and the other at $x = 0.600$ m, as shown in Fig. 21.36. Find the position on the x-axis where the net force on a small charge $+q$ would be zero.

Figure 21.36 Exercise 21.24.



Section 21.4 Electric Field and Electric Forces

21.25. A proton is placed in a uniform electric field of 2.75×10^3 N/C. Calculate: (a) the magnitude of the electric force felt by the proton; (b) the proton's acceleration; (c) the proton's speed after 1.00 μs in the field, assuming it starts from rest.

21.26. A particle has charge -3.00 nC. (a) Find the magnitude and direction of the electric field due to this particle at a point 0.250 m directly above it. (b) At what distance from this particle does its electric field have a magnitude of 12.0 N/C?

21.27. A proton is traveling horizontally to the right at 4.50×10^6 m/s. (a) Find the magnitude and direction of the weakest electric field that can bring the proton uniformly to rest over a distance of 3.20 cm. (b) How much time does it take the proton to stop after entering the field? (c) What minimum field (magnitude and direction) would be needed to stop an electron under the conditions of part (a)?

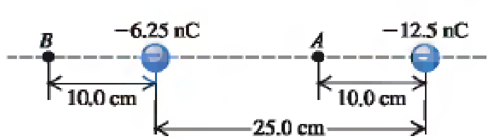
21.28. An electron is released from rest in a uniform electric field. The electron accelerates vertically upward, traveling 4.50 m in the first 3.00 μs after it is released. (a) What are the magnitude and direction of the electric field? (b) Are we justified in ignoring the effects of gravity? Justify your answer quantitatively.

21.29. (a) What must the charge (sign and magnitude) of a 1.45-g particle be for it to remain stationary when placed in a downward-directed electric field of magnitude 650 N/C? (b) What is the magnitude of an electric field in which the electric force on a proton is equal in magnitude to its weight?

21.30. (a) What is the electric field of an iron nucleus at a distance of 6.00×10^{-10} m from the nucleus? The atomic number of iron is 26. Assume that the nucleus may be treated as a point charge. (b) What is the electric field of a proton at a distance of 5.29×10^{-11} m from the proton? (This is the radius of the electron orbit in the Bohr model for the ground state of the hydrogen atom.)

21.31. Two point charges are separated by 25.0 cm (Fig. 21.37). Find the net electric field these charges produce at (a) point A and (b) point B. (c) What would be the manitude and direction of the electric force this combination of charges would produce on a proton at A?

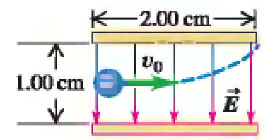
Figure 21.37 Exercise 21.31.



21.32. Electric Field of the Earth. The earth has a net electric charge that causes a field at points near its surface equal to 150 N/C and directed in toward the center of the earth. (a) What magnitude and sign of charge would a 60-kg human have to acquire to overcome his or her weight by the force exerted by the earth's electric field? (b) What would be the force of repulsion between two people each with the charge calculated in part (a) and separated by a distance of 100 m? Is use of the earth's electric field a feasible means of flight? Why or why not?

21.33. An electron is projected with an initial speed $v_0 = 1.60 \times 10^6$ m/s into the uniform field between the parallel plates in Fig. 21.38. Assume that the field between the plates is uniform and directed vertically downward, and that the field outside the plates is zero. The electron enters the field at a point midway between the plates. (a) If the electron just misses the upper plate as it emerges from the field, find the magnitude of the electric field. (b) Suppose that in Fig. 21.38 the electron is replaced by a proton with the same initial speed v_0. Would the proton hit one of the plates? If the proton would not hit one of the plates, what would be the magnitude and direction of its vertical displacement as it exits the region between the plates? (c) Compare the paths traveled by the electron and the proton and explain the differences. (d) Discuss whether it is reasonable to ignore the effects of gravity for each particle.

Figure 21.38 Exercise 21.33.



21.34. Point charge $q_1 = -5.00$ nC is at the origin and point charge $q_2 = +3.00$ nC is on the x-axis at $x = 3.00$ cm. Point P is on the y-axis at $y = 4.00$ cm. (a) Calculate the electric fields $\vec{E}_1$ and $\vec{E}_2$ at point P due to the charges q_1 and q_2. Express your results in terms of unit vectors (see Example 21.6). (b) Use the results of part (a) to obtain the resultant field at P, expressed in unit vector form.

21.35. In Exercise 21.33, what is the speed of the electron as it emerges from the field?

21.36. (a) Calculate the magnitude and direction (relative to the $+x$-axis) of the electric field in Example 21.6. (b) A -2.5-nC point charge is placed at the point P in Fig. 21.19. Find the magnitude and direction of (i) the force that the -8.0-nC charge at the origin exerts on this charge and (ii) the force that this charge exerts on the -8.0-nC charge at the origin.

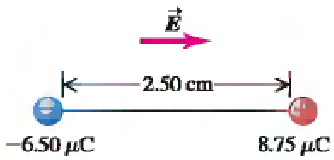
21.37. (a) For the electron in Examples 21.7 and 21.8, compare the weight of the electron to the magnitude of the electric force on the electron. Is it appropriate to ignore the gravitational force on the electron in these examples? Explain. (b) A particle with charge $+e$ is placed at rest between the charged plates in Fig. 21.20. What must the mass of this object be if it is to remain at rest? Give your answer in kilograms and in multiples of the electron mass. (c) Does the answer to part (b) depend on where between the plates the object is placed? Why or why not?

21.38. A uniform electric field exists in the region between two oppositely charged plane parallel plates. A proton is released from rest at the surface of the positively charged plate and strikes the surface of the opposite plate, 1.60 cm distant from the first, in a time interval of 1.50×10^{-6} s. (a) Find the magnitude of the electric field. (b) Find the speed of the proton when it strikes the negatively charged plate.

21.39. A point charge is at the origin. With this point charge as the source point, what is the unit vector $\hat{r}$ in the direction of (a) the field point at $x = 0$, $y = -1.35$ m; (b) the field point at $x = 12.0$ cm, $y = 12.0$ cm; (c) the field point at $x = -1.10$ m, $y = 2.60$ m? Express your results in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

21.40. A $+8.75\text{-}\mu\text{C}$ point charge is glued down on a horizontal frictionless table. It is tied to a $-6.50\text{-}\mu\text{C}$ point charge by a light, nonconducting 2.50-cm wire. A uniform electric field of magnitude 1.85×10^8 N/C is directed parallel to the wire, as shown in Fig. 21.39. (a) Find the tension in the wire. (b) What would the tension be if both charges were negative?

Figure 21.39 Exercise 21.40.



21.41. (a) An electron is moving east in a uniform electric field of 1.50 N/C directed to the west. At point A, the velocity of the electron is 4.50×10^5 m/s toward the east. What is the speed of the electron when it reaches point B, 0.375 m east of point A? (b) A proton is moving in the uniform electric field of part (a). At point A, the velocity of the proton is 1.90×10^4 m/s, east. What is the speed of the proton at point B?

21.42. Electric Field in the Nucleus. Protons in the nucleus are of the order of 10^{-15} m (1 fm) apart. (a) What is the magnitude of the electric field produced by a proton at a distance of 1.50 fm from it? (b) How does this field compare in magnitude to the field in Example 21.7?

Section 21.5 Electric-Field Calculations

21.43. Two positive point charges q are placed on the x-axis, one at $x = a$ and one at $x = -a$. (a) Find the magnitude and direction of the electric field at $x = 0$. (b) Derive an expression for the electric field at points on the x-axis. Use your result to graph the x-component of the electric field as a function of x, for values of x between $-4a$ and $+4a$.

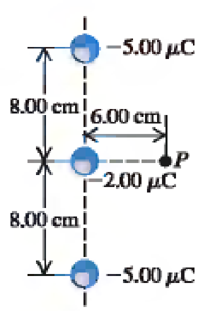
21.44. Two particles having charges $q_1 = 0.500$ nC and $q_2 = 8.00$ nC are separated by a distance of 1.20 m. At what point along the line connecting the two charges is the total electric field due to the two charges equal to zero?

21.45. A +2.00-nC point charge is at the origin, and a second −5.00-nC point charge is on the x-axis at $x = 0.800$ m. (a) Find the electric field (magnitude and direction) at each of the following points on the x-axis: (i) $x = 0.200$ m; (ii) $x = 1.20$ m; (iii) $x = -0.200$ m. (b) Find the net electric force that the two charges would exert on an electron placed at each point in part (a).

21.46. Repeat Exercise 21.44, but now let $q_1 = -4.00$ nC.

21.47. Three negative point charges lie along a line as shown in Fig. 21.40. Find the magnitude and direction of the electric field this combination of charges produces at point P, which lies 6.00 cm. from the $-2.00\text{-}\mu\text{C}$ charge measured perpendicular to the line connecting the three charges.

Figure 21.40 Exercise 21.47.



21.46. A positive point charge q is placed at $x = a$, and a negative point charge $-q$ is placed at $x = -a$. (a) Find the magnitude and direction of the electric field at $x = 0$. (b) Derive an expression for the electric field at points on the x-axis. Use your result to graph the x-component of the electric field as a function of x, for values of x between $-4a$ and $+4a$.

21.49. In a rectangular coordinate system a positive point charge $q = 6.00 \times 10^{-9}$ C is placed at the point $x = +0.150$ m, $y = 0$, and an identical point charge is placed at $x = -0.150$ m, $y = 0$. Find the x- and y-components, the magnitude, and the direction of the electric field at the following points: (a) the origin; (b) $x = 0.300$ m, $y = 0$; (c) $x = 0.150$ m, $y = -0.400$ m; (d) $x = 0$, $y = 0.200$ m.

21.50. A point charge $q_1 = -4.00$ nC is at the point $x = 0.600$ m, $y = 0.800$ m, and a second point charge $q_2 = +6.00$ nC is at the point $x = 0.600$ m, $y = 0$. Calculate the magnitude and direction of the net electric field at the origin due to these two point charges.

21.51. Repeat Exercise 21.49 for the case where the point charge at $x = +0.150$ m, $y = 0$ is positive and the other is negative, each with magnitude 6.00×10^{-9} C.

21.52. A very long, straight wire has charge per unit length 1.50×10^{-10} C/m. At what distance from the wire is the electric-field magnitude equal to 2.50 N/C?

21.53. Positive electric charge is distributed along the y-axis with charge per unit length λ. (a) Consider the case where charge is distributed only between the points $y = a$ and $y = -a$. For points on the $+x$-axis, graph the x-component of the electric field as a function of x for values of x between $x = a/2$ and $x = 4a$. (b) Consider instead the case where charge is distributed along the entire y-axis with the same charge per unit length λ. Using the same graph as in part (a), plot the x-component of the electric field as a function of x for values of x between $x = a/2$ and $x = 4a$. Label which graph refers to which situation.

21.54. A straight, nonconducting plastic wire 8.50 cm long carries a charge density of +175 nC/m distributed uniformly along its length. It is lying on a horizontal tabletop. (a) Find the magnitude and direction of the electric field this wire produces at a point 6.00 cm directly above its midpoint. (b) If the wire is now bent into a circle lying flat on the table, find the magnitude and direction of the electric field it produces at a point 6.00 cm directly above its center.

21.55. A ring-shaped conductor with radius $a = 2.50$ cm has a total positive charge $Q = +0.125$ nC uniformly distributed around it, as shown in Fig. 21.24. The center of the ring is at the origin of coordinates O. (a) What is the electric field (magnitude and direction) at point P, which is on the x-axis at $x = 40.0$ cm? (b) A point charge $q = -2.50\ \mu\text{C}$ is placed at the point P described in part (a). What are the magnitude and direction of the force exerted *by* the charge q *on* the ring?

21.56. A charge of −6.50 nC is spread uniformly over the surface of one face of a nonconducting disk of radius 1.25 cm. (a) Find the magnitude and direction of the electric field this disk produces at a point P on the axis of the disk a distance of 2.00 cm from its center. (b) Suppose that the charge were all pushed away from the center and distributed uniformly on the outer rim of the disk. Find the magnitude and direction of the electric field at point P. (c) If the charge is all brought to the center of the disk, find the magnitude and direction of the electric field at point P. (d) Why is the field in part (a) stronger than the field in part (b)? Why is the field in part (c) the strongest of the three fields?

21.57. Two horizontal, infinite, plane sheets of charge are separated by a distance d. The lower sheet has negative charge with uniform surface charge density $-\sigma < 0$. The upper sheet has positive charge with uniform surface charge density $\sigma > 0$. What is the electric field (magnitude, and direction if the field is nonzero) (a) above the upper sheet, (b) below the lower sheet, (c) between the sheets?

Section 21.6 Electric Field Lines

21.58. Infinite sheet A carries a positive uniform charge density σ, and sheet B, which is to the right of A and parallel to it, carries a uniform negative charge density -2σ. (a) Sketch the electric field lines for this pair of sheets. Include the region between the sheets as well as the regions to the left of A and to the right of B. (b) Repeat part (a) for the case in which sheet B carries a charge density of $+2\sigma$.

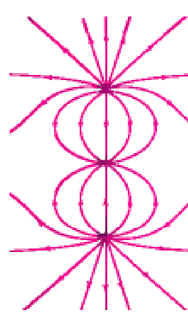
21.58. Suppose the charge shown in Fig. 21.29a is fixed in position. A small, positively charged particle is then placed at some point in the figure and released. Will the trajectory of the particle follow an electric field line? Why or why not? Suppose instead that the particle is placed at some point in Fig. 21.29b and released (the positive and negative charges shown in the figure are fixed in position). Will its trajectory follow an electric field line? Again, why or why not? Explain any differences between your answers for the two different situations.

21.60. Sketch the electric field lines for a disk of radius R with a positive uniform surface charge density σ. Use what you know about the electric field very close to the disk and very far from the disk to make your sketch.

21.61. (a) Sketch the electric field lines for an infinite line of charge. You may find it helpful to show the field lines in a plane containing the line of charge in one sketch and the field lines in a plane perpendicular to the line of charge in a second sketch. (b) Explain how your sketches show (i) that the magnitude E of the electric field depends only on the distance r from the line of charge and (ii) that E decreases like $1/r$.

21.62. Figure 21.41 shows some of the electric field lines due to three point charges arranged along the vertical axis. All three charges have the same magnitude. (a) What are the signs of the three charges? Explain your reasoning. (b) At what point(s) is the magnitude of the electric field the smallest? Explain your reasoning. Explain how the fields produced by each individual point charge combine to give a small net field at this point or points.

Figure 21.41 Exercise 21.62.



Section 21.7 Electric Dipoles

21.63. Point charges $q_1 = -4.5\text{ nC}$ and $q_2 = +4.5\text{ nC}$ are separated by 3.1 mm, forming an electric dipole. (a) Find the electric dipole moment (magnitude and direction). (b) The charges are in a uniform electric field whose direction makes an angle of 36.9° with the line connecting the charges. What is the magnitude of this field if the torque exerted on the dipole has magnitude $7.2 \times 10^{-9}\text{ N}\cdot\text{m}$?

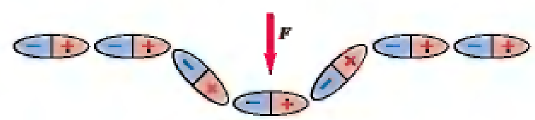
21.64. The ammonia molecule (NH_3) has a dipole moment of $5.0 \times 10^{-30}\text{ C}\cdot\text{m}$. Ammonia molecules in the gas phase are placed in a uniform electric field $\vec{E}$ with magnitude $1.6 \times 10^{6}\text{ N/C}$. (a) What is the change in electric potential energy when the dipole moment of a molecule changes its orientation with respect to $\vec{E}$ from parallel to perpendicular? (b) At what absolute temperature T is the average translational kinetic energy $\frac{3}{2}kT$ of a molecule equal to the change in potential energy calculated in part (a)? (*Note:* Above this temperature, thermal agitation prevents the dipoles from aligning with the electric field.)

21.65. In Example 21.15, the approximate result $E \cong p/2\pi\epsilon_0 y^3$ was derived for the electric field of a dipole at points on the dipole axis. (a) Rederive this result by putting the fractions in the expression for E_y over a common denominator, as described in Example 21.15. (b) Explain why the approximate result also gives the correct approximate expression for E_y for $y < 0$.

21.66. The dipole moment of the water molecule (H_2O) is $6.17 \times 10^{-30}\text{ C}\cdot\text{m}$. Consider a water molecule located at the origin whose dipole moment $\vec{p}$ points in the $+x$-direction. A chlorine ion (Cl^-), of charge $-1.60 \times 10^{-19}\text{ C}$, is located at $x = 3.00 \times 10^{-9}\text{ m}$. Find the magnitude and direction of the electric force that the water molecule exerts on the chlorine ion. Is this force attractive or repulsive? Assume that x is much larger than the separation d between the charges in the dipole, so that the approximate expression for the electric field along the dipole axis derived in Example 21.15 can be used.

21.67. Surface Tension. The surface of a polar liquid, such as water, can be viewed as a series of dipoles strung together in the stable arrangement in which the dipole moment vectors are parallel to the surface and all point in the same direction. Suppose now that something presses inward on the surface, distorting the dipoles as shown in Fig. 21.42. (a) Show that the two slanted dipoles exert a net upward force on the dipole between them, and hence oppose the downward external force. (b) Show that the dipoles attract each other and hence resist being separated. The force between dipoles opposes penetration of the liquid's surface and is a simple model for surface tension (see Section 14.3 and Fig. 14.15).

Figure 21.42 Exercise 21.67.



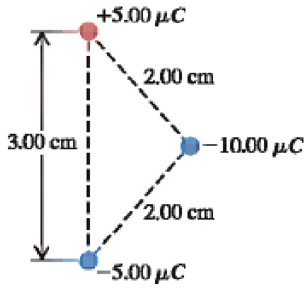
21.66. Consider the electric dipole of Example 21.15. (a) Derive an expression for the magnitude of the electric field produced by the dipole at a point on the x-axis in Fig. 21.34. What is the direction of this electric field? (b) How does the electric field at points on the x-axis depend on x when x is very large?

21.69. Torque on a Dipole. An electric dipole with dipole moment $\vec{p}$ is in a uniform electric field $\vec{E}$. (a) Find the orientations of the dipole for which the torque on the dipole is zero. (b) Which of the orientations in part (a) is stable, and which is unstable? (*Hint:* Consider a small displacement away from the equilibrium position and see what happens.) (c) Show that for the stable orientation in part (b), the dipole's own electric field tends to oppose the external field.

21.70. A dipole consisting of charges $\pm e$, 220 nm apart, is placed between two very large (essentially infinite) sheets carrying equal but opposite charge densities of $125\ \mu\text{C/m}^2$. (a) What is the maximum potential energy this dipole can have due to the sheets, and how should it be oriented relative to the sheets to attain this value? (b) What is the maximum torque the sheets can exert on the dipole, and how should it be oriented relative to the sheets to attain this value? (c) What net force do the two sheets exert on the dipole?

21.71. Three charges are at the corners of an isosceles triangle as shown in Fig. 21.43. The $\pm 5.00\text{-}\mu C$ charges form a dipole. (a) Find the force (magnitude and direction) the $-10.00\text{-}\mu C$ charge exerts on the dipole. (b) For an axis perpendicular to the line connecting the $\pm 5.00\text{-}\mu C$ charges at the midpoint of this line, find the torque (magnitude and direction) exerted on the dipole by the $-10.00\text{-}\mu C$ charge.

Figure 21.43 Exercise 21.71.



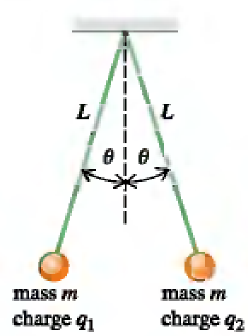
Problems

21.72. A charge $q_1 = +5.00$ nC is placed at the origin of an xy-coordinate system, and a charge $q_2 = -2.00$ nC is placed on the positive x-axis at $x = 4.00$ cm. (a) If a third charge $q_3 = +6.00$ nC is now placed at the point $x = 4.00$ cm, $y = 3.00$ cm, find the x- and y-components of the total force exerted on this charge by the other two. (b) Find the magnitude and direction of this force.

21.73. Two positive point charges Q are held fixed on the x-axis at $x = a$ and $x = -a$. A third positive point charge q, with mass m, is placed on the x-axis away from the origin at a coordinate x such that $|x| \ll a$. The charge q, which is free to move along the x-axis, is then released. (a) Find the frequency of oscillation of the charge q. (*Hint:* Review the definition of simple harmonic motion in Section 13.2. Use the binomial expansion $(1 + z)^n = 1 + nz + n(n-1)z^2/2 + \cdots$, valid for the case $|z| < 1$.) (b) Suppose instead that the charge q were placed on the y-axis at a coordinate y such that $|y| \ll a$, and then released. If this charge is free to move anywhere in the xy-plane, what will happen to it? Explain your answer.

21.74. Two identical spheres with mass m are hung from silk threads of length L, as shown in Fig. 21.44. Each sphere has the same charge, so $q_1 = q_2 = q$. The radius of each sphere is very small compared to the distance between the spheres, so they may be treated as point charges. Show that if the angle θ is small, the equilibrium separation d between the spheres is $d = (q^2L/2\pi\epsilon_0 mg)^{1/3}$. (*Hint:* If θ is small, then $\tan\theta \cong \sin\theta$.)

Figure 21.44 Problems 21.74, 21.75, and 21.76.



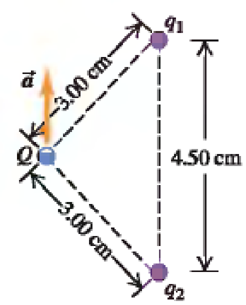
21.75. Two small spheres with mass $m = 15.0$ g are hung by silk threads of length $L = 1.20$ m from a common point (Fig. 21.44). When the spheres are given equal quantities of negative charge, so that $q_1 = q_2 = q$, each thread hangs at $\theta = 25.0°$ from the vertical. (a) Draw a diagram showing the forces on each sphere. Treat the spheres as point charges. (b) Find the magnitude of q. (c) Both threads are now shortened to length $L = 0.600$ m, while the charges q_1 and q_2 remain unchanged. What new angle will each thread make with the vertical? (*Hint:* This part of the problem can be solved numerically by using trial values for θ and adjusting the values of θ until a self-consistent answer is obtained.)

21.76. Two identical spheres are each attached to silk threads of length $L = 0.500$ m and hung from a common point (Fig. 21.44). Each sphere has mass $m = 8.00$ g. The radius of each sphere is very small compared to the distance between the spheres, so they may be treated as point charges. One sphere is given positive charge q_1, and the other a different positive charge q_2; this causes the spheres to separate so that when the spheres are in equilibrium, each thread makes an angle $\theta = 20.0°$ with the vertical. (a) Draw a free-body diagram for each sphere when in equilibrium, and label all the forces that act on each sphere. (b) Determine the magnitude of the electrostatic force that acts on each sphere, and determine the tension in each thread. (c) Based on the information you have been given, what can you say about the magnitudes of q_1 and q_2? Explain your answers. (d) A small wire is now connected between the spheres, allowing charge to be transferred from one sphere to the other until the two spheres have equal charges; the wire is then removed. Each thread now makes an angle of 30.0° with the vertical. Determine the original charges. (*Hint:* The total charge on the pair of spheres is conserved.)

21.77. Sodium chloride (NaCl, ordinary table salt) is made up of positive sodium ions (Na^+) and negative chloride ions (Cl^-). (a) If a point charge with the same charge and mass as all the Na^+ ions in 0.100 mol of NaCl is 2.00 cm from a point charge with the same charge and mass as all the Cl^- ions, what is the magnitude of the attractive force between these two point charges? (b) If the positive point charge in part (a) is held in place and the negative point charge is released from rest, what is its initial acceleration? (See Appendix D for atomic masses.) (c) Does it seem reasonable that the ions in NaCl could be separated in this way? Why or why not? (In fact, when sodium chloride dissolves in water, it breaks up into Na^+ and Cl^- ions. However, in this situation there are additional electric forces exerted by the water molecules on the ions.)

21.78. Two point charges q_1 and q_2 are held in place 4.50 cm apart. Another point charge $Q = -1.75\ \mu C$ of mass 5.00 g is initially located 3.00 cm from each of these charges (Fig. 21.45) and released from rest. You observe that the initial acceleration of Q is 324 m/s² upward, parallel to the line connecting the two point charges. Find q_1 and q_2.

Figure 21.45 Problem 21.78.



21.79. Three identical point charges q are placed at each of three corners of a square of side L. Find the magnitude and direction of the net force on a point charge $-3q$ placed (a) at the center of the square and (b) at the vacant corner of the square. In each case, draw a free-body diagram showing the forces exerted on the $-3q$ charge by each of the other three charges.

21.60. Three point charges are placed on the y-axis: a charge q at $y = a$, a charge $-2q$ at the origin, and a charge q at $y = -a$. Such an arrangement is called an electric quadrupole. (a) Find the magnitude and direction of the electric field at points on the positive x-axis. (b) Use the binomial expansion to find an approximate expression for the electric field valid for $x \gg a$. Contrast this behavior to that of the electric field of a point charge and that of the electric field of a dipole.

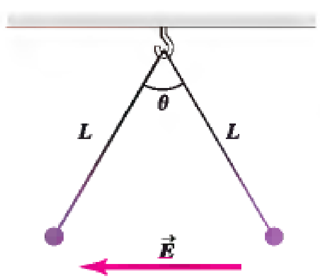
21.61. Strength of the Electric Force. Imagine two 1.0-g bags of protons, one at the earth's north pole and the other at the south pole. (a) How many protons are in each bag? (b) Calculate the gravitational attraction and the electrical repulsion that each bag exerts on the other. (c) Are the forces in part (b) large enough for you to feel if you were holding one of the bags?

21.82. Electric Force Within the Nucleus. Typical dimensions of atomic nuclei are of the order of 10^{-15} m (1 fm). (a) If two protons in a nucleus are 2.0 fm apart, find the magnitude of the electric force each one exerts on the other. Express the answer in newtons and in pounds. Would this force be large enough for a person to feel? (b) Since the protons repel each other so strongly, why don't they shoot out of the nucleus?

21.63. If Atoms Were Not Neutral . . . Because the charges on the electron and proton have the same absolute value, atoms are electrically neutral. Suppose this were not precisely true, and the absolute value of the charge of the electron were less than the charge of the proton by 0.00100%. (a) Estimate what the net charge of this textbook would be under these circumstances. Make any assumptions you feel are justified, but state clearly what they are. (*Hint:* Most of the atoms in this textbook have equal numbers of electrons, protons, and neutrons.) (b) What would be the magnitude of the electric force between two textbooks placed 5.0 m apart? Would this force be attractive or repulsive? Estimate what the acceleration of each book would be if the books were 5.0 m apart and there were no nonelectrical forces on them. (c) Discuss how the fact that ordinary matter is stable shows that the absolute values of the charges on the electron and proton must be identical to a *very* high level of accuracy.

21.84. Two tiny balls of mass m carry equal but opposite charges of magnitude q. They are tied to the same ceiling hook by light strings of length L. When a horizontal uniform electric field E is turned on, the balls hang with an angle θ between the strings (Fig. 21.46). (a) Which ball (the right or the left) is positive, and which is negative? (b) Find the angle θ between the strings in terms of E, q, m, and g. (c) As the electric field is gradually increased in strength, what does your result from part (b) give for the largest possible angle θ?

Figure 21.46 Problem 21.84.



21.85. Two small, copper spheres each have radius 1.00 mm. (a) How many atoms does each sphere contain? (b) Assume that each copper atom contains 29 protons and 29 electrons. We know that electrons and protons have charges of exactly the same magnitude, but let's explore the effect of small differences (see also Problem 21.83). If the charge of a proton is $+e$ and the magnitude of the charge of an electron is 0.100% smaller, what is the net charge of each sphere and what force would one sphere exert on the other if they were separated by 1.00 m?

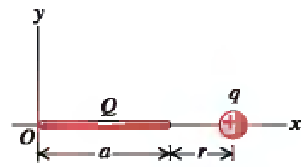
21.88. Operation of an Inkjet Printer. In an inkjet printer, letters are built up by squirting drops of ink at the paper from a rapidly moving nozzle. The ink drops, which have a mass of 1.4×10^{-8} g each, leave the nozzle and travel toward the paper at 20 m/s, passing through a charging unit that gives each drop a positive charge q by removing some electrons from it. The drops then pass between parallel deflecting plates 2.0 cm long where there is a uniform vertical electric field with magnitude 8.0×10^4 N/C. If a drop is to be deflected 0.30 mm by the time it reaches the end of the deflection plates, what magnitude of charge must be given to the drop?

21.67. A proton is projected into a uniform electric field that points vertically upward and has magnitude E. The initial velocity of the proton has a magnitude v_0 and is directed at an angle α below the horizontal. (a) Find the maximum distance h_{max} that the proton descends vertically below its initial elevation. You can ignore gravitational forces. (b) After what horizontal distance d does the proton return to its original elevation? (c) Sketch the trajectory of the proton. (d) Find the numerical values of h_{max} and d if $E = 500$ N/C, $v_0 = 4.00 \times 10^5$ m/s, and $\alpha = 30.0°$.

21.88. A negative point charge $q_1 = -4.00$ nC is on the x-axis at $x = 0.60$ m. A second point charge q_2 is on the x-axis at $x = -1.20$ m. What must the sign and magnitude of q_2 be for the net electric field at the origin to be (a) 50.0 N/C in the $+x$-direction and (b) 50.0 N/C in the $-x$-direction?

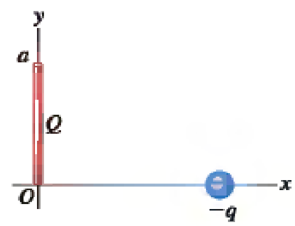
21.88. Positive charge Q is distributed uniformly along the x-axis from $x = 0$ to $x = a$. A positive point charge q is located on the positive x-axis at $x = a + r$, a distance r to the right of the end of Q (Fig. 21.47). (a) Calculate the x- and y-components of the electric field produced by the charge distribution Q at points on the positive x-axis where $x > a$. (b) Calculate the force (magnitude and direction) that the charge distribution Q exerts on q. (c) Show that if $r \gg a$, the magnitude of the force in part (b) is approximately $Qq/4\pi\epsilon_0 r^2$. Explain why this result is obtained.

Figure 21.47 Problem 21.89.



21.90. Positive charge Q is distributed uniformly along the positive y-axis between $y = 0$ and $y = a$. A negative point charge $-q$ lies on the positive x-axis, a distance x from the origin (Fig. 21.48). (a) Calculate the x- and y-components of the electric field produced by the charge distribution Q at points on the positive x-axis. (b) Calculate the x- and y-components of the force that the charge distribution Q exerts on q. (c) Show that if $x \gg a$, $F_x \cong -Qq/4\pi\epsilon_0 x^2$ and $F_y \cong +Qqa/8\pi\epsilon_0 x^3$. Explain why this result is obtained.

Figure 21.48 Problem 21.90.



21.91. A charged line like that shown in Fig. 21.25 extends from $y = 2.50$ cm to $y = -2.50$ cm. The total charge distributed uniformly along the line is -9.00 nC. (a) Find the electric field (magnitude and direction) on the x-axis at $x = 10.0$ cm. (b) Is the magnitude of the electric field you calculated in part (a) larger or smaller than the electric field 10.0 cm from a point charge that has the same total charge as this finite line of charge? In terms of the approximation used to derive $E = Q/4\pi\epsilon_0 x^2$ for a point charge from Eq. (21.9), explain why this is so. (c) At what distance x does the result for the finite line of charge differ by 1.0% from that for the point charge?

21.82. A Parallel Universe. Imagine a parallel universe in which the electric force has the same properties as in our universe but there is no gravity. In this parallel universe, the sun carries charge Q, the earth carries charge $-Q$, and the electric attraction between them keeps the earth in orbit. The earth in the parallel universe has the same mass, the same orbital radius, and the same orbital period as in our universe. Calculate the value of Q. (Consult Appendix F as needed.)

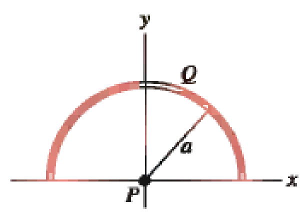
21.93. A uniformly charged disk like the disk in Fig. 21.26 has radius 2.50 cm and carries a total charge of 4.0×10^{-12} C. (a) Find the electric field (magnitude and direction) on the x-axis at $x = 20.0$ cm. (b) Show that for $x \gg R$, Eq. (21.11) becomes $E = Q/4\pi\epsilon_0 x^2$, where Q is the total charge on the disk. (c) Is the magnitude of the electric field you calculated in part (a) larger or smaller than the electric field 20.0 cm from a point charge that has the same total charge as this disk? In terms of the approximation used in part (b) to derive $E = Q/4\pi\epsilon_0 x^2$ for a point charge from Eq. (21.11), explain why this is so. (d) What is the percent difference between the electric fields produced by the finite disk and by a point charge with the same charge at $x = 20.0$ cm and at $x = 10.0$ cm?

21.94. (a) Let $f(x)$ be an even function of x so that $f(x) = f(-x)$. Show that $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$. (*Hint:* Write the integral from $-a$ to a as the sum of the integral from $-a$ to 0 and the integral from 0 to a. In the first integral, make the change of variable $x' = -x$.) (b) Let $g(x)$ be an odd function of x so that $g(x) = -g(-x)$. Use the method given in the hint for part (a) to show that $\int_{-a}^{a} g(x)dx = 0$. (c) Use the result of part (b) to show why E_y in Example 21.11 (Section 21.5) is zero.

21.95. Positive charge $+Q$ is distributed uniformly along the $+x$-axis from $x = 0$ to $x = a$. Negative charge $-Q$ is distributed uniformly along the $-x$-axis from $x = 0$ to $x = -a$. (a) A positive point charge q lies on the positive y-axis, a distance y from the origin. Find the force (magnitude and direction) that the positive and negative charge distributions together exert on q. Show that this force is proportional to y^{-3} for $y \gg a$. (b) Suppose instead that the positive point charge q lies on the positive x-axis, a distance $x > a$ from the origin. Find the force (magnitude and direction) that the charge distribution exerts on q. Show that this force is proportional to x^{-3} for $x \gg a$.

21.96. Positive charge Q is uniformly distributed around a semicircle of radius a (Fig. 21.49). Find the electric field (magnitude and direction) at the center of curvature P.

Figure 21.49 Problem 21.96.

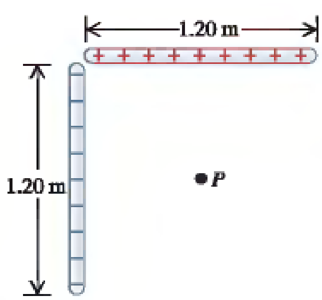


21.97. Negative charge $-Q$ is distributed uniformly around a quarter-circle of radius a that lies in the first quadrant, with the center of curvature at the origin. Find the x- and y-components of the net electric field at the origin.

21.96. A small sphere with mass m carries a positive charge q and is attached to one end of a silk fiber of length L. The other end of the fiber is attached to a large vertical insulating sheet that has a positive surface charge density σ. Show that when the sphere is in equilibrium, the fiber makes an angle equal to arctan $(q\sigma/2mg\epsilon_0)$ with the vertical sheet.

21.99. Two 1.20-m nonconducting wires meet at a right angle. One segment carries $+2.50\ \mu$C of charge distributed uniformly along its length, and the other carries $-2.50\ \mu$C distributed uniformly along it, as shown in Fig. 21.50. (a) Find the magnitude and direction of the electric field these wires produce at point P, which is 60.0 cm from each wire. (b) If an electron is released at P, what are the magnitude and direction of the net force that these wires exert on it?

Figure 21.50 Problem 21.99.



21.100. Two very large parallel sheets are 5.00 cm apart. Sheet A carries a uniform surface charge density of -9.50μC/m^2, and sheet B, which is to the right of A, carries a uniform charge of $-11.6\ \mu$C/m^2. Assume the sheets are large enough to be treated as infinite. Find the magnitude and direction of the net electric field these sheets produce at a point (a) 4.00 cm to the right of sheet A; (b) 4.00 cm to the left of sheet A; (c) 4.00 cm to the right of sheet B.

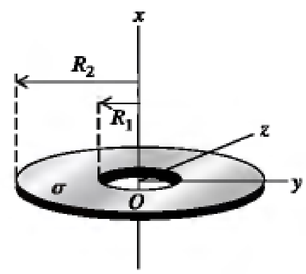
21.101. Repeat Problem 21.100 for the case where sheet B is positive.

21.102. Two very large horizontal sheets are 4.25 cm apart and carry equal but opposite uniform surface charge densities of magnitude σ. You want to use these sheets to hold stationary in the region between them an oil droplet of mass 324 μg that carries an excess of five electrons. Assuming that the drop is in vacuum, (a) which way should the electric field between the plates point, and (b) what should σ be?

21.103. An infinite sheet with positive charge per unit area σ lies in the xy-plane. A second infinite sheet with negative charge per unit area $-\sigma$ lies in the yz-plane. Find the net electric field at all points that do not lie in either of these planes. Express your answer in terms of the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

21.104. A thin disk with a circular hole at its center, called an *annulus*, has inner radius R_1 and outer radius R_2 (Fig. 21.51). The disk has a uniform positive surface charge density σ on its surface. (a) Determine the total electric charge on the annulus. (b) The annulus lies in the yz-plane, with its center at the origin. For an arbitrary point on the x-axis (the axis of the annulus), find the magnitude and direction of the electric field $\vec{E}$. Consider points both above and below the annulus in Fig. 21.51. (c) Show that at points on the x-axis that are sufficiently close to the origin, the magnitude of the electric field is approximately proportional to the distance between the center of the annulus and the point. How close is "sufficiently close"? (d) A point particle with mass m and negative charge $-q$ is free to move along the x-axis (but cannot move off the axis). The particle is originally placed at rest at $x = 0.01R_1$ and released. Find the frequency of oscillation of the particle. (*Hint:* Review Section 13.2. The annulus is held stationary.)

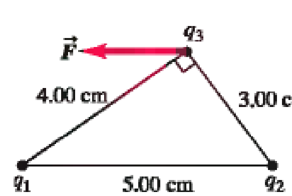
Figure 21.51 Problem 21.104.



Challenge Problems

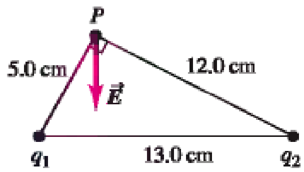
21.105. Three charges are placed as shown in Fig. 21.52. The magnitude of q_1 is 2.00 μC, but its sign and the value of the charge q_2 are not known. Charge q_3 is $+4.00\ \mu$C, and the net force $\vec{F}$ on q_3 is entirely in the negative x-direction. (a) Considering the different possible signs of q_1 and there are four possible force diagrams representing the forces $\vec{F}_1$ and $\vec{F}_2$ that q_1 and q_2 exert on q_3. Sketch these four possible force configurations. (b) Using the sketches from part (a) and the direction of $\vec{F}$, deduce the signs of the charges q_1 and q_2. (c) Calculate the magnitude of q_2. (d) Determine F, the magnitude of the net force on q_3.

Figure 21.52 Challenge Problem 21.105.



21.106. Two charges are placed as shown in Fig. 21.53. The magnitude of q_1 is 3.00 μC, but its sign and the value of the charge q_2 are not known. The direction of the net electric field $\vec{E}$ at point P is entirely in the negative y-direction. (a) Considering the different possible signs of q_1 and q_2, there are four possible diagrams that could represent the electric fields $\vec{E}_1$ and $\vec{E}_2$ produced by q_1 and q_2. Sketch the four possible electric field configurations. (b) Using the sketches from part (a) and the direction of $\vec{E}$, deduce the signs of q_1 and q_2. (c) Determine the magnitude of $\vec{E}$.

Figure 21.53 Challenge Problem 21.106.



21.107. Two thin rods of length L lie along the x-axis, one between $x = a/2$ and $x = a/2 + L$ and the other between $x = -a/2$ and $x = -a/2 - L$. Each rod has positive charge Q distributed uniformly along its length. (a) Calculate the electric field produced by the second rod at points along the positive x-axis. (b) Show that the magnitude of the force that one rod exerts on the other is

$$F = \frac{Q^2}{4\pi\epsilon_0 L^2}\ln\left[\frac{(a+L)^2}{a(a+2L)}\right]$$

(c) Show that if $a \gg L$, the magnitude of this force reduces to $F = Q^2/4\pi\epsilon_0 a^2$. (*Hint:* Use the expansion $\ln(1+z) = z - z^2/2 + z^3/3 - \cdots$, valid for $|z| \ll 1$. Carry *all* expansions to at least order L^2/a^2.) Interpret this result.

22 GAUSS'S LAW

LEARNING GOALS

By studying this chapter, you will learn:

- How you can determine the amount of charge within a closed surface by examining the electric field on the surface.
- What is meant by electric flux, and how to calculate it.
- How Gauss's law relates the electric flux through a closed surface to the charge enclosed by the surface.
- How to use Gauss's law to calculate the electric field due to a symmetrical charge distribution.
- Where the charge is located on a charged conductor.

? This child acquires an electric charge by touching the charged metal sphere. The charged hairs on the child's head repel and stand out. If the child stands *inside* a large, charged metal sphere, will her hair stand on end?

Often, there are both an easy way and a hard way to do a job; the easy way may involve nothing more than using the right tools. In physics, an important tool for simplifying problems is the *symmetry properties* of systems. Many physical systems have symmetry; for example, a cylindrical body doesn't look any different after you've rotated it around its axis, and a charged metal sphere looks just the same after you've turned it about any axis through its center.

Gauss's law is part of the key to using symmetry considerations to simplify electric-field calculations. For example, the field of a straight-line or plane-sheet charge distribution, which we derived in Section 21.5 using some fairly strenuous integrations, can be obtained in a few lines with the help of Gauss's law. But Gauss's law is more than just a way to make certain calculations easier. Indeed, it is a fundamental statement about the relationship between electric charges and electric fields. Among other things, Gauss's law can help us understand how electric charge distributes itself over conducting bodies.

Here's what Gauss's law is all about. Given any general distribution of charge, we surround it with an imaginary surface that encloses the charge. Then we look at the electric field at various points on this imaginary surface. Gauss's law is a relationship between the field at *all* the points on the surface and the total charge enclosed within the surface. This may sound like a rather indirect way of expressing things, but it turns out to be a tremendously useful relationship. Above and beyond its use as a calculational tool, Gauss's law can help us gain deeper insights into electric fields. We will make use of these insights repeatedly in the next several chapters as we pursue our study of electromagnetism.

22.1 Charge and Electric Flux

The discussion of Gauss's law in this section is based on and inspired by the innovative ideas of Ruth W. Chabay and Bruce A. Sherwood in *Electric and Magnetic Interactions* (John Wiley & Sons, 1994).

In Chapter 21 we asked the question, "Given a charge distribution, what is the electric field produced by that distribution at a point P?" We saw that the answer could be found by representing the distribution as an assembly of point charges, each of which produces an electric field $\vec{E}$ given by Eq. (21.7). The total field at P is then the vector sum of the fields due to all the point charges.

But there is an alternative relationship between charge distributions and electric fields. To discover this relationship, let's stand the question of Chapter 21 on its head and ask, "If the electric field pattern is known in a given region, what can we determine about the charge distribution in that region?"

Here's an example. Consider the box shown in Fig. 22.1a, which may or may not contain electric charge. We'll imagine that the box is made of a material that has no effect on any electric fields; it's of the same breed as the massless rope and the frictionless incline. Better still, let the box represent an *imaginary* surface that may or may not enclose some charge. We'll refer to the box as a **closed surface** because it completely encloses a volume. How can you determine how much (if any) electric charge lies within the box?

Knowing that a charge distribution produces an electric field and that an electric field exerts a force on a test charge, you move a test charge q_0 around the vicinity of the box. By measuring the force $\vec{F}$ experienced by the test charge at different positions, you make a three-dimensional map of the electric field $\vec{E} = \vec{F}/q_0$ outside the box. In the case shown in Fig. 22.1b, the map turns out to be the same as that of the electric field produced by a positive point charge (Fig. 21.29a). From the details of the map, you can find the exact value of the point charge inside the box.

To determine the contents of the box, we actually need to measure $\vec{E}$ only on the *surface* of the box. In Fig. 22.2a there is a single positive point charge inside the box, and in Fig. 22.2b there are two such charges. The field patterns on the surfaces of the boxes are different in detail, but in both cases the electric field points out of the box. Figures 22.2c and 22.2d show cases with one and two negative point charges, respectively, inside the box. Again, the details of $\vec{E}$ on the surface of the box are different, but in both cases the field points into the box.

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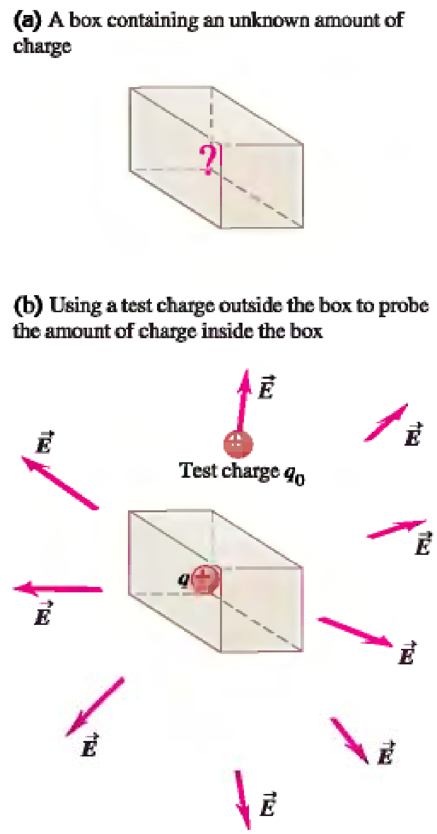
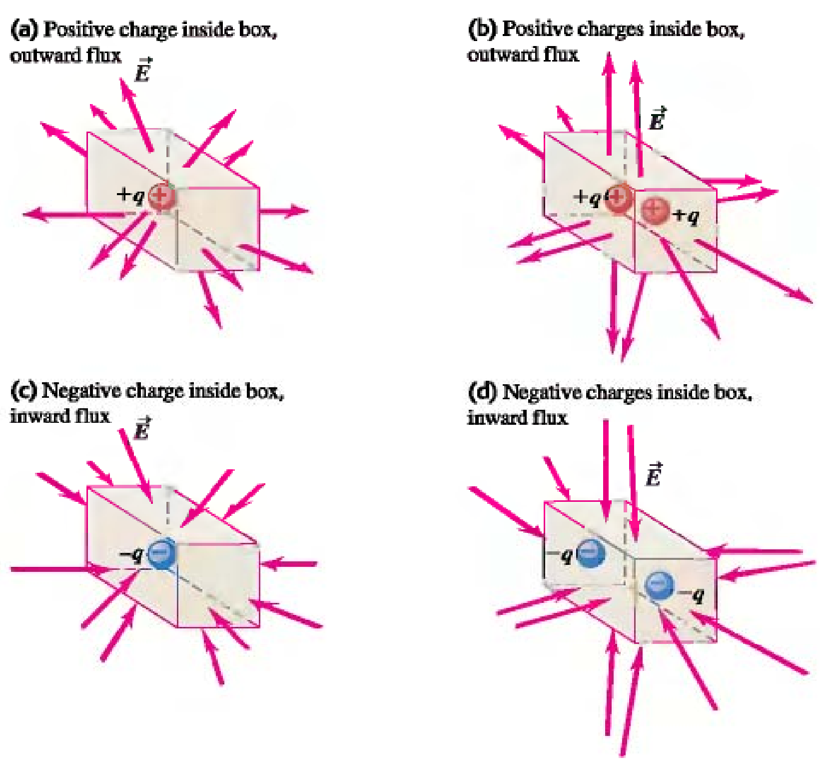
11.7 Electric Flux

22.1 How can you measure the charge inside a box without opening it?



22.2 The electric field on the surface of boxes containing (a) a single positive point charge, (b) two positive point charges, (c) a single negative point charge, or (d) two negative point charges.



Electric Flux and Enclosed Charge

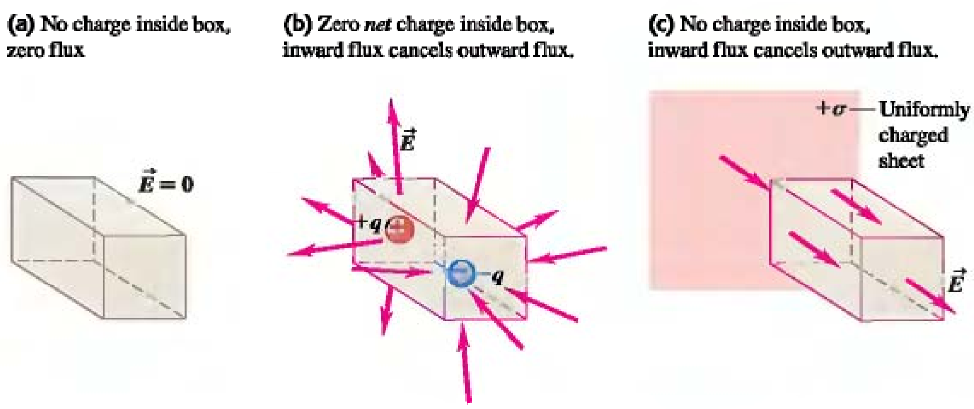
In Section 21.4 we mentioned the analogy between electric-field vectors and the velocity vectors of a fluid in motion. This analogy can be helpful, even though an electric field does not actually "flow." Using this analogy, in Figs. 22.2a and 22.2b, in which the electric field vectors point out of the surface, we say that there is an outward **electric flux.** (The word "flux" comes from a Latin word meaning "flow.") In Figs. 22.2c and 22.2d the $\vec{E}$ vectors point into the surface, and the electric flux is *inward.*

Figure 22.2 suggests a simple relationship: Positive charge inside the box goes with an outward electric flux through the box's surface, and negative charge inside goes with an inward electric flux. What happens if there is *zero* charge inside the box? In Fig. 22.3a the box is empty and $\vec{E} = \mathbf{0}$ everywhere, so there is no electric flux into or out of the box. In Fig. 22.3b, one positive and one negative point charge of equal magnitude are enclosed within the box, so the *net* charge inside the box is zero. There is an electric field, but it "flows into" the box on half of its surface and "flows out of" the box on the other half. Hence there is no *net* electric flux into or out of the box.

The box is again empty in Fig. 22.3c. However, there is charge present *outside* the box; the box has been placed with one end parallel to a uniformly charged infinite sheet, which produces a uniform electric field perpendicular to the sheet (as we learned in Example 21.12 of Section 21.5). On one end of the box, $\vec{E}$ points into the box; on the opposite end, $\vec{E}$ points out of the box; and on the sides, $\vec{E}$ is parallel to the surface and so points neither into nor out of the box. As in Fig. 22.3b, the inward electric flux on one part of the box exactly compensates for the outward electric flux on the other part. So in all of the cases shown in Fig. 22.3, there is no *net* electric flux through the surface of the box, and no *net* charge is enclosed in the box.

Figures 22.2 and 22.3 demonstrate a connection between the *sign* (positive, negative, or zero) of the *net* charge enclosed by a closed surface and the sense (outward, inward, or none) of the net electric flux through the surface. There is also a connection between the *magnitude* of the net charge inside the closed surface and the *strength* of the net "flow" of $\vec{E}$ over the surface. In both Figs. 22.4a and 22.4b there is a single point charge inside the box, but in Fig. 22.4b the magnitude of the charge is twice as great, and so $\vec{E}$ is everywhere twice as great in magnitude as in Fig. 22.4a. If we keep in mind the fluid-flow analogy, this means that the net outward electric flux is also twice as great in Fig. 22.4b as in Fig. 22.4a. This suggests that the net electric flux through the surface of the box is *directly proportional* to the magnitude of the net charge enclosed by the box.

22.3 Three cases in which there is zero *net* charge inside a box and no net electric flux through the surface of the box. (a) An empty box with $\vec{E} = \mathbf{0}$. (b) A box containing one positive and one equal-magnitude negative point charge. (c) An empty box immersed in a uniform electric field.



This conclusion is independent of the size of the box. In Fig. 22.4c the point charge $+q$ is enclosed by a box with twice the linear dimensions of the box in Fig.22.4a. The magnitude of the electric field of a point charge decreases with distance according to $1/r^2$, so the average magnitude of $\vec{E}$ on each face of the large box in Fig. 22.4c is just $\frac{1}{4}$ of the average magnitude on the corresponding face in Fig. 22.4a. But each face of the large box has exactly four times the area of the corresponding face of the small box. Hence the outward electric flux is the *same* for the two boxes if we *define* electric flux as follows: For each face of the box, take the product of the average perpendicular component of $\vec{E}$ and the area of that face; then add up the results from all faces of the box. With this definition the net electric flux due to a single point charge inside the box is independent of the size of the box and depends only on the net charge inside the box.

We have seen that there is a relationship between the net amount of charge inside a closed surface and the electric flux through that surface. For the special cases of a closed surface in the shape of a rectangular box and charge distributions made up of point charges or infinite charged sheets, we have found:

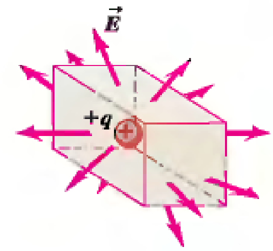
1. Whether there is a net outward or inward electric flux through a closed surface depends on the sign of the enclosed charge.
2. Charges *outside* the surface do not give a net electric flux through the surface.
3. The net electric flux is directly proportional to the net amount of charge enclosed within the surface but is otherwise independent of the size of the closed surface.

These observations are a qualitative statement of *Gauss's law.*

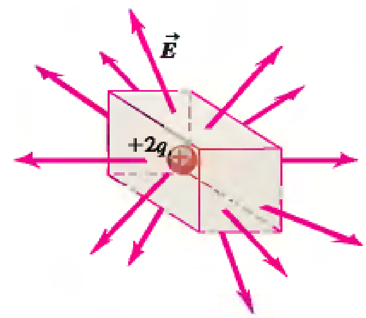
Do these observations hold true for other kinds of charge distributions and for closed surfaces of arbitrary shape? The answer to these questions will prove to be yes. But to explain why this is so, we need a precise mathematical statement of what we mean by electric flux. This is developed in the next section.

22.4 (a) A box enclosing a positive point charge $+q$. (b) Doubling the charge causes the magnitude of $\vec{E}$ to double, and it doubles the electric flux through the surface. (c) If the charge stays the same but the dimensions of the box are doubled, the flux stays the same. The magnitude of $\vec{E}$ on the surface decreases by a factor of $\frac{1}{4}$, but the area through which $\vec{E}$ "flows" increases by a factor of 4.

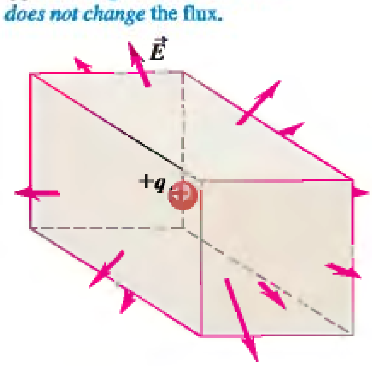
(a) A box containing a charge



(b) Doubling the enclosed charge doubles the flux.



(c) Doubling the box dimensions *does not change* the flux.



Test Your Understanding of Section 22.1 If all of the dimensions of the box in Fig. 22.2a are increased by a factor of 3, what effect will this change have on the electric flux through the box? (i) The flux will be $3^2 = 9$ times greater; (ii) the flux will be 3 times greater; (iii) the flux will be unchanged; (iv) the flux will be $\left(\frac{1}{3}\right)$ as great; (v) the flux will be $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$ as great; (vi) not enough information is given to decide.

22.2 Calculating Electric Flux

In the preceding section we introduced the concept of *electric flux*. Qualitatively, the electric flux through a surface is a description of whether the electric field $\vec{E}$ points into or out of the surface. We used this to give a rough qualitative statement of Gauss's law: The net electric flux through a closed surface is directly proportional to the net charge inside that surface. To be able to make full use of this law, we need to know how to *calculate* electric flux. To do this, let's again make use of the analogy between an electric field $\vec{E}$ and the field of velocity vectors $\vec{v}$ in a flowing fluid. (Again, keep in mind that this is only an analogy; an electric field is *not* a flow.)

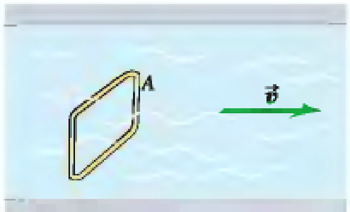
Flux: Fluid-Flow Analogy

Figure 22.5 shows a fluid flowing steadily from left to right. Let's examine the volume flow rate dV/dt (in, say, cubic meters per second) through the wire rectangle with area A. When the area is perpendicular to the flow velocity $\vec{v}$ (Fig. 22.5a) and the flow velocity is the same at all points in the fluid, the volume flow rate dV/dt is the area A multiplied by the flow speed v:

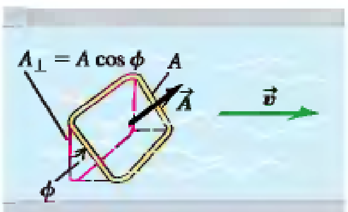
$$\frac{dV}{dt} = vA$$

22.5 The volume flow rate of fluid through the wire rectangle **(a)** is vA when the area of the rectangle is perpendicular to $\vec{v}$ and **(b)** is $vA\cos\phi$ when the rectangle is tilted at an angle ϕ.

(a) A wire rectangle in a fluid



(b) The wire rectangle tilted by an angle ϕ



When the rectangle is tilted at an angle ϕ (Fig. 22.5b) so that its face is not perpendicular to $\vec{v}$, the area that counts is the silhouette area that we see when we look in the direction of $\vec{v}$. This area, which is outlined in red and labeled $A_\perp$ in Fig. 22.5b, is the *projection* of the area A onto a surface perpendicular to $\vec{v}$. Two sides of the projected rectangle have the same length as the original one, but the other two are foreshortened by a factor of $\cos\phi$, so the projected area $A_\perp$ is equal to $A\cos\phi$. Then the volume flow rate through A is

$$\frac{dV}{dt} = vA\cos\phi$$

If $\phi = 90°$, $dV/dt = 0$; the wire rectangle is edge-on to the flow, and no fluid passes through the rectangle.

Also, $v\cos\phi$ is the component of the vector $\vec{v}$ perpendicular to the plane of the area A. Calling this component $v_\perp$, we can rewrite the volume flow rate as

$$\frac{dV}{dt} = v_\perp A$$

We can express the volume flow rate more compactly by using the concept of *vector area* $\vec{A}$, a vector quantity with magnitude A and a direction perpendicular to the plane of the area we are describing. The vector area $\vec{A}$ describes both the size of an area and its orientation in space. In terms of $\vec{A}$, we can write the volume flow rate of fluid through the rectangle in Fig. 22.5b as a scalar (dot) product:

$$\frac{dV}{dt} = \vec{v}\cdot\vec{A}$$

Flux of a Uniform Electric Field

Using the analogy between electric field and fluid flow, we now define electric flux in the same way as we have just defined the volume flow rate of a fluid; we simply replace the fluid velocity $\vec{v}$ by the electric field $\vec{E}$. The symbol that we use for electric flux is Φ_E (the capital Greek letter phi; the subscript E is a reminder that this is *electric* flux). Consider first a flat area A perpendicular to a uniform electric field $\vec{E}$ (Fig. 22.6a). We define the electric flux through this area to be the product of the field magnitude E and the area A:

$$\Phi_E = EA$$

Roughly speaking, we can picture Φ_E in terms of the field lines passing through A. Increasing the area means that more lines of $\vec{E}$ pass through the area, increasing the flux; stronger field means more closely spaced lines of $\vec{E}$ and therefore more lines per unit area, so again the flux increases.

If the area A is flat but not perpendicular to the field $\vec{E}$, then fewer field lines pass through it. In this case the area that counts is the silhouette area that we see when looking in the direction of $\vec{E}$. This is the area $A_\perp$ in Fig. 22.6b and is equal to $A\cos\phi$ (compare to Fig. 22.5b). We generalize our definition of electric flux for a uniform electric field to

$$\Phi_E = EA\cos\phi \qquad \text{(electric flux for uniform } \vec{E}\text{, flat surface)} \qquad (22.1)$$

Since $E\cos\phi$ is the component of $\vec{E}$ perpendicular to the area, we can rewrite Eq. (22.1) as

$$\Phi_E = E_\perp A \qquad \text{(electric flux for uniform } \vec{E}\text{, flat surface)} \qquad (22.2)$$

In terms of the vector area $\vec{A}$ perpendicular to the area, we can write the electric flux as the scalar product of $\vec{E}$ and $\vec{A}$:

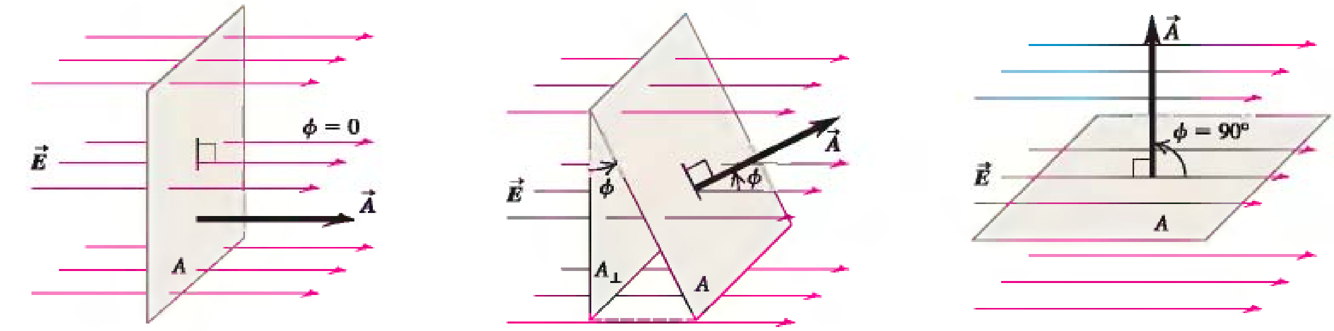
$$\Phi_E = \vec{E}\cdot\vec{A} \qquad \text{(electric flux for uniform } \vec{E}\text{, flat surface)} \qquad (22.3)$$

22.6 A flat surface in a uniform electric field. The electric flux Φ_E through the surface equals the scalar product of the electric field $\vec{E}$ and the area vector $\vec{A}$.

(a) Surface is face-on to electric field:
- $\vec{E}$ and $\vec{A}$ are parallel (the angle between $\vec{E}$ and $\vec{A}$ is $\phi = 0$).
- The flux $\Phi_E = \vec{E} \cdot \vec{A} = EA$.

(b) Surface is tilted from a face-on orientation by an angle ϕ:
- The angle between $\vec{E}$ and $\vec{A}$ is ϕ.
- The flux $\Phi_E = \vec{E} \cdot \vec{A} = EA \cos \phi$.

(c) Surface is edge-on to electric field:
- $\vec{E}$ and $\vec{A}$ are perpendicular (the angle between $\vec{E}$ and $\vec{A}$ is $\phi = 90°$).
- The flux $\Phi_E = \vec{E} \cdot \vec{A} = EA \cos 90° = 0$.



Equations (22.1), (22.2), and (22.3) express the electric flux for a *flat* surface and a *uniform* electric field in different but equivalent ways. The SI unit for electric flux is $1\ \mathrm{N \cdot m^2/C}$. Note that if the area is edge-on to the field, $\vec{E}$ and $\vec{A}$ are perpendicular and the flux is zero (Fig. 22.6c).

We can represent the direction of a vector area $\vec{A}$ by using a *unit vector* $\hat{n}$ perpendicular to the area; $\hat{n}$ stands for "normal." Then

$$\vec{A} = A\hat{n} \tag{22.4}$$

A surface has two sides, so there are two possible directions for $\hat{n}$ and $\vec{A}$. We must always specify which direction we choose. In Section 22.1 we related the charge inside a *closed* surface to the electric flux through the surface. With a closed surface we will always choose the direction of $\hat{n}$ to be *outward*, and we will speak of the flux *out of* a closed surface. Thus what we called "outward electric flux" in Section 22.1 corresponds to a *positive* value of Φ_E, and what we called "inward electric flux" corresponds to a *negative* value of Φ_E.

Flux of a Nonuniform Electric Field

What happens if the electric field $\vec{E}$ isn't uniform but varies from point to point over the area A ? Or what if A is part of a curved surface? Then we divide A into many small elements dA, each of which has a unit vector $\hat{n}$ perpendicular to it and a vector area $d\vec{A} = \hat{n}\,dA$. We calculate the electric flux through each element and integrate the results to obtain the total flux:

$$\Phi_E = \int E\cos\phi\, dA = \int E_\perp\, dA = \int \vec{E} \cdot d\vec{A} \quad \text{(general definition of electric flux)} \tag{22.5}$$

We call this integral the **surface integral** of the component $E_\perp$ over the area, or the surface integral of $\vec{E} \cdot d\vec{A}$. The various forms of the integral all express the same thing in different terms. In specific problems, one form is sometimes more convenient than another. Example 22.3 at the end of this section illustrates the use of Eq. (22.5).

In Eq. (22.5) the electric flux $\int E_\perp\, dA$ is equal to the *average* value of the perpendicular component of the electric field, multiplied by the area of the surface. This is the same definition of electric flux that we were led to in Section 22.1, now expressed more mathematically. In the next section we will see the connection between the total electric flux through *any* closed surface, no matter what its shape, and the amount of charge enclosed within that surface.

Example 22.1 Electric flux through a disk

A disk with radius 0.10 m is oriented with its normal unit vector $\hat{n}$ at an angle of 30° to a uniform electric field $\vec{E}$ with magnitude 2.0×10^3 N/C (Fig. 22.7). (Since this isn't a closed surface, it has no "inside" or "outside." That's why we have to specify the direction of $\hat{n}$ in the figure.) (a) What is the electric flux through the disk? (b) What is the flux through the disk if it is turned so that its normal is perpendicular to $\vec{E}$? (c) What is the flux through the disk if its normal is parallel to $\vec{E}$?

SOLUTION

IDENTIFY: This problem is about a flat surface in a uniform electric field, so we can apply the ideas of this section.

SET UP: The orientation of the disk is like that of the rectangle in Fig. 22.6b. We calculate the electric flux using Eq. (22.1).

EXECUTE: (a) The area is $A = \pi(0.10\text{ m})^2 = 0.0314\text{ m}^2$ and the angle between $\vec{E}$ and $\vec{A} = A\hat{n}$ is $\phi = 30°$, so

$$\begin{aligned}\Phi_E &= EA\cos\phi = (2.0 \times 10^3\text{ N/C})(0.0314\text{ m}^2)(\cos 30°)\\ &= 54\text{ N}\cdot\text{m}^2/\text{C}\end{aligned}$$

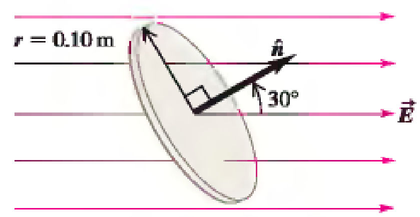
(b) The normal to the disk is now perpendicular to $\vec{E}$, so $\phi = 90°$, $\cos\phi = 0$, and $\Phi_E = 0$. There is no flux through the disk.

(c) The normal to the disk is parallel to $\vec{E}$, so $\phi = 0$, $\cos\phi = 1$, and the flux has its maximum possible value. From Eq. (22.1),

$$\begin{aligned}\Phi_E &= EA\cos\phi = (2.0 \times 10^3\text{ N/C})(0.0314\text{ m}^2)(1)\\ &= 63\text{ N}\cdot\text{m}^2/\text{C}\end{aligned}$$

EVALUATE: As a check on our results, note that the answer to part (a) is smaller than the answer to part (c). Is this as it should be?

22.7 The electric flux Φ_E through a disk depends on the angle between its normal $\hat{n}$ and the electric field $\vec{E}$.



Example 22.2 Electric flux through a cube

A cube of side L is placed in a region of uniform electric field $\vec{E}$. Find the electric flux through each face of the cube and the total flux through the cube when (a) it is oriented with two of its faces perpendicular to the field $\vec{E}$, as in Fig. 22.8a; and (b) when the cube is turned by an angle θ, as in Fig. 22.8b.

SOLUTION

IDENTIFY: In this problem we are to find the electric flux through each face of the cube as well as the total flux (the sum of the fluxes through the six faces).

SET UP: Since $\vec{E}$ is uniform and each of the six faces of the cube is a flat surface, we find the flux through each face using Eqs. (22.3) and (22.4). We then calculate the total flux through the cube by adding the six individual fluxes.

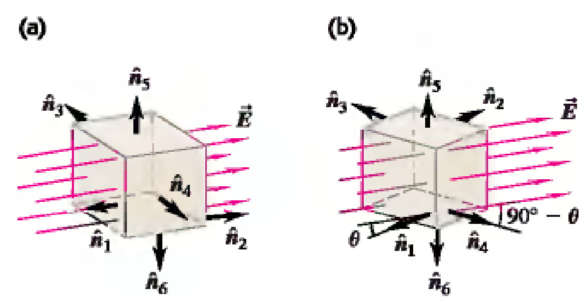
EXECUTE: (a) The unit vectors for each face ($\hat{n}_1$ through $\hat{n}_6$) are shown in the figure; the direction of each unit vector is *outward* from the closed surface of the cube. The angle between $\vec{E}$ and $\hat{n}_1$ is 180°; the angle between $\vec{E}$ and $\hat{n}_2$ is 0°; and the angle between $\vec{E}$ and each of the other four unit vectors is 90°. Each face of the cube has area L^2, so the fluxes through each of the faces are

$$\begin{aligned}\Phi_{E1} &= \vec{E}\cdot\hat{n}_1 A = EL^2\cos 180° = -EL^2\\ \Phi_{E2} &= \vec{E}\cdot\hat{n}_2 A = EL^2\cos 0° = +EL^2\\ \Phi_{E3} &= \Phi_{E4} = \Phi_{E5} = \Phi_{E6} = EL^2\cos 90° = 0\end{aligned}$$

The flux is negative on face 1, where $\vec{E}$ is directed into the cube, and positive on face 2, where $\vec{E}$ is directed out of the cube. The *total* flux through the cube is the sum of the fluxes through the six faces:

$$\begin{aligned}\Phi_E &= \Phi_{E1} + \Phi_{E2} + \Phi_{E3} + \Phi_{E4} + \Phi_{E5} + \Phi_{E6}\\ &= -EL^2 + EL^2 + 0 + 0 + 0 + 0 = 0\end{aligned}$$

22.8 Electric flux of a uniform field $\vec{E}$ through a cubical box of side L in two orientations.



(b) The fluxes through faces 1 and 3 are negative, since $\vec{E}$ is directed into those faces; the field is directed out of faces 2 and 4, so the fluxes through those faces are positive. We find

$$\begin{aligned}\Phi_{E1} &= \vec{E}\cdot\hat{n}_1 A = EL^2\cos(180° - \theta) = -EL^2\cos\theta\\ \Phi_{E2} &= \vec{E}\cdot\hat{n}_2 A = +EL^2\cos\theta\\ \Phi_{E3} &= \vec{E}\cdot\hat{n}_3 A = EL^2\cos(90° + \theta) = -EL^2\sin\theta\\ \Phi_{E4} &= \vec{E}\cdot\hat{n}_4 A = EL^2\cos(90° - \theta) = +EL^2\sin\theta\\ \Phi_{E5} &= \Phi_{E6} = EL^2\cos 90° = 0\end{aligned}$$

The total flux $\Phi_E = \Phi_{E1} + \Phi_{E2} + \Phi_{E3} + \Phi_{E4} + \Phi_{E5} + \Phi_{E6}$ through the surface of the cube is again zero.

EVALUATE: It's no surprise that the total flux is zero for both orientations. We came to this same conclusion in our discussion of Fig. 22.3c in Section 22.1. There we observed that there was zero net flux of a uniform electric field through a closed surface that contains no electric charge.

Example 22.3 Electric flux through a sphere

A positive point charge $q = 3.0\ \mu\text{C}$ is surrounded by a sphere with radius 0.20 m centered on the charge (Fig. 22.9). Find the electric flux through the sphere due to this charge.

SOLUTION

IDENTIFY: Here the surface is not flat and the electric field is not uniform, so we must use the general definition of electric flux.

SET UP: We use Eq. (22.5) to calculate the electric flux (our target variable). Because the sphere is centered on the point charge, at any point on the spherical surface, $\vec{E}$ is directed out of the sphere perpendicular to the surface. The positive direction for both $\hat{n}$ and $E_\perp$ is outward, so $E_\perp = E$ and the flux through a surface element dA is $\vec{E} \cdot d\vec{A} = E\,dA$. This greatly simplifies the integral in Eq. (22.5).

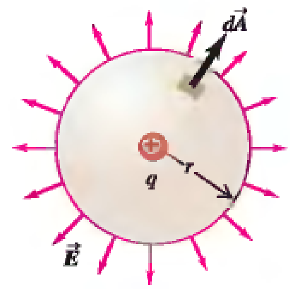
EXECUTE: At any point on the sphere the magnitude of $\vec{E}$ is

$$E = \frac{q}{4\pi\epsilon_0 r^2} = (9.0 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\frac{3.0 \times 10^{-6}\ \text{C}}{(0.20\ \text{m})^2}$$
$$= 6.75 \times 10^5\ \text{N/C}$$

Because E is the same at every point, it can be taken outside the integral $\Phi_E = \int E\,dA$ in Eq. (22.5). What remains is the integral $\int dA$, which is just the total area $A = 4\pi r^2$ of the spherical surface. Thus the total flux out of the sphere is

$$\Phi_E = EA = (6.75 \times 10^5\ \text{N/C})(4\pi)(0.20\ \text{m})^2$$
$$= 3.4 \times 10^5\ \text{N}\cdot\text{m}^2/\text{C}$$

22.9 Electric flux through a sphere centered on a point charge.



EVALUATE: Notice that we divided by $r^2 = (0.20\ \text{m})^2$ to find E, then multiplied by $r^2 = (0.20\ \text{m})^2$ to find Φ_E; hence the radius r of the sphere cancels out of the result for Φ_E. We would have obtained the same flux with a sphere of radius 2.0 m or 200 m. We came to essentially the same conclusion in our discussion of Fig. 22.4 in Section 22.1, where we considered rectangular closed surfaces of two different sizes enclosing a point charge. There we found that the flux of $\vec{E}$ was independent of the size of the surface; the same result holds true for a spherical surface. Indeed, the flux through *any* surface enclosing a single point charge is independent of the shape or size of the surface, as we'll soon see.

Test Your Understanding of Section 22.2 Rank the following surfaces in order from most positive to most negative electric flux. (i) a flat rectangular surface with vector area $\vec{A} = (6.0\ \text{m}^2)\hat{\imath}$ in a uniform electric field $\vec{E} = (4.0\ \text{N/C})\hat{\jmath}$; (ii) a flat circular surface with vector area $\vec{A} = (3.0\ \text{m}^2)\hat{\jmath}$ in a uniform electric field $\vec{E} = (4.0\ \text{N/C})\hat{\imath} + (2.0\ \text{N/C})\hat{\jmath}$; (iii) a flat square surface with vector area $\vec{A} = (3.0\ \text{m}^2)\hat{\imath} + (7.0\ \text{m}^2)\hat{\jmath}$ in a uniform electric field $\vec{E} = (4.0\ \text{N/C})\hat{\imath} - (2.0\ \text{N/C})\hat{\jmath}$; (iv) a flat oval surface with vector area $\vec{A} = (3.0\ \text{m}^2)\hat{\imath} - (7.0\ \text{m}^2)\hat{\jmath}$ in a uniform electric field $\vec{E} = (4.0\ \text{N/C})\hat{\imath} - (2.0\ \text{N/C})\hat{\jmath}$.

MP

22.3 Gauss's Law

Gauss's law is an alternative to Coulomb's law. While completely equivalent to Coulomb's law, Gauss's law provides a different way to express the relationship between electric charge and electric field. It was formulated by Carl Friedrich Gauss (1777–1855), one of the greatest mathematicians of all time. Many areas of mathematics bear the mark of his influence, and he made equally significant contributions to theoretical physics (Fig. 22.10).

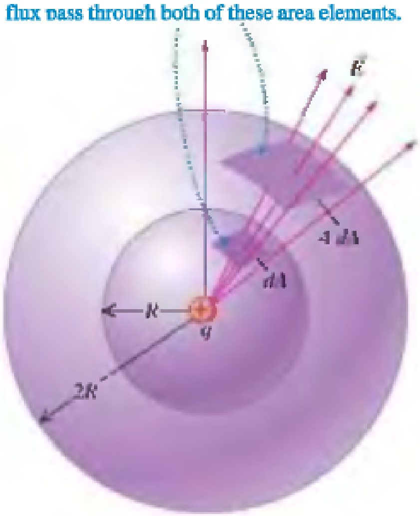
22.10 Carl Friedrich Gauss helped develop several branches of mathematics, including differential geometry, real analysis, and number theory. The "bell curve" of statistics is one of his inventions. Gauss also made state-of-the-art investigations of the earth's magnetism and calculated the orbit of the first asteroid to be discovered.

Point Charge Inside a Spherical Surface

Gauss's law states that the total electric flux through any closed surface (a surface enclosing a definite volume) is proportional to the total (net) electric charge inside the surface. In Section 22.1 we observed this relationship qualitatively for certain special cases; now we'll develop it more rigorously. We'll start with the field of a single positive point charge q. The field lines radiate out equally in all directions. We place this charge at the center of an imaginary spherical surface with radius R. The magnitude E of the electric field at every point on the surface is given by

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}$$

22.11 Projection of an element of area dA of a sphere of radius R onto a concentric sphere of radius $2R$. The projection multiplies each linear dimension by 2, so the area element on the larger sphere is 4 dA.



At each point on the surface, $\vec{E}$ is perpendicular to the surface, and its magnitude is the same at every point, just as in Example 22.3 (Section 22.2). The total electric flux is the product of the field magnitude E and the total area $A = 4\pi R^2$ of the sphere:

$$\Phi_E = EA = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}(4\pi R^2) = \frac{q}{\epsilon_0} \tag{22.6}$$

The flux is independent of the radius R of the sphere. It depends only on the charge q enclosed by the sphere.

We can also interpret this result in terms of field lines. Figure 22.11 shows two spheres with radii R and $2R$ centered on the point charge q. Every field line that passes through the smaller sphere also passes through the larger sphere, so the total flux through each sphere is the same.

What is true of the entire sphere is also true of any portion of its surface. In Fig.22.11 an area dA is outlined on the sphere of radius R and then projected onto the sphere of radius $2R$ by drawing lines from the center through points on the boundary of dA. The area projected on the larger sphere is clearly 4 dA. But since the electric field due to a point charge is inversely proportional to r^2, the field magnitude is $\frac{1}{4}$ as great on the sphere of radius $2R$ as on the sphere of radius R. Hence the electric flux is the same for both areas and is independent of the radius of the sphere.

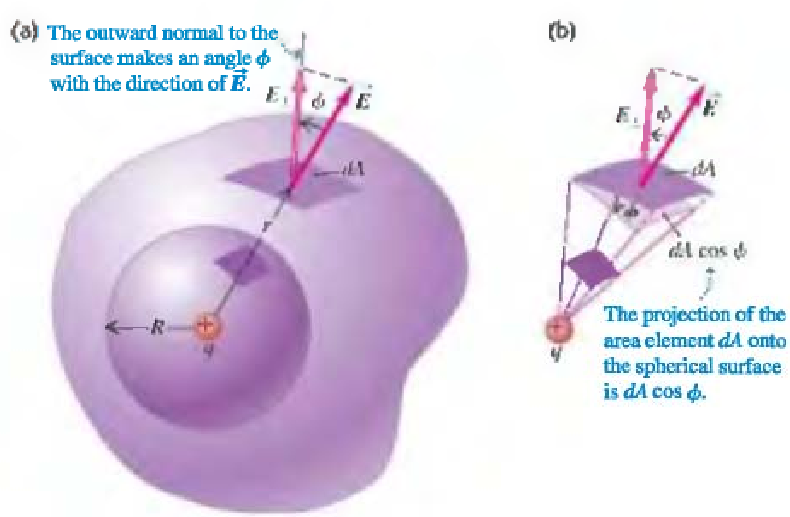
Point Charge Inside a Nonspherical Surface

This projection technique shows us how to extend this discussion to nonspherical surfaces. Instead of a second sphere, let us surround the sphere of radius R by a surface of irregular shape, as in Fig. 22.12a. Consider a small element of area dA on the irregular surface; we note that this area is *larger* than the corresponding element on a spherical surface at the same distance from q. If a normal to dA makes an angle ϕ with a radial line from q, two sides of the area projected onto the spherical surface are foreshortened by a factor $\cos\phi$ (Fig. 22.12b). The other two sides are unchanged. Thus the electric flux through the spherical surface element is equal to the flux $E\,dA\cos\phi$ through the corresponding irregular surface element.

We can divide the entire irregular surface into elements dA, compute the electric flux $E\,dA\cos\phi$ for each, and sum the results by integrating, as in Eq. (22.5). Each of the area elements projects onto a corresponding spherical surface element. Thus the *total* electric flux through the irregular surface, given by any of the forms of Eq. (22.5), must be the same as the total flux through a sphere, which Eq.(22.6) shows is equal to q/ϵ_0. Thus, for the irregular surface,

$$\Phi_E = \oint \vec{E}\cdot d\vec{A} = \frac{q}{\epsilon_0} \tag{22.7}$$

22.12 Calculating the electric flux through a nonspherical surface.



Equation (22.7) holds for a surface of *any* shape or size, provided only that it is a *closed* surface enclosing the charge q. The circle on the integral sign reminds us that the integral is always taken over a *closed* surface.

The area elements $d\vec{A}$ and the corresponding unit vectors $\hat{n}$ always point *out of* the volume enclosed by the surface. The electric flux is then positive in areas where the electric field points out of the surface and negative where it points inward. Also, $E_\perp$ is positive at points where $\vec{E}$ points out of the surface and negative at points where $\vec{E}$ points into the surface.

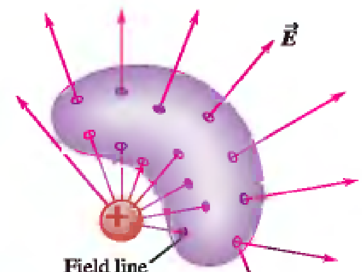
If the point charge in Fig. 22.12 is negative, the $\vec{E}$ field is directed radially *inward;* the angle ϕ is then greater than 90°, its cosine is negative, and the integral in Eq. (22.7) is negative. But since q is also negative, Eq. (22.7) still holds.

For a closed surface enclosing *no* charge,

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = 0$$

This is the mathematical statement that when a region contains no charge, any field lines caused by charges *outside* the region that enter on one side must leave again on the other side. (In Section 22.1 we came to the same conclusion by considering the special case of a rectangular box in a uniform field.) Figure 22.13 illustrates this point. ***Electric field lines can begin or end inside a region of space only when there is charge in that region.***

22.13 A point charge *outside* a closed surface that encloses no charge. If an electric field line from the external charge enters the surface at one point, it must leave at another.



General Form of Gauss's Law

Now comes the final step in obtaining the general form of Gauss's law. Suppose the surface encloses not just one point charge q but several charges q_1, q_2, q_3, ... The total (resultant) electric field $\vec{E}$ at any point is the vector sum of the $\vec{E}$ fields of the individual charges. Let Q_{encl} be the *total* charge enclosed by the surface: $Q_{encl} = q_1 + q_2 + q_3 + \cdots$. Also let $\vec{E}$ be the *total* field at the position of the surface area element $d\vec{A}$, and let $E_\perp$ be its component perpendicular to the plane of that element (that is, parallel to $d\vec{A}$). Then we can write an equation like Eq. (22.7) for each charge and its corresponding field and add the results. When we do, we obtain the general statement of Gauss's law:

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0} \quad \text{(Gauss's law)} \tag{22.8}$$

The total electric flux through a closed surface is equal to the total (net) electric charge inside the surface, divided by ϵ_0.

CAUTION **Gaussian surfaces are imaginary** Remember that the closed surface in Gauss's law is *imaginary;* there need not be any material object at the position of the surface. We often refer to a closed surface used in Gauss's law as a **Gaussian surface.**

Using the definition of Q_{encl} and the various ways to express electric flux given in Eq. (22.5), we can express Gauss's law in the following equivalent forms:

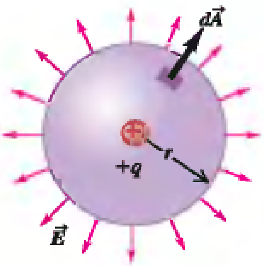
$$\Phi_E = \oint E\cos\phi \, dA = \oint E_\perp \, dA = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0} \quad \text{(various forms of Gauss's law)} \tag{22.9}$$

As in Eq. (22.5), the various forms of the integral all express the same thing, the total electric flux through the Gaussian surface, in different terms. One form is sometimes more convenient than another.

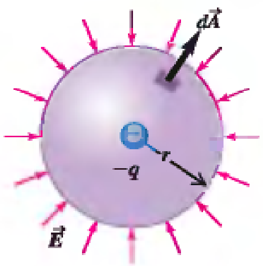
As an example, Fig. 22.14a shows a spherical Gaussian surface of radius r around a positive point charge $+q$. The electric field points out of the Gaussian surface, so at every point on the surface $\vec{E}$ is in the same direction as $d\vec{A}$, $\phi = 0$, and

22.14 Spherical Gaussian surfaces around (a) a positive point charge and (b) a negative point charge.

(a) Gaussian surface around positive charge: positive (outward) flux



(b) Gaussian surface around negative charge: negative (inward) flux



$E_\perp$ is equal to the field magnitude $E = q/4\pi\epsilon_0 r^2$. Since E is the same at all points on the surface, we can take it outside the integral in Eq. (22.9). Then the remaining integral is $\int dA = A = 4\pi r^2$, the area of the sphere. Hence Eq. (22.9) becomes

$$\Phi_E = \oint E_\perp\, dA = \oint \frac{q}{4\pi\epsilon_0 r^2}\, dA = \frac{q}{4\pi\epsilon_0 r^2}\oint dA = \frac{q}{4\pi\epsilon_0 r^2}\, 4\pi r^2 = \frac{q}{\epsilon_0}$$

The enclosed charge Q_{encl} is just the charge $+q$, so this agrees with Gauss's law. If the Gaussian surface encloses a *negative* point charge as in Fig. 22.14b, then $\vec{E}$ points *into* the surface at each point in the direction opposite $d\vec{A}$. Then $\phi = 180°$ and $E_\perp$ is equal to the negative of the field magnitude: $E_\perp = -E = -|-q|/4\pi\epsilon_0 r^2 = -q/4\pi\epsilon_0 r^2$. Equation (22.9) then becomes

$$\Phi_E = \oint E_\perp\, dA = \oint \left(\frac{-q}{4\pi\epsilon_0 r^2}\right) dA = \frac{-q}{4\pi\epsilon_0 r^2}\oint dA = \frac{-q}{4\pi\epsilon_0 r^2}\, 4\pi r^2 = \frac{-q}{\epsilon_0}$$

This again agrees with Gauss's law because the enclosed charge in Fig. 22.14b is $Q_{\text{encl}} = -q$.

In Eqs. (22.8) and (22.9), Q_{encl} is always the algebraic sum of all the positive and negative charges enclosed by the Gaussian surface, and $\vec{E}$ is the *total* field at each point on the surface. Also note that in general, this field is caused partly by charges inside the surface and partly by charges outside. But as Fig. 22.13 shows, the outside charges do *not* contribute to the total (net) flux through the surface. So Eqs. (22.8) and (22.9) are correct even when there are charges outside the surface that contribute to the electric field at the surface. When $Q_{\text{encl}} = 0$, the total flux through the Gaussian surface must be zero, even though some areas may have positive flux and others may have negative flux (see Fig. 22.3b).

Gauss's law is the definitive answer to the question we posed at the beginning of Section 22.1: "If the electric field pattern is known in a given region, what can we determine about the charge distribution in that region?" It provides a relationship between the electric field on a closed surface and the charge distribution within that surface. But in some cases we can use Gauss's law to answer the reverse question: "If the charge distribution is known, what can we determine about the electric field that the charge distribution produces?" Gauss's law may seem like an unappealing way to address this question, since it may look as though evaluating the integral in Eq. (22.8) is a hopeless task. Sometimes it is, but other times it is surprisingly easy. Here's an example in which *no* integration is involved at all; we'll work out several more examples in the next section.

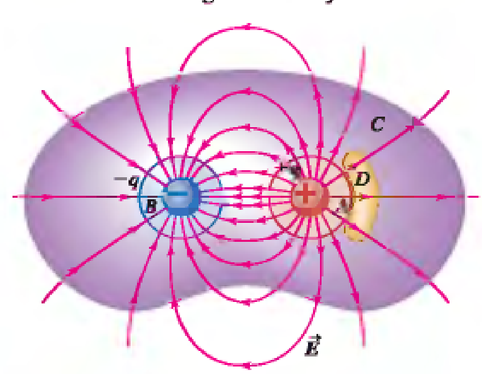
Conceptual Example 22.4 Electric flux and enclosed charge

Figure 22.15 shows the field produced by two point charges $+q$ and $-q$ of equal magnitude but opposite sign (an electric dipole). Find the electric flux through each of the closed surfaces A, B, C, and D.

SOLUTION

The definition of electric flux given in Eq. (22.5) involves a surface integral, and so it might seem that integration is called for. But Gauss's law says that the total electric flux through a closed surface is equal to the total enclosed charge divided by ϵ_0. By inspection of Fig. 22.15, surface A (shown in red) encloses the positive charge, so $Q_{\text{encl}} = +q$; surface B (shown in blue) encloses the negative charge, so $Q_{\text{encl}} = -q$; surface C (shown in yellow), which encloses *both* charges, has $Q_{\text{encl}} = +q + (-q) = 0$; and surface D (shown in purple), which has no charges enclosed within it, also has $Q_{\text{encl}} = 0$. Hence without having to do any integration, we can

22.15 The net number of field lines leaving a closed surface is proportional to the total charge enclosed by that surface.



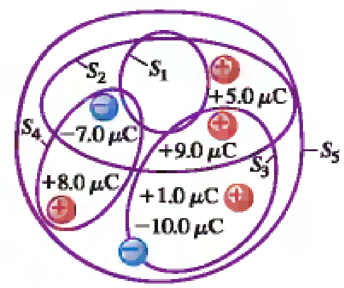
conclude that the total fluxes for the various surfaces are $\Phi_E = +q/\epsilon_0$ for surface A, $\Phi_E = -q/\epsilon_0$ for surface B, and $\Phi_E = 0$ for both surface C and surface D.

These results depend only on the charges enclosed within each Gaussian surface, not on the precise shapes of the surfaces. For example, compare surface C to the rectangular surface shown in Fig. 22.3b, which also encloses both charges of an electric dipole. In that case as well, we concluded that the net flux of $\vec{E}$ was zero; the inward flux on one part of the surface exactly compensates for the outward flux on the remainder of the surface.

We can draw similar conclusions by examining the electric field lines. Surface A encloses only the positive charge; in Fig. 22.15, 18 lines are depicted crossing A in an outward direction. Surface B encloses only the negative charge; it is crossed by these same 18 lines, but in an inward direction. Surface C encloses *both* charges. It is intersected by lines at 16 points; at 8 intersections the lines are outward, and at 8 they are inward. The *net* number of lines crossing in an outward direction is zero, and the net charge inside the surface is also zero. Surface D is intersected at 6 points; at 3 points the lines are outward, and at the other 3 they are inward. The net number of lines crossing in an outward direction and the total charge enclosed are both zero. There are points on the surfaces where $\vec{E}$ is not perpendicular to the surface, but this doesn't affect the counting of the field lines.

Test Your Understanding of Section 22.3 Figure 22.16 shows six point charges that all lie in the same plane. Five Gaussian surfaces—S_1, S_2, S_3, S_4, and S_5—each enclose part of this plane, and Fig. 22.16 shows the intersection of each surface with the plane. Rank these five surfaces in order of the electric flux through them, from most positive to most negative.

22.16 Five Gaussian surfaces and six point charges.



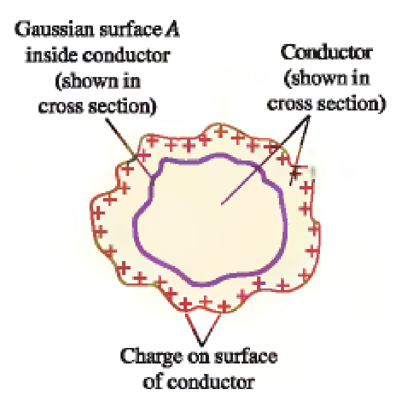
22.4 Applications of Gauss's Law

Gauss's law is valid for *any* distribution of charges and for *any* closed surface. Gauss's law can be used in two ways. If we know the charge distribution, and if it has enough symmetry to let us evaluate the integral in Gauss's law, we can find the field. Or if we know the field, we can use Gauss's law to find the charge distribution, such as charges on conducting surfaces.

In this section we present examples of both kinds of applications. As you study them, watch for the role played by the symmetry properties of each system. We will use Gauss's law to calculate the electric fields caused by several simple charge distributions; the results are collected in a table in the chapter summary.

In practical problems we often encounter situations in which we want to know the electric field caused by a charge distribution on a conductor. These calculations are aided by the following remarkable fact: *When excess charge is placed on a solid conductor and is at rest, it resides entirely on the surface, not in the interior of the material.* (By *excess* we mean charges other than the ions and free electrons that make up the neutral conductor.) Here's the proof. We know from Section 21.4 that in an electrostatic situation (with all charges at rest) the electric field $\vec{E}$ at every point in the interior of a conducting material is zero. If $\vec{E}$ were *not* zero, the excess charges would move. Suppose we construct a Gaussian surface inside the conductor, such as surface A in Fig. 22.17. Because $\vec{E} = 0$ everywhere on this surface, Gauss's law requires that the net charge inside the surface is zero. Now imagine shrinking the surface like a collapsing balloon until it encloses a region so small that we may consider it as a point P; then the charge at that point must be zero. We can do this anywhere inside the conductor, so *there can be no excess charge at any point within a solid conductor; any excess charge must reside on the conductor's surface.* (This result is for a *solid* conductor. In the next section we'll discuss what can happen if the conductor has cavities in its interior.) We will make use of this fact frequently in the examples that follow.

22.17 Under electrostatic conditions (charges not in motion), any excess charge on a solid conductor resides entirely on the conductor's surface.



Problem-Solving Strategy 22.1 Gauss's Law

IDENTIFY *the relevant concepts:* Gauss's law is most useful in situations where the charge distribution has spherical or cylindrical symmetry or is distributed uniformly over a plane. In these situations we determine the direction of $\vec{E}$ from the symmetry of the charge distribution. If we are given the charge distribution, we can use Gauss's law to find the magnitude of $\vec{E}$. Alternatively, if we are given the field, we can use Gauss's law to determine the details of the charge distribution. In either case, begin your analysis by asking the question: What is the symmetry?

SET UP *the problem* using the following steps:

1. Select the surface that you will use with Gauss's law. We often call it a *Gaussian surface.* If you are trying to find the field at a particular point, then that point must lie on your Gaussian surface.
2. The Gaussian surface does not have to be a real physical surface, such as a surface of a solid body. Often the appropriate surface is an imaginary geometric surface; it may be in empty space, embedded in a solid body, or both.
3. Usually you can evaluate the integral in Gauss's law (without using a computer) only if the Gaussian surface and the charge distribution have some symmetry property. If the charge distribution has cylindrical or spherical symmetry, choose the Gaussian surface to be a coaxial cylinder or a concentric sphere, respectively.

EXECUTE *the solution* as follows:

1. Carry out the integral in Eq. (22.9). This may look like a daunting task, but the symmetry of the charge distribution and your careful choice of a Gaussian surface make it straightforward.
2. Often you can think of the closed Gaussian surface as being made up of several separate surfaces, such as the sides and ends of a cylinder. The integral $\oint E_\perp \, dA$ over the entire closed surface is always equal to the sum of the integrals over all the separate surfaces. Some of these integrals may be zero, as in points 4 and 5 below.
3. If $\vec{E}$ is *perpendicular* (normal) at every point to a surface with area A, if it points *outward* from the interior of the surface, and if it also has the same *magnitude* at every point on the surface, then $E_\perp = E =$ constant, and $\int E_\perp \, dA$ over that surface is equal to EA. If instead $\vec{E}$ is perpendicular and *inward,* then $E_\perp = -E$ and $\int E_\perp \, dA = -EA$.
4. If $\vec{E}$ is *tangent* to a surface at every point, then $E_\perp = 0$ and the integral over that surface is zero.
5. If $\vec{E} = \mathbf{0}$ at every point on a surface, the integral is zero.
6. In the integral $\oint E_\perp \, dA$, $E_\perp$ is always the perpendicular component of the *total* electric field at each point on the closed Gaussian surface. In general, this field may be caused partly by charges within the surface and partly by charges outside it. Even when there is *no* charge within the surface, the field at points on the Gaussian surface is not necessarily zero. In that case, however, the *integral* over the Gaussian surface—that is, the total electric flux through the Gaussian surface—is always zero.
7. Once you have evaluated the integral, use Eq. (22.9) to solve for your target variable.

EVALUATE *your answer:* Often your result will be a *function* that describes how the magnitude of the electric field varies with position. Examine this function with a critical eye to see whether it makes sense.

Example 22.5 Field of a charged conducting sphere

We place positive charge q on a solid conducting sphere with radius R (Fig. 22.18). Find $\vec{E}$ at any point inside or outside the sphere.

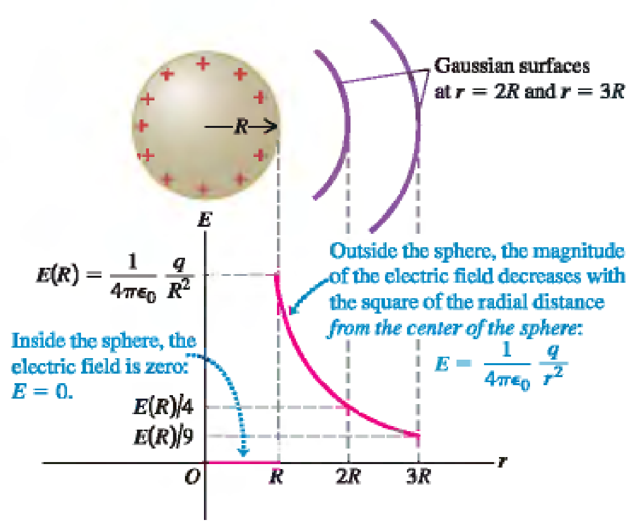
SOLUTION

IDENTIFY: As we discussed earlier in this section, all the charge must be on the surface of the sphere. The system has spherical symmetry.

SET UP: To take advantage of the symmetry, we take as our Gaussian surface an imaginary sphere of radius r centered on the conductor. To calculate the field outside the conductor, we take r to be greater than the conductor's radius R; to calculate the field inside, we take r to be less than R. In either case, the point where we want to calculate $\vec{E}$ lies on the Gaussian surface.

EXECUTE: The role of symmetry deserves careful discussion before we do any calculations. When we say that the system is spherically symmetric, we mean that if we rotate it through any angle about any axis through the center, the system after rotation is indistinguishable from the original unrotated system. The charge is free to move on the conductor, and there is nothing about the con-

22.18 Calculating the electric field of a conducting sphere with positive charge q. Outside the sphere, the field is the same as if all of the charge were concentrated at the center of the sphere.



ductor that would make it tend to concentrate more in some regions than others. So we conclude that the charge is distributed *uniformly* over the surface.

Symmetry also shows that the direction of the electric field must be *radial*, as shown in Fig. 22.18. If we again rotate the system, the field pattern of the rotated system must be identical to that of the original system. If the field had a component at some point that was perpendicular to the radial direction, that component would have to be different after at least some rotations. Thus there can't be such a component, and the field must be radial. For the same reason the magnitude E of the field can depend only on the distance r from the center and must have the same value at all points on a spherical surface concentric with the conductor.

Our choice of a sphere as a Gaussian surface takes advantage of these symmetry properties. We first consider the field outside the conductor, so we choose $r > R$. The entire conductor is within the Gaussian surface, so the enclosed charge is q. The area of the Gaussian surface is $4\pi r^2$; $\vec{E}$ is uniform over the surface and perpendicular to it at each point. The flux integral $\oint E_\perp \, dA$ in Gauss's law is therefore just $E(4\pi r^2)$, and Eq. (22.8) gives

$$E(4\pi r^2) = \frac{q}{\epsilon_0} \quad \text{and}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \quad \text{(outside a charged conducting sphere)}$$

This expression for the field at any point *outside* the sphere $(r > R)$ is the same as for a point charge; the field due to the charged sphere is the same as though the entire charge were concentrated at its center. Just outside the surface of the sphere, where $r = R$,

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}$$

(at the surface of a charged conducting sphere)

CAUTION **Flux can be positive or negative** Remember that we have chosen the charge q to be *positive*. If the charge is negative, the electric field is radially *inward* instead of radially outward, and the electric flux through the Gaussian surface is negative. The electric field magnitudes outside and at the surface of the sphere are given by the same expressions as above, except that q denotes the *magnitude* (absolute value) of the charge.

To find $\vec{E}$ inside the conductor, we use a spherical Gaussian surface with radius $r < R$. The spherical symmetry again tells us that $E(4\pi r^2) = Q_{\text{encl}}/\epsilon_0$. But because all of the charge is on the surface of the conductor, our Gaussian surface (which lies entirely within the conductor) encloses *no* charge. So $Q_{\text{encl}} = 0$ and, therefore, the electric field inside the conductor is zero.

EVALUATE: We already knew that $\vec{E} = \mathbf{0}$ inside the conductor, as it must be inside any solid conductor when the charges are at rest. Figure 22.18 shows E as a function of the distance r from the center of the sphere. Note that in the limit as $R \to 0$, the sphere becomes a point charge; there is then only an "outside," and the field is everywhere given by $E = q/4\pi\epsilon_0 r^2$. Thus we have deduced Coulomb's law from Gauss's law. (In Section 22.3 we deduced Gauss's law from Coulomb's law, so this completes the demonstration of their logical equivalence.)

We can also use this method for a conducting spherical *shell* (a spherical conductor with a concentric spherical hole in the center) if there is no charge inside the hole. We use a spherical Gaussian surface with radius r less than the radius of the hole. If there *were* a field inside the hole, it would have to be radial and spherically symmetric as before, so $E = Q_{\text{encl}}/4\pi\epsilon_0 r^2$. But now there is no enclosed charge, so $Q_{\text{encl}} = 0$ and $E = 0$ inside the hole.

Can you use this same technique to find the electric field in the interspace between a charged sphere and a concentric hollow conducting sphere that surrounds it?

Example 22.6 Field of a line charge

Electric charge is distributed uniformly along an infinitely long, thin wire. The charge per unit length is λ (assumed positive). Find the electric field. (This is an approximate representation of the field of a uniformly charged *finite* wire, provided that the distance from the field point to the wire is much less than the length of the wire.)

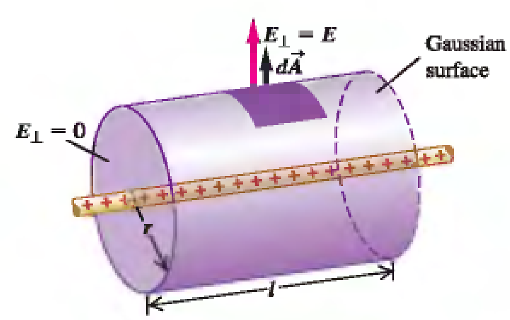
SOLUTION

IDENTIFY: The system has *cylindrical* symmetry. The field must point away from the positive charges. To determine the direction of $\vec{E}$ more precisely, as well as how its magnitude can depend on position, we use symmetry as in Example 22.5.

SET UP: Cylindrical symmetry means that we can rotate the system through any angle about its axis, and we can shift it by any amount along the axis; in each case the resulting system is indistinguishable from the original. Hence $\vec{E}$ at each point can't change when either of these operations is carried out. The field can't have any component parallel to the wire; if it did, we would have to explain why the field lines that begin on the wire pointed in one direction parallel to the wire and not the other. Also, the field can't have any component tangent to a circle in a plane perpendicular to the wire with its center on the wire. If it did, we would have to explain why the component pointed in one direction around the wire rather than the other. All that's left is a component radially outward from the wire at each point. So the field lines outside a uniformly charged, infinite wire are *radial* and lie in planes perpendicular to the wire. The field *magnitude* can depend only on the radial distance from the wire.

These symmetry properties suggest that we use as a Gaussian surface a *cylinder* with arbitrary radius r and arbitrary length l, with its ends perpendicular to the wire (Fig. 22.19).

22.19 A coaxial cylindrical Gaussian surface is used to find the electric field outside an infinitely long, charged wire.



Continued

EXECUTE: We break the surface integral for the flux Φ_E into an integral over each flat end and one over the curved side walls. There is no flux through the ends because $\vec{E}$ lies in the plane of the surface and $E_\perp = 0$. To find the flux through the side walls, note that $\vec{E}$ is perpendicular to the surface at each point, so $E = E_\perp$; by symmetry, E has the same value everywhere on the walls. The area of the side walls is $2\pi rl$. (To make a paper cylinder with radius r and height l, you need a paper rectangle with width $2\pi r$, height l, and area $2\pi rl$.) Hence the total flux Φ_E through the entire cylinder is the sum of the flux through the side walls, which is $(E)(2\pi rl)$, and the zero flux through the two ends. Finally, we need the total enclosed charge, which is the charge per unit length multiplied by the length of wire inside the Gaussian surface, or $Q_{encl} = \lambda l$. From Gauss's law, Eq. (22.8),

$$\Phi_E = (E)(2\pi rl) = \frac{\lambda l}{\epsilon_0} \quad \text{and}$$

$$E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r} \qquad \text{(field of an infinite line of charge)}$$

This is the same result that we found in Example 21.11 (Section 21.5) by much more laborious means.

We have assumed that λ is *positive*. If it is *negative*, $\vec{E}$ is directed radially inward toward the line of charge, and in the above expression for the field magnitude E we must interpret λ as the *magnitude* (absolute value) of the charge per unit length.

EVALUATE: Note that although the *entire* charge on the wire contributes to the field, only the part of the total charge that is within the Gaussian surface is considered when we apply Gauss's law. This may seem strange; it looks as though we have somehow obtained the right answer by ignoring part of the charge and the field of a *short* wire of length l would be the same as that of a very long wire. But we *do* include the entire charge on the wire when we make use of the *symmetry* of the problem. If the wire is short, the symmetry with respect to shifts along the axis is not present, and the field is not uniform in magnitude over our Gaussian surface. Gauss's law is then no longer useful and *cannot* be used to find the field; the problem is best handled by the integration technique used in Example 21.11.

We can use a Gaussian surface like that in Fig. 22.19 to show that the field at points outside a long, uniformly charged cylinder is the same as though all the charge were concentrated on a line along its axis. We can also calculate the electric field in the space between a charged cylinder and a coaxial hollow conducting cylinder surrounding it. We leave these calculations to you (see Problems 22.37 and 22.40).

Example 22.7 Field of an infinite plane sheet of charge

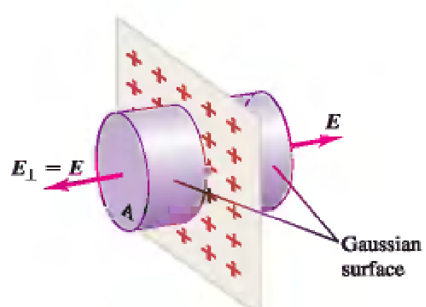
Find the electric field caused by a thin, flat, infinite sheet on which there is a uniform positive charge per unit area σ.

SOLUTION

IDENTIFY: The field must point away from the positively charged sheet. As in Examples 22.5 and 22.6, before doing calculations we use the symmetry (in this case, *planar* symmetry) to learn more about the direction and position dependence of $\vec{E}$.

SET UP: Planar symmetry means that the charge distribution doesn't change if we slide it in any direction parallel to the sheet. From this we conclude that at each point, $\vec{E}$ is perpendicular to the sheet. The symmetry also tells us that the field must have the same magnitude E at any given distance on either side of the sheet. To take advantage of these symmetry properties, we use as our Gaussian surface a cylinder with its axis perpendicular to the sheet of charge, with ends of area A (Fig. 22.20).

22.20 A cylindrical Gaussian surface is used to find the field of an infinite plane sheet of charge.



EXECUTE: The charged sheet passes through the middle of the cylinder's length, so the cylinder ends are equidistant from the sheet. At each end of the cylinder, $\vec{E}$ is perpendicular to the surface and $E_\perp$ is equal to E; hence the flux through each end is $+EA$.

Because $\vec{E}$ is perpendicular to the charged sheet, it is parallel to the curved *side* walls of the cylinder, so $E_\perp$ at these walls is zero and there is no flux through these walls. The total flux integral in Gauss's law is then $2EA$ (EA from each end and zero from the side walls). The net charge within the Gaussian surface is the charge per unit area multiplied by the sheet area enclosed by the surface, or $Q_{encl} = \sigma A$. Hence Gauss's law, Eq. (22.8), gives

$$2EA = \frac{\sigma A}{\epsilon_0} \quad \text{and}$$

$$E = \frac{\sigma}{2\epsilon_0} \qquad \text{(field of an infinite sheet of charge)}$$

This is the same result that we found in Example 21.12 (Section 21.5) using a much more complex calculation. The field is uniform and directed perpendicular to the plane of the sheet. Its magnitude is *independent* of the distance from the sheet. The field lines are therefore straight, parallel to each other, and perpendicular to the sheet.

If the charge density is negative, $\vec{E}$ is directed *toward* the sheet, the flux through the Gaussian surface in Fig. 22.20 is negative, and σ in the expression $E = \sigma/2\epsilon_0$ denotes the magnitude (absolute value) of the charge density.

EVALUATE: The assumption that the sheet is infinitely large is an idealization; nothing in nature is really infinitely large. But the result $E = \sigma/2\epsilon_0$ is a good approximation for points that are close to the sheet (compared to the sheet's dimensions) and not too near its edges. At such points, the field is very nearly uniform and perpendicular to the plane.

Example 22.8 Field between oppositely charged parallel conducting plates

Two large plane parallel conducting plates are given charges of equal magnitude and opposite sign; the charge per unit area is $+\sigma$ for one and $-\sigma$ for the other. Find the electric field in the region between the plates.

SOLUTION

IDENTIFY: The field between and around the plates is approximately as shown in Fig. 22.21a. Because opposite charges attract, most of the charge accumulates at the opposing faces of the plates. A small amount of charge resides on the *outer* surfaces of the plates, and there is some spreading or "fringing" of the field at the edges. But if the plates are very large in comparison to the distance between them, the amount of charge on the outer surfaces is negligibly small, and the fringing can be neglected except near the edges. In this case we can assume that the field is uniform in the interior region between plates, as in Fig. 22.21b, and that the charges are distributed uniformly over the opposing surfaces.

SET UP: To exploit this symmetry, we can use the shaded Gaussian surfaces S_1, S_2, S_3, and S_4. These surfaces are cylinders with ends of area A like the one shown in perspective in Fig. 22.20; they are shown in a side view in Fig. 22.21b. One end of each surface lies within one of the conducting plates.

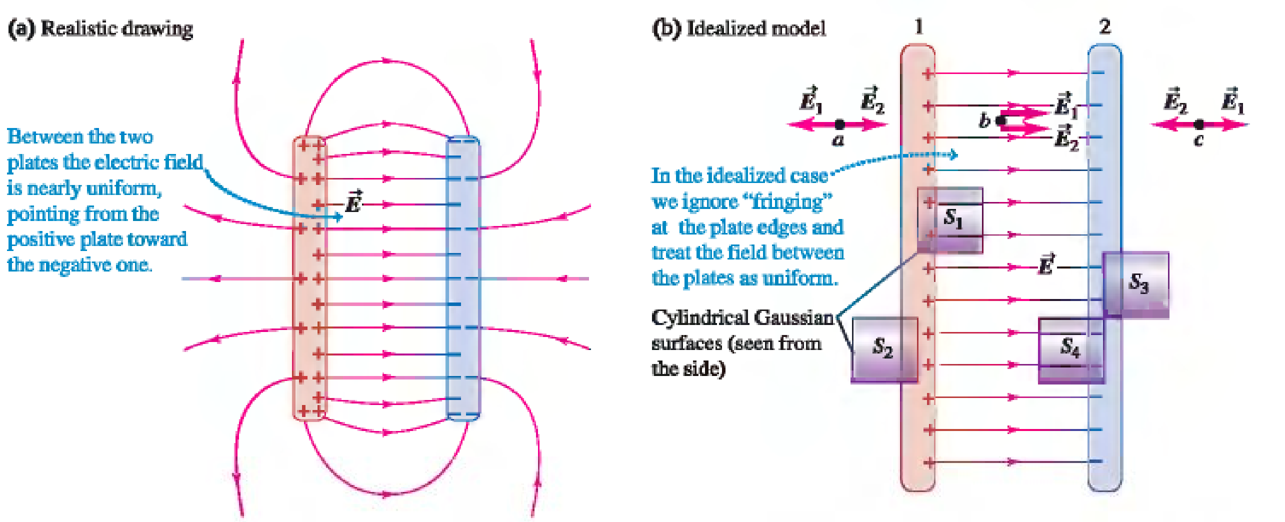
EXECUTE: For the surface labeled S_1, the left-hand end is within plate 1 (the positive plate). Since the field is zero within the volume of any solid conductor under electrostatic conditions, there is no electric flux through this end. The electric field between the plates is perpendicular to the right-hand end, so on that end, $E_\perp$ is equal to E and the flux is EA; this is positive, since $\vec{E}$ is directed out of the Gaussian surface. There is no flux through the side walls of the cylinder, since these walls are parallel to $\vec{E}$. So the total flux integral in Gauss's law is EA. The net charge enclosed by the cylinder is σA, so Eq. (22.8) yields

$$EA = \frac{\sigma A}{\epsilon_0} \quad \text{and} \quad E = \frac{\sigma}{\epsilon_0} \quad \text{(field between oppositely charged conducting plates)}$$

The field is uniform and perpendicular to the plates, and its magnitude is independent of the distance from either plate. This same result can be obtained by using the Gaussian surface S_4; furthermore, the surfaces S_2 and S_3 can be used to show that $E = 0$ to the left of plate 1 and to the right of plate 2. We leave these calculations to you (see Exercise 22.27).

EVALUATE: We obtained the same results in Example 21.13 (Section 21.5) by using the principle of superposition of electric fields. The fields due to the two sheets of charge (one on each plate) are $\vec{E}_1$ and $\vec{E}_2$; from Example 22.7, both of these have magnitude $\sigma/2\epsilon_0$. The total (resultant) electric field at any point is the vector sum $\vec{E} = \vec{E}_1 + \vec{E}_2$. At points a and c in Fig. 22.21b, $\vec{E}_1$ and $\vec{E}_2$ have opposite directions, and their resultant is zero. This is also true at every point within the material of each plate, consistent with the requirement that with charges at rest there can be no field within a solid conductor. At any point b between the plates, $\vec{E}_1$ and $\vec{E}_2$ have the same direction; their resultant has magnitude $E = \sigma/\epsilon_0$, just as we found above using Gauss's law.

22.21 Electric field between oppositely charged parallel plates.



Example 22.9 Field of a uniformly charged sphere

Positive electric charge Q is distributed uniformly *throughout the volume* of an *insulating* sphere with radius R. Find the magnitude of the electric field at a point P a distance r from the center of the sphere.

SOLUTION

IDENTIFY: As in Example 22.5, the system is spherically symmetric. Hence we can use the conclusions of that example about the direction and magnitude of $\vec{E}$.

SET UP: To make use of the symmetry, we choose as our Gaussian surface a sphere with radius r, concentric with the charge distribution.

EXECUTE: From symmetry the magnitude E of the electric field has the same value at every point on the Gaussian surface, and the direction of $\vec{E}$ is radial at every point on the surface, so $E_\perp = E$. Hence the total electric flux through the Gaussian surface is the product of E and the total area of the surface $A = 4\pi r^2$, that is, $\Phi_E = 4\pi r^2 E$.

The amount of charge enclosed within the Gaussian surface depends on the radius r. Let's first find the field magnitude *inside* the charged sphere of radius R; the magnitude E is evaluated at the radius of the Gaussian surface, so we choose $r < R$. The volume charge density ρ is the charge Q divided by the volume of the entire charged sphere of radius R:

$$\rho = \frac{Q}{4\pi R^3/3}$$

The volume V_{encl} enclosed by the Gaussian surface is $\frac{4}{3}\pi r^3$, so the total charge Q_{encl} enclosed by that surface is

$$Q_{\text{encl}} = \rho V_{\text{encl}} = \left(\frac{Q}{4\pi R^3/3}\right)\left(\frac{4}{3}\pi r^3\right) = Q\frac{r^3}{R^3}$$

Then Gauss's law, Eq. (22.8), becomes

$$4\pi r^2 E = \frac{Q}{\epsilon_0}\frac{r^3}{R^3} \quad \text{or}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3} \qquad \text{(field inside a uniformly charged sphere)}$$

The field magnitude is proportional to the distance r of the field point from the center of the sphere. At the center $(r = 0)$, $E = 0$.

To find the field magnitude *outside* the charged sphere, we use a spherical Gaussian surface of radius $r > R$. This surface encloses the entire charged sphere, so $Q_{\text{encl}} = Q$, and Gauss's law gives

$$4\pi r^2 E = \frac{Q}{\epsilon_0} \quad \text{or}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2} \qquad \text{(field outside a uniformly charged sphere)}$$

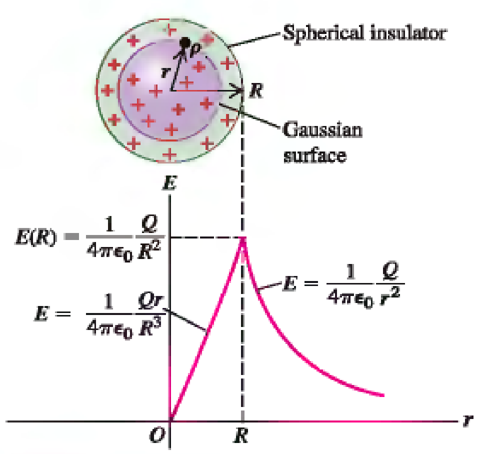
For *any* spherically symmetric charged body the electric field outside the body is the same as though the entire charge were concentrated at the center. (We made this same observation in Example 22.5.)

Figure 22.22 shows a graph of E as a function of r for this problem. For $r < R$, E is directly proportional to r, and for $r > R$, E varies as $1/r^2$. If the charge is negative instead of positive, $\vec{E}$ is radially *inward* and Q in the expressions for E is interpreted as the magnitude (absolute value) of the charge.

EVALUATE: Notice that if we set $r = R$ in either of the two expressions for E (inside or outside the sphere), we get the same result $E = Q/4\pi\epsilon_0 R^2$ for the magnitude of the field at the surface of the sphere. This is because the magnitude E is a *continuous* function of r. By contrast, for the charged conducting sphere of Example 22.5 the electric-field magnitude is *discontinuous* at $r = R$ (it jumps from $E = 0$ just inside the sphere to $E = Q/4\pi\epsilon_0 R^2$ just outside the sphere). In general, the electric field $\vec{E}$ is discontinuous in magnitude, direction, or both wherever there is a *sheet* of charge, such as at the surface of a charged conducting sphere (Example 22.5), at the surface of an infinite charged sheet (Example 22.7), or at the surface of a charged conducting plate (Example 22.8).

The general technique used in this example can be applied to *any* spherically symmetric distribution of charge, whether it is uniform or not. Such charge distributions occur within many atoms and atomic nuclei, which is why Gauss's law is a useful tool in atomic and nuclear physics.

22.22 The magnitude of the electric field of a uniformly charged insulating sphere. Compare this with the field for a conducting sphere (Fig. 22.18).



Example 22.10 Field of a hollow charged sphere

A thin-walled, hollow sphere of radius 0.250 m has an unknown amount of charge distributed uniformly over its surface. At a distance of 0.300 m from the center of the sphere, the electric field points directly toward the center of the sphere and has magnitude 1.80×10^2 N/C. How much charge is on the sphere?

SOLUTION

IDENTIFY: The charge distribution is spherically symmetric. As in Examples 22.5 and 22.9, it follows that the electric field is radial everywhere and its magnitude is a function only of the radial distance r from the center of the sphere.

SET UP: We again use a spherical Gaussian surface that is concentric with the charge distribution and that passes through the point of interest at $r = 0.300$ m.

EXECUTE: The charge distribution is the same as if the charge were on the surface of a 0.250-m-radius conducting sphere. Hence we can borrow the results of Example 22.5. A key difference from that example is that because the electric field here is directed toward the sphere, the charge must be *negative*. Furthermore, because the electric field is directed into the Gaussian surface, $E_\perp = -E$ and the flux is $\oint E_\perp\, dA = -E(4\pi r^2)$.

By Gauss's law, the flux is equal to the charge q on the sphere (all of which is enclosed by the Gaussian surface) divided by ϵ_0. Solving for q, we find

$$\begin{aligned} q &= -E(4\pi\epsilon_0 r^2) = -(1.80 \times 10^2\,\mathrm{N/C})(4\pi) \\ &\quad \times (8.854 \times 10^{-12}\,\mathrm{C^2/N \cdot m^2})(0.300\,\mathrm{m})^2 \\ &= -8.01 \times 10^{-10}\,\mathrm{C} = -0.801\,\mathrm{nC} \end{aligned}$$

EVALUATE: To determine the charge, we had to know the electric field at *all* points on the Gaussian surface so that we could calculate the flux integral. This was possible here because the charge distribution is highly symmetric. If the charge distribution is irregular or lacks symmetry, however, Gauss's law is not very useful for calculating the charge distribution from the field, or vice versa.

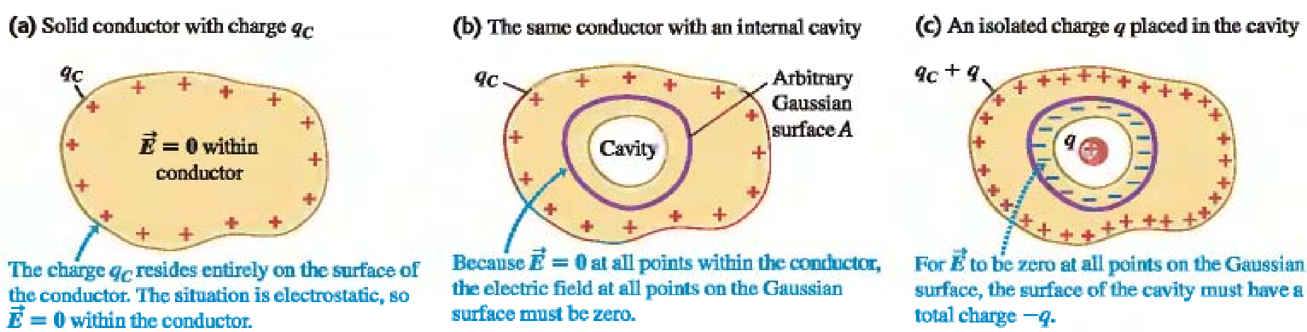
Test Your Understanding of Section 22.4 You place a known amount of charge Q on the irregularly shaped conductor shown in Fig. 22.17. If you know the size and shape of the conductor, can you use Gauss's law to calculate the electric field at an arbitrary position outside the conductor?

22.5 Charges on Conductors

We have learned that in an electrostatic situation (in which there is no net motion of charge) the electric field at every point within a conductor is zero and that any excess charge on a solid conductor is located entirely on its surface (Fig. 22.23a). But what if there is a *cavity* inside the conductor (Fig. 22.23b)? If there is no charge within the cavity, we can use a Gaussian surface such as A (which lies completely within the material of the conductor) to show that the *net* charge on the *surface of the cavity* must be zero, because $\vec{E} = \mathbf{0}$ everywhere on the Gaussian surface. In fact, we can prove in this situation that there can't be any charge *anywhere* on the cavity surface. We will postpone detailed proof of this statement until Chapter 23.

Suppose we place a small body with a charge q inside a cavity within a conductor (Fig. 22.23c). The conductor is uncharged and is insulated from the charge q. Again $\vec{E} = \mathbf{0}$ everywhere on surface A, so according to Gauss's law the *total* charge inside this surface must be zero. Therefore there must be a charge $-q$ distributed on the surface of the cavity, drawn there by the charge q inside the cavity. The *total* charge on the conductor must remain zero, so a charge $+q$ must appear either on its outer surface or inside the material. But we showed in Section 22.4 that in an electrostatic situation there can't be any excess charge within the material of a conductor. So we conclude that the charge $+q$ must appear on the outer surface. By the same reasoning, if the conductor originally had a charge q_C, then the total charge on the outer surface must be $q_C + q$ after the charge q is inserted into the cavity.

22.23 Finding the electric field within a charged conductor.



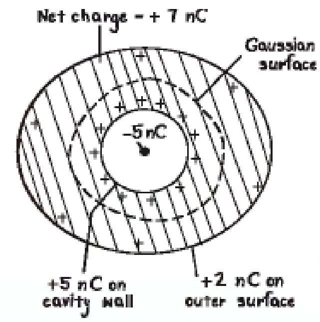
Conceptual Example 22.11 A conductor with a cavity

A solid conductor with a cavity carries a total charge of $+7\ \text{nC}$. Within the cavity, insulated from the conductor, is a point charge of $-5\ \text{nC}$. How much charge is on each surface (inner and outer) of the conductor?

SOLUTION

Figure 22.24 shows the situation. If the charge in the cavity is $q = -5\ \text{nC}$, the charge on the inner cavity surface must be $-q = -(-5\ \text{nC}) = +5\ \text{nC}$. The conductor carries a *total* charge of $+7\ \text{nC}$, none of which is in the interior of the material. If $+5\ \text{nC}$ is on the inner surface of the cavity, then there must be $(+7\ \text{nC}) - (+5\ \text{nC}) = +2\ \text{nC}$ on the outer surface of the conductor.

22.24 Our sketch for this problem. There is zero electric field inside the bulk conductor and hence zero flux through the Gaussian surface shown, so the charge on the cavity wall must be the opposite of the point charge.

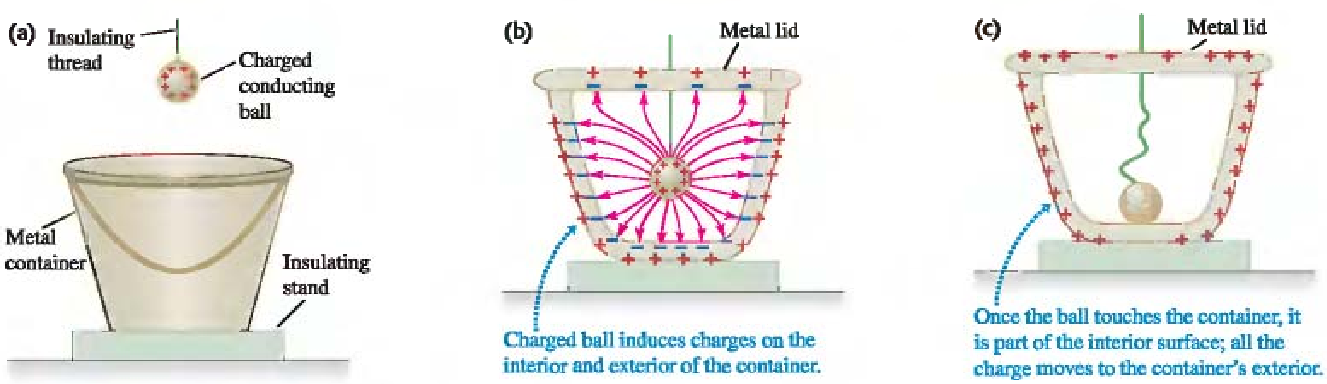


Testing Gauss's Law Experimentally

We can now consider a historic experiment, shown in Fig. 22.25. We mount a conducting container, such as a metal pail with a lid, on an insulating stand. The container is initially uncharged. Then we hang a charged metal ball from an insulating thread (Fig. 22.25a), lower it into the pail, and put the lid on (Fig. 22.25b). Charges are induced on the walls of the container, as shown. But now we let the ball *touch* the inner wall (Fig. 22.25c). The surface of the ball becomes, in effect, part of the cavity surface. The situation is now the same as Fig. 22.23b; if Gauss's law is correct, the net charge on the cavity surface must be zero. Thus the ball must lose all its charge. Finally, we pull the ball out; we find that it has indeed lost all its charge.

This experiment was performed in the 19th century by the English scientist Michael Faraday, using a metal icepail with a lid, and it is called **Faraday's icepail experiment.** (Similar experiments were carried out in the 18th century by Benjamin Franklin in America and Joseph Priestley in England, although with much less precision.) The result confirms the validity of Gauss's law and therefore of Coulomb's law. Faraday's result was significant because Coulomb's experimental method, using a torsion balance and dividing of charges, was not very precise; it is very difficult to confirm the $1/r^2$ dependence of the electrostatic force with great precision by direct force measurements. By contrast, experiments like Faraday's test the validity of Gauss's law, and therefore of Coulomb's law, with much greater precision.

22.25 **(a)** A charged conducting ball suspended by an insulating thread outside a conducting container on an insulating stand. **(b)** The ball is lowered into the container, and the lid is put on. **(c)** The ball is touched to the inner surface of the container.

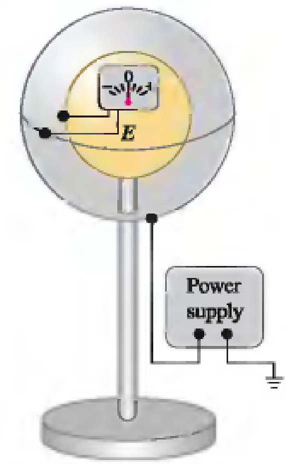


A modern version of Faraday's experiment is shown in Fig. 22.26. The details of the box labeled "Power supply" aren't important; its job is to place charge on the outer sphere and remove it, on demand. The inner box with a dial is a sensitive *electrometer*, an instrument that can detect motion of extremely small amounts of charge between the outer and inner spheres. If Gauss's law is correct, there can never be any charge on the inner surface of the outer sphere. If so, there should be no flow of charge between spheres while the outer sphere is being charged and discharged. The fact that no flow is actually observed is a very sensitive confirmation of Gauss's law and therefore of Coulomb's law. The precision of the experiment is limited mainly by the electrometer, which can be astonishingly sensitive. Experiments have shown that the exponent 2 in the $1/r^2$ of Coulomb's law does not differ from precisely 2 by more than 10^{-16}. So there is no reason to suspect that it is anything other than exactly 2.

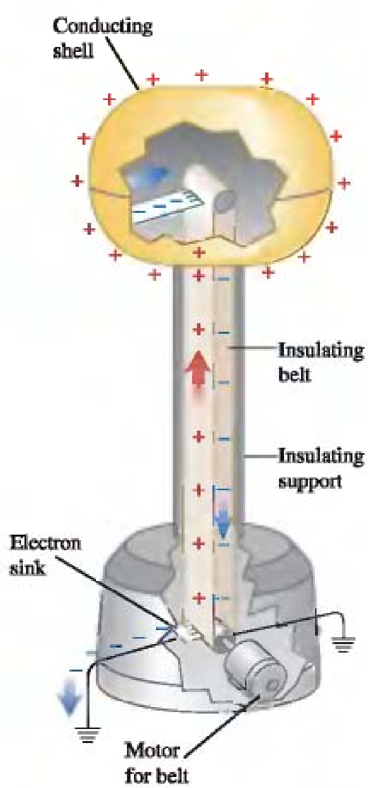
The same principle behind Faraday's icepail experiment is used in a *Van de Graaff electrostatic generator* (Fig. 22.27). The charged conducting sphere of Fig. 22.26 is replaced by a charged belt that continuously carries charge to the inside of a conducting shell, only to have it carried away to the outside surface of the shell. As a result, the charge on the shell and the electric field around it can become very large very rapidly. The Van de Graaff generator is used as an accelerator of charged particles and for physics demonstrations.

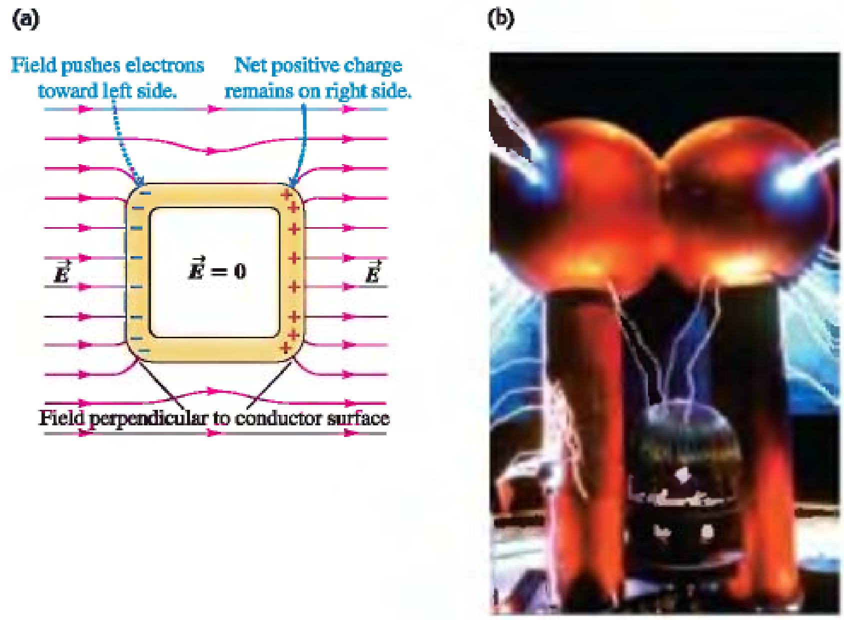
This principle also forms the basis for *electrostatic shielding*. Suppose we have a very sensitive electronic instrument that we want to protect from stray electric fields that might cause erroneous measurements. We surround the instrument with a conducting box, or we line the walls, floor, and ceiling of the room with a conducting material such as sheet copper. The external electric field redistributes the free electrons in the conductor, leaving a net positive charge on the outer

22.26 The outer spherical shell can be alternately charged and discharged by the power supply. If there were any flow of charge between the inner and outer shells, it would be detected by the electrometer inside the inner shell.



22.27 Cutaway view of the essential parts of a Van de Graaff electrostatic generator. The electron sink at the bottom draws electrons from the belt, giving it a positive charge; at the top the belt attracts electrons away from the conducting shell, giving the shell a positive charge.





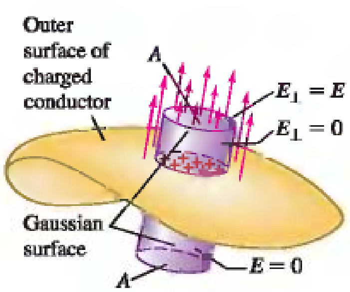
22.28 (a) A conducting box (a Faraday cage) immersed in a uniform electric field. The field of the induced charges on the box combines with the uniform field to give zero total field inside the box. (b) Electrostatic shielding can protect you from a dangerous electric discharge.

surface in some regions and a net negative charge in others (Fig. 22.28). This charge distribution causes an additional electric field such that the *total* field at every point inside the box is zero, as Gauss's law says it must be. The charge distribution on the box also alters the shapes of the field lines near the box, as the figure shows. Such a setup is often called a *Faraday cage*. The same physics tells you that one of the safest places to be in a lightning storm is inside an automobile; if the car is struck by lightning, the charge tends to remain on the metal skin of the vehicle, and little or no electric field is produced inside the passenger compartment.

Field at the Surface of a Conductor

Finally, we note that there is a direct relationship between the $\vec{E}$ field at a point just outside any conductor and the surface charge density σ at that point. In general, σ varies from point to point on the surface. We will show in Chapter 23 that at any such point, the direction of $\vec{E}$ is always *perpendicular* to the surface (see Fig. 22.28a).

To find a relationship between σ at any point on the surface and the perpendicular component of the electric field at that point, we construct a Gaussian surface in the form of a small cylinder (Fig. 22.29). One end face, with area A, lies within the conductor and the other lies just outside. The electric field is zero at all points within the conductor. Outside the conductor the component of $\vec{E}$ perpendicular to the side walls of the cylinder is zero, and over the end face the perpendicular component is equal to $E_\perp$. (If σ is positive, the electric field points out of the conductor and $E_\perp$ is positive; if σ is negative, the field points inward and $E_\perp$ is negative.) Hence the total flux through the surface is $E_\perp A$. The charge enclosed within the Gaussian surface is σA, so from Gauss's law,



22.29 The field just outside a charged conductor is perpendicular to the surface, and its perpendicular component $E_\perp$ is equal to σ/ϵ_0.

$$E_\perp A = \frac{\sigma A}{\epsilon_0} \quad \text{and} \quad E_\perp = \frac{\sigma}{\epsilon_0} \quad \text{(field at the surface of a conductor)} \tag{22.10}$$

We can check this with the results we have obtained for spherical, cylindrical, and plane surfaces.

We showed in Example 22.8 that the field magnitude between two infinite flat oppositely charged conducting plates also equals σ/ϵ_0. In this case the field magnitude is the same at *all* distances from the plates, but in all other cases it decreases with increasing distance from the surface.

Conceptual Example 22.12 Field at the surface of a conducting sphere

Verify Eq. (22.10) for a conducting sphere with radius R and total charge q.

SOLUTION

In Example 22.5 (Section 22.4) we showed that the electric field just outside the surface is

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}$$

The surface charge density is uniform and equal to q divided by the surface area of the sphere:

$$\sigma = \frac{q}{4\pi R^2}$$

Comparing these two expressions, we see that $E = \sigma/\epsilon_0$, as Eq. (22.10) states.

Example 22.13 Electric field of the earth

The earth (a conductor) has a net electric charge. The resulting electric field near the surface can be measured with sensitive electronic instruments; its average value is about 150 N/C, directed toward the center of the earth. (a) What is the corresponding surface charge density? (b) What is the *total* surface charge of the earth?

SOLUTION

IDENTIFY: We are given the electric field magnitude at the surface of the conducting earth, and we are asked to calculate the surface charge density and the total charge on the entire surface of the earth.

SET UP: Given the perpendicular electric field, we determine the surface charge density σ using Eq. (22.10). The total surface charge on the earth is then the product of σ and the earth's surface area.

EXECUTE: (a) We know from the direction of the field that σ is negative (corresponding to $\vec{E}$ being directed *into* the surface, so $E_\perp$ is negative). From Eq. (22.10),

$$\begin{aligned}\sigma = \epsilon_0 E_\perp &= (8.85 \times 10^{-12}\ \mathrm{C^2/N\cdot m^2})(-150\ \mathrm{N/C})\\ &= -1.33 \times 10^{-9}\ \mathrm{C/m^2} = -1.33\ \mathrm{nC/m^2}\end{aligned}$$

(b) The earth's surface area is $4\pi R_E^2$, where $R_E = 6.38 \times 10^6$ m is the radius of the earth (see Appendix F). The total charge Q is the product $4\pi R_E^2\sigma$, or

$$\begin{aligned}Q &= 4\pi(6.38 \times 10^6\ \mathrm{m})^2(-1.33 \times 10^{-9}\ \mathrm{C/m^2})\\ &= -6.8 \times 10^5\ \mathrm{C} = -680\ \mathrm{kC}\end{aligned}$$

EVALUATE: You can check our result in part (b) using the result of Example 22.5. Solving for Q, we find

$$\begin{aligned}Q &= 4\pi\epsilon_0 R^2 E_\perp\\ &= \frac{1}{9.0 \times 10^9\ \mathrm{N\cdot m^2/C^2}}(6.38 \times 10^6\ \mathrm{m})^2(-150\ \mathrm{N/C})\\ &= -6.8 \times 10^5\ \mathrm{C}\end{aligned}$$

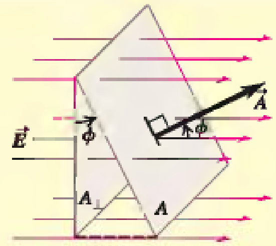
One electron has a charge of -1.60×10^{-19} C. Hence this much excess negative electric charge corresponds to there being $(-6.8 \times 10^5\ \mathrm{C})/(-1.60 \times 10^{-19}\ \mathrm{C}) = 4.2 \times 10^{24}$ excess electrons on the earth, or about 7 moles of excess electrons. This is compensated by an equal *deficiency* of electrons in the earth's upper atmosphere, so the combination of the earth and its atmosphere is electrically neutral.

Test Your Understanding of Section 22.5 A hollow conducting sphere has no net charge. There is a positive point charge q at the center of the spherical cavity within the sphere. You connect a conducting wire from the outside of the sphere to ground. Will you measure an electric field outside the sphere?

CHAPTER 22 SUMMARY

Electric flux: Electric flux is a measure of the "flow" of electric field through a surface. It is equal to the product of an area element and the perpendicular component of $\vec{E}$, integrated over a surface. (See Examples 22.1–22.3.)

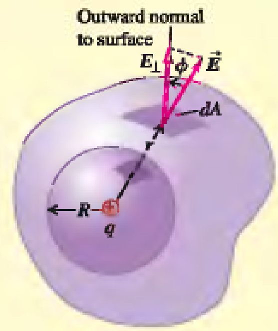
$$\Phi_E = \int E\cos\phi\, dA$$
$$= \int E_\perp\, dA = \int \vec{E}\cdot d\vec{A} \quad (22.5)$$



Gauss's law: Gauss's law states that the total electric flux through a closed surface, which can be written as the surface integral of the component of $\vec{E}$ normal to the surface, equals a constant times the total charge Q_{encl} enclosed by the surface. Gauss's law is logically equivalent to Coulomb's law, but its use greatly simplifies problems with a high degree of symmetry. (See Examples 22.4–22.10.)

When excess charge is placed on a conductor and is at rest, it resides entirely on the surface, and $\vec{E} = \mathbf{0}$ everywhere in the material of the conductor. (See Examples 22.11–22.13.)

$$\Phi_E = \oint E\cos\phi\, dA$$
$$= \oint E_\perp\, dA = \oint \vec{E}\cdot d\vec{A}$$
$$= \frac{Q_{encl}}{\epsilon_0} \quad (22.8), (22.9)$$



Electric field of various symmetric charge distributions: The following table lists electric fields caused by several symmetric charge distributions. In the table, q, Q, λ, and σ refer to the *magnitudes* of the quantities.

Charge Distribution	Point in Electric Field	Electric Field Magnitude
Single point charge q	Distance r from q	$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$
Charge q on surface of conducting sphere with radius R	Outside sphere, $r > R$	$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$
	Inside sphere, $r < R$	$E = 0$
Infinite wire, charge per unit length λ	Distance r from wire	$E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}$
Infinite conducting cylinder with radius R, charge per unit length λ	Outside cylinder, $r > R$	$E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}$
	Inside cylinder, $r < R$	$E = 0$
Solid insulating sphere with radius R, charge Q distributed uniformly throughout volume	Outside sphere, $r > R$	$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$
	Inside sphere, $r < R$	$E = \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}$
Infinite sheet of charge with uniform charge per unit area σ	Any point	$E = \frac{\sigma}{2\epsilon_0}$
Two oppositely charged conducting plates with surface charge densities $+\sigma$ and $-\sigma$	Any point between plates	$E = \frac{\sigma}{\epsilon_0}$

Key Terms

closed surface, *751*
electric flux, *752*
surface integral, *755*
Gauss's law, *757*
Gaussian surface, *759*
Faraday's icepail experiment, *768*

Answer to Chapter Opening Question

No. The electric field inside a cavity within a conductor is zero, so there is no electric effect on the child. (See Section 22.5.)

Answers to Test Your Understanding Questions

22.1 Answer: (iii) Each part of the surface of the box will be three times farther from the charge $+q$, so the electric field will be $(\frac{1}{3})^2 = \frac{1}{9}$ as strong. But the area of the box will increase by a factor of $3^2 = 9$. Hence the electric flux will be multiplied by a factor of $(\frac{1}{9})(9) = 1$. In other words, the flux will be unchanged.

22.2 Answer: (iv), (ii), (i), (iii) In each case the electric field in uniform, so the flux is $\Phi_E = \vec{E} \cdot \vec{A}$. We use the relationships for the scalar products of unit vectors: $\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = 1, \hat{\imath} \cdot \hat{\jmath} = 0$. In case (i) we have $\Phi_E = (4.0\,\mathrm{N/C})(6.0\,\mathrm{m}^2)\hat{\imath} \cdot \hat{\jmath} = 0$ (the electric field and vector area are perpendicular, so there is zero flux). In case (ii) we have $\Phi_E [(4.0\,\mathrm{N/C})\hat{\imath} + (2.0\,\mathrm{N/C})\hat{\jmath}] \cdot (3.0\,\mathrm{m}^2)\hat{\jmath} = (2.0\,\mathrm{N/C}) \cdot (3.0\,\mathrm{m}^2) = 6.0\,\mathrm{N \cdot m^2/C}$. Similarly, in case (iii) we have $\Phi_E = [(4.0\,\mathrm{N/C})\hat{\imath} - (2.0\,\mathrm{N/C})\hat{\jmath}] \cdot [(3.0\,\mathrm{m}^2)\hat{\imath} + (7.0\,\mathrm{m}^2)\hat{\jmath}] = (4.0\,\mathrm{N/C})(3.0\,\mathrm{m}^2) - (2.0\,\mathrm{N/C})(7.0\,\mathrm{m}^2) = -2\,\mathrm{N \cdot m^2/C}$, and in case (iv) we have $\Phi_E = [(4.0\,\mathrm{N/C})\hat{\imath} - (2.0\,\mathrm{N/C})\hat{\jmath}] \cdot [(3.0\,\mathrm{m}^2)\hat{\imath} - (7.0\,\mathrm{m}^2)\hat{\jmath}] = (4.0\,\mathrm{N/C})(3.0\,\mathrm{m}^2) + (2.0\,\mathrm{N/C}) \cdot (7.0\,\mathrm{m}^2) = 26\,\mathrm{N \cdot m^2/C}$.

22.3 Answer: S_2, S_5, S_4, S_1 and S_3 (tie) Gauss's law tells us that the flux through a closed surface is proportional to the amount of charge enclosed within that surface. So an ordering of these surfaces by their fluxes is the same as an ordering by the amount of enclosed charge. Surface S_1 encloses no charge, surface S_2 encloses $9.0\,\mu\mathrm{C} + 5.0\,\mu\mathrm{C} + (-7.0\,\mu\mathrm{C}) = 7.0\,\mu\mathrm{C}$, surface S_3 encloses $9.0\,\mu\mathrm{C} + 1.0\,\mu\mathrm{C} + (-10.0\,\mu\mathrm{C}) = 0$, surface S_4 encloses $8.0\,\mu\mathrm{C} + (-7.0\,\mu\mathrm{C}) = 1.0\,\mu\mathrm{C}$, and surface S_5 encloses $8.0\,\mu\mathrm{C} + (-7.0\,\mu\mathrm{C}) + (-10.0\,\mu\mathrm{C}) + (1.0\,\mu\mathrm{C}) + (9.0\,\mu\mathrm{C}) + (5.0\,\mu\mathrm{C}) = 6.0\,\mu\mathrm{C}$.

22.4 Answer: no You might be tempted to draw a Gaussian surface that is an enlarged version of the conductor, with the same shape and placed so that it completely encloses the conductor. While you know the flux through this Gaussian surface (by Gauss's law, it's $\Phi_E = Q/\epsilon_0$), the direction of the electric field need not be perpendicular to the surface and the magnitude of the field need not be the same at all points on the surface. It's not possible to do the flux integral $\oint E_\perp \, dA$, and we can't calculate the electric field. Gauss's law is useful for calculating the electric field only when the charge distribution is *highly* symmetric.

22.5 Answer: no Before you connect the wire to the sphere, the presence of the point charge will induce a charge $-q$ on the inner surface of the hollow sphere and a charge q on the outer surface (the net charge on the sphere is zero). There will be an electric field outside the sphere due to the charge on the outer surface. Once you touch the conducting wire to the sphere, however, electrons will flow from ground to the outer surface of the sphere to neutralize the charge there (see Fig. 21.7c). As a result the sphere will have no charge on its outer surface and no electric field outside.

PROBLEMS

For instructor-assigned homework, go to www.masteringphysics.com

Discussion Questions

Q22.1. A rubber balloon has a single point charge in its interior. Does the electric flux through the balloon depend on whether or not it is fully inflated? Explain your reasoning.

Q22.2. Suppose that in Fig. 22.15 both charges were positive. What would be the fluxes through each of the four surfaces in the example?

Q22.3. In Fig. 22.15, suppose a third point charge were placed outside the purple Gaussian surface C. Would this affect the electric flux through any of the surfaces A, B, C, or D in the figure? Why or why not?

Q22.4. A certain region of space bounded by an imaginary closed surface contains no charge. Is the electric field always zero everywhere on the surface? If not, under what circumstances is it zero on the surface?

Q22.5. A spherical Gaussian surface encloses a point charge q. If the point charge is moved from the center of the sphere to a point away from the center, does the electric field at a point on the surface change? Does the total flux through the Gaussian surface change? Explain.

Q22.6. You find a sealed box on your doorstep. You suspect that the box contains several charged metal spheres packed in insulating material. How can you determine the total net charge inside the box without opening the box? Or isn't this possible?

Q22.7. During the flow of electric current in a conducting wire, one or more electrons from each atom are free to move along the wire, somewhat like water flowing through a pipe. Would you expect to find an electric field outside a wire carrying such a steady flow of electrons? Explain.

Q22.8. If the electric field of a point charge were proportional to $1/r^3$ instead of $1/r^2$, would Gauss's law still be valid? Explain your reasoning. (*Hint:* Consider a spherical Gaussian surface centered on a single point charge.)

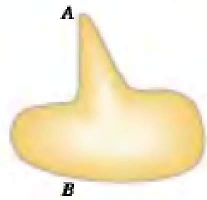
Q22.9. Suppose the disk in Example 22.1 (Section 22.2), instead of having its normal vector oriented at just two or three particular angles to the electric field, began to rotate continuously, so that its normal vector was first parallel to the field, then perpendicular to it, then opposite to it, and so on. Sketch a graph of the resulting electric flux versus time, for an entire rotation of 360°.

Q22.10. In a conductor, one or more electrons from each atom are free to roam throughout the volume of the conductor. Does this contradict the statement that any excess charge on a solid conductor must reside on its surface? Why or why not?

Q22.11. You charge up the van de Graaff generator shown in Fig. 22.27, and then bring an identical but uncharged hollow conducting sphere near it, without letting the two spheres touch. Sketch the distribution of charges on the second sphere. What is the net flux through the second sphere? What is the electric field inside the second sphere?

Q22.12. The magnitude of $\vec{E}$ at the surface of an irregularly shaped solid conductor must be greatest in regions where the surface curves most sharply, such as point A in Fig. 22.30, and must be least in flat regions such as point B in Fig. 22.30. Explain why this must be so by considering how electric field lines must be arranged near a conducting surface. How does the surface charge density compare at points A and B? Explain.

Figure 22.30 Question Q22.12.



Q22.13. A lightning rod is a rounded copper rod mounted on top of a building and welded to a heavy copper cable running down into the ground. Lightning rods are used to protect houses and barns from lightning; the lightning current runs through the copper rather than through the building. Why? Why should the end of the rod be rounded? (*Hint:* The answer to Discussion Question Q22.12 may be helpful.)

Q22.14. A solid conductor has a cavity in its interior. Would the presence of a point charge inside the cavity affect the electric field outside the conductor? Why or why not? Would the presence of a point charge outside the conductor affect the electric field inside the cavity? Again, why or why not?

Q22.15. Explain this statement: "In a static situation, the electric field at the surface of a conductor can have no component parallel to the surface because this would violate the condition that the charges on the surface are at rest." Would this same statement be valid for the electric field at the surface of an *insulator?* Explain your answer and the reason for any differences between the cases of a conductor and an insulator.

Q22.16. A solid copper sphere has a net positive charge. The charge is distributed uniformly over the surface of the sphere, and the electric field inside the sphere is zero. Then a negative point charge outside the sphere is brought close to the surface of the sphere. Is all the net charge on the sphere still on its surface? If so, is this charge still distributed uniformly over the surface? If it is not uniform, how is it distributed? Is the electric field inside the sphere still zero? In each case justify your answers.

Q22.17. Some modern aircraft are made primarily of composite materials that do not conduct electricity. The U.S. Federal Aviation Administration requires that such aircraft have conducting wires embedded in their surfaces to provide protection when flying near thunderstorms. Explain the physics behind this requirement.

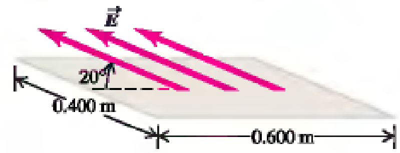
Exercises

Section 22.2 Calculating Electric Flux

22.1. A flat sheet of paper of area $0.250\ \text{m}^2$ is oriented so that the normal to the sheet is at an angle of $60°$ to a uniform electric field of magnitude 14 N/C. (a) Find the magnitude of the electric flux through the sheet. (b) Does the answer to part (a) depend on the shape of the sheet? Why or why not? (c) For what angle ϕ between the normal to the sheet and the electric field is the magnitude of the flux through the sheet (i) largest and (ii) smallest? Explain your answers.

22.2. A flat sheet is in the shape of a rectangle with sides of lengths 0.400 m and 0.600 m. The sheet is immersed in a uniform electric field of magnitude 75.0 N/C that is directed at $20°$ from the plane of the sheet (Fig. 22.31). Find the magnitude of the electric flux through the sheet.

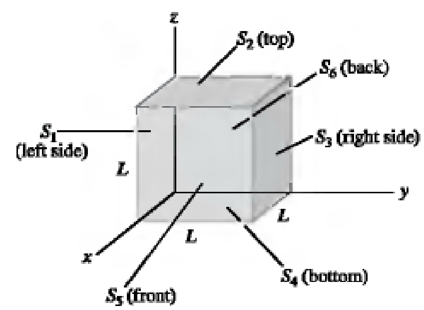
Figure 22.31 Exercise 22.2.



22.3. You measure an electric field of 1.25×10^6 N/C at a distance of 0.150 m from a point charge. (a) What is the electric flux through a sphere at that distance from the charge? (b) What is the magnitude of the charge?

22.4. A cube has sides of length $L = 0.300$ m. It is placed with one corner at the origin as shown in Fig. 22.32. The electric field is not uniform but is given by $\vec{E} = (-5.00\ \text{N/C}\cdot\text{m})x\hat{\imath} + (3.00\ \text{N/C}\cdot\text{m})z\hat{k}$. (a) Find the electric flux through each of the six cube faces S_1, S_2, S_3, S_4, S_5, and S_6. (b) Find the total electric charge inside the cube.

Figure 22.32 Exercises 22.4 and 22.6; Problem 22.32.



22.5. A hemispherical surface with radius r in a region of uniform electric field $\vec{E}$ has its axis aligned parallel to the direction of the field. Calculate the flux through the surface.

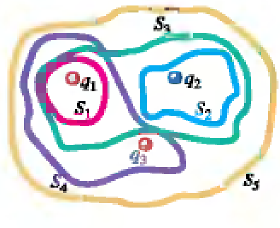
22.6. The cube in Fig. 22.32 has sides of length $L = 10.0$ cm. The electric field is uniform, has magnitude $E = 4.00 \times 10^5$ N/C, and is parallel to the xy-plane at an angle of $36.9°$ measured from the $+x$-axis toward the $+y$-axis. (a) What is the electric flux through each of the six cube faces S_1, S_2, S_3, S_4, S_5, and S_6? (b) What is the total electric flux through all faces of the cube?

22.7. It was shown in Example 21.11 (Section 21.5) that the electric field due to an infinite line of charge is perpendicular to the line and has magnitude $E = \lambda/2\pi\epsilon_0 r$. Consider an imaginary cylinder with radius $r = 0.250$ m and length $l = 0.400$ m that has an infinite line of positive charge running along its axis. The charge per unit length on the line is $\lambda = 6.00\ \mu\text{C/m}$. (a) What is the electric flux through the cylinder due to this infinite line of charge? (b) What is the flux through the cylinder if its radius is increased to $r = 0.500$ m? (c) What is the flux through the cylinder if its length is increased to $l = 0.800$ m?

Section 22.3 Gauss's Law

22.8. The three small spheres shown in Fig. 22.33 carry charges $q_1 = 4.00$ nC, $q_2 = -7.80$ nC, and $q_3 = 2.40$ nC. Find the net electric flux through each of the following closed surfaces shown in cross section in the figure: (a) S_1; (b) S_2; (c) S_3; (d) S_4; (e) S_5. (f) Do your answers to parts (a)–(e) depend on how the charge is distributed over each small sphere? Why or why not?

Figure 22.33 Exercise 22.8.



Surface	What it encloses
S_1	q_1
S_2	q_2
S_3	q_1 and q_2
S_4	q_1 and q_3
S_5	q_1 and q_2 and q_3

22.9. A charged paint is spread in a very thin uniform layer over the surface of a plastic sphere of diameter 12.0 cm, giving it a charge of $-15.0\ \mu$C. Find the electric field (a) just inside the paint layer; (b) just outside the paint layer; (c) 5.00 cm outside the surface of the paint layer.

22.10. A point charge $q_1 = 4.00$ nC is located on the x-axis at $x = 2.00$ m, and a second point charge $q_2 = -6.00$ nC is on the y-axis at $y = 1.00$ m. What is the total electric flux due to these two point charges through a spherical surface centered at the origin and with radius (a) 0.500 m, (b) 1.50 m, (c) 2.50 m?

22.11. In a certain region of space, the electric field $\vec{E}$ is uniform. (a) Use Gauss's law to prove that this region of space must be electrically neutral; that is, the volume charge density ρ must be zero. (b) Is the converse true? That is, in a region of space where there is no charge, must $\vec{E}$ be uniform? Explain.

22.12. (a) In a certain region of space, the volume charge density ρ has a uniform positive value. Can $\vec{E}$ be uniform in this region? Explain. (b) Suppose that in this region of uniform positive ρ there is a "bubble" within which $\rho = 0$. Can $\vec{E}$ be uniform within this bubble? Explain.

22.13. A 9.60-μC point charge is at the center of a cube with sides of length 0.500 m. (a) What is the electric flux through one of the six faces of the cube? (b) How would your answer to part (a) change if the sides were 0.250 m long? Explain.

22.14. Electric Fields in an Atom. The nuclei of large atoms, such as uranium, with 92 protons, can be modeled as spherically symmetric spheres of charge. The radius of the uranium nucleus is approximately 7.4×10^{-15} m. (a) What is the electric field this nucleus produces just outside its surface? (b) What magnitude of electric field does it produce at the distance of the electrons, which is about 1.0×10^{-10} m? (c) The electrons can be modeled as forming a uniform shell of negative charge. What net electric field do they produce at the location of the nucleus?

22.15. A point charge of $+5.00\ \mu$C is located on the x-axis at $x = 4.00$ m, next to a spherical surface of radius 3.00 m centered at the origin. (a) Calculate the magnitude of the electric field at $x = 3.00$ m. (b) Calculate the magnitude of the electric field at $x = -3.00$ m. (c) According to Gauss's law, the net flux through the sphere is zero because it contains no charge. Yet the field due to the external charge is much stronger on the near side of the sphere (i.e., at $x = 3.00$ m) than on the far side (at $x = -3.00$ m). How, then, can the flux into the sphere (on the near side) equal the flux out of it (on the far side)? Explain. A sketch will help.

Section 22.4 Applications of Gauss's Law and Section 22.5 Charges on Conductors

22.16. A solid metal sphere with radius 0.450 m carries a net charge of 0.250 nC. Find the magnitude of the electric field (a) at a point 0.100 m outside the surface of the sphere and (b) at a point inside the sphere, 0.100 m below the surface.

22.17. On a humid day, an electric field of 2.00×10^4 N/C is enough to produce sparks about an inch long. Suppose that in your physics class, a van de Graaff generator (see Fig. 22.27) with a sphere radius of 15.0 cm is producing sparks 6 inches long. (a) Use Gauss's law to calculate the amount of charge stored on the surface of the sphere before you bravely discharge it with your hand. (b) Assume all the charge is concentrated at the center of the sphere, and use Coulomb's law to calculate the electric field at the surface of the sphere.

22.18. Some planetary scientists have suggested that the planet Mars has an electric field somewhat similar to that of the earth, producing a net electric flux of $3.63 \times 10^{16}\ \text{N} \cdot \text{m}^2/\text{C}$ at the planet's surface. Calculate: (a) the total electric charge on the planet; (b) the electric field at the planet's surface (refer to the astronomical data inside the back cover); (c) the charge density on Mars, assuming all the charge is uniformly distributed over the planet's surface.

22.19. How many excess electrons must be added to an isolated spherical conductor 32.0 cm in diameter to produce an electric field of 1150 N/C just outside the surface?

22.20. The electric field 0.400 m from a very long uniform line of charge is 840 N/C. How much charge is contained in a 2.00-cm section of the line?

22.21. A very long uniform line of charge has charge per unit length 4.80 μC/m and lies along the x-axis. A second long uniform line of charge has charge per unit length $-2.40\ \mu$C/m and is parallel to the x-axis at $y = 0.400$ m. What is the net electric field (magnitude and direction) at the following points on the y-axis: (a) $y = 0.200$ m and (b) $y = 0.600$ m?

22.22. (a) At a distance of 0.200 cm from the center of a charged conducting sphere with radius 0.100 cm, the electric field is 480 N/C. What is the electric field 0.600 cm from the center of the sphere? (b) At a distance of 0.200 cm from the axis of a very long charged conducting cylinder with radius 0.100 cm, the electric field is 480 N/C. What is the electric field 0.600 cm from the axis of the cylinder? (c) At a distance of 0.200 cm from a large uniform sheet of charge, the electric field is 480 N/C. What is the electric field 1.20 cm from the sheet?

22.23. A hollow, conducting sphere with an outer radius of 0.250 m and an inner radius of 0.200 m has a uniform surface charge density of $+6.37 \times 10^{-6}\ \text{C/m}^2$. A charge of $-0.500\ \mu$C is now introduced into the cavity inside the sphere. (a) What is the new charge density on the outside of the sphere? (b) Calculate the strength of the electric field just outside the sphere. (c) What is the electric flux through a spherical surface just inside the inner surface of the sphere?

22.24. A point charge of $-2.00\ \mu$C is located in the center of a spherical cavity of radius 6.50 cm inside an insulating charged solid. The charge density in the solid is $\rho = 7.35 \times 10^{-4}\ \text{C/m}^3$. Calculate the electric field inside the solid at a distance of 9.50 cm from the center of the cavity.

22.25. The electric field at a distance of 0.145 m from the surface of a solid insulating sphere with radius 0.355 m is 1750 N/C. (a) Assuming the sphere's charge is uniformly distributed, what is the charge density inside it? (b) Calculate the electric field inside the sphere at a distance of 0.200 m from the center.

22.26. A conductor with an inner cavity, like that shown in Fig. 22.23c, carries a total charge of $+5.00$ nC. The charge within the cavity, insulated from the conductor, is -6.00 nC. How much charge is on (a) the inner surface of the conductor and (b) the outer surface of the conductor?

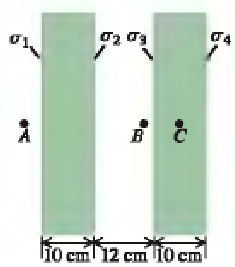
22.27. Apply Gauss's law to the Gaussian surfaces S_2, S_3, and S_4 in Fig. 22.21b to calculate the electric field between and outside the plates.

22.26. A square insulating sheet 80.0 cm on a side is held horizontally. The sheet has 7.50 nC of charge spread uniformly over its area. (a) Calculate the electric field at a point 0.100 mm above the center of the sheet. (b) Estimate the electric field at a point 100 m above the center of the sheet. (c) Would the answers to parts (a) and (b) be different if the sheet were made of a conducting material? Why or why not?

22.29. An infinitely long cylindrical conductor has radius R and uniform surface charge density σ. (a) In terms of σ and R, what is the charge per unit length λ for the cylinder? (b) In terms of σ, what is the magnitude of the electric field produced by the charged cylinder at a distance $r > R$ from its axis? (c) Express the result of part (b) in terms of λ and show that the electric field outside the cylinder is the same as if all the charge were on the axis. Compare your result to the result for a line of charge in Example 22.6 (Section 22.4).

22.30. Two very large, nonconducting plastic sheets, each 10.0 cm thick, carry uniform charge densities $\sigma_1, \sigma_2, \sigma_3$, and σ_4 on their surfaces, as shown in Fig. 22.34. These surface charge densities have the values $\sigma_1 = -6.00\ \mu C/m^2$, $\sigma_2 = +5.00\ \mu C/m^2$, $\sigma_3 = +2.00\ \mu C/m^2$, and $\sigma_4 = +4.00\ \mu C/m^2$. Use Gauss's law to find the magnitude and direction of the electric field at the following points, far from the edges of these sheets: (a) point A, 5.00 cm from the left face of the left-hand sheet; (b) point B, 1.25 cm from the inner surface of the right-hand sheet; (c) point C, in the middle of the right-hand sheet.

Figure 22.34 Exercise 22.30.



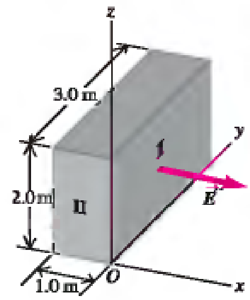
22.31. A negative charge $-Q$ is placed inside the cavity of a hollow metal solid. The outside of the solid is grounded by connecting a conducting wire between it and the earth. (a) Is there any excess charge induced on the inner surface of the piece of metal? If so, find its sign and magnitude. (b) Is there any excess charge on the outside of the piece of metal? Why or why not? (c) Is there an electric field in the cavity? Explain. (d) Is there an electric field within the metal? Why or why not? Is there an electric field outside the piece of metal? Explain why or why not. (e) Would someone outside the solid measure an electric field due to the charge $-Q$? Is it reasonable to say that the grounded conductor has *shielded* the region from the effects of the charge $-Q$? In principle, could the same thing be done for gravity? Why or why not?

Problems

22.32. A cube has sides of length L. It is placed with one corner at the origin as shown in Fig. 22.32. The electric field is uniform and given by $\vec{E} = -B\hat{\imath} + C\hat{\jmath} - D\hat{k}$, where B, C, and D are positive constants. (a) Find the electric flux through each of the six cube faces S_1, S_2, S_3, S_4, S_5, and S_6. (b) Find the electric flux through the entire cube.

22.33. The electric field $\vec{E}$ in Fig. 22.35 is everywhere parallel to the x-axis, so the components E_y and E_z are zero. The x-component of the field E_x depends on x but not on y and z. At points in the yz-plane (where $x = 0$), $E_x = 125$ N/C. (a) What is the electric flux through surface I in Fig. 22.35? (b) What is the electric flux through surface II? (c) The volume shown in the figure is a small section of a very large insulating slab 1.0 m thick. If there is a total charge of -24.0 nC within the volume shown, what are the magnitude and direction of $\vec{E}$ at the face opposite surface I? (d) Is the electric field produced only by charges within the slab, or is the field also due to charges outside the slab? How can you tell?

Figure 22.35 Problem 22.33.



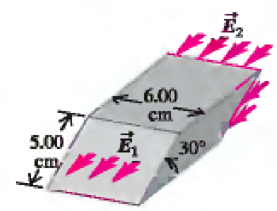
22.34. A flat, square surface with sides of length L is described by the equations

$$x = L \qquad (0 \le y \le L, 0 \le z \le L)$$

(a) Draw this square and show the x-, y-, and z-axes. (b) Find the electric flux through the square due to a positive point charge q located at the origin $(x = 0, y = 0, z = 0)$. (*Hint:* Think of the square as part of a cube centered on the origin.)

22.35. The electric field $\vec{E}_1$ at one face of a parallelepiped is uniform over the entire face and is directed out of the face. At the opposite face, the electric field $\vec{E}_2$ is also uniform over the entire face and is directed into that face (Fig. 22.36). The two faces in question are inclined at 30.0° from the horizontal, while $\vec{E}_1$ and $\vec{E}_2$ are both horizontal; $\vec{E}_1$ has a magnitude of 2.50×10^4 N/C, and $\vec{E}_2$ has a magnitude of 7.00×10^4 N/C. (a) Assuming that no other electric field lines cross the surfaces of the parallelepiped, determine the net charge contained within. (b) Is the electric field produced only by the charges within the parallelepiped, or is the field also due to charges outside the parallelepiped? How can you tell?

Figure 22.36 Problem 22.35.



22.36. A long line carrying a uniform linear charge density $+50.0\ \mu C/m$ runs parallel to and 10.0 cm from the surface of a large, flat plastic sheet that has a uniform surface charge density of $-100\ \mu C/m^2$ on one side. Find the location of all points where an α particle would feel no force due to this arrangement of charged objects.

22.37. The Coaxial Cable. A long coaxial cable consists of an inner cylindrical conductor with radius a and an outer coaxial cylinder with inner radius b and outer radius c. The outer cylinder is mounted on insulating supports and has no net charge. The inner cylinder has a uniform positive charge per unit length λ. Calculate the electric field (a) at any point between the cylinders a distance r from the axis and (b) at any point outside the outer cylinder. (c) Graph the magnitude of the electric field as a function of the distance r from the axis of the cable, from $r = 0$ to $r = 2c$. (d) Find the charge per unit length on the inner surface and on the outer surface of the outer cylinder.

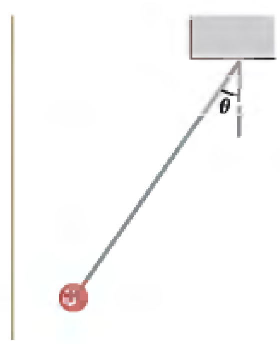
22.38. A very long conducting tube (hollow cylinder) has inner radius a and outer radius b. It carries charge per unit length $+\alpha$, where α is a positive constant with units of C/m. A line of charge lies along the axis of the tube. The line of charge has charge per unit length $+\alpha$. (a) Calculate the electric field in terms of α and the distance r from the axis of the tube for (i) $r < a$; (ii) $a < r < b$; (iii) $r > b$. Show your results in a graph of E as a function of r. (b) What is the charge per unit length on (i) the inner surface of the tube and (ii) the outer surface of the tube?

22.38. Repeat Problem 22.38, but now let the conducting tube have charge per unit length $-\alpha$. As in Problem 22.38, the line of charge has charge per unit length $+\alpha$.

22.40. A very long, solid cylinder with radius R has positive charge uniformly distributed throughout it, with charge per unit volume ρ. (a) Derive the expression for the electric field inside the volume at a distance r from the axis of the cylinder in terms of the charge density ρ. (b) What is the electric field at a point outside the volume in terms of the charge per unit length λ in the cylinder? (c) Compare the answers to parts (a) and (b) for $r = R$. (d) Graph the electric-field magnitude as a function of r from $r = 0$ to $r = 3R$.

22.41. A small sphere with a mass of 0.002 g and carrying a charge of 5.00×10^{-8} C hangs from a thread near a very large, charged conducting sheet, as shown in Fig. 22.37. The charge density on the sheet is $2.50 \times 10^{-9}\ \mathrm{C/m^2}$. Find the angle of the thread.

Figure 22.37 Problem 22.41.

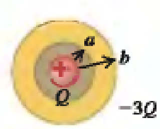


22.42. A Sphere in a Sphere. A solid conducting sphere carrying charge q has radius a. It is inside a concentric hollow conducting sphere with inner radius b and outer radius c. The hollow sphere has no net charge. (a) Derive expressions for the electric-field magnitude in terms of the distance r from the center for the regions $r < a$, $a < r < b$, $b < r < c$, and $r > c$. (b) Graph the magnitude of the electric field as a function of r from $r = 0$ to $r = 2c$. (c) What is the charge on the inner surface of the hollow sphere? (d) On the outer surface? (e) Represent the charge of the small sphere by four plus signs. Sketch the field lines of the system within a spherical volume of radius 2 c.

22.43. A solid conducting sphere with radius R that carries positive charge Q is concentric with a very thin insulating shell of radius $2R$ that also carries charge Q. The charge Q is distributed uniformly over the insulating shell. (a) Find the electric field (magnitude and direction) in each of the regions $0 < r < R$, $R < r < 2R$, and $r > 2R$. (b) Graph the electric-field magnitude as a function of r.

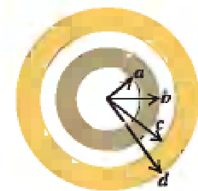
22.44. A conducting spherical shell with inner radius a and outer radius b has a positive point charge Q located at its center. The total charge on the shell is $-3Q$, and it is insulated from its surroundings (Fig. 22.38). (a) Derive expressions for the electric-field magnitude in terms of the distance r from the center for the regions $r < a$, $a < r < b$, and $r > b$. (b) What is the surface charge density on the inner surface of the conducting shell? (c) What is the surface charge density on the outer surface of the conducting shell? (d) Sketch the electric field lines and the location of all charges. (e) Graph the electric-field magnitude as a function of r.

Figure 22.38 Problem 22.44.



22.45. Concentric Spherical Shells. A small conducting spherical shell with inner radius a and outer radius b is concentric with a larger conducting spherical shell with inner radius c and outer radius d (Fig. 22.39). The inner shell has total charge $+2q$, and the outer shell has charge $+4q$. (a) Calculate the electric field (magnitude and direction) in terms of q and the distance r from the common center of the two shells for (i) $r < a$; (ii) $a < r < b$; (iii) $b < r < c$; (iv) $c < r < d$; (v) $r > d$. Show your results in a graph of the radial component of $\vec{E}$ as a function of r. (b) What is the total charge on the (i) inner surface of the small shell; (ii) outer surface of the small shell; (iii) inner surface of the large shell; (iv) outer surface of the large shell?

Figure 22.39 Problem 22.45.



22.46. Repeat Problem 22.45, but now let the outer shell have charge $-2q$. As in Problem 22.45, the inner shell has charge $+2q$.

22.47. Repeat Problem 22.45, but now let the outer shell have charge $-4q$. As in Problem 22.45, the inner shell has charge $+2q$.

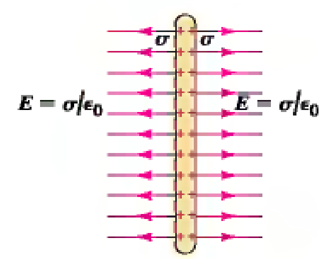
22.48. A solid conducting sphere with radius R carries a positive total charge Q. The sphere is surrounded by an insulating shell with inner radius R and outer radius $2R$. The insulating shell has a uniform charge density ρ. (a) Find the value of ρ so that the net charge of the entire system is zero. (b) If ρ has the value found in part (a), find the electric field (magnitude and direction) in each of the regions $0 < r < R$, $R < r < 2R$, and $r > 2R$. Show your results in a graph of the radial component of $\vec{E}$ as a function of r. (c) As a general rule, the electric field is discontinuous only at locations where there is a thin sheet of charge. Explain how your results in part (b) agree with this rule.

22.48. Negative charge $-Q$ is distributed uniformly over the surface of a thin spherical insulating shell with radius R. Calculate the force (magnitude and direction) that the shell exerts on a positive point charge q located (a) a distance $r > R$ from the center of the shell (outside the shell) and (b) a distance $r < R$ from the center of the shell (inside the shell).

22.50. (a) How many excess electrons must be distributed uniformly within the volume of an isolated plastic sphere 30.0 cm in diameter to produce an electric field of 1150 N/C just outside the surface of the sphere? (b) What is the electric field at a point 10.0 cm outside the surface of the sphere?

22.51. A single isolated, large conducting plate (Fig. 22.40) has a charge per unit area σ on its surface. Because the plate is a conductor, the electric field at its surface is perpendicular to the surface and has magnitude $E = \sigma/\epsilon_0$. (a) In Example 22.7 (Section 22.4) it was shown that the field caused by a large, uniformly charged sheet with charge per unit area σ has magnitude $E = \sigma/2\epsilon_0$, exactly *half* as much as for a charged conducting plate. Why is there a difference? (b) Regarding the charge distribution on the conducting plate as being two sheets of charge (one on each surface), each with charge per unit area σ, use the result of Example 22.7 and the principle of superposition to show that $E = 0$ inside the plate and $E = \sigma/\epsilon_0$ outside the plate.

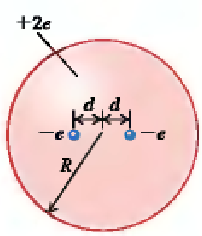
Figure 22.40 Problem 22.51.



22.52. Thomson's Model of the Atom. In the early years of the 20th century, a leading model of the structure of the atom was that of the English physicist J. J. Thomson (the discoverer of the electron). In Thomson's model, an atom consisted of a sphere of positively charged material in which were embedded negatively charged electrons, like chocolate chips in a ball of cookie dough. Consider such an atom consisting of one electron with mass m and charge $-e$, which may be regarded as a point charge, and a uniformly charged sphere of charge $+e$ and radius R. (a) Explain why the equilibrium position of the electron is at the center of the nucleus. (b) In Thomson's model, it was assumed that the positive material provided little or no resistance to the motion of the electron. If the electron is displaced from equilibrium by a distance less than R, show that the resulting motion of the electron will be simple harmonic, and calculate the frequency of oscillation. (*Hint:* Review the definition of simple harmonic motion in Section 13.2. If it can be shown that the net force on the electron is of this form, then it follows that the motion is simple harmonic. Conversely, if the net force on the electron does not follow this form, the motion is not simple harmonic.) (c) By Thomson's time, it was known that excited atoms emit light waves of only certain frequencies. In his model, the frequency of emitted light is the same as the oscillation frequency of the electron or electrons in the atom. What would the radius of a Thomson-model atom have to be for it to produce red light of frequency 4.57×10^{14} Hz? Compare your answer to the radii of real atoms, which are of the order of 10^{-10} m (see Appendix F for data about the electron). (d) If the electron were displaced from equilibrium by a distance greater than R, would the electron oscillate? Would its motion be simple harmonic? Explain your reasoning. (*Historical note:* In 1910, the atomic nucleus was discovered, proving the Thomson model to be incorrect. An atom's positive charge is not spread over its volume as Thomson supposed, but is concentrated in the tiny nucleus of radius 10^{-14} to 10^{-15} m.)

22.53. Thomson's Model of the Atom, Continued. Using Thomson's (outdated) model of the atom described in Problem 22.52, consider an atom consisting of two electrons, each of charge $-e$, embedded in a sphere of charge $+2e$ and radius R. In equilibrium, each electron is a distance d from the center of the atom (Fig. 22.41). Find the distance d in terms of the other properties of the atom.

Figure 22.41 Problem 22.53.



22.54. A Uniformly Charged Slab. A slab of insulating material has thickness $2d$ and is oriented so that its faces are parallel to the yz-plane and given by the planes $x = d$ and $x = -d$. The y- and z-dimensions of the slab are very large compared to d and may be treated as essentially infinite. The slab has a uniform positive charge density ρ. (a) Explain why the electric field due to the slab is zero at the center of the slab $(x = 0)$. (b) Using Gauss's law, find the electric field due to the slab (magnitude and direction) at all points in space.

22.55. A Nonuniformly Charged Slab. Repeat Problem 22.54, but now let the charge density of the slab be given by $\rho(x) = \rho_0(x/d)^2$, where ρ_0 is a positive constant.

22.56. Can Electric Forces Alone Give Stable Equilibrium? In Chapter 21, several examples were given of calculating the force exerted on a point charge by other point charges in its surroundings. (a) Consider a positive point charge $+q$. Give an example of how you would place two other point charges of your choosing so that the net force on charge $+q$ will be zero. (b) If the net force on charge $+q$ is zero, then that charge is in equilibrium. The equilibrium will be *stable* if, when the charge $+q$ is displaced slightly in *any* direction from its position of equilibrium, the net force on the charge pushes it back toward the equilibrium position. For this to be the case, what must the direction of the electric field $\vec{E}$ be due to the other charges at points surrounding the equilibrium position of $+q$? (c) Imagine that the charge $+q$ is moved very far away, and imagine a small Gaussian surface centered on the position where $+q$ was in equilibrium. By applying Gauss's law to this surface, show that it is *impossible* to satisfy the condition for stability described in part (b). In other words, a charge $+q$ cannot be held in stable equilibrium by electrostatic forces alone. This result is known as *Earnshaw's theorem.* (d) Parts (a)–(c) referred to the equilibrium of a positive point charge $+q$. Prove that Earnshaw's theorem also applies to a negative point charge $-q$.

22.57. A nonuniform, but spherically symmetric, distribution of charge has a charge density $\rho(r)$ given as follows:

$$\rho(r) = \rho_0(1 - r/R) \quad \text{for } r \le R$$
$$\rho(r) = 0 \quad \text{for } r \ge R$$

where $\rho_0 = 3Q/\pi R^3$ is a positive constant. (a) Show that the total charge contained in the charge distribution is Q. (b) Show that the electric field in the region $r \ge R$ is identical to that produced by a point charge Q at $r = 0$. (c) Obtain an expression for the electric field in the region $r \le R$. (d) Graph the electric-field magnitude E as a function of r. (e) Find the value of r at which the electric field is maximum, and find the value of that maximum field.

22.58. A nonuniform, but spherically symmetric, distribution of charge has a charge density $\rho(r)$ given as follows:

$$\rho(r) = \rho_0(1 - 4r/3R) \quad \text{for } r \le R$$
$$\rho(r) = 0 \quad \text{for } r \ge R$$

where ρ_0 is a positive constant. (a) Find the total charge contained in the charge distribution. (b) Obtain an expression for the electric field in the region $r \ge R$. (c) Obtain an expression for the electric field in the region $r \le R$. (d) Graph the electric-field magnitude E as a function of r. (e) Find the value of r at which the electric field is maximum, and find the value of that maximum field.

22.59. Gauss's Law for Gravitation. The gravitational force between two point masses separated by a distance r is proportional to $1/r^2$, just like the electric force between two point charges. Because of this similarity between gravitational and electric interactions, there is also a Gauss's law for gravitation. (a) Let $\vec{g}$ be the acceleration due to gravity caused by a point mass m at the origin, so that $\vec{g} = -(Gm/r^2)\hat{r}$. Consider a spherical Gaussian surface with radius r centered on this point mass, and show that the flux of $\vec{g}$ through this surface is given by

$$\oint \vec{g} \cdot d\vec{A} = -4\pi Gm$$

(b) By following the same logical steps used in Section 22.3 to obtain Gauss's law for the electric field, show that the flux of $\vec{g}$ through *any* closed surface is given by

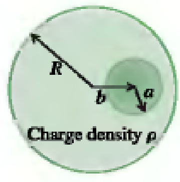
$$\oint \vec{g} \cdot d\vec{A} = -4\pi GM_{\text{encl}}$$

where M_{encl} is the total mass enclosed within the closed surface.

22.60. Applying Gauss's Law for Gravitation. Using Gauss's law for gravitation (derived in part (b) of Problem 22.59), show that the following statements are true: (a) For any spherically symmetric mass distribution with total mass M, the acceleration due to gravity outside the distribution is the same as though all the mass were concentrated at the center. (*Hint:* See Example 22.5 in Section 22.4.) (b) At any point inside a spherically symmetric shell of mass, the acceleration due to gravity is zero. (*Hint:* See Example 22.5.) (c) If we could drill a hole through a spherically symmetric planet to its center, and if the density were uniform, we would find that the magnitude of $\vec{g}$ is directly proportional to the distance r from the center. (*Hint:* See Example 22.9 in Section 22.4.) We proved these results in Section 12.6 using some fairly strenuous analysis; the proofs using Gauss's law for gravitation are *much* easier.

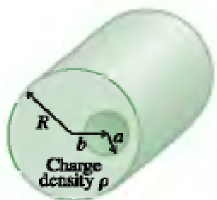
22.61. (a) An insulating sphere with radius a has a uniform charge density ρ. The sphere is not centered at the origin but at $\vec{r} = \vec{b}$. Show that the electric field inside the sphere is given by $\vec{E} = \rho(\vec{r} - \vec{b})/3\epsilon_0$. (b) An insulating sphere of radius R has a spherical hole of radius a located within its volume and centered a distance b from the center of the sphere, where $a < b < R$ (a cross section of the sphere is shown in Fig. 22.42). The solid part of the sphere has a uniform volume charge density ρ. Find the magnitude and direction of the electric field $\vec{E}$ inside the hole, and show that $\vec{E}$ is uniform over the entire hole. [*Hint:* Use the principle of superposition and the result of part (a).]

Figure 22.42 Problem 22.61.



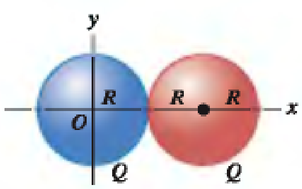
22.62. A very long, solid insulating cylinder with radius R has a cylindrical hole with radius a bored along its entire length. The axis of the hole is a distance b from the axis of the cylinder, where $a < b < R$ (Fig. 22.43). The solid material of the cylinder has a uniform volume charge density ρ. Find the magnitude and direction of the electric field $\vec{E}$ inside the hole, and show that $\vec{E}$ is uniform over the entire hole. (*Hint:* See Problem 22.61.)

Figure 22.43 Problem 22.62.



22.63. Positive charge Q is distributed uniformly over each of two spherical volumes with radius R. One sphere of charge is centered at the origin and the other at $x = 2R$ (Fig. 22.44). Find the magnitude and direction of the net electric field due to these two distributions of charge at the following points on the x-axis: (a) $x = 0$; (b) $x = R/2$; (c) $x = R$; (d) $x = 3R$.

Figure 22.44 Problem 22.63.



22.64. Repeat Problem 22.63, but now let the left-hand sphere have positive charge Q and let the right-hand sphere have negative charge $-Q$.

22.65. Electric Field Inside a Hydrogen Atom. A hydrogen atom is made up of a proton of charge $+Q = 1.60 \times 10^{-19}$ C and an electron of charge $-Q = -1.60 \times 10^{-19}$ C. The proton may be regarded as a point charge at $r = 0$, the center of the atom. The motion of the electron causes its charge to be "smeared out" into a spherical distribution around the proton, so that the electron is equivalent to a charge per unit volume of

$$\rho(r) = -\frac{Q}{\pi a_0^3} e^{-2r/a_0}$$

where $a_0 = 5.29 \times 10^{-11}$ m is called the *Bohr radius.* (a) Find the total amount of the hydrogen atom's charge that is enclosed within a sphere with radius r centered on the proton. Show that as $r \to \infty$, the enclosed charge goes to zero. Explain this result. (b) Find the electric field (magnitude and direction) caused by the charge of the hydrogen atom as a function of r. (c) Graph the electric-field magnitude E as a function of r.

Challenge Problems

22.66. A region in space contains a total positive charge Q that is distributed spherically such that the volume charge density $\rho(r)$ is given by

$$\begin{aligned} \rho(r) &= \alpha && \text{for } r \le R/2 \\ \rho(r) &= 2\alpha(1 - r/R) && \text{for } R/2 \le r \le R \\ \rho(r) &= 0 && \text{for } r \ge R \end{aligned}$$

Here α is a positive constant having units of C/m^3. (a) Determine α in terms of Q and R. (b) Using Gauss's law, derive an expression for the magnitude of $\vec{E}$ as a function of r. Do this separately for all three regions. Express your answers in terms of the total charge Q. Be sure to check that your results agree on the boundaries of the regions. (c) What fraction of the total charge is contained within the region $r \le R/2$? (d) If an electron with charge $q' = -e$ is oscillating back and forth about $r = 0$ (the center of the distribution) with an amplitude less than $R/2$, show that the motion is simple harmonic. (*Hint:* Review the discussion of simple harmonic motion in Section 13.2. If, and only if, the net force on the electron is proportional to its displacement from equilibrium, then the motion is simple harmonic.) (e) What is the period of the motion in part (d)? (f) If the amplitude of the motion described in part (e) is greater than $R/2$, is the motion still simple harmonic? Why or why not?

22.67. A region in space contains a total positive charge Q that is distributed spherically such that the volume charge density $\rho(r)$ is given by

$$\begin{aligned} \rho(r) &= 3\alpha r/(2R) && \text{for } r \le R/2 \\ \rho(r) &= \alpha[1 - (r/R)^2] && \text{for } R/2 \le r \le R \\ \rho(r) &= 0 && \text{for } r \ge R \end{aligned}$$

Here α is a positive constant having units of C/m^3. (a) Determine α in terms of Q and R. (b) Using Gauss's law, derive an expression for the magnitude of the electric field as a function of r. Do this separately for all three regions. Express your answers in terms of the total charge Q. (c) What fraction of the total charge is contained within the region $R/2 \le r \le R$? (d) What is the magnitude of $\vec{E}$ at $r = R/2$? (e) If an electron with charge $q' = -e$ is released from rest at any point in any of the three regions, the resulting motion will be oscillatory but not simple harmonic. Why? (See Challenge Problem 22.66.)

23 ELECTRIC POTENTIAL

LEARNING GOALS

By studying this chapter, you will learn:

- How to calculate the electric potential energy of a collection of charges.
- The meaning and significance of electric potential.
- How to calculate the electric potential that a collection of charges produces at a point in space.
- How to use equipotential surfaces to visualize how the electric potential varies in space.
- How to use electric potential to calculate the electric field.

? In one type of welding, electric charge flows between the welding tool and the metal pieces that are to be joined together. This produces a glowing arc whose high temperature fuses the pieces together. Why must the tool be held close to the pieces being welded?

This chapter is about energy associated with electrical interactions. Every time you turn on a light, a CD player, or an electric appliance, you are making use of electrical energy, an indispensable ingredient of our technological society. In Chapters 6 and 7 we introduced the concepts of *work* and *energy* in the context of mechanics; now we'll combine these concepts with what we've learned about electric charge, electric forces, and electric fields. Just as the energy concept made it possible to solve some kinds of mechanics problems very simply, using energy ideas makes it easier to solve a variety of problems in electricity.

When a charged particle moves in an electric field, the field exerts a force that can do *work* on the particle. This work can always be expressed in terms of electric potential energy. Just as gravitational potential energy depends on the height of a mass above the earth's surface, electric potential energy depends on the position of the charged particle in the electric field. We'll describe electric potential energy using a new concept called *electric potential,* or simply *potential.* In circuits, a difference in potential from one point to another is often called *voltage.* The concepts of potential and voltage are crucial to understanding how electric circuits work and have equally important applications to electron beams used in cancer radiotherapy, high-energy particle accelerators, and many other devices.

23.1 Electric Potential Energy

The concepts of work, potential energy, and conservation of energy proved to be extremely useful in our study of mechanics. In this section we'll show that these concepts are just as useful for understanding and analyzing electrical interactions.

Let's begin by reviewing three essential points from Chapters 6 and 7. First, when a force $\vec{F}$ acts on a particle that moves from point a to point b, the work $W_{a\to b}$ done by the force is given by a *line integral:*

$$W_{a\to b} = \int_a^b \vec{F}\cdot d\vec{l} = \int_a^b F\cos\phi\, dl \qquad \text{(work done by a force)} \qquad (23.1)$$

where $d\vec{l}$ is an infinitesimal displacement along the particle's path and ϕ is the angle between $\vec{F}$ and $d\vec{l}$ at each point along the path.

Second, if the force $\vec{F}$ is *conservative,* as we defined the term in Section 7.3, the work done by $\vec{F}$ can always be expressed in terms of a **potential energy** U. When the particle moves from a point where the potential energy is U_a to a point where it is U_b, the change in potential energy is $\Delta U = U_b - U_a$ and the work $W_{a\to b}$ done by the force is

$$W_{a\to b} = U_a - U_b = -(U_b - U_a) = -\Delta U \quad \text{(work done by a conservative force)} \tag{23.2}$$

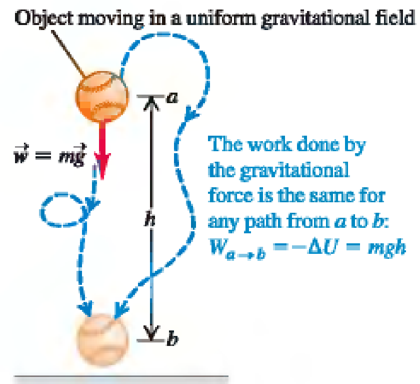
When $W_{a\to b}$ is positive, U_a is greater than U_b, ΔU is negative, and the potential energy *decreases.* That's what happens when a baseball falls from a high point (a) to a lower point (b) under the influence of the earth's gravity; the force of gravity does positive work, and the gravitational potential energy decreases (Fig. 23.1). When a tossed ball is moving upward, the gravitational force does negative work during the ascent, and the potential energy increases.

Third, the work–energy theorem says that the change in kinetic energy $\Delta K = K_b - K_a$ during any displacement is equal to the *total* work done on the particle. If the only work done on the particle is done by conservative forces, then Eq. (23.2) gives the total work, and $K_b - K_a = -(U_b - U_a)$. We usually write this as

$$K_a + U_a = K_b + U_b \tag{23.3}$$

That is, the total mechanical energy (kinetic plus potential) is *conserved* under these circumstances.

23.1 The work done on a baseball moving in a uniform gravitational field.



Electric Potential Energy in a Uniform Field

Let's look at an electrical example of these basic concepts. In Fig. 23.2 a pair of charged parallel metal plates sets up a uniform, downward electric field with magnitude E. The field exerts a downward force with magnitude $F = q_0E$ on a positive test charge q_0. As the charge moves downward a distance d from point a to point b, the force on the test charge is constant and independent of its location. So the work done by the electric field is the product of the force magnitude and the component of displacement in the (downward) direction of the force:

$$W_{a\to b} = Fd = q_0Ed \tag{23.4}$$

This work is positive, since the force is in the same direction as the net displacement of the test charge.

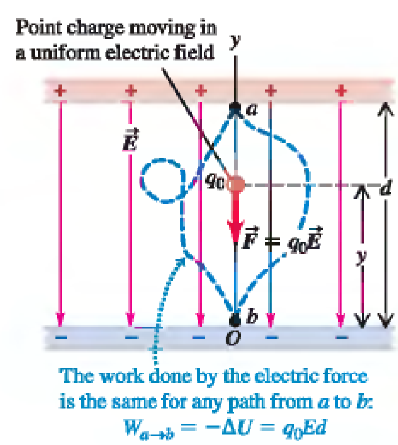
The y-component of the electric force, $F_y = -q_0E$, is constant, and there is no x- or z-component. This is exactly analogous to the gravitational force on a mass m near the earth's surface; for this force, there is a constant y-component $F_y = -mg$ and the x- and z-components are zero. Because of this analogy, we can conclude that the force exerted on q_0 by the uniform electric field in Fig. 23.2 is *conservative,* just as is the gravitational force. This means that the work $W_{a\to b}$ done by the field is independent of the path the particle takes from a to b. We can represent this work with a *potential-energy* function U, just as we did for gravitational potential energy in Section 7.1. The potential energy for the gravitational force $F_y = -mg$ was $U = mgy$; hence the potential energy for the electric force $F_y = -q_0E$ is

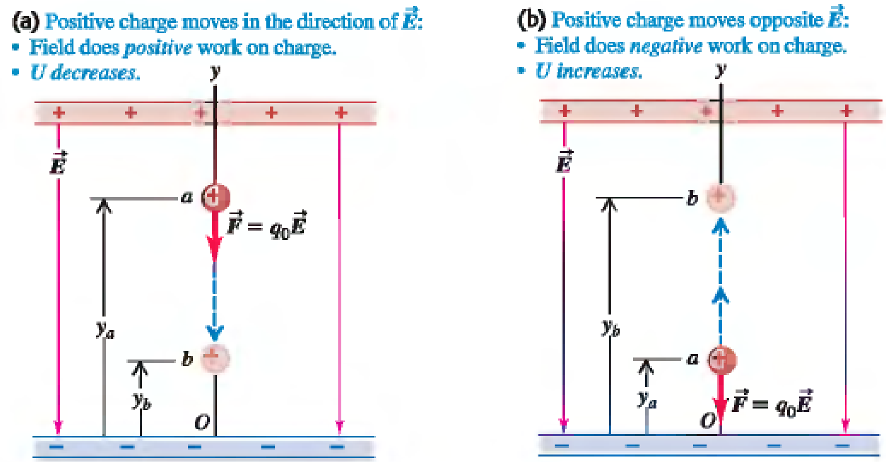
$$U = q_0Ey \tag{23.5}$$

When the test charge moves from height y_a to height y_b, the work done on the charge by the field is given by

$$W_{a\to b} = -\Delta U = -(U_b - U_a) = -(q_0Ey_b - q_0Ey_a) = q_0E(y_a - y_b) \tag{23.6}$$

23.2 The work done on a point charge moving in a uniform electric field. Compare with Fig. 23.1.





23.3 A positive charge moving (a) in the direction of the electric field $\vec{E}$ and (b) in the direction opposite $\vec{E}$.

When y_a is greater than y_b (Fig. 23.3a), the positive test charge q_0 moves downward, in the same direction as $\vec{E}$; the displacement is in the same direction as the force $\vec{F} = q_0\vec{E}$, so the field does positive work and U decreases. [In particular, if $y_a - y_b = d$ as in Fig. 23.2, Eq. (23.6) gives $W_{a\to b} = q_0Ed$, in agreement with Eq. (23.4).] When y_a is less than y_b (Fig. 23.3b), the positive test charge q_0 moves upward, in the opposite direction to $\vec{E}$; the displacement is opposite the force, the field does negative work, and U increases.

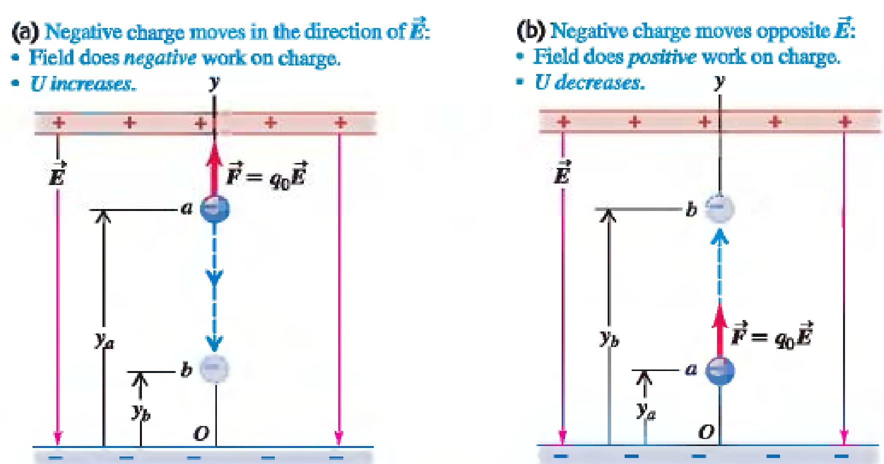
If the test charge q_0 is *negative*, the potential energy increases when it moves with the field and decreases when it moves against the field (Fig. 23.4).

Whether the test charge is positive or negative, the following general rules apply: *U increases* if the test charge q_0 moves in the direction *opposite* the electric force $\vec{F} = q_0\vec{E}$ (Figs. 23.3b and 23.4a); *U decreases* if q_0 moves in the *same* direction as $\vec{F} = q_0\vec{E}$ (Figs. 23.3a and 23.4b). This is the same behavior as for gravitational potential energy, which increases if a mass m moves upward (opposite the direction of the gravitational force) and decreases if m moves downward (in the same direction as the gravitational force).

CAUTION **Electric potential energy** The relationship between electric potential energy change and motion in an electric field is an important one that we'll use often. It's also a relationship that takes a little effort to truly understand. Take the time to review the preceding paragraph thoroughly and to study Figs. 23.3 and 23.4 carefully. Doing so now will help you tremendously later!

Electric Potential Energy of Two Point Charges

The idea of electric potential energy isn't restricted to the special case of a uniform electric field. Indeed, we can apply this concept to a point charge in *any* electric field caused by a static charge distribution. Recall from Chapter 21 that



23.4 A negative charge moving (a) in the direction of the electric field $\vec{E}$ and (b) in the direction opposite $\vec{E}$. Compare with Fig. 23.3.

we can represent any charge distribution as a collection of point charges. Therefore it's useful to calculate the work done on a test charge q_0 moving in the electric field caused by a single, stationary point charge q.

We'll consider first a displacement along the *radial* line in Fig. 23.5, from point a to point b. The force on q_0 is given by Coulomb's law, and its radial component is

$$F_r = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r^2} \tag{23.7}$$

If q and q_0 have the same sign (+ or −) the force is repulsive and F_r is positive; if the two charges have opposite signs, the force is attractive and F_r is negative. The force is *not* constant during the displacement, and we have to integrate to calculate the work $W_{a\to b}$ done on q_0 by this force as q_0 moves from a to b. We find

$$W_{a\to b} = \int_{r_a}^{r_b} F_r dr = \int_{r_a}^{r_b} \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r^2} dr = \frac{qq_0}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) \tag{23.8}$$

The work done by the electric force for this particular path depends only on the endpoints.

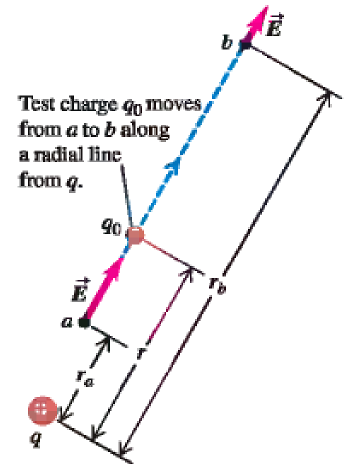
In fact, the work is the same for *all possible* paths from a to b. To prove this, we consider a more general displacement (Fig. 23.6) in which a and b do not lie on the same radial line. From Eq. (23.1) the work done on q_0 during this displacement is given by

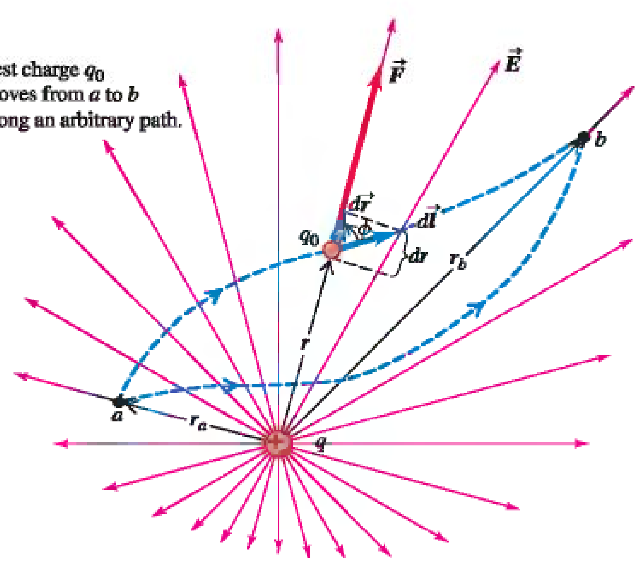
$$W_{a\to b} = \int_{r_a}^{r_b} F\cos\phi \, dl = \int_{r_a}^{r_b} \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r^2}\cos\phi \, dl$$

But the figure shows that $\cos\phi \, dl = dr$. That is, the work done during a small displacement $d\vec{l}$ depends only on the change dr in the distance r between the charges, which is the *radial component* of the displacement. Thus Eq. (23.8) is valid even for this more general displacement; the work done on q_0 by the electric field $\vec{E}$ produced by q depends only on r_a and r_b, not on the details of the path. Also, if q_0 returns to its starting point a by a different path, the total work done in the round-trip displacement is zero (the integral in Eq. (23.8) is from r_a back to r_a). These are the needed characteristics for a conservative force, as we defined it in Section 7.3. Thus the force on q_0 is a *conservative* force.

We see that Eqs. (23.2) and (23.8) are consistent if we define $qq_0/4\pi\epsilon_0 r_a$ to be the potential energy U_a when q_0 is at point a, a distance r_a from q, and we define $qq_0/4\pi\epsilon_0 r_b$ to be the potential energy U_b when q_0 is at point b, a distance r_b from

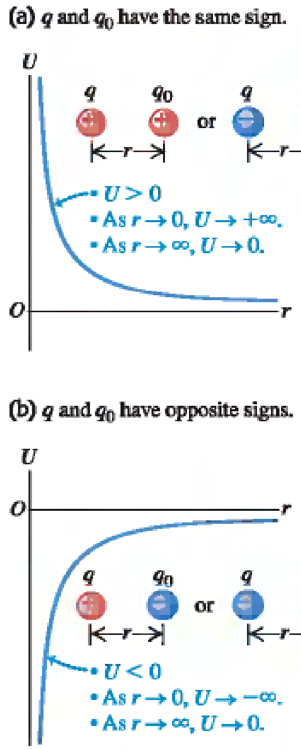
23.5 Test charge q_0 moves along a straight line extending radially from charge q. As it moves from a to b, the distance varies from r_a to r_b.





23.6 The work done on charge q_0 by the electric field of charge q does not depend on the path taken, but only on the distances r_a and r_b.

23.7 Graphs of the potential energy U of two point charges q and q_0 versus their separation r.



q. Thus the potential energy U when the test charge q_0 is at *any* distance r from charge q is

$$U = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r} \quad \text{(electric potential energy of two point charges } q \text{ and } q_0\text{)} \tag{23.9}$$

Note that we have *not* assumed anything about the signs of q and q_0; Eq. (23.9) is valid for any combination of signs. The potential energy is positive if the charges q and q_0 have the same sign (Fig. 23.7a) and negative if they have opposite signs (Fig. 23.7b).

CAUTION **Electric potential energy vs. electric force** Be careful not to confuse Eq. (23.9) for the potential energy of two point charges with the similar expression in Eq. (23.7) for the radial component of the electric force that one charge exerts on the other. The potential energy U is proportional to $1/r$, while the force component F_r is proportional to $1/r^2$.

Potential energy is always defined relative to some reference point where $U = 0$. In Eq. (23.9), U is zero when q and q_0 are infinitely far apart and $r = \infty$. Therefore U represents the work that would be done on the test charge q_0 by the field of q if q_0 moved from an initial distance r to infinity. If q and q_0 have the same sign, the interaction is repulsive, this work is positive, and U is positive at any finite separation (Fig. 23.7a). If the charges have opposite signs, the interaction is attractive, the work done is negative, and U is negative (Fig. 23.7b).

We emphasize that the potential energy U given by Eq. (23.9) is a *shared* property of the two charges q and q_0; it is a consequence of the *interaction* between these two bodies. If the distance between the two charges is changed from r_a to r_b, the change in potential energy is the same whether q is held fixed and q_0 is moved or q_0 is held fixed and q is moved. For this reason, we never use the phrase "the electric potential energy *of* a point charge." (Likewise, if a mass m is at a height h above the earth's surface, the gravitational potential energy is a shared property of the mass m and the earth. We emphasized this in Sections 7.1 and 12.3.)

Gauss's law tells us that the electric field outside any spherically symmetric charge distribution is the same as though all the charge were concentrated at the center. Therefore Eq. (23.9) also holds if the test charge q_0 is outside any spherically symmetric charge distribution with total charge q at a distance r from the center.

Example 23.1 Conservation of energy with electric forces

A positron (the antiparticle of the electron) has a mass of 9.11×10^{-31} kg and a charge $+e = +1.60 \times 10^{-19}$ C. Suppose a positron moves in the vicinity of an alpha particle, which has a charge $+2e = 3.20 \times 10^{-19}$ C. The alpha particle is more than 7000 times as massive as the positron, so we assume that it is at rest in some inertial frame of reference. When the positron is 1.00×10^{-10} m from the alpha particle, it is moving directly away from the alpha particle at a speed of 3.00×10^6 m/s. (a) What is the positron's speed when the two particles are 2.00×10^{-10} m apart? (b) What is the positron's speed when it is very far away from the alpha particle? (c) How would the situation change if the moving particle were an electron (same mass as the positron but opposite charge)?

SOLUTION

IDENTIFY: The electric force between the positron and the alpha particle is conservative, so mechanical energy (kinetic plus potential) is conserved.

SET UP: The kinetic and potential energies at any two points a and b are related by Eq. (23.3), $K_a + U_a = K_b + U_b$, and the potential energy at any distance r is given by Eq. (23.9). We are given complete information about the system at a point a where the two charges are 1.00×10^{-10} m apart. We use Eqs. (23.3) and (23.9) to find the speed at two different values of r in parts (a) and (b), and for the case where the charge $+e$ is replaced by $-e$ in part (c).

EXECUTE: (a) In this part, $r_b = 2.00 \times 10^{-10}$ m and we want to find the final speed v_b of the positron. This appears in the expression for the final kinetic energy, $K_b = \frac{1}{2}mv_b^2$; solving the energy-conservation equation for K_b, we have

$$K_b = K_a + U_a - U_b$$

The values of the energies on the right-hand side of this expression are

$$K_a = \frac{1}{2}mv_a^2 = \frac{1}{2}(9.11 \times 10^{-31}\,\mathrm{kg})(3.00 \times 10^6\,\mathrm{m/s})^2$$
$$= 4.10 \times 10^{-18}\,\mathrm{J}$$
$$U_a = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r_a}$$
$$= (9.0 \times 10^9\,\mathrm{N \cdot m^2/C^2})\frac{(3.20 \times 10^{-19}\,\mathrm{C})(1.60 \times 10^{-19}\,\mathrm{C})}{1.00 \times 10^{-10}\,\mathrm{m}}$$
$$= 4.61 \times 10^{-18}\,\mathrm{J}$$
$$U_b = (9.0 \times 10^9\,\mathrm{N \cdot m^2/C^2})\frac{(3.20 \times 10^{-19}\,\mathrm{C})(1.60 \times 10^{-19}\,\mathrm{C})}{2.00 \times 10^{-10}\,\mathrm{m}}$$
$$= 2.30 \times 10^{-18}\,\mathrm{J}$$

Hence the final kinetic energy is

$$K_b = \frac{1}{2}mv_b^2 = K_a + U_a - U_b$$
$$= 4.10 \times 10^{-18}\,\mathrm{J} + 4.61 \times 10^{-18}\,\mathrm{J} - 2.30 \times 10^{-18}\,\mathrm{J}$$
$$= 6.41 \times 10^{-18}\,\mathrm{J}$$

and the final speed of the positron is

$$v_b = \sqrt{\frac{2K_b}{m}} = \sqrt{\frac{2(6.41 \times 10^{-18}\,\mathrm{J})}{9.11 \times 10^{-31}\,\mathrm{kg}}} = 3.8 \times 10^6\,\mathrm{m/s}$$

The force is repulsive, so the positron speeds up as it moves away from the stationary alpha particle.

(b) When the final positions of the positron and alpha particle are very far apart, the separation r_b approaches infinity and the final potential energy U_b approaches zero. Then the final kinetic energy of the positron is

$$K_b = K_a + U_a - U_b = 4.10 \times 10^{-18}\,\mathrm{J} + 4.61 \times 10^{-18}\,\mathrm{J} - 0$$
$$= 8.71 \times 10^{-18}\,\mathrm{J}$$

and its final speed is

$$v_b = \sqrt{\frac{2K_b}{m}} = \sqrt{\frac{2(8.71 \times 10^{-18}\,\mathrm{J})}{9.11 \times 10^{-31}\,\mathrm{kg}}} = 4.4 \times 10^6\,\mathrm{m/s}$$

Comparing to part (a), we see that as the positron moves from $r = 2.00 \times 10^{-10}$ m to infinity, the additional work done on it by the electric field of the alpha particle increases the speed by only about 16%. This is because the electric force decreases rapidly with distance.

(c) If the moving charge is negative, the force on it is attractive rather than repulsive, and we expect it to slow down rather than speed up. The only difference in the above calculations is that both potential-energy quantities are negative. From part (a), at a distance $r_b = 2.00 \times 10^{-10}$ m we have

$$K_b = K_a + U_a - U_b$$
$$= 4.10 \times 10^{-18}\,\mathrm{J} + (-4.61 \times 10^{-18}\,\mathrm{J}) - (-2.30 \times 10^{-18}\,\mathrm{J})$$
$$= 1.79 \times 10^{-18}\,\mathrm{J}$$
$$v_b = \sqrt{\frac{2K_b}{m}} = 2.0 \times 10^6\,\mathrm{m/s}$$

From part (b), at $r_b = \infty$ the kinetic energy of the electron would seem to be

$$K_b = K_a + U_a - U_b$$
$$= 4.10 \times 10^{-18}\,\mathrm{J} + (-4.61 \times 10^{-18}\,\mathrm{J}) - 0$$
$$= -5.1 \times 10^{-19}\,\mathrm{J}$$

But kinetic energies can *never* be negative! This result means that the electron can never reach $r_b = \infty$; the attractive force brings the electron to a halt at a finite distance from the alpha particle. The electron will then begin to move back toward the alpha particle. You can solve for the distance r_b at which the electron comes momentarily to rest by setting K_b equal to zero in the equation for conservation of mechanical energy.

EVALUATE: It's useful to compare our calculations with Fig. 23.7. In parts (a) and (b), the charges have the same sign; since $r_b > r_a$, the potential energy U_b is less than U_a. In part (c), the charges have opposite signs; since $r_b > r_a$, the potential energy U_b is greater (that is, less negative) than U_a.

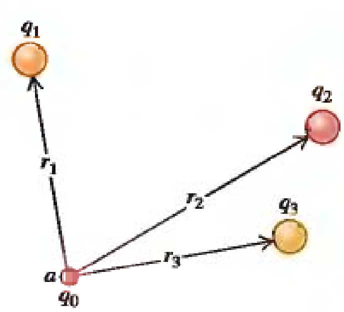
Electric Potential Energy with Several Point Charges

Suppose the electric field $\vec{E}$ in which charge q_0 moves is caused by *several* point charges $q_1, q_2, q_3, \ldots$ at distances $r_1, r_2, r_3, \ldots$ from q_0, as in Fig. 23.8. For example, q_0 could be a positive ion moving in the presence of other ions (Fig. 23.9). The total electric field at each point is the *vector sum* of the fields due to the individual charges, and the total work done on q_0 during any displacement is the sum of the contributions from the individual charges. From Eq. (23.9) we conclude that the potential energy associated with the test charge q_0 at point a in Fig. 23.8 is the *algebraic* sum (*not* a vector sum):

$$U = \frac{q_0}{4\pi\epsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \cdots\right) = \frac{q_0}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i} \qquad \text{(point charge } q_0 \text{ and collection of charges } q_i\text{)} \qquad (23.10)$$

When q_0 is at a different point b, the potential energy is given by the same expression, but $r_1, r_2, \ldots$ are the distances from $q_1, q_2, \ldots$ to point b. The work

23.8 The potential energy associated with a charge q_0 at point a depends on the other charges q_1, q_2, and q_3 and on their distances r_1, r_2, and r_3 from point a.



23.9 This ion engine for spacecraft uses electric forces to eject a stream of positive xenon ions (Xe^+) at speeds in excess of 30 km/s. The thrust produced is very low (about 0.09 newton) but can be maintained continuously for days, in contrast to chemical rockets, which produce a large thrust for a short time (see Fig. 8.33). Such ion engines have been used for maneuvering interplanetary spacecraft.

done on charge q_0 when it moves from a to b along any path is equal to the difference $U_a - U_b$ between the potential energies when q_0 is at a and at b.

We can represent *any* charge distribution as a collection of point charges, so Eq. (23.10) shows that we can always find a potential-energy function for *any* static electric field. It follows that **for every electric field due to a static charge distribution, the force exerted by that field is conservative.**

Equations (23.9) and (23.10) define U to be zero when all the distances $r_1, r_2, \ldots$ are infinite—that is, when the test charge q_0 is very far away from all the charges that produce the field. As with any potential-energy function, the point where $U = 0$ is arbitrary; we can always add a constant to make U equal zero at any point we choose. In electrostatics problems it's usually simplest to choose this point to be at infinity. When we analyze electric circuits in Chapters 25 and 26, other choices will be more convenient.

Equation (23.10) gives the potential energy associated with the presence of the test charge q_0 in the $\vec{E}$ field produced by $q_1, q_2, q_3, \ldots$. But there is also potential energy involved in assembling these charges. If we start with charges $q_1, q_2, q_3, \ldots$ all separated from each other by infinite distances and then bring them together so that the distance between q_i and q_j is r_{ij}, the *total* potential energy U is the sum of the potential energies of interaction for each pair of charges. We can write this as

$$U = \frac{1}{4\pi\epsilon_0}\sum_{i<j}\frac{q_i q_j}{r_{ij}} \tag{23.11}$$

This sum extends over all *pairs* of charges; we don't let $i = j$ (because that would be an interaction of a charge with itself), and we include only terms with $i < j$ to make sure that we count each pair only once. Thus, to account for the interaction between q_3 and q_4, we include a term with $i = 3$ and $j = 4$ but not a term with $i = 4$ and $j = 3$.

Interpreting Electric Potential Energy

As a final comment, here are two viewpoints on electric potential energy. We have defined it in terms of the work done *by the electric field* on a charged particle moving in the field, just as in Chapter 7 we defined potential energy in terms of the work done by gravity or by a spring. When a particle moves from point a to point b, the work done on it by the electric field is $W_{a\to b} = U_a - U_b$. Thus the potential-energy difference $U_a - U_b$ equals *the work that is done by the electric force when the particle moves from a to b.* When U_a is greater than U_b, the field does positive work on the particle as it "falls" from a point of higher potential energy (a) to a point of lower potential energy (b).

An alternative but equivalent viewpoint is to consider how much work we would have to do to "raise" a particle from a point b where the potential energy is U_b to a point a where it has a greater value U_a (pushing two positive charges closer together, for example). To move the particle slowly (so as not to give it any kinetic energy), we need to exert an additional external force $\vec{F}_{ext}$ that is equal and opposite to the electric-field force and does positive work. The potential-energy difference $U_a - U_b$ is then defined as *the work that must be done by an external force to move the particle slowly from b to a against the electric force.* Because $\vec{F}_{ext}$ is the negative of the electric-field force and the displacement is in the opposite direction, this definition of the potential difference $U_a - U_b$ is equivalent to that given above. This alternative viewpoint also works if U_a is less than U_b, corresponding to "lowering" the particle; an example is moving two positive charges away from each other. In this case, $U_a - U_b$ is again equal to the work done by the external force, but now this work is negative.

We will use both of these viewpoints in the next section to interpret what is meant by electric *potential,* or potential energy per unit charge.

Example 23.2 A system of point charges

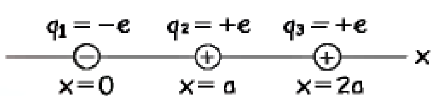
Two point charges are located on the x-axis, $q_1 = -e$ at $x = 0$ and $q_2 = +e$ at $x = a$. (a) Find the work that must be done by an external force to bring a third point charge $q_3 = +e$ from infinity to $x = 2a$. (b) Find the total potential energy of the system of three charges.

SOLUTION

IDENTIFY: This problem involves the relationship between the work done to move a point charge and the change in potential energy. It also involves the expression for the potential energy of a collection of point charges.

SET UP: Figure 23.10 shows the final arrangement of the three charges. To find the work required to bring q_3 in from infinity, we use Eq. (23.10) to find the potential energy associated with q_3 in the presence of q_1 and q_2. We then use Eq. (23.11) to find the total potential energy of the system.

23.10 Our sketch of the situation after the third charge has been brought in from infinity.



EXECUTE: (a) The work that must be done on q_3 by an external force $\vec{F}_{ext}$ is equal to the difference between two quantities: the potential energy U associated with q_3 when it is at $x = 2a$ and the potential energy when it is infinitely far away. The second of these is zero, so the work that must be done is equal to U. The distances between the charges are $r_{13} = 2a$ and $r_{23} = a$, so from Eq. (23.10),

$$W = U = \frac{q_3}{4\pi\epsilon_0}\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right) = \frac{+e}{4\pi\epsilon_0}\left(\frac{-e}{2a} + \frac{+e}{a}\right) = \frac{+e^2}{8\pi\epsilon_0 a}$$

If q_3 is brought in from infinity along the $+x$-axis, it is attracted by q_1 but is repelled more strongly by q_2; hence positive work must be done to push q_3 to the position at $x = 2a$.

(b) The total potential energy of the assemblage of three charges is given by Eq. (23.11):

$$U = \frac{1}{4\pi\epsilon_0}\sum_{i<j}\frac{q_i q_j}{r_{ij}} = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$
$$= \frac{1}{4\pi\epsilon_0}\left(\frac{(-e)(e)}{a} + \frac{(-e)(e)}{2a} + \frac{(e)(e)}{a}\right) = \frac{-e^2}{8\pi\epsilon_0 a}$$

EVALUATE: Since our result in part (b) is negative, the system has lower potential energy than it would if the three charges were infinitely far apart. An external force would have to do *negative* work to bring the three charges from infinity to assemble this entire arrangement and would have to do *positive* work to move the three charges back to infinity.

Test Your Understanding of Section 23.1 Consider the system of three point charges in Example 21.4 (Section 21.3) and shown in Fig. 21.14. (a) What is the sign of the total potential energy of this system? (i) positive; (ii) negative; (iii) zero. (b) What is the sign of the total amount of work you would have to do to move these charges infinitely far from each other? (i) positive; (ii) negative; (iii) zero.

23.2 Electric Potential

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11.13 Electrical Potential Energy and Potential

In Section 23.1 we looked at the potential energy U associated with a test charge q_0 in an electric field. Now we want to describe this potential energy on a "per unit charge" basis, just as electric field describes the force per unit charge on a charged particle in the field. This leads us to the concept of *electric potential*, often called simply *potential*. This concept is very useful in calculations involving energies of charged particles. It also facilitates many electric-field calculations because electric potential is closely related to the electric field $\vec{E}$. When we need to determine an electric field, it is often easier to determine the potential first and then find the field from it.

Potential is *potential energy per unit charge*. We define the potential V at any point in an electric field as the potential energy U *per unit charge* associated with a test charge q_0 at that point:

$$V = \frac{U}{q_0} \quad \text{or} \quad U = q_0 V \tag{23.12}$$

Potential energy and charge are both scalars, so potential is a scalar quantity. From Eq. (23.12) its units are found by dividing the units of energy by those of charge. The SI unit of potential, called one **volt** (1 V) in honor of the Italian scientist and electrical experimenter Alessandro Volta (1745–1827), equals 1 joule per coulomb:

$$1\text{ V} = 1\text{ volt} = 1\text{ J/C} = 1\text{ joule/coulomb}$$

Let's put Eq. (23.2), which equates the work done by the electric force during a displacement from a to b to the quantity $-\Delta U = -(U_b - U_a)$, on a "work per unit charge" basis. We divide this equation by q_0, obtaining

$$\frac{W_{a\to b}}{q_0} = -\frac{\Delta U}{q_0} = -\left(\frac{U_b}{q_0} - \frac{U_a}{q_0}\right) = -(V_b - V_a) = V_a - V_b \quad (23.13)$$

where $V_a = U_a/q_0$ is the potential energy per unit charge at point a and similarly for V_b. We call V_a and V_b the *potential at point a* and *potential at point b*, respectively. Thus the work done per unit charge by the electric force when a charged body moves from a to b is equal to the potential at a minus the potential at b.

The difference $V_a - V_b$ is called the *potential of a with respect to b*; we sometimes abbreviate this difference as $V_{ab} = V_a - V_b$ (note the order of the subscripts). This is often called the potential difference between a and b, but that's ambiguous unless we specify which is the reference point. In electric circuits, which we will analyze in later chapters, the potential difference between two points is often called **voltage** (Fig. 23.11). Equation (23.13) then states: **V_{ab}, the potential of a with respect to b, equals the work done by the electric force when a UNIT charge moves from a to b.**

23.11 The voltage of this battery equals the difference in potential $V_{ab} = V_a - V_b$ between its positive terminal (point a) and its negative terminal (point b).

Another way to interpret the potential difference V_{ab} in Eq. (23.13) is to use the alternative viewpoint mentioned at the end of Section 23.1. In that viewpoint, $U_a - U_b$ is the amount of work that must be done by an *external* force to move a particle of charge q_0 slowly from b to a against the electric force. The work that must be done *per unit charge* by the external force is then $(U_a - U_b)/q_0 = V_a - V_b = V_{ab}$. In other words: **$V_{ab}$, the potential of a with respect to b, equals the work that must be done to move a UNIT charge slowly from b to a against the electric force.**

An instrument that measures the difference of potential between two points is called a *voltmeter*. In Chapter 26 we will discuss the principle of the common type of moving-coil voltmeter. There are also much more sensitive potential-measuring devices that use electronic amplification. Instruments that can measure a potential difference of 1 μV are common, and sensitivities down to 10^{-12} V can be attained.

Calculating Electric Potential

To find the potential V due to a single point charge q, we divide Eq. (23.9) by q_0:

$$V = \frac{U}{q_0} = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \quad \text{(potential due to a point charge)} \quad (23.14)$$

where r is the distance from the point charge q to the point at which the potential is evaluated. If q is positive, the potential that it produces is positive at all points; if q is negative, it produces a potential that is negative everywhere. In either case, V is equal to zero at $r = \infty$, an infinite distance from the point charge. Note that potential, like electric field, is independent of the test charge q_0 that we use to define it.

Similarly, we divide Eq. (23.10) by q_0 to find the potential due to a collection of point charges:

$$V = \frac{U}{q_0} = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i} \quad \text{(potential due to a collection of point charges)} \quad (23.15)$$

In this expression, r_i is the distance from the ith charge, q_i, to the point at which V is evaluated. Just as the electric field due to a collection of point charges is the *vector* sum of the fields produced by each charge, the electric potential due to a collection of point charges is the *scalar* sum of the potentials due to each charge. When we have a continuous distribution of charge along a line, over a surface, or through a volume, we divide the charge into elements dq, and the sum in Eq. (23.15) becomes an integral:

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r} \quad \text{(potential due to a continuous distribution of charge)} \tag{23.16}$$

where r is the distance from the charge element dq to the field point where we are finding V. We'll work out several examples of such cases. The potential defined by Eqs. (23.15) and (23.16) is zero at points that are infinitely far away from *all* the charges. Later we'll encounter cases in which the charge distribution itself extends to infinity. We'll find that in such cases we cannot set $V = 0$ at infinity, and we'll need to exercise care in using and interpreting Eqs. (23.15) and (23.16).

CAUTION **What is electric potential?** Before getting too involved in the details of how to calculate electric potential, you should stop and remind yourself what potential is. The electric *potential* at a certain point is the potential energy that would be associated with a *unit* charge placed at that point. That's why potential is measured in joules per coulomb, or volts. Keep in mind, too, that there doesn't have to be a charge at a given point for a potential V to exist at that point. (In the same way, an electric field can exist at a given point even if there's no charge there to respond to it.)

Finding Electric Potential from Electric Field

When we are given a collection of point charges, Eq. (23.15) is usually the easiest way to calculate the potential V. But in some problems in which the electric field is known or can be found easily, it is easier to determine V from $\vec{E}$. The force $\vec{F}$ on a test charge q_0 can be written as $\vec{F} = q_0\vec{E}$, so from Eq. (23.1) the work done by the electric force as the test charge moves from a to b is given by

$$W_{a\to b} = \int_a^b \vec{F}\cdot d\vec{l} = \int_a^b q_0\vec{E}\cdot d\vec{l}$$

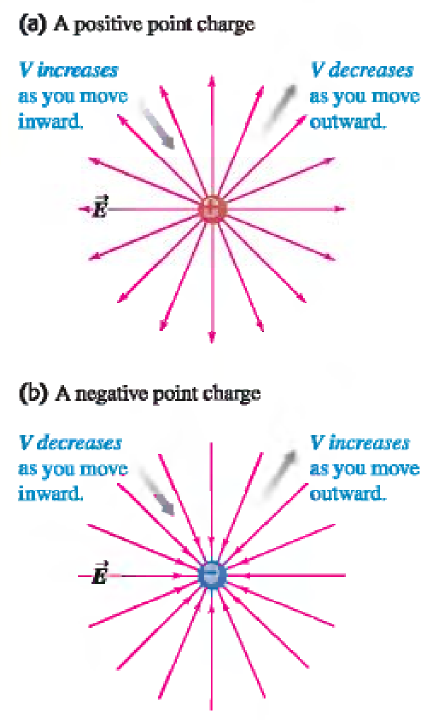
If we divide this by q_0 and compare the result with Eq. (23.13), we find

$$V_a - V_b = \int_a^b \vec{E}\cdot d\vec{l} = \int_a^b E\cos\phi\, dl \quad \text{(potential difference as an integral of } \vec{E}\text{)} \tag{23.17}$$

The value of $V_a - V_b$ is independent of the path taken from a to b, just as the value of $W_{a\to b}$ is independent of the path. To interpret Eq. (23.17), remember that $\vec{E}$ is the electric force per unit charge on a test charge. If the line integral $\int_a^b \vec{E}\cdot d\vec{l}$ is positive, the electric field does positive work on a positive test charge as it moves from a to b. In this case the electric potential energy decreases as the test charge moves, so the potential energy per unit charge decreases as well; hence V_b is less than V_a and $V_a - V_b$ is positive.

As an illustration, consider a positive point charge (Fig. 23.12a). The electric field is directed away from the charge, and $V = q/4\pi\epsilon_0 r$ is positive at any finite distance from the charge. If you move away from the charge, in the direction of $\vec{E}$, you move toward lower values of V; if you move toward the charge, in the direction opposite $\vec{E}$, you move toward greater values of V. For the negative point charge in Fig. 23.12b, $\vec{E}$ is directed toward the charge and $V = q/4\pi\epsilon_0 r$ is negative at any finite distance from the charge. In this case, if you move toward the charge, you are moving in the direction of $\vec{E}$ and in the direction of decreasing (more negative) V. Moving away from the charge, in the direction opposite $\vec{E}$,

23.12 If you move in the direction of $\vec{E}$, electric potential V decreases; if you move in the direction opposite $\vec{E}$, V increases.



moves you toward increasing (less negative) values of V. The general rule, valid for *any* electric field, is: Moving *with* the direction of $\vec{E}$ means moving in the direction of *decreasing* V, and moving *against* the direction of $\vec{E}$ means moving in the direction of *increasing* V.

Also, a positive test charge q_0 experiences an electric force in the direction of $\vec{E}$, toward lower values of V; a negative test charge experiences a force opposite $\vec{E}$, toward higher values of V. Thus a positive charge tends to "fall" from a high-potential region to a lower-potential region. The opposite is true for a negative charge.

Notice that Eq. (23.17) can be rewritten as

$$V_a - V_b = -\int_b^a \vec{E} \cdot d\vec{l} \qquad (23.18)$$

This has a negative sign compared to the integral in Eq. (23.17), and the limits are reversed; hence Eqs. (23.17) and (23.18) are equivalent. But Eq. (23.18) has a slightly different interpretation. To move a unit charge slowly against the electric force, we must apply an *external* force per unit charge equal to $-\vec{E}$, equal and opposite to the electric force per unit charge $\vec{E}$. Equation (23.18) says that $V_a - V_b = V_{ab}$, the potential of a with respect to b, equals the work done per unit charge by this external force to move a unit charge from b to a. This is the same alternative interpretation we discussed under Eq. (23.13).

Equations (23.17) and (23.18) show that the unit of potential difference (1 V) is equal to the unit of electric field (1 N/C) multiplied by the unit of distance (1 m). Hence the unit of electric field can be expressed as 1 *volt per meter* (1 V/m), as well as 1 N/C:

$$1\text{ V/m} = 1\text{ volt/meter} = 1\text{ N/C} = 1\text{ newton/coulomb}$$

In practice, the volt per meter is the usual unit of electric-field magnitude.

Electron Volts

The magnitude e of the electron charge can be used to define a unit of energy that is useful in many calculations with atomic and nuclear systems. When a particle with charge q moves from a point where the potential is V_b to a point where it is V_a, the change in the potential energy U is

$$U_a - U_b = q(V_a - V_b) = qV_{ab}$$

If the charge q equals the magnitude e of the electron charge, 1.602×10^{-19} C, and the potential difference is $V_{ab} = 1$ V, the change in energy is

$$U_a - U_b = (1.602 \times 10^{-19}\text{ C})(1\text{ V}) = 1.602 \times 10^{-19}\text{ J}$$

This quantity of energy is defined to be 1 **electron volt** (1 eV):

$$1\text{ eV} = 1.602 \times 10^{-19}\text{ J}$$

The multiples meV, keV, MeV, GeV, and TeV are often used.

CAUTION **Electron volts vs. volts** Remember that the electron volt is a unit of energy, *not* a unit of potential or potential difference!

When a particle with charge e moves through a potential difference of 1 volt, the change in potential *energy* is 1 eV. If the charge is some multiple of e—say Ne—the change in potential energy in electron volts is N times the potential difference in volts. For example, when an alpha particle, which has charge $2e$, moves between two points with a potential difference of 1000 V, the change in potential energy is $2(1000\text{ eV}) = 2000$ eV. To confirm this, we write

$$\begin{aligned} U_a - U_b = qV_{ab} &= (2e)(1000\text{ V}) = (2)(1.602 \times 10^{-19}\text{ C})(1000\text{ V}) \\ &= 3.204 \times 10^{-16}\text{ J} = 2000\text{ eV} \end{aligned}$$

Although we have defined the electron volt in terms of *potential* energy, we can use it for *any* form of energy, such as the kinetic energy of a moving particle. When we speak of a "one-million-electron-volt proton," we mean a proton with a kinetic energy of one million electron volts $(1\ \text{MeV})$, equal to $(10^6)(1.602 \times 10^{-19}\ \text{J}) = 1.602 \times 10^{-13}\ \text{J}$ (Fig. 23.13).

23.13 This accelerator at the Fermi National Accelerator Laboratory in Illinois gives protons a kinetic energy of 400 MeV $(4 \times 10^8\ \text{eV})$. Additional acceleration stages increase their kinetic energy to 980 GeV, or 0.98 TeV $(9.8 \times 10^{11}\ \text{eV})$.

Example 23.3 Electric force and electric potential

A proton (charge $+e = 1.602 \times 10^{-19}$ C) moves in a straight line from point a to point b inside a linear accelerator, a total distance $d = 0.50$ m. The electric field is uniform along this line, with magnitude $E = 1.5 \times 10^7\ \text{V/m} = 1.5 \times 10^7\ \text{N/C}$ in the direction from a to b. Determine (a) the force on the proton; (b) the work done on it by the field; (c) the potential difference $V_a - V_b$.

SOLUTION

IDENTIFY: This problem uses the relationship between electric field (which we are given) and electric force (which is one of our target variables). It also uses the relationship among force, work, and potential energy difference.

SET UP: We are given the electric field, so it is straightforward to find the electric force on the proton. Calculating the work done on the proton by this force is also straightforward because $\vec{E}$ is uniform, which means that the force is constant. Once the work is known, we find the potential difference using Eq. (23.13).

EXECUTE: (a) The force on the proton is in the same direction as the electric field, and its magnitude is

$$F = qE = (1.602 \times 10^{-19}\ \text{C})(1.5 \times 10^7\ \text{N/C}) = 2.4 \times 10^{-12}\ \text{N}$$

(b) The force is constant and in the same direction as the displacement, so the work done on the proton is

$$W_{a \to b} = Fd = (2.4 \times 10^{-12}\ \text{N})(0.50\ \text{m}) = 1.2 \times 10^{-12}\ \text{J}$$
$$= (1.2 \times 10^{-12}\ \text{J})\frac{1\ \text{eV}}{1.602 \times 10^{-19}\ \text{J}}$$
$$= 7.5 \times 10^6\ \text{eV} = 7.5\ \text{MeV}$$

(c) From Eq. (23.13) the potential difference is the work per unit charge, which is

$$V_a - V_b = \frac{W_{a \to b}}{q} = \frac{1.2 \times 10^{-12}\ \text{J}}{1.602 \times 10^{-19}\ \text{C}} = 7.5 \times 10^6\ \text{J/C}$$
$$= 7.5 \times 10^6\ \text{V} = 7.5\ \text{MV}$$

We can get this same result even more easily by remembering that 1 electron volt equals 1 volt multiplied by the charge e. Since the work done is 7.5×10^6 eV and the charge is e, the potential difference is $(7.5 \times 10^6\ \text{eV})/e = 7.5 \times 10^6\ \text{V}$.

EVALUATE: We can check our result in part (c) by using Eq. (23.17) or (23.18) to calculate an integral of the electric field. The angle ϕ between the constant field $\vec{E}$ and the displacement is zero, so Eq. (23.17) becomes

$$V_a - V_b = \int_a^b E\cos\phi\, dl = \int_a^b E\, dl = E\int_a^b dl$$

The integral of dl from a to b is just the distance d, so we again find

$$V_a - V_b = Ed = (1.5 \times 10^7\ \text{V/m})(0.50\ \text{m}) = 7.5 \times 10^6\ \text{V}$$

Example 23.4 Potential due to two point charges

An electric dipole consists of two point charges, $q_1 = +12\text{ nC}$ and $q_2 = -12\text{ nC}$, placed 10 cm apart (Fig. 23.14). Compute the potentials at points a, b, and c by adding the potentials due to either charge, as in Eq. (23.15).

SOLUTION

IDENTIFY: This is the same arrangement of charges as in Example 21.9 (Section 21.5). In that example we calculated electric *field* at each point by doing a *vector* sum. Our target variable in this problem is the electric *potential* V at three points.

SET UP: To find V at each point, we do the *algebraic* sum in Eq. (23.15):

$$V = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i}$$

EXECUTE: At point a the potential due to the positive charge q_1 is

$$\begin{aligned}\frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1} &= (9.0\times10^9\text{ N}\cdot\text{m}^2/\text{C}^2)\frac{12\times10^{-9}\text{ C}}{0.060\text{ m}}\\ &= 1800\text{ N}\cdot\text{m/C}\\ &= 1800\text{ J/C} = 1800\text{ V}\end{aligned}$$

and the potential due to the negative charge q_2 is

$$\begin{aligned}\frac{1}{4\pi\epsilon_0}\frac{q_2}{r_2} &= (9.0\times10^9\text{ N}\cdot\text{m}^2/\text{C}^2)\frac{(-12\times10^{-9}\text{ C})}{0.040\text{ m}}\\ &= -2700\text{ N}\cdot\text{m/C}\\ &= -2700\text{ J/C} = -2700\text{ V}\end{aligned}$$

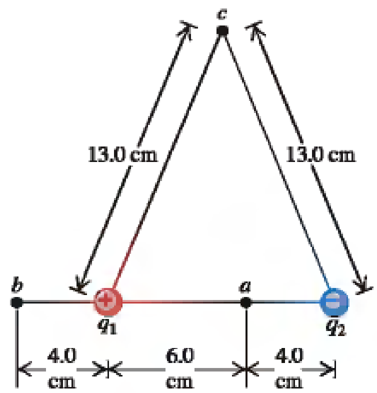
The potential V_a at point a is the sum of these:

$$V_a = 1800\text{ V} + (-2700\text{ V}) = -900\text{ V}$$

By similar calculations you can show that at point b the potential due to the positive charge is $+2700$ V, the potential due to the negative charge is -770 V, and

$$V_b = 2700\text{ V} + (-770\text{ V}) = 1930\text{ V}$$

23.14 What are the potentials at points a, b, and c due to this electric dipole?



At point c the potential due to the positive charge is

$$\frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1} = (9.0\times10^9\text{ N}\cdot\text{m}^2/\text{C}^2)\frac{12\times10^{-9}\text{ C}}{0.13\text{ m}} = 830\text{ V}$$

The potential due to the negative charge is -830 V, and the total potential is zero:

$$V_c = 830\text{ V} + (-830\text{ V}) = 0$$

The potential is also equal to zero at infinity (infinitely far from both charges).

EVALUATE: Comparing this example with Example 21.9 shows that it's much easier to calculate electric potential (a scalar) than electric field (a vector). We'll take advantage of this simplification whenever possible.

Example 23.5 Potential and potential energy

Compute the potential energy associated with a point charge of $+4.0$ nC if it is placed at points a, b, and c in Fig. 23.14.

SOLUTION

IDENTIFY: We know the value of the electric potential at each of these points, and we need to find the potential energy for a point charge placed at each point.

SET UP: For any point charge q, the associated potential energy is $U = qV$. We use the values of V from Example 23.4.

EXECUTE: At point a,

$$U_a = qV_a = (4.0\times10^{-9}\text{ C})(-900\text{ J/C}) = -3.6\times10^{-6}\text{ J}$$

At point b,

$$U_b = qV_b = (4.0\times10^{-9}\text{ C})(1930\text{ J/C}) = 7.7\times10^{-6}\text{ J}$$

At point c,

$$U_c = qV_c = 0$$

All of these values correspond to U and V being zero at infinity.

EVALUATE: Note that *no* net work is done on the 4.0-nC charge if it moves from point c to infinity *by any path*. In particular, let the path be along the perpendicular bisector of the line joining the other two charges q_1 and q_2 in Fig. 23.14. As shown in Example 21.9 (Section 21.5), at points on the bisector the direction of $\vec{E}$ is perpendicular to the bisector. Hence the force on the 4.0-nC charge is perpendicular to the path, and no work is done in any displacement along it.

Example 23.6 Finding potential by integration

By integrating the electric field as in Eq. (23.17), find the potential at a distance r from a point charge q.

SOLUTION

IDENTIFY: This problem asks us to find the electric potential from the electric field.

SET UP: To find the potential V at a distance r from the point charge, we let point a in Eq. (23.17) be at distance r and let point b be at infinity (Fig. 23.15). As usual, we choose the potential to be zero at an infinite distance from the charge.

EXECUTE: To carry out the integral, we can choose any path we like between points a and b. The most convenient path is a straight radial line as shown in Fig. 23.15, so that $d\vec{l}$ is in the radial direction and has magnitude dr. If q is positive, $\vec{E}$ and $d\vec{l}$ are always parallel, so $\phi = 0$ and Eq. (23.17) becomes

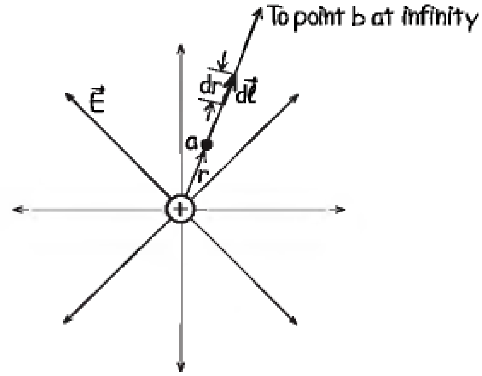
$$V - 0 = \int_r^{\infty} E\,dr = \int_r^{\infty} \frac{q}{4\pi\epsilon_0 r^2}dr$$

$$= -\frac{q}{4\pi\epsilon_0 r}\bigg|_r^{\infty} = 0 - \left(-\frac{q}{4\pi\epsilon_0 r}\right)$$

$$V = \frac{q}{4\pi\epsilon_0 r}$$

This agrees with Eq. (23.14). If q is negative, $\vec{E}$ is radially inward while $d\vec{l}$ is still radially outward, so $\phi = 180°$. Since $\cos 180° = -1$, this adds a minus sign to the above result. However, the field magnitude E is always positive, and since q is negative, we must write $E = |q|/4\pi\epsilon_0 r = -q/4\pi\epsilon_0 r$, giving another minus sign. The two minus signs cancel, and the above result for V is valid for point charges of either sign.

23.15 Calculating the potential by integrating $\vec{E}$ for a single point charge.



EVALUATE: We can get the same result by using Eq. (21.7) for the electric field, which is valid for either sign of q, and writing $d\vec{l} = \hat{r}dr$:

$$V - 0 = V = \int_r^{\infty} \vec{E} \cdot d\vec{l}$$

$$= \int_r^{\infty} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}\cdot\hat{r}\,dr = \int_r^{\infty}\frac{q}{4\pi\epsilon_0 r^2}dr$$

$$V = \frac{q}{4\pi\epsilon_0 r}$$

Example 23.7 Moving through a potential difference

In Fig. 23.16 a dust particle with mass $m = 5.0 \times 10^{-9}$ kg $=$ 5.0 μg and charge $q_0 = 2.0$ nC starts from rest at point a and moves in a straight line to point b. What is its speed v at point b?

SOLUTION

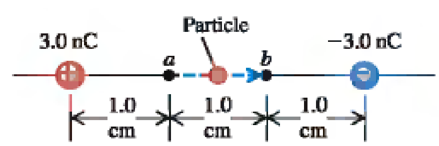
IDENTIFY: This problem involves the change in speed and hence kinetic energy of the particle, so we can use an energy approach. This problem would be difficult to solve without using energy techniques, since the force that acts on the particle varies in magnitude as the particle moves from a to b.

SET UP: Only the conservative electric force acts on the particle, so mechanical energy is conserved:

$$K_a + U_a = K_b + U_b$$

EXECUTE: For this situation, $K_a = 0$ and $K_b = \frac{1}{2}mv^2$. We get the potential energies (U) from the potentials (V) using Eq. (23.12):

23.16 The particle moves from point a to point b; its acceleration is not constant.



$U_a = q_0 V_a$ and $U_b = q_0 V_b$. Substituting these into the energy-conservation equation and solving for v, we find

$$0 + q_0 V_a = \frac{1}{2}mv^2 + q_0 V_b$$

$$v = \sqrt{\frac{2q_0(V_a - V_b)}{m}}$$

Continued

We calculate the potentials using Eq. (23.15), just as we did in Example 23.4:

$$V_a = (9.0 \times 10^9\ \mathrm{N \cdot m^2/C^2}) \times \left(\frac{3.0 \times 10^{-9}\ \mathrm{C}}{0.010\ \mathrm{m}} + \frac{(-3.0 \times 10^{-9}\ \mathrm{C})}{0.020\ \mathrm{m}}\right) = 1350\ \mathrm{V}$$

$$V_b = (9.0 \times 10^9\ \mathrm{N \cdot m^2/C^2}) \times \left(\frac{3.0 \times 10^{-9}\ \mathrm{C}}{0.020\ \mathrm{m}} + \frac{(-3.0 \times 10^{-9}\ \mathrm{C})}{0.010\ \mathrm{m}}\right) = -1350\ \mathrm{V}$$

$$V_a - V_b = (1350\ \mathrm{V}) - (-1350\ \mathrm{V}) = 2700\ \mathrm{V}$$

Finally,

$$v = \sqrt{\frac{2(2.0 \times 10^{-9}\ \mathrm{C})(2700\ \mathrm{V})}{5.0 \times 10^{-9}\ \mathrm{kg}}} = 46\ \mathrm{m/s}$$

EVALUATE: Our result makes sense: The positive test charge gains speed as it moves away from the positive charge and toward the negative charge. To check unit consistency in the final line of the calculation, note that $1\ \mathrm{V} = 1\ \mathrm{J/C}$, so the numerator under the radical has units of J or $\mathrm{kg \cdot m^2/s^2}$.

We can use exactly this same method to find the speed of an electron accelerated across a potential difference of 500 V in an oscilloscope tube or 20 kV in a TV picture tube. The end-of-chapter problems include several examples of such calculations.

Test Your Understanding of Section 23.2 If the electric *potential* at a certain point is zero, does the electric *field* at that point have to be zero? (*Hint:* Consider point *c* in Example 23.4 and Example 21.9.)

23.3 Calculating Electric Potential

When calculating the potential due to a charge distribution, we usually follow one of two routes. If we know the charge distribution, we can use Eq. (23.15) or (23.16). Or if we know how the electric field depends on position, we can use Eq. (23.17), defining the potential to be zero at some convenient place. Some problems require a combination of these approaches.

As you read through these examples, compare them with the related examples of calculating electric *field* in Section 21.5. You'll see how much easier it is to calculate scalar electric potentials than vector electric fields. The moral is clear: Whenever possible, solve problems using an energy approach (using electric potential and electric potential energy) rather than a dynamics approach (using electric fields and electric forces).

Problem-Solving Strategy 23.1 Calculating Electric Potential

IDENTIFY *the relevant concepts:* Remember that potential is *potential energy per unit charge.* Understanding this statement can get you a long way.

SET UP *the problem* using the following steps:

1. Make a drawing that clearly shows the locations of the charges (which may be point charges or a continuous distribution of charge) and your choice of coordinate axes.
2. Indicate on your drawing the position of the point at which you want to calculate the electric potential V. Sometimes this position will be an arbitrary one (say, a point a distance r from the center of a charged sphere).

EXECUTE *the solution* as follows:

1. To find the potential due to a collection of point charges, use Eq. (23.15). If you are given a continuous charge distribution, devise a way to divide it into infinitesimal elements and then use Eq. (23.16). Carry out the integration, using appropriate limits to include the entire charge distribution. In the integral, be careful about which geometric quantities vary and which are held constant.
2. If you are given the electric field, or if you can find it using any of the methods presented in Chapter 21 or 22, it may be easier to use Eq. (23.17) or (23.18) to calculate the potential difference between points a and b. When appropriate, make use of your freedom to define V to be zero at some convenient place, and choose this place to be point b. (For point charges, this will usually be at infinity. For other distributions of charge—especially those that themselves extend to infinity—it may be convenient or necessary to define V_b to be zero at some finite distance from the charge distribution. This is just like defining U to be zero at ground level in gravitational problems.) Then the potential at any other point, say a, can by found from Eq. (23.17) or (23.18) with $V_b = 0$.
3. Remember that potential is a *scalar* quantity, not a *vector.* It doesn't have components! However, you may have to use components of the vectors $\vec{E}$ and $d\vec{l}$ when you use Eq. (23.17) or (23.18).

EVALUATE *your answer:* Check whether your answer agrees with your intuition. If your result gives V as a function of position, make a graph of this function to see whether it makes sense. If you know the electric field, you can make a rough check of your result for V by verifying that V decreases if you move in the direction of $\vec{E}$.

Example 23.8 A charged conducting sphere

A solid conducting sphere of radius R has a total charge q. Find the potential everywhere, both outside and inside the sphere.

SOLUTION

IDENTIFY: We used Gauss's law in Example 22.5 (Section 22.4) to find the electric field at all points for this charge distribution. We can use that result to determine the potential at all points.

SET UP: We choose the origin at the center of the sphere. Since we know E at all values of the distance r from the center of the sphere, we can determine V as a function of r.

EXECUTE: From Example 22.5, at all points *outside* the sphere the field is the same as if the sphere were removed and replaced by a point charge q. We take $V = 0$ at infinity, as we did for a point charge. Then the potential at a point outside the sphere at a distance r from its center is the same as the potential due to a point charge q at the center:

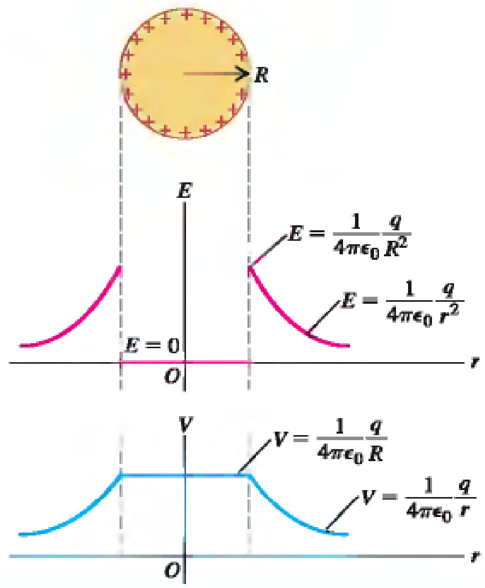
$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

The potential at the surface of the sphere is $V_{\text{surface}} = q/4\pi\epsilon_0 R$.

Inside the sphere, $\vec{E}$ is zero everywhere; otherwise, charge would move within the sphere. Hence if a test charge moves from any point to any other point inside the sphere, no work is done on that charge. This means that the potential is the same at every point inside the sphere and is equal to its value $q/4\pi\epsilon_0 R$ at the surface.

EVALUATE: Figure 23.17 shows the field and potential as a function of r for a positive charge q. In this case the electric field points radially away from the sphere. As you move away from the sphere, in the direction of $\vec{E}$, V decreases (as it should). The electric field at the surface has magnitude $E_{\text{surface}} = |q|/4\pi\epsilon_0 R^2$.

23.17 Electric field magnitude E and potential V at points inside and outside a positively charged spherical conductor.



Ionization and Corona Discharge

The results of Example 23.8 have numerous practical consequences. One consequence relates to the maximum potential to which a conductor in air can be raised. This potential is limited because air molecules become *ionized,* and air becomes a conductor, at an electric-field magnitude of about 3×10^6 V/m. Assume for the moment that q is positive. When we compare the expressions in Example 23.8 for the potential V_{surface} and field magnitude E_{surface} at the surface of a charged conducting sphere, we note that $V_{\text{surface}} = E_{\text{surface}}R$. Thus, if E_m represents the electric-field magnitude at which air becomes conductive (known as the *dielectric strength* of air), then the maximum potential V_m to which a spherical conductor can be raised is

$$V_m = RE_m$$

For a conducting sphere 1 cm in radius in air, $V_m = (10^{-2}\text{ m})(3 \times 10^6\text{ V/m}) = 30{,}000\text{ V}$. No amount of "charging" could raise the potential of a conducting sphere of this size in air higher than about 30,000 V; attempting to raise the potential further by adding extra charge would cause the surrounding air to become ionized and conductive, and the extra added charge would leak into the air.

To attain even higher potentials, high-voltage machines such as Van de Graaff generators use spherical terminals with very large radii (see Fig. 22.27 and the photograph that opens Chapter 22). For example, a terminal of radius $R = 2$ m has a maximum potential $V_m = (2\text{ m})(3 \times 10^6\text{ V/m}) = 6 \times 10^6\text{ V} = 6\text{ MV}$. Such machines are sometimes placed in pressurized tanks filled with a gas such as sulfur hexafluoride (SF_6) that has a larger value of E_m than does air and, therefore, can withstand even larger fields without becoming conductive.

23.18 The metal mast at the top of the Empire State Building acts as a lightning rod. It is struck by lightning as many as 500 times each year.

Our result in Example 23.8 also explains what happens with a charged conductor with a very *small* radius of curvature, such as a sharp point or thin wire. Because the maximum potential is proportional to the radius, even relatively small potentials applied to sharp points in air produce sufficiently high fields just outside the point to ionize the surrounding air, making it become a conductor. The resulting current and its associated glow (visible in a dark room) are called *corona*. Laser printers and photocopying machines use corona from fine wires to spray charge on the imaging drum (see Fig. 21.2).

A large-radius conductor is used in situations where it's important to *prevent* corona. An example is the metal ball at the end of a car radio antenna, which prevents the static that would be caused by corona. Another example is the blunt end of a metal lightning rod (Fig. 23.18). If there is an excess charge in the atmosphere, as happens during thunderstorms, a substantial charge of the opposite sign can build up on this blunt end. As a result, when the atmospheric charge is discharged through a lightning bolt, it tends to be attracted to the charged lightning rod rather than to other nearby structures that could be damaged. (A conducting wire connecting the lightning rod to the ground then allows the acquired charge to dissipate harmlessly.) A lightning rod with a sharp end would allow less charge buildup and hence would be less effective.

Example 23.9 Oppositely charged parallel plates

Find the potential at any height y between the two oppositely charged parallel plates discussed in Section 23.1 (Fig. 23.19).

SOLUTION

IDENTIFY: From Section 23.1 we know the electric *potential energy* U for a test charge q_0 as a function of y. Our goal here is to find the electric *potential* V due to the charges on the plates as a function of y.

SET UP: From Eq. (23.5), $U = q_0Ey$ at a point a distance y above the bottom plate. We use this expression to determine the potential V at such a point.

EXECUTE: The potential $V(y)$ at coordinate y is the potential energy per unit charge:

$$V(y) = \frac{U(y)}{q_0} = \frac{q_0Ey}{q_0} = Ey$$

We have chosen $U(y)$, and therefore $V(y)$, to be zero at point b, where $y = 0$. Even if we choose the potential to be different from zero at b, it is still true that

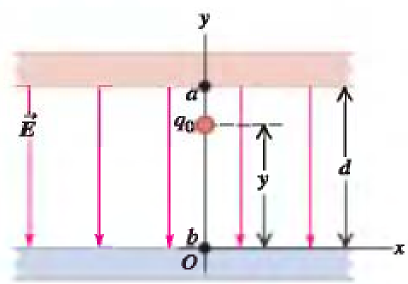
$$V(y) - V_b = Ey$$

The potential decreases as we move in the direction of $\vec{E}$ from the upper to the lower plate. At point a, where $y = d$ and $V(y) = V_a$,

$$V_a - V_b = Ed \quad \text{and} \quad E = \frac{V_a - V_b}{d} = \frac{V_{ab}}{d}$$

where V_{ab} is the potential of the positive plate with respect to the negative plate. That is, the electric field equals the potential difference between the plates divided by the distance between them. For a given potential difference V_{ab}, the smaller the distance d between the two plates, the greater the magnitude E of the electric field. (This relationship between E and V_{ab} holds *only* for the planar geometry we have described. It does *not* work for situations such as concentric cylinders or spheres in which the electric field is not uniform.)

23.19 The charged parallel plates from Fig. 23.2



EVALUATE: Our result tells us how to measure the charge density on the charges on the two plates in Fig. 23.19. In Example 22.8 (Section 22.4), we derived the expression $E = \sigma/\epsilon_0$ for the electric field E between two conducting plates having surface charge densities $+\sigma$ and $-\sigma$. Setting this expression equal to $E = V_{ab}/d$ gives

$$\sigma = \frac{\epsilon_0 V_{ab}}{d}$$

The surface charge density on the positive plate is directly proportional to the potential difference between the plates, and its value σ can be determined by measuring V_{ab}. This technique is useful because no instruments are available that read surface charge density directly. On the negative plate the surface charge density is $-\sigma$.

CAUTION **"Zero potential" is arbitrary** You might think that if a conducting body has zero potential, it must necessarily also have zero net charge. But that just isn't so! As an example, the plate at $y = 0$ in Fig. 23.19 has zero potential ($V = 0$) but has a nonzero charge per unit area $-\sigma$. Remember that there's nothing particularly special about the place where potential is zero; we can *define* this place to be wherever we want it to be.

Example 23.10 An infinite line charge or charged conducting cylinder

Find the potential at a distance r from a very long line of charge with linear charge density (charge per unit length) λ.

SOLUTION

IDENTIFY: One approach to this problem is to divide the line of charge into infinitesimal elements, as we did in Example 21.11 (Section 21.5) to find the electric field produced by such a line. We could then integrate as in Eq. (23.16) to find the net potential V. In this case, however, our task is greatly simplified because we already know the electric field.

SET UP: In both Example 21.11 and Example 22.6 (Section 22.4), we found that the electric field at a distance r from a long straight-line charge (Fig. 23.20a) has only a radial component, given by

$$E_r = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}$$

We use this expression to find the potential by integrating $\vec{E}$ as in Eq. (23.17).

EXECUTE: Since the field has only a radial component, the scalar product $\vec{E}\cdot d\vec{l}$ is equal to $E_r dr$. Hence the potential of any point a with respect to any other point b, at radial distances r_a and r_b from the line of charge, is

$$V_a - V_b = \int_a^b \vec{E}\cdot d\vec{l} = \int_a^b E_r dr = \frac{\lambda}{2\pi\epsilon_0}\int_{r_a}^{r_b}\frac{dr}{r} = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{r_b}{r_a}$$

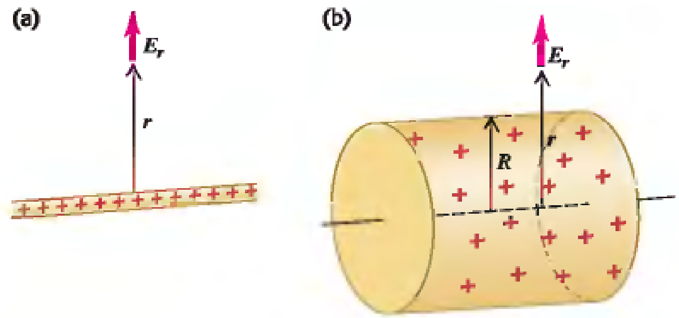
If we take point b at infinity and set $V_b = 0$, we find that V_a is *infinite:*

$$V_a = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{\infty}{r_a} = \infty$$

This shows that if we try to define V to be zero at infinity, then V must be infinite at *any* finite distance from the line charge. This is *not* a useful way to define V for this problem! The difficulty is that the charge distribution itself extends to infinity.

To get around this difficulty, remember that we can define V to be zero at any point we like. We set $V_b = 0$ at point b at an arbitrary radial distance r_0. Then the potential $V = V_a$ at point a at a radial distance r is given by $V - 0 = (\lambda/2\pi\epsilon_0)\ln(r_0/r)$, or

$$V = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{r_0}{r}$$



23.20 Electric field outside (a) a long positively charged wire and (b) a long, positively charged cylinder.

EVALUATE: According to our result, if λ is positive, then V decreases as r increases. This is as it should be: V decreases as we move in the direction of $\vec{E}$.

From Example 22.6, the expression for E_r with which we started also applies outside a long charged conducting cylinder with charge per unit length λ (Fig. 23.20b). Hence our result also gives the potential for such a cylinder, but only for values of r (the distance from the cylinder axis) equal to or greater than the radius R of the cylinder. If we choose r_0 to be the cylinder radius R, so that $V = 0$ when $r = R$, then at any point for which $r > R$,

$$V = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{R}{r}$$

Inside the cylinder, $\vec{E} = 0$, and V has the same value (zero) as on the cylinder's surface.

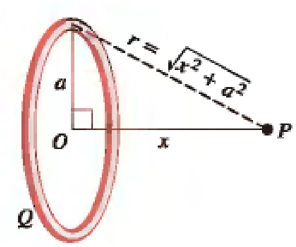
Example 23.11 A ring of charge

Electric charge is distributed uniformly around a thin ring of radius a, with total charge Q (Fig. 23.21). Find the potential at a point P on the ring axis at a distance x from the center of the ring.

SOLUTION

IDENTIFY: We already know the electric field at all points along the x-axis from Example 21.10 (Section 21.5), so we could solve the problem by integrating $\vec{E}$ as in Eq. (23.17) to find V along this axis. Alternatively, we could divide the ring up into infinitesimal segments and use Eq. (23.16) to find V.

SET UP: Figure 23.21 shows that it's far easier to find V on the axis by using the infinitesimal-segment approach. That's because



23.21 All the charge in a ring of charge Q is the same distance r from a point P on the ring axis.

Continued

all parts of the ring (that is, all elements of the charge distribution) are the same distance r from point P.

EXECUTE: Figure 23.21 shows that the distance from each charge element dq on the ring to the point P is $r = \sqrt{x^2 + a^2}$. Hence we can take the factor $1/r$ outside the integral in Eq. (23.16), and

$$V = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{r} = \frac{1}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2+a^2}}\int dq = \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{x^2+a^2}}$$

Potential is a *scalar* quantity; there is no need to consider components of vectors in this calculation, as we had to do when we found the electric field at P. So the potential calculation is a lot simpler than the field calculation.

EVALUATE: When x is much larger than a, the above expression for V becomes approximately equal to $V = Q/4\pi\epsilon_0 x$. This corresponds to the potential of a point charge Q at distance x. So when we are very far away from a charged ring, it looks like a point charge. (We drew a similar conclusion about the electric field of a ring in Example 21.10.)

These results for V can also be found by integrating the expression for E_x found in Example 21.10 (see Problem 23.69).

Example 23.12 A line of charge

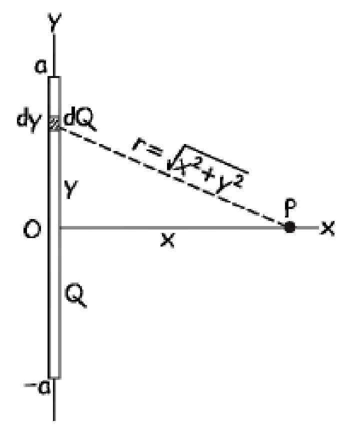
Electric charge Q is distributed uniformly along a line or thin rod of length $2a$. Find the potential at a point P along the perpendicular bisector of the rod at a distance x from its center.

SOLUTION

IDENTIFY: This is the same situation as in Example 21.11 (Section 21.5), where we found an expression for the electric field $\vec{E}$ at an arbitrary point on the x-axis. We could integrate $\vec{E}$ using Eq. (23.17) to find V. Instead, we'll integrate over the charge distribution using Eq. (23.16) to get a bit more experience with this approach.

SET UP: Figure 23.22 shows the situation. Unlike the situation in Example 23.11, each charge element dQ is a different distance from point P.

23.22 Our sketch for this problem.



EXECUTE: As in Example 21.11, the element of charge dQ corresponding to an element of length dy on the rod is given by $dQ = (Q/2a)dy$. The distance from dQ to P is $\sqrt{x^2 + y^2}$, and the contribution dV that it makes to the potential at P is

$$dV = \frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\frac{dy}{\sqrt{x^2+y^2}}$$

To get the potential at P due to the entire rod, we integrate dV over the length of the rod from $y = -a$ to $y = a$:

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\int_{-a}^{a}\frac{dy}{\sqrt{x^2+y^2}}$$

You can look up the integral in a table. The final result is

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\ln\left(\frac{\sqrt{a^2+x^2}+a}{\sqrt{a^2+x^2}-a}\right)$$

EVALUATE: We can check our result by letting x approach infinity. In this limit the point P is infinitely far from all of the charge, so we expect V to approach zero; we invite you to verify that it does so.

As in Example 23.11, this problem is simpler than finding $\vec{E}$ at point P because potential is a scalar quantity and no vector calculations are involved.

Test Your Understanding of Section 23.3 If the electric *field* at a certain point is zero, does the electric *potential* at that point have to be zero? (*Hint:* Consider the center of the ring in Example 23.11 and Example 21.10.)

23.4 Equipotential Surfaces

Field lines (see Section 21.6) help us visualize electric fields. In a similar way, the potential at various points in an electric field can be represented graphically by *equipotential surfaces.* These use the same fundamental idea as topographic maps like those used by hikers and mountain climbers (Fig. 23.23). On a topographic map, contour lines are drawn through points that are all at the same elevation. Any number of these could be drawn, but typically only a few contour lines are shown at equal spacings of elevation. If a mass m is moved over the terrain along such a contour line, the gravitational potential energy mgy does not change because the elevation y is constant. Thus contour lines on a topographic map are really curves of constant gravitational potential energy. Contour lines are close together in regions where the terrain is steep and there are large changes in elevation over a small horizontal distance; the contour lines are farther apart where the terrain is gently sloping. A ball allowed to roll downhill will experience the greatest downhill gravitational force where contour lines are closest together.

By analogy to contour lines on a topographic map, an **equipotential surface** is a three-dimensional surface on which the *electric potential* V is the same at every point. If a test charge q_0 is moved from point to point on such a surface, the *electric* potential energy q_0V remains constant. In a region where an electric field is present, we can construct an equipotential surface through any point. In diagrams we usually show only a few representative equipotentials, often with equal potential differences between adjacent surfaces. No point can be at two different potentials, so equipotential surfaces for different potentials can never touch or intersect.

23.23 Contour lines on a topographic map are curves of constant elevation and hence of constant gravitational potential energy.

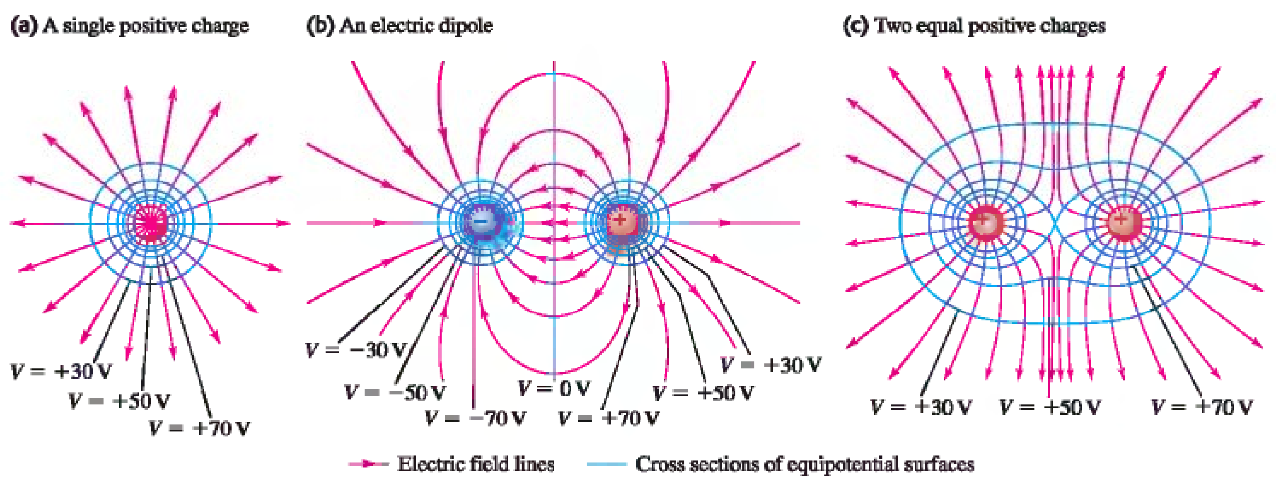
Equipotential Surfaces and Field Lines

Because potential energy does not change as a test charge moves over an equipotential surface, the electric field can do no work on such a charge. It follows that $\vec{E}$ must be perpendicular to the surface at every point so that the electric force $q_0\vec{E}$ is always perpendicular to the displacement of a charge moving on the surface. **Field lines and equipotential surfaces are always mutually perpendicular.** In general, field lines are curves, and equipotentials are curved surfaces. For the special case of a *uniform* field, in which the field lines are straight, parallel, and equally spaced, the equipotentials are parallel *planes* perpendicular to the field lines.

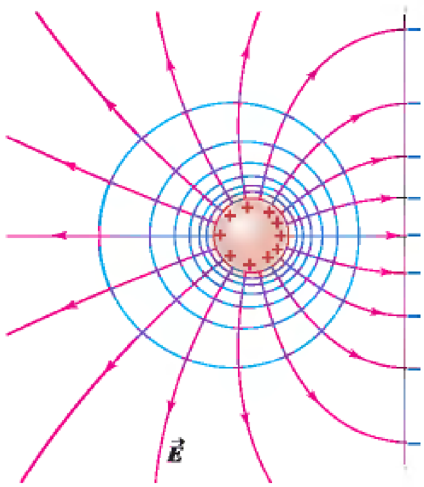
Figure 23.24 shows three arrangements of charges. The field lines in the plane of the charges are represented by red lines, and the intersections of the equipotential surfaces with this plane (that is, cross sections of these surfaces) are shown as blue lines. The actual equipotential surfaces are three-dimensional. At each crossing of an equipotential and a field line, the two are perpendicular.

In Fig. 23.24 we have drawn equipotentials so that there are equal potential differences between adjacent surfaces. In regions where the magnitude of $\vec{E}$ is large, the equipotential surfaces are close together because the field does a rela-

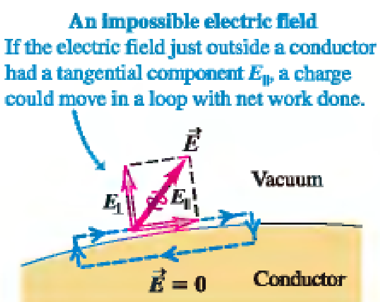
23.24 Cross sections of equipotential surfaces (blue lines) and electric field lines (red lines) for assemblies of point charges. There are equal potential differences between adjacent surfaces. Compare these diagrams to those in Fig. 21.29, which showed only the electric field lines.



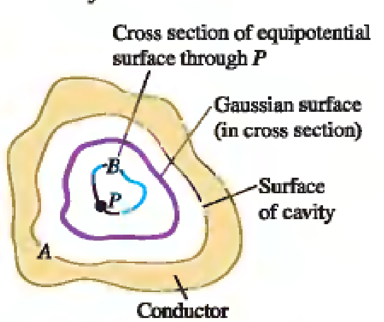
23.25 When charges are at rest, a conducting surface is always an equipotential surface. Field lines are perpendicular to a conducting surface.



23.26 At all points on the surface of a conductor, the electric field must be perpendicular to the surface. If $\vec{E}$ had a tangential component, a net amount of work would be done on a test charge by moving it around a loop as shown here—which is impossible because the electric force is conservative.



23.27 A cavity in a conductor. If the cavity contains no charge, every point in the cavity is at the same potential, the electric field is zero everywhere in the cavity, and there is no charge anywhere on the surface of the cavity.



tively large amount of work on a test charge in a relatively small displacement. This is the case near the point charge in Fig. 23.24a or between the two point charges in Fig. 23.24b; note that in these regions the field lines are also closer together. This is directly analogous to the downhill force of gravity being greatest in regions on a topographic map where contour lines are close together. Conversely, in regions where the field is weaker, the equipotential surfaces are farther apart; this happens at larger radii in Fig. 23.24a, to the left of the negative charge or the right of the positive charge in Fig. 23.24b, and at greater distances from both charges in Fig.23.24c. (It may appear that two equipotential surfaces intersect at the center of Fig.23.24c, in violation of the rule that this can never happen. In fact this is a single figure-8–shaped equipotential surface.)

CAUTION ***E* need not be constant over an equipotential surface** On a given equipotential surface, the potential V has the same value at every point. In general, however, the electric-field magnitude E is *not* the same at all points on an equipotential surface. For instance, on the equipotential surface labeled "$V = -30$ V" in Fig. 23.24b, the magnitude E is less to the left of the negative charge than it is between the two charges. On the figure-8–shaped equipotential surface in Fig. 23.24c, $E = 0$ at the middle point halfway between the two charges; at any other point on this surface, E is nonzero.

Equipotentials and Conductors

Here's an important statement about equipotential surfaces: **When all charges are at rest, the surface of a conductor is always an equipotential surface.** Since the electric field $\vec{E}$ is always perpendicular to an equipotential surface, we can prove this statement by proving that **when all charges are at rest, the electric field just outside a conductor must be perpendicular to the surface at every point** (Fig. 23.25). We know that $\vec{E} = \mathbf{0}$ everywhere inside the conductor; otherwise, charges would move. In particular, at any point just inside the surface the component of $\vec{E}$ tangent to the surface is zero. It follows that the tangential component of $\vec{E}$ is also zero just *outside* the surface. If it were not, a charge could move around a rectangular path partly inside and partly outside (Fig. 23.26) and return to its starting point with a net amount of work having been done on it. This would violate the conservative nature of electrostatic fields, so the tangential component of $\vec{E}$ just outside the surface must be zero at every point on the surface. Thus $\vec{E}$ is perpendicular to the surface at each point, proving our statement.

Finally, we can now prove a theorem that we quoted without proof in Section 22.5. The theorem is as follows: In an electrostatic situation, if a conductor contains a cavity and if no charge is present inside the cavity, then there can be no net charge *anywhere* on the surface of the cavity. This means that if you're inside a charged conducting box, you can safely touch any point on the inside walls of the box without being shocked. To prove this theorem, we first prove that *every point in the cavity is at the same potential.* In Fig. 23.27 the conducting surface A of the cavity is an equipotential surface, as we have just proved. Suppose point P in the cavity is at a different potential; then we can construct a different equipotential surface B including point P.

Now consider a Gaussian surface, shown in Fig. 23.27, between the two equipotential surfaces. Because of the relationship between $\vec{E}$ and the equipotentials, we know that the field at every point between the equipotentials is from A toward B, or else at every point it is from B toward A, depending on which equipotential surface is at higher potential. In either case the flux through this Gaussian surface is certainly not zero. But then Gauss's law says that the charge enclosed by the Gaussian surface cannot be zero. This contradicts our initial assumption that there is *no* charge in the cavity. So the potential at P *cannot* be different from that at the cavity wall.

The entire region of the cavity must therefore be at the same potential. But for this to be true, *the electric field inside the cavity must be zero everywhere.*

Finally, Gauss's law shows that the electric field at any point on the surface of a conductor is proportional to the surface charge density σ at that point. We conclude that *the surface charge density on the wall of the cavity is zero at every point.* This chain of reasoning may seem tortuous, but it is worth careful study.

CAUTION **Equipotantial surfaces vs. Gaussian surfaces** Don't confuse equipotential surfaces with the Gaussian surfaces we encountered in Chapter 22. Gaussian surfaces have relevance only when we are using Gauss's law, and we can choose *any* Gaussian surface that's convenient. We are *not* free to choose the shape of equipotential surfaces; the shape is determined by the charge distribution.

Test Your Understanding of Section 23.4 Would the shapes of the equipotential surfaces in Fig. 23.24 change if the sign of each charge were reversed?

23.5 Potential Gradient

ActivPhysics Online 11.12.3 Electric Potential, Field, and Force

Electric field aud potential are closely related. Equation (23.17), restated here, expresses one aspect of that relationship:

$$V_a - V_b = \int_a^b \vec{E} \cdot d\vec{l}$$

If we know $\vec{E}$ at various points, we can use this equation to calculate potential differences. In this section we show how to turn this around; if we know the potential V at various points, we can use it to determine $\vec{E}$. Regarding V as a function of the coordinates (x, y, z) of a point in space, we will show that the components of $\vec{E}$ are directly related to the *partial derivatives* of V with respect to x, y, and z.

In Eq. (23.17), $V_a - V_b$ is the potential of a with respect to b—that is, the change of potential encountered on a trip from b to a. We can write this as

$$V_a - V_b = \int_b^a dV = -\int_a^b dV$$

where dV is the infinitesimal change of potential accompanying an infinitesimal element $d\vec{l}$ of the path from b to a. Comparing to Eq. (23.17), we have

$$-\int_a^b dV = \int_a^b \vec{E} \cdot d\vec{l}$$

These two integrals must be equal for *any* pair of limits a and b, and for this to be true the *integrands* must be equal. Thus, for *any* infinitesimal displacement $d\vec{l}$,

$$-dV = \vec{E} \cdot d\vec{l}$$

To interpret this expression, we write $\vec{E}$ and $d\vec{l}$ in terms of their components: $\vec{E} = \hat{\imath} E_x + \hat{\jmath} E_y + \hat{k} E_z$ and $d\vec{l} = \hat{\imath}\, dx + \hat{\jmath}\, dy + \hat{k}\, dz$. Then we have

$$-dV = E_x dx + E_y dy + E_z dz$$

Suppose the displacement is parallel to the x-axis, so $dy = dz = 0$. Then $-dV = E_x dx$ or $E_x = -(dV/dx)_{y,\,z\,\text{constant}}$, where the subscript reminds us that only x varies in the derivative; recall that V is in general a function of x, y, and z. But this is just what is meant by the partial derivative $\partial V/\partial x$. The y- and z-components of $\vec{E}$ are related to the corresponding derivatives of V in the same way, so we have

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z} \qquad \text{(components of } \vec{E} \text{ in terms of } V) \qquad (23.19)$$

This is consistent with the units of electric field being V/m. In terms of unit vectors we can write $\vec{E}$ as

$$\vec{E} = -\left(\hat{\imath}\frac{\partial V}{\partial x} + \hat{\jmath}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right) \qquad (\vec{E} \text{ in terms of } V) \tag{23.20}$$

In vector notation the following operation is called the **gradient** of the function f:

$$\vec{\nabla} f = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)f \tag{23.21}$$

The operator denoted by the symbol $\vec{\nabla}$ is called "grad" or "del." Thus in vector notation,

$$\vec{E} = -\vec{\nabla} V \tag{23.22}$$

This is read "$\vec{E}$ is the negative of the gradient of V" or "$\vec{E}$ equals negative grad V." The quantity $\vec{\nabla} V$ is called the *potential gradient.*

At each point, the potential gradient points in the direction in which V *increases* most rapidly with a change in position. Hence at each point the direction of $\vec{E}$ is the direction in which V *decreases* most rapidly and is always perpendicular to the equipotential surface through the point. This agrees with our observation in Section 23.2 that moving in the direction of the electric field means moving in the direction of decreasing potential.

Equation (23.22) doesn't depend on the particular choice of the zero point for V. If we were to change the zero point, the effect would be to change V at every point by the same amount; the derivatives of V would be the same.

If $\vec{E}$ is radial with respect to a point or an axis and r is the distance from the point or the axis, the relationship corresponding to Eqs. (23.19) is

$$E_r = -\frac{\partial V}{\partial r} \qquad \text{(radial electric field)} \tag{23.23}$$

Often we can compute the electric field caused by a charge distribution in either of two ways: directly, by adding the $\vec{E}$ fields of point charges, or by first calculating the potential and then taking its gradient to find the field. The second method is often easier because potential is a *scalar* quantity, requiring at worst the integration of a scalar function. Electric field is a *vector* quantity, requiring computation of components for each element of charge and a separate integration for each component. Thus, quite apart from its fundamental significance, potential offers a very useful computational technique in field calculations. Below, we present two examples in which a knowledge of V is used to find the electric field.

We stress once more that if we know $\vec{E}$ as a function of position, we can calculate V using Eq. (23.17) or (23.18), and if we know V as a function of position, we can calculate $\vec{E}$ using Eq. (23.19), (23.20), or (23.23). Deriving V from $\vec{E}$ requires integration, and deriving $\vec{E}$ from V requires differentiation.

Example 23.13 Potential and field of a point charge

From Eq. (23.14) the potential at a radial distance r from a point charge q is $V = q/4\pi\epsilon_0 r$. Find the vector electric field from this expression for V.

SOLUTION

IDENTIFY: This problem uses the relationship between the electric potential as a function of position and the electric field vector.

SET UP: By symmetry, the electric field has only a radial component E_r. We use Eq. (23.23) to find this component.

EXECUTE: From Eq. (23.23),

$$E_r = -\frac{\partial V}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

so the vector electric field is

$$\vec{E} = \hat{r}E_r = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

EVALUATE: Our result agrees with Eq. (21.7), as it must.

An alternative approach is to ignore the radial symmetry, write the radial distance as $r = \sqrt{x^2 + y^2 + z^2}$, and take the derivatives of V with respect to x, y, and z as in Eq. (23.20). We find

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{x^2+y^2+z^2}}\right) = -\frac{1}{4\pi\epsilon_0}\frac{qx}{(x^2+y^2+z^2)^{3/2}}$$
$$= -\frac{qx}{4\pi\epsilon_0 r^3}$$

and similarly

$$\frac{\partial V}{\partial y} = -\frac{qy}{4\pi\epsilon_0 r^3} \qquad \frac{\partial V}{\partial z} = -\frac{qz}{4\pi\epsilon_0 r^3}$$

From Eq. (23.20), the electric field is

$$\vec{E} = -\left[\hat{\imath}\left(-\frac{qx}{4\pi\epsilon_0 r^3}\right) + \hat{\jmath}\left(-\frac{qy}{4\pi\epsilon_0 r^3}\right) + \hat{k}\left(-\frac{qz}{4\pi\epsilon_0 r^3}\right)\right]$$
$$= \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\left(\frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{r}\right) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

This approach gives us the same answer, but with a bit more effort. Clearly it's best to exploit the symmetry of the charge distribution whenever possible.

Example 23.14 Potential and field of a ring of charge

In Example 23.11 (Section 23.3) we found that for a ring of charge with radius a and total charge Q, the potential at a point P on the ring axis a distance x from the center is

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{x^2 + a^2}}$$

Find the electric field at P.

SOLUTION

IDENTIFY: We are given V as a function of x along the x-axis, and we wish to find the electric field at a point on this axis.

SET UP: From the symmetry of the charge distribution shown in Fig. 23.21, the electric field along the symmetry axis of the ring can have only an x-component. We find it using the first of Eqs. (23.19).

EXECUTE: The x-component of the electric field is

$$E_x = -\frac{\partial V}{\partial x} = \frac{1}{4\pi\epsilon_0}\frac{Qx}{(x^2 + a^2)^{3/2}}$$

EVALUATE: This agrees with the result that we obtained in Example 21.10 (Section 21.5).

CAUTION **Don't use expressions where they don't apply** In this example, V does not appear to be a function of y or z, but it would *not* be correct to conclude that $\partial V/\partial y = \partial V/\partial z = 0$ and $E_y = E_z = 0$ everywhere. The reason is that our expression for V is valid *only for points on the x-axis*, where $y = z = 0$. Hence our expression for E_x is likewise valid on the x-axis only. If we had the complete expression for V valid at *all* points in space, then we could use it to find the components of $\vec{E}$ at any point using Eq. (23.19).

Test Your Understanding of Section 23.5 In a certain region of space the potential is given by $V = A + Bx + Cy^3 + Dxy$, where A, B, C, and D are positive constants. Which of these statements about the electric field $\vec{E}$ in this region of space is correct? (There may be more than one correct answer.) (i) Increasing the value of A will increase the value of $\vec{E}$ at all points; (ii) increasing the value of A will decrease the value of $\vec{E}$ at all points; (iii) $\vec{E}$ has no z-component; (iv) the electric field is zero at the origin $(x = 0, y = 0, z = 0)$.

CHAPTER 23 SUMMARY

Electric potential energy: The electric force caused by any collection of charges at rest is a conservative force. The work W done by the electric force on a charged particle moving in an electric field can be represented by the change in a potential-energy function U.

The electric potential energy for two point charges q and q_0 depends on their separation r. The electric potential energy for a charge q_0 in the presence of a collection of charges q_1, q_2, q_3 depends on the distance from q_0 to each of these other charges. (See Examples 23.1 and 23.2.)

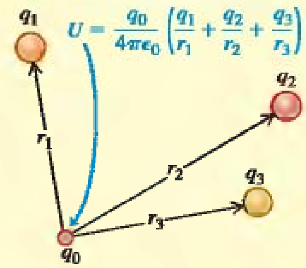
$$W_{a \to b} = U_a - U_b \tag{23.2}$$

$$U = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r} \tag{23.9}$$

(two point charges)

$$U = \frac{q_0}{4\pi\epsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \cdots\right) = \frac{q_0}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i} \tag{23.10}$$

(q_0 in presence of other point charges)



Electric potential: Potential, denoted by V, is potential energy per unit charge. The potential difference between two points equals the amount of work that would be required to move a unit positive test charge between those points. The potential V due to a quantity of charge can be calculated by summing (if the charge is a collection of point charges) or by integrating (if the charge is a distribution). (See Examples 23.3, 23.4, 23.5, 23.7, 23.11, and 23.12.)

The potential difference between two points a and b, also called the potential of a with respect to b, is given by the line integral of $\vec{E}$. The potential at a given point can be found by first finding $\vec{E}$ and then carrying out this integral. (See Examples 23.6, 23.8, 23.9, and 23.10.)

$$V = \frac{U}{q_0} = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \tag{23.14}$$

(due to a point charge)

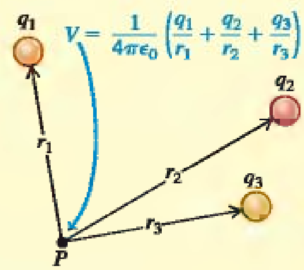
$$V = \frac{U}{q_0} = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i} \tag{23.15}$$

(due to a collection of point charges)

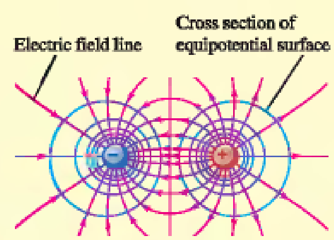
$$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r} \tag{23.16}$$

(due to a charge distribution)

$$V_a - V_b = \int_a^b \vec{E} \cdot d\vec{l} = \int_a^b E\cos\phi \, dl \tag{23.17}$$



Equipotential surfaces: An equipotential surface is a surface on which the potential has the same value at every point. At a point where a field line crosses an equipotential surface, the two are perpendicular. When all charges are at rest, the surface of a conductor is always an equipotential surface and all points in the interior of a conductor are at the same potential. When a cavity within a conductor contains no charge, the entire cavity is an equipotential region and there is no surface charge anywhere on the surface of the cavity.



Finding electric field from electric potentiel: If the potential V is known as a function of the coordinates x, y, and z, the components of electric field $\vec{E}$ at any point are given by partial derivatives of V. (See Examples 23.13 and 23.14.)

$$E_x = -\frac{\partial V}{\partial x} \quad E_y = -\frac{\partial V}{\partial y} \quad E_z = -\frac{\partial V}{\partial z} \tag{23.19}$$

$$\vec{E} = -\left(\hat{\imath}\frac{\partial V}{\partial x} + \hat{\jmath}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right) \tag{23.20}$$

(vector form)

Key Terms

(electric) potential energy, *781*
(electric) potential, *787*
volt, *787*
voltage, *788*
electron volt, *790*
equipotential surface, *799*
gradient, *802*

Answer to Chapter Opening Question ?

A large, constant potential difference V_{ab} is maintained between the welding tool (a) and the metal pieces to be welded (b). From Example 23.9 (Section 23.3) the electric field between two conductors separated by a distance d has magnitude $E = V_{ab}/d$. Hence d must be small in order for the field magnitude E to be large enough to ionize the gas between the conductors a and b (see Section 23.3) and produce an arc through this gas.

Answers to Test Your Understanding Questions

23.1 Answers: (a) (i), (b) (ii) The three charges q_1, q_2, and q_3 are all positive, so all three of the terms in the sum in Eq. (23.11)—q_1q_2/r_{12}, q_1q_3/r_{13}, and q_2q_3/r_{23}—are positive. Hence the total electric potential energy U is positive. This means that it would take positive work to bring the three charges from infinity to the positions shown in Fig. 21.14, and hence *negative* work to move the three charges from these positions back to infinity.

23.2 Answer: no If $V = 0$ at a certain point, $\vec{E}$ does *not* have to be zero at that point. An example is point c in Figs. 21.23 and 23.14, for which there is an electric field in the $+x$-direction (see Example 21.9 in Section 21.5) even though $V = 0$ (see Example 23.4). This isn't a surprising result because V and $\vec{E}$ are quite different quantities: V is the net amount of work required to bring a unit charge from infinity to the point in question, whereas $\vec{E}$ is the electric force that acts on a unit charge when it arrives at that point.

23.3 Answer: no If $\vec{E} = \mathbf{0}$ at a certain point, V does *not* have to be zero at that point. An example is point O at the center of the charged ring in Figs. 21.24 and 23.21. From Example 21.10 (Section 21.5), the electric field is zero at O because the electric-field contributions from different parts of the ring completely cancel. From Example 23.11, however, the potential at O is *not* zero: This point corresponds to $x = 0$, so $V = (1/4\pi\epsilon_0)(Q/a)$. This value of V corresponds to the work that would have to be done to move a unit positive test charge along a path from infinity to point O; it is nonzero because the charged ring repels the test charge, so positive work must be done to move the test charge toward the ring.

23.4 Answer: no If the positive charges in Fig. 23.24 were replaced by negative charges, and vice versa, the equipotential surfaces would be the same but the sign of the potential would be reversed. For example, the surfaces in Fig. 23.24b with potential $V = +30\text{ V}$ and $V = -50\text{ V}$ would have potential $V = -30\text{ V}$ and $V = +50\text{ V}$, respectively.

23.5 Answer: (iii) From Eqs. (23.19), the components of the electric field are $E_x = -\partial V/\partial x = B + Dy$, $E_y = -\partial V/\partial y = 3Cy^2 + Dx$, and $E_z = -\partial V/\partial z = 0$. The value of A has no effect, which means that we can add a constant to the electric potential at all points without changing $\vec{E}$ or the potential difference between two points. The potential does not depend on z, so the z-component of $\vec{E}$ is zero. Note that at the origin the electric field is not zero because it has a nonzero x-component: $E_x = B$, $E_y = 0$, $E_z = 0$.

PROBLEMS

For instructor-assigned homework, go to www.masteringphysics.com

Discussion Questions

Q23.1. A student asked, "Since electrical potential is always proportional to potential energy, why bother with the concept of potential at all?" How would you respond?

Q23.2. The potential (relative to a point at infinity) midway between two charges of equal magnitude and opposite sign is zero. Is it possible to bring a test charge from infinity to this midpoint in such a way that no work is done in any part of the displacement? If so, describe how it can be done. If it is not possible, explain why.

Q23.3. Is it possible to have an arrangement of two point charges separated by a finite distance such that the electric potential energy of the arrangement is the same as if the two charges were infinitely far apart? Why or why not? What if there are three charges? Explain your reasoning.

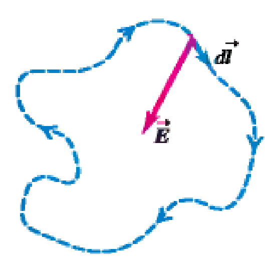
Q23.4. Since potential can have any value you want depending on the choice of the reference level of zero potential, how does a voltmeter know what to read when you connect it between two points?

Q23.5. If $\vec{E}$ is zero everywhere along a certain path that leads from point A to point B, what is the potential difference between those two points? Does this mean that $\vec{E}$ is zero everywhere along *any* path from A to B? Explain.

Q23.6. If $\vec{E}$ is zero throughout a certain region of space, is the potential necessarily also zero in this region? Why or why not? If not, what *can* be said about the potential?

Q23.7. If you carry out the integral of the electric field $\int \vec{E} \cdot d\vec{l}$ for a *closed* path like that shown in Fig. 23.28, the integral will *always* be equal to zero, independent of the shape of the path and independent of where charges may be located relative to the path. Explain why.

Figure 23.28 Question Q23.7.



Q23.8. The potential difference between the two terminals of an AA battery (used in flashlights and portable stereos) is 1.5 V. If two AA batteries are placed end to end with the positive terminal of one battery touching the negative terminal of the other, what is the potential difference between the terminals at the exposed ends of the combination? What if the two positive terminals are touching each other? Explain your reasoning.

Q23.9. It is easy to produce a potential difference of several thousand volts between your body and the floor by scuffing your shoes across a nylon carpet. When you touch a metal doorknob, you get a mild shock. Yet contact with a power line of comparable voltage would probably be fatal. Why is there a difference?

Q23.10. If the electric potential at a single point is known, can $\vec{E}$ at that point be determined? If so, how? If not, why not?

Q23.11. Because electric field lines and equipotential surfaces are always perpendicular, two equipotential surfaces can never cross; if they did, the direction of $\vec{E}$ would be ambiguous at the crossing points. Yet two equipotential surfaces appear to cross at the center of Fig. 23.24c. Explain why there is no ambiguity about the direction of $\vec{E}$ in this particular case.

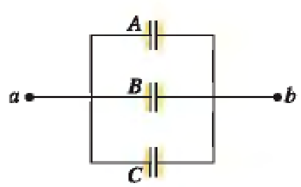
Q23.12. The electric field due to a very large sheet of charge is independent of the distance from the sheet, yet the fields due to the individual point charges on the sheet all obey an inverse-square law. Why doesn't the field of the sheet get weaker at greater distances?

Q23.13. We often say that if point A is at a higher potential than point B, A is at positive potential and B is at negative potential. Does it necessarily follow that a point at positive potential is positively charged, or that a point at negative potential is negatively charged? Illustrate your answers with clear, simple examples.

Q23.14. A conducting sphere is to be charged by bringing in positive charge a little at a time until the total charge is Q. The total work required for this process is alleged to be proportional to Q^2. Is this correct? Why or why not?

Q23.15. Three pairs of parallel metal plates (A, B, and C) are connected as shown in Fig. 23.29, and a battery maintains a potential of 1.5 V across ab. What can you say about the potential difference across each pair of plates? Why?

Figure 23.29 Question Q23.15.



Q23.16. A conducting sphere is placed between two charged parallel plates such as those shown in Fig. 23.2. Does the electric field inside the sphere depend on precisely where between the plates the sphere is placed? What about the electric potential inside the sphere? Do the answers to these questions depend on whether or not there is a net charge on the sphere? Explain your reasoning.

Q23.17. A conductor that carries a net charge Q has a hollow, empty cavity in its interior. Does the potential vary from point to point within the material of the conductor? What about within the cavity? How does the potential inside the cavity compare to the potential within the material of the conductor?

Q23.18. A high-voltage dc power line falls on a car, so the entire metal body of the car is at a potential of 10,000 V with respect to the ground. What happens to the occupants (a) when they are sitting in the car and (b) when they step out of the car? Explain your reasoning.

Q23.19. When a thunderstorm is approaching, sailors at sea sometimes observe a phenomenon called "St. Elmo's fire," a bluish flickering light at the tips of masts. What causes this? Why does it occur at the tips of masts? Why is the effect most pronounced when the masts are wet? (*Hint:* Seawater is a good conductor of electricity.)

Q23.20. A positive point charge is placed near a very large conducting plane. A professor of physics asserted that the field caused by this configuration is the same as would be obtained by removing the plane and placing a negative point charge of equal magnitude in the mirror-image position behind the initial position of the plane. Is this correct? Why or why not? (*Hint:* Inspect Fig. 23.24b.)

Q23.21. In electronics it is customary to define the potential of ground (thinking of the earth as a large conductor) as zero. Is this consistent with the fact that the earth has a net electric charge that is not zero? (Refer to Exercise 21.32.)

Exercises

Section 23.1 Electric Potential Energy

23.1. A point charge $q_1 = +2.40\ \mu\text{C}$ is held stationary at the origin. A second point charge $q_2 = -4.30\ \mu\text{C}$ moves from the point $x = 0.150$ m, $y = 0$ to the point $x = 0.250$ m, $y = 0.250$ m. How much work is done by the electric force on q_2?

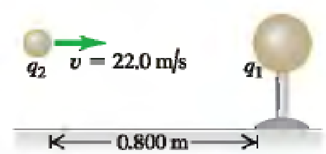
23.2. A point charge q_1 is held stationary at the origin. A second charge q_2 is placed at point a, and the electric potential energy of the pair of charges is $+5.4 \times 10^{-8}$ J. When the second charge is moved to point b, the electric force on the charge does -1.9×10^{-8} J of work. What is the electric potential energy of the pair of charges when the second charge is at point b?

23.3. Energy of the Nucleus. How much work is needed to assemble an atomic nucleus containing three protons (such as Be) if we model it as an equilateral triangle of side 2.00×10^{-15} m with a proton at each vertex? Assume the protons started from very far away.

23.4. (a) How much work would it take to push two protons very slowly from a separation of 2.00×10^{-10} m (a typical atomic distance) to 3.00×10^{-15} m (a typical nuclear distance)? (b) If the protons are both released from rest at the closer distance in part (a), how fast are they moving when they reach their original separation?

23.5. A small metal sphere, carrying a net charge of $q_1 = -2.80\ \mu\text{C}$, is held in a stationary position by insulating supports. A second small metal sphere, with a net charge of $q_2 = -7.80\ \mu\text{C}$ and mass 1.50 g, is projected toward q_1. When the two spheres are 0.800 m apart, q_2 is moving toward q_1 with speed 22.0 m/s (Fig. 23.30). Assume that the two spheres can be treated as point charges. You can ignore the force of gravity. (a) What is the speed of q_2 when the spheres are 0.400 m apart? (b) How close does q_2 get to q_1?

Figure 23.30 Exercise 23.5.



23.6. How far from a -7.20-μC point charge must a $+2.30$-μC point charge be placed for the electric potential energy U of the pair of charges to be -0.400 J? (Take U to be zero when the charges have infinite separation.)

23.7. A point charge $Q = +4.60\ \mu\text{C}$ is held fixed at the origin. A second point charge $q = +1.20\ \mu\text{C}$ with mass of 2.80×10^{-4} kg is placed on the x-axis, 0.250 m from the origin. (a) What is the electric potential energy U of the pair of charges? (Take U to be zero when the charges have infinite separation.) (b) The second point charge is released from rest. What is its speed when its distance from the origin is (i) 0.500 m; (ii) 5.00 m; (iii) 50.0 m?

23.8. Three equal 1.20-μC point charges are placed at the corners of an equilateral triangle whose sides are 0.500 m long. What is the potential energy of the system? (Take as zero the potential energy of the three charges when they are infinitely far apart.)

23.9. A point charge $q_1 = 4.00$ nC is placed at the origin, and a second point charge $q_2 = -3.00$ nC is placed on the x-axis at $x = +20.0$ cm. A third point charge $q_3 = 2.00$ nC is to be placed on the x-axis between q_1 and q_2. (Take as zero the potential energy of the three charges when they are infinitely far apart.) (a) What is the potential energy of the system of the three charges if q_3 is placed at $x = +10.0$ cm? (b) Where should q_3 be placed to make the potential energy of the system equal to zero?

23.10. Four electrons are located at the corners of a square 10.0 nm on a side, with an alpha particle at its midpoint. How much work is needed to move the alpha particle to the midpoint of one of the sides of the square?

23.11. Three point charges, which initially are infinitely far apart, are placed at the corners of an equilateral triangle with sides d. Two of the point charges are identical and have charge q. If zero net work is required to place the three charges at the corners of the triangle, what must the value of the third charge be?

23.12. Two protons are aimed directly toward each other by a cyclotron accelerator with speeds of 1000 km/s, measured relative to the earth. Find the maximum electrical force that these protons will exert on each other.

Section 23.2 Electric Potential

23.13. A uniform electric field is directed due east. Point B is 2.00 m west of point A, point C is 2.00 m east of point A, and point D is 2.00 m south of A. For each point, B, C, and D, is the potential at that point larger, smaller, or the same as at point A? Give the reasoning behind your answers.

23.14. Identical point charges $q = +5.00\ \mu$C are placed at opposite corners of a square. The length of each side of the square is 0.200 m. A point charge $q_0 = -2.00\ \mu$C is placed at one of the empty corners. How much work is done on q_0 by the electric force when q_0 is moved to the other empty corner?

23.15. A small particle has charge $-5.00\ \mu$C and mass 2.00×10^{-4} kg. It moves from point A, where the electric potential is $V_A = +200$ V, to point B, where the electric potential is $V_B = +800$ V. The electric force is the only force acting on the particle. The particle has speed 5.00 m/s at point A. What is its speed at point B? Is it moving faster or slower at B than at A? Explain.

23.16. A particle with a charge of $+4.20$ nC is in a uniform electric field $\vec{E}$ directed to the left. It is released from rest and moves to the left; after it has moved 6.00 cm, its kinetic energy is found to be $+1.50 \times 10^{-6}$ J. (a) What work was done by the electric force? (b) What is the potential of the starting point with respect to the end point? (c) What is the magnitude of $\vec{E}$?

23.17. A charge of 28.0 nC is placed in a uniform electric field that is directed vertically upward and has a magnitude of 4.00×10^4 V/m. What work is done by the electric force when the charge moves (a) 0.450 m to the right; (b) 0.670 m upward; (c) 2.60 m at an angle of 45.0° downward from the horizontal?

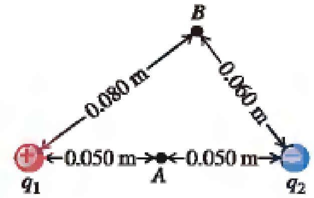
23.18. Two stationary point charges $+3.00$ nC and $+2.00$ nC are separated by a distance of 50.0 cm. An electron is released from rest at a point midway between the two charges and moves along the line connecting the two charges. What is the speed of the electron when it is 10.0 cm from the $+3.00$-nC charge?

23.19. A point charge has a charge of 2.50×10^{-11} C. At what distance from the point charge is the electric potential (a) 90.0 V and (b) 30.0 V? Take the potential to be zero at an infinite distance from the charge.

23.20. Two charges of equal magnitude Q are held a distance d apart. Consider only points on the line passing through both charges. (a) If the two charges have the same sign, find the location of all points (if there are any) at which (i) the potential (relative to infinity) is zero (is the electric field zero at these points?), and (ii) the electric field is zero (is the potential zero at these points?). (b) Repeat part (a) for two charges having opposite signs.

23.21. Two point charges $q_1 = +2.40$ nC and $q_2 = -6.50$ nC are 0.100 m apart. Point A is midway between them; point B is 0.080 m from q_1 and 0.060 m from q_2 (Fig. 23.31). Take the electric potential to be zero at infinity. Find (a) the potential at point A; (b) the potential at point B; (c) the work done by the electric field on a charge of 2.50 nC that travels from point B to point A.

Figure 23.31 Exercise 23.21.



23.22. Two positive point charges, each of magnitude q, are fixed on the y-axis at the points $y = +a$ and $y = -a$. Take the potential to be zero at an infinite distance from the charges. (a) Show the positions of the charges in a diagram. (b) What is the potential V_0 at the origin? (c) Show that the potential at any point on the x-axis is

$$V = \frac{1}{4\pi\epsilon_0} \frac{2q}{\sqrt{a^2 + x^2}}$$

(d) Graph the potential on the x-axis as a function of x over the range from $x = -4a$ to $x = +4a$. (e) What is the potential when $x \gg a$? Explain why this result is obtained.

23.23. A positive charge $+q$ is located at the point $x = 0$, $y = -a$, and a negative charge $-q$ is located at the point $x = 0$, $y = +a$. (a) Show the positions of the charges in a diagram. (b) Derive an expression for the potential V at points on the x-axis as a function of the coordinate x. Take V to be zero at an infinite distance from the charges. (c) Graph V at points on the x-axis as a function of x over the range from $x = -4a$ to $x = +4a$. (d) What is the answer to part (b) if the two charges are interchanged so that $+q$ is at $y = +a$ and $-q$ is at $y = -a$?

23.24. Consider the arrangement of charges described in Exercise 23.23. (a) Derive an expression for the potential V at points on the y-axis as a function of the coordinate y. Take V to be zero at an infinite distance from the charges. (b) Graph V at points on the y-axis as a function of y over the range from $y = -4a$ to $y = +4a$. (c) Show that for $y \gg a$, the potential at a point on the positive y-axis is given by $V = -(1/4\pi\epsilon_0)2qa/y^2$. (d) What are the answers to parts (a) and (c) if the two charges are interchanged so that $+q$ is at $y = +a$ and $-q$ is at $y = -a$?

23.25. A positive charge q is fixed at the point $x = 0$, $y = 0$, and a negative charge $-2q$ is fixed at the point $x = a$, $y = 0$. (a) Show the positions of the charges in a diagram. (b) Derive an expression for the potential V at points on the x-axis as a function of the coordinate x. Take V to be zero at an infinite distance from the charges. (c) At which positions on the x-axis is $V = 0$? (d) Graph V at points on the x-axis as a function of x in the range from $x = -2a$ to $x = +2a$. (e) What does the answer to part (b) become when $x \gg a$? Explain why this result is obtained.

23.26. Consider the arrangement of point charges described in Exercise 23.25. (a) Derive an expression for the potential V at points on the y-axis as a function of the coordinate y. Take V to be zero at an infinite distance from the charges. (b) At which positions on the y-axis is $V = 0$? (c) Graph V at points on the y-axis as a function of y in the range from $y = -2a$ to $y = +2a$. (d) What does the answer to part (a) become when $y \gg a$? Explain why this result is obtained.

23.27. Before the advent of solid-state electronics, vacuum tubes were widely used in radios and other devices. A simple type of vacuum tube known as a *diode* consists essentially of two electrodes within a highly evacuated enclosure. One electrode, the *cathode*, is maintained at a high temperature and emits electrons from its surface. A potential difference of a few hundred volts is maintained between the cathode and the other electrode, known as the *anode*, with the anode at the higher potential. Suppose that in a particular vacuum tube the potential of the anode is 295 V higher than that of the cathode. An electron leaves the surface of the cathode with zero initial speed. Find its speed when it strikes the anode.

23.28. At a certain distance from a point charge, the potential and electric-field magnitude due to that charge are 4.98 V and 12.0 V/m, respectively. (Take the potential to be zero at infinity.) (a) What is the distance to the point charge? (b) What is the magnitude of the charge? (c) Is the electric field directed toward or away from the point charge?

23.29. A uniform electric field has magnitude E and is directed in the negative x-direction. The potential difference between point a (at $x = 0.60$ m) and point b (at $x = 0.90$ m) is 240 V. (a) Which point, a or b, is at the higher potential? (b) Calculate the value of E. (c) A negative point charge $q = -0.200\ \mu\text{C}$ is moved from b to a. Calculate the work done on the point charge by the electric field.

23.30. For each of the following arrangements of two point charges, find all the points along the line passing through both charges for which the electric potential V is zero (take $V = 0$ infinitely far from the charges) and for which the electric field E is zero: (a) charges $+Q$ and $+2Q$ separated by a distance d, and (b) charges $-Q$ and $+2Q$ separated by a distance d. (c) Are both V and E zero at the same places? Explain.

23.31. (a) An electron is to be accelerated from 3.00×10^6 m/s to 8.00×10^6 m/s. Through what potential difference must the electron pass to accomplish this? (b) Through what potential difference must the electron pass if it is to be slowed from 8.00×10^6 m/s to a halt?

Section 23.3 Calculating Electric Potential

23.32. A total electric charge of 3.50 nC is distributed uniformly over the surface of a metal sphere with a radius of 24.0 cm. If the potential is zero at a point at infinity, find the value of the potential at the following distances from the center of the sphere: (a) 48.0 cm; (b) 24.0 cm; (c) 12.0 cm.

23.33. A uniformly charged thin ring has radius 15.0 cm and total charge +24.0 nC. An electron is placed on the ring's axis a distance 30.0 cm from the center of the ring and is constrained to stay on the axis of the ring. The electron is then released from rest. (a) Describe the subsequent motion of the electron. (b) Find the speed of the electron when it reaches the center of the ring.

23.34. An infinitely long line of charge has linear charge density 5.00×10^{-12} C/m. A proton (mass 1.67×10^{-27} kg, charge $+1.60 \times 10^{-19}$ C) is 18.0 cm from the line and moving directly toward the line at 1.50×10^3 m/s. (a) Calculate the proton's initial kinetic energy. (b) How close does the proton get to the line of charge? (*Hint:* See Example 23.10.)

23.35. A very long wire carries a uniform linear charge density λ. Using a voltmeter to measure potential difference, you find that when one probe of the meter is placed 2.50 cm from the wire and the other probe is 1.00 cm farther from the wire, the meter reads 575 V. (a) What is λ? (b) If you now place one probe at 3.50 cm from the wire and the other probe 1.00 cm farther away, will the voltmeter read 575 V? If not, will it read more or less than 575 V? Why? (c) If you place both probes 3.50 cm from the wire but 17.0 cm from each other, what will the voltmeter read?

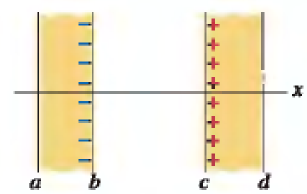
23.36. A very long insulating cylinder of charge of radius 2.50 cm carries a uniform linear density of 15.0 nC/m. If you put one probe of a voltmeter at the surface, how far from the surface must the other probe be placed so that the voltmeter reads 175 V?

23.37. A very long insulating cylindrical shell of radius 6.00 cm carries charge of linear density 8.50 μC/m spread uniformly over its outer surface. What would a voltmeter read if it were connected between (a) the surface of the cylinder and a point 4.00 cm above the surface, and (b) the surface and a point 1.00 cm from the central axis of the cylinder?

23.38. A ring of diameter 8.00 cm is fixed in place and carries a charge of +5.00 μC uniformly spread over its circumference. (a) How much work does it take to move a tiny +3.00-μC charged ball of mass 1.50 g from very far away to the center of the ring? (b) Is it necessary to take a path along the axis of the ring? Why? (c) If the ball is slightly displaced from the center of the ring, what will it do and what is the maximum speed it will reach?

23.38. Two very large, parallel metal plates carry charge densities of the same magnitude but opposite signs (Fig. 23.32). Assume they are close enough together to be treated as ideal infinite plates. Taking the potential to be zero at the left surface of the negative plate, sketch a graph of the potential as a function of x. Include *all* regions from the left of the plates to the right of the plates.

Figure 23.32 Exercise 23.39.



23.40. Two large, parallel conducting plates carrying opposite charges of equal magnitude are separated by 2.20 cm. (a) If the surface charge density for each plate has magnitude 47.0 nC/m^2, what is the magnitude of $\vec{E}$ in the region between the plates? (b) What is the potential difference between the two plates? (c) If the separation between the plates is doubled while the surface charge density is kept constant at the value in part (a), what happens to the magnitude of the electric field and to the potential difference?

23.41. Two large, parallel, metal plates carry opposite charges of equal magnitude. They are separated by 45.0 mm, and the potential difference between them is 360 V. (a) What is the magnitude of the electric field (assumed to be uniform) in the region between the plates? (b) What is the magnitude of the force this field exerts on a particle with charge +2.40 nC? (c) Use the results of part (b) to compute the work done by the field on the particle as it moves from the higher-potential plate to the lower. (d) Compare the result of part (c) to the change of potential energy of the same charge, computed from the electric potential.

23.42. (a) How much excess charge must be placed on a copper sphere 25.0 cm in diameter so that the potential of its center, relative to infinity, is 1.50 kV? (b) What is the potential of the sphere's surface relative to infinity?

23.43. (a) Show that V for a spherical shell of radius R, that has charge q distributed uniformly over its surface, is the same as V for a solid conductor with radius R and charge q. (b) You rub an inflated balloon on the carpet and it acquires a potential that is 1560 V lower than its potential before it became charged. If the charge is uniformly distributed over the surface of the balloon and if the radius of the balloon is 15 cm, what is the net charge on the balloon? (c) In light of its 1200-V potential difference relative to you, do you think this balloon is dangerous? Explain.

23.44. The electric field at the surface of a charged, solid, copper sphere with radius 0.200 m is 3800 N/C, directed toward the center of the sphere. What is the potential at the center of the sphere, if we take the potential to be zero infinitely far from the sphere?

Section 23.4 Equipotential Surfaces and Section 23.5 Potential Gradient

23.45. A potential difference of 480 V is established between large, parallel, metal plates. Let the potential of one plate be 480 V and the other be 0 V. The plates are separated by $d = 1.70$ cm. (a) Sketch the equipotential surfaces that correspond to 0, 120, 240, 360, and 480 V. (b) In your sketch, show the electric field lines. Does your sketch confirm that the field lines and equipotential surfaces are mutually perpendicular?

23.46. A very large plastic sheet carries a uniform charge density of -6.00 nC/m^2 on one face. (a) As you move away from the sheet along a line perpendicular to it, does the potential increase or decrease? How do you know, without doing any calculations? Does your answer depend on where you choose the reference point for potential? (b) Find the spacing between equipotential surfaces that differ from each other by 1.00 V. What type of surfaces are these?

23.47. In a certain region of space, the electric potential is $V(x, y, z) = Axy - Bx^2 + Cy$, where $A, B,$ and C are positive constants. (a) Calculate the x-, y-, and z-components of the electric field. (b) At which points is the electric field equal to zero?

23.48. The potential due to a point charge Q at the origin may be written as

$$V = \frac{Q}{4\pi\epsilon_0 r} = \frac{Q}{4\pi\epsilon_0\sqrt{x^2 + y^2 + z^2}}$$

(a) Calculate E_x, E_y, and E_z using Eqs. (23.19). (b) Show that the results of part (a) agrees with Eq. (21.7) for the electric field of a point charge.

23.48. A metal sphere with radius r_a is supported on an insulating stand at the center of a hollow, metal, spherical shell with radius r_b. There is charge $+q$ on the inner sphere and charge $-q$ on the outer spherical shell. (a) Calculate the potential $V(r)$ for (i) $r < r_a$; (ii) $r_a < r < r_b$; (iii) $r > r_b$. (*Hint:* The net potential is the sum of the potentials due to the individual spheres.) Take V to be zero when r is infinite. (b) Show that the potential of the inner sphere with respect to the outer is

$$V_{ab} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

(c) Use Eq. (23.23) and the result from part (a) to show that the electric field at any point between the spheres has magnitude

$$E(r) = \frac{V_{ab}}{(1/r_a - 1/r_b)}\frac{1}{r^2}$$

(d) Use Eq. (23.23) and the result from part (a) to find the electric field at a point outside the larger sphere at a distance r from the center, where $r > r_b$. (e) Suppose the charge on the outer sphere is not $-q$ but a negative charge of different magnitude, say $-Q$. Show that the answers for parts (b) and (c) are the same as before but the answer for part (d) is different.

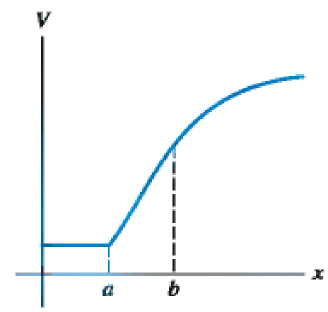
23.50. A metal sphere with radius $r_a = 1.20$ cm is supported on an insulating stand at the center of a hollow, metal, spherical shell with radius $r_b = 9.60$ cm. Charge $+q$ is put on the inner sphere and charge $-q$ on the outer spherical shell. The magnitude of q is chosen to make the potential difference between the spheres 500 V, with the inner sphere at higher potential. (a) Use the result of Exercise 23.49(b) to calculate q. (b) With the help of the result of Exercise 23.49(a), sketch the equipotential surfaces that correspond to 500, 400, 300, 200, 100, and 0 V. (c) In your sketch, show the electric field lines. Are the electric field lines and equipotential surfaces mutually perpendicular? Are the equipotential surfaces closer together when the magnitude of $\vec{E}$ is largest?

23.51. A very long cylinder of radius 2.00 cm carries a uniform charge density of 1.50 nC/m. (a) Describe the shape of the equipotential surfaces for this cylinder. (b) Taking the reference level for the zero of potential to be the surface of the cylinder, find the radius of equipotential surfaces having potentials of 10.0 V, 20.0 V, and 30.0 V. (c) Are the equipotential surfaces equally spaced? If not, do they get closer together or farther apart as r increases?

Problems

23.52. Figure 23.33 shows the potential of a charge distribution as a function of x. Sketch a graph of the electric field E_x over the region shown.

Figure 23.33 Problem 23.52.



23.53. A particle with charge $+7.60$ nC is in a uniform electric field directed to the left. Another force, in addition to the electric force, acts on the particle so that when it is released from rest, it moves to the right. After it has moved 8.00 cm, the additional force has done 6.50×10^{-5} J of work and the particle has 4.35×10^{-5} J of kinetic energy. (a) What work was done by the electric force? (b) What is the potential of the starting point with respect to the end point? (c) What is the magnitude of the electric field?

23.54. In the *Bohr model* of the hydrogen atom, a single electron revolves around a single proton in a circle of radius r. Assume that the proton remains at rest. (a) By equating the electric force to the electron mass times its acceleration, derive an expression for the electron's speed. (b) Obtain an expression for the electron's kinetic energy, and show that its magnitude is just half that of the electric potential energy. (c) Obtain an expression for the total energy, and evaluate it using $r = 5.29 \times 10^{-11}$ m. Give your numerical result in joules and in electron volts.

23.55. A vacuum tube diode (see Exercise 23.27) consists of concentric cylindrical electrodes, the negative cathode and the positive anode. Because of the accumulation of charge near the cathode, the electric potential between the electrodes is not a linear function of the position, even with planar geometry, but is given by

$$V(x) = Cx^{4/3}$$

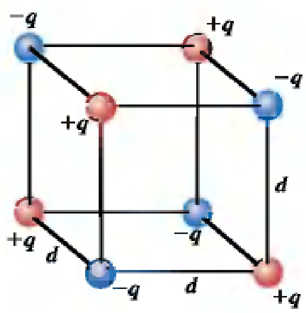
where x is the distance from the cathode and C is a constant, characteristic of a particular diode and operating conditions. Assume that the distance between the cathode and anode is 13.0 mm and the potential difference between electrodes is 240 V. (a) Determine the value of C. (b) Obtain a formula for the electric field between the electrodes as a function of x. (c) Determine the force on an electron when the electron is halfway between the electrodes.

23.56. Two oppositely charged identical insulating spheres, each 50.0 cm in diameter and carrying a uniform charge of magnitude 175 μC, are placed 1.00 m apart center to center (Fig. 23.34). (a) If a voltmeter is connected between the nearest points (a and b) on their surfaces, what will it read? (b) Which point, a or b, is at the higher potential? How can you know this without any calculations?

Figure 23.34 Problem 23.56.

23.57. An Ionic Crystal. Figure 23.35 shows eight point charges arranged at the corners of a cube with sides of length d. The values of the charges are $+q$ and $-q$, as shown. This is a model of one cell of a cubic ionic crystal. In sodium chloride (NaCl), for instance, the positive ions are Na^+ and the negative ions are Cl^-. (a) Calculate the potential energy U of this arrangement. (Take as zero the potential energy of the eight charges when they are infinitely far apart.) (b) In part (a), you should have found that $U < 0$. Explain the relationship between this result and the observation that such ionic crystals exist in nature.

Figure 23.35 Problem 23.57.

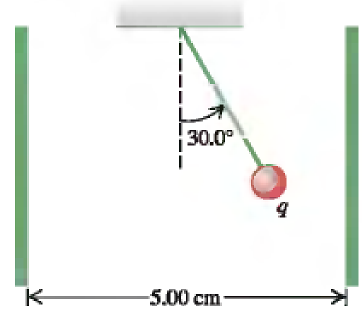


23.58. (a) Calculate the potential energy of a system of two small spheres, one carrying a charge of 2.00 μC and the other a charge of -3.50 μC, with their centers separated by a distance of 0.250 m. Assume zero potential energy when the charges are infinitely separated. (b) Suppose that one of the spheres is held in place and the other sphere, which has a mass of 1.50 g, is shot away from it. What minimum initial speed would the moving sphere need in order to escape completely from the attraction of the fixed sphere? (To escape, the moving sphere would have to reach a velocity of zero when it was infinitely distant from the fixed sphere.)

23.59. The H_2^+ Ion. The H_2^+ ion is composed of two protons, each of charge $+e = 1.60 \times 10^{-19}$ C, and an electron of charge $-e$ and mass 9.11×10^{-31} kg. The separation between the protons is 1.07×10^{-10} m. The protons and the electron may be treated as point charges. (a) Suppose the electron is located at the point midway between the two protons. What is the potential energy of the interaction between the electron and the two protons? (Do not include the potential energy due to the interaction between the two protons.) (b) Suppose the electron in part (a) has a velocity of magnitude 1.50×10^6 m/s in a direction along the perpendicular bisector of the line connecting the two protons. How far from the point midway between the two protons can the electron move? Because the masses of the protons are much greater than the electron mass, the motions of the protons are very slow and can be ignored. (*Note:* A realistic description of the electron motion requires the use of quantum mechanics, not Newtonian mechanics.)

23.60. A small sphere with mass 1.50 g hangs by a thread between two parallel vertical plates 5.00 cm apart (Fig. 23.36). The plates are insulating and have uniform surface charge densities $+\sigma$ and $-\sigma$. The charge on the sphere is $q = 8.90 \times 10^{-6}$ C. What potential difference between the plates will cause the thread to assume an angle of 30.0° with the vertical?

Figure 23.36 Problem 23.60.



23.61. Coaxial Cylinders. A long metal cylinder with radius a is supported on an insulating stand on the axis of a long, hollow, metal tube with radius b. The positive charge per unit length on the inner cylinder is λ, and there is an equal negative charge per unit length on the outer cylinder. (a) Calculate the potential $V(r)$ for (i) $r < a$; (ii) $a < r < b$; (iii) $r > b$. (*Hint:* The net potential is the sum of the potentials due to the individual conductors.) Take $V = 0$ at $r = b$. (b) Show that the potential of the inner cylinder with respect to the outer is

$$V_{ab} = \frac{\lambda}{2\pi\epsilon_0} \ln\frac{b}{a}$$

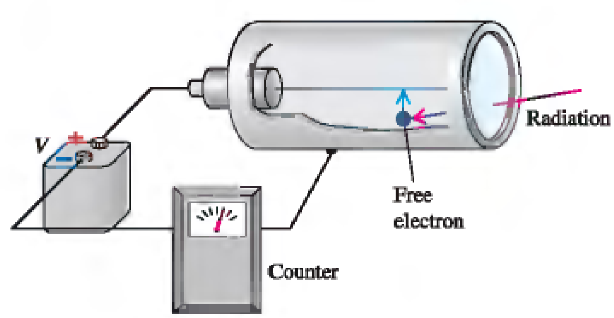
(c) Use Eq. (23.23) and the result from part (a) to show that the electric field at any point between the cylinders has magnitude

$$E(r) = \frac{V_{ab}}{\ln(b/a)} \frac{1}{r}$$

(d) What is the potential difference between the two cylinders if the outer cylinder has no net charge?

23.62. A *Geiger counter* detects radiation such as alpha particles by using the fact that the radiation ionizes the air along its path. A thin wire lies on the axis of a hollow metal cylinder and is insulated from it (Fig. 23.37). A large potential difference is established between the wire and the outer cylinder, with the wire at higher potential; this sets up a strong electric field directed radially outward. When ionizing radiation enters the device, it ionizes a few air molecules. The free electrons produced are accelerated by the electric field toward the wire and, on the way there, ionize many more air molecules. Thus a current pulse is produced that can be detected by appropriate electronic circuitry and converted to an audible "click." Suppose the radius of the central wire is 145 μm and the radius of the hollow cylinder is 1.80 cm. What potential difference between the wire and the cylinder produces an

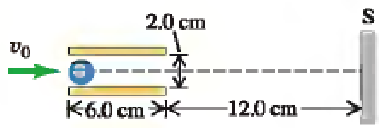
Figure 23.37 Problem 23.62.



electric field of 2.00×10^4 V/m at a distance of 1.20 cm from the axis of the wire? (The wire and cylinder are both very long in comparison to their radii, so the results of Problem 23.61 apply.)

23.63. Deflection in a CRT. Cathode-ray tubes (CRTs) are often found in oscilloscopes and computer monitors. In Fig. 23.38 an electron with an initial speed of 6.50×10^6 m/s is projected along the axis midway between the deflection plates of a cathode-ray tube. The uniform electric field between the plates has a magnitude of 1.10×10^3 V/m and is upward. (a) What is the force (magnitude and direction) on the electron when it is between the plates? (b) What is the acceleration of the electron (magnitude and direction) when acted on by the force in part (a)? (c) How far below the axis has the electron moved when it reaches the end of the plates? (d) At what angle with the axis is it moving as it leaves the plates? (e) How far below the axis will it strike the fluorescent screen S?

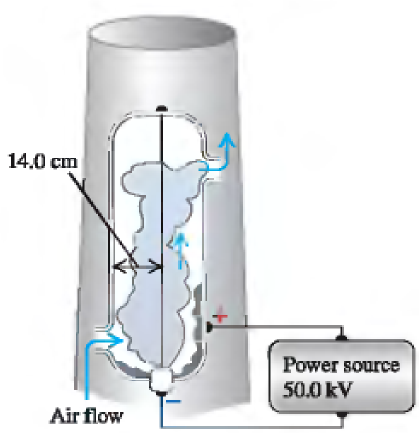
Figure 23.38 Problem 23.63.



23.64. Deflecting Plates of an Oscilloscope. The vertical deflecting plates of a typical classroom oscilloscope are a pair of parallel square metal plates carrying equal but opposite charges. Typical dimensions are about 3.0 cm on a side, with a separation of about 5.0 mm. The plates are close enough that we can ignore fringing at the ends. Under these conditions: (a) how much charge is on each plate, and (b) how strong is the electric field between the plates? (c) If an electron is ejected at rest from the negative plates, how fast is it moving when it reaches the positive plate?

23.65. *Electrostatic precipitators* use electric forces to remove pollutant particles from smoke, in particular in the smokestacks of coal-burning power plants. One form of precipitator consists of a vertical, hollow, metal cylinder with a thin wire, insulated from the cylinder, running along its axis (Fig. 23.39). A large potential difference is established between the wire and the outer cylinder, with the wire at lower potential. This sets up a strong radial electric field directed inward. The field produces a region of ionized air near the wire. Smoke enters the precipitator at the bottom, ash and dust in it pick up electrons, and the charged pollutants are accelerated toward the outer cylinder wall by the electric field. Suppose the radius of the central wire is 90.0 μm, the radius of the cylinder is 14.0 cm, and a potential difference of 50.0 kV is established between the wire and the cylinder. Also assume that the wire and cylinder are both very long in comparison to the cylinder radius, so the results of Problem 23.61 apply. (a) What is the magnitude of the electric field midway between the wire and the cylinder wall? (b) What magnitude of charge must a 30.0-μg ash particle have if the electric field computed in part (a) is to exert a force ten times the weight of the particle?

Figure 23.39 Problem 23.65.



23.66. A disk with radius R has uniform surface charge density σ. (a) By regarding the disk as a series of thin concentric rings, calculate the electric potential V at a point on the disk's axis a distance x from the center of the disk. Assume that the potential is zero at infinity. (*Hint:* Use the result of Example 23.11 in Section 23.3.) (b) Calculate $-\partial V/\partial x$. Show that the result agrees with the expression for E_x calculated in Example 21.12 (Section 21.5).

23.67. (a) From the expression for E obtained in Problem 22.40, find the expressions for the electric potential V as a function of r, both inside and outside the cylinder. Let $V = 0$ at the surface of the cylinder. In each case, express your result in terms of the charge per unit length λ of the charge distribution. (b) Graph V and E as functions of r from $r = 0$ to $r = 3R$.

23.68. Alpha particles ($mass = 6.7 \times 10^{-27}$ kg, charge $= +2e$) are shot directly at a gold foil target. We can model the gold nucleus as a uniform sphere of charge and assume that the gold does not move. (a) If the radius of the gold nucleus is 5.6×10^{-15} m, what minimum speed do the alpha particles need when they are far away to reach the surface of the gold nucleus? (Ignore relativistic effects.) (b) Give good physical reasons why we can ignore the effects of the orbital electrons when the alpha particle is (i) outside the electron orbits and (ii) inside the electron orbits.

23.69. For the ring of charge described in Example 23.11 (Section 23.3), integrate the expression for E_x found in Example 21.10 (Section 21.5) to find the potential at point P on the ring's axis. Assume that $V = 0$ at infinity. Compare your result to that obtained in Example 23.11 using Eq. (23.16).

23.70. A thin insulating rod is bent into a semicircular arc of radius a, and a total electric charge Q is distributed uniformly along the rod. Calculate the potential at the center of curvature of the arc if the potential is assumed to be zero at infinity.

23.71. Self-Energy of a Sphere of Charge. A solid sphere of radius R contains a total charge Q distributed uniformly throughout its volume. Find the energy needed to assemble this charge by bringing infinitesimal charges from far away. This energy is called the "self-energy" of the charge distribution. (*Hint:* After you have assembled a charge q in a sphere of radius r, how much energy would it take to add a spherical shell of thickness dr having charge dq? Then integrate to get the total energy.)

23.72. (a) From the expression for E obtained in Example 22.9 (Section 22.4), find the expression for the electric potential V as a function of r both inside and outside the uniformly charged sphere. Assume that $V = 0$ at infinity. (b) Graph V and E as functions of r from $r = 0$ to $r = 3R$.

23.73. A solid insulating sphere with radius R has charge Q uniformly distributed throughout its volume. (a) Use the results of Problem 23.72 to find the magnitude of the potential difference between the surface of the sphere and its center. (b) Which is at higher potential, the surface or the center, if (i) Q is positive and (ii) Q is negative?

23.74. An insulating spherical shell with inner radius 25.0 cm and outer radius 60.0 cm carries a charge of $+150.0\ \mu C$ uniformly distributed over its outer surface (see Exercise 23.43). Point a is at the center of the shell, point b is on the inner surface, and point c is on the outer surface. (a) What will a voltmeter read if it is connected between the following points: (i) a and b; (ii) b and c; (iii) c and infinity; (iv) a and c? (b) Which is at higher potential: (i) a or b; (ii) b or c; (iii) a or c? (c) Which, if any, of the answers would change sign if the charges were $-150\ \mu C$?

23.75. Exercise 23.43 shows that, outside a spherical shell with uniform surface charge, the potential is the same as if all the charge were concentrated into a point charge at the center of the sphere. (a) Use this result to show that for two uniformly charged insulating shells, the force they exert on each other and their mutual electrical energy are the same as if all the charge were concentrated at their centers. (*Hint:* See Section 12.6.) (b) Does this same result hold for solid insulating spheres, with charge distributed uniformly throughout their volume? (c) Does this same result hold for the force between two charged conducting shells? Between two charged solid conductors? Explain.

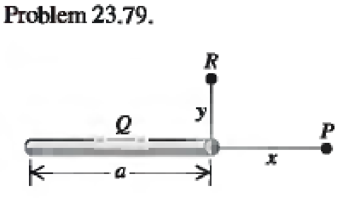
23.76. Two plastic spheres, each carrying charge uniformly distributed throughout its interior, are initially placed in contact and then released. One sphere is 60.0 cm in diameter, has mass 50.0 g and contains $-10.0\ \mu C$ of charge. The other sphere is 40.0 cm in diameter, has mass 150.0 g, and contains $-30.0\ \mu C$ of charge. Find the maximum acceleration and the maximum speed achieved by each sphere (relative to the fixed point of their initial location in space), assuming that no other forces are acting on them. (*Hint:* The uniformly distributed charges behave as though they were concentrated at the centers of the two spheres.)

23.77. Use the electric field calculated in Problem 22.43 to calculate the potential difference between the solid conducting sphere and the thin insulating shell.

23.78. Consider a solid conducting sphere inside a hollow conducting sphere, with radii and charges specified in Problem 22.42. Take $V = 0$ as $r \to \infty$. Use the electric field calculated in Problem 22.42 to calculate the potential V at the following values of r: (a) $r = c$ (at the outer surface of the hollow sphere); (b) $r = b$ (at the inner surface of the hollow sphere); (c) $r = a$ (at the surface of the solid sphere); (d) $r = 0$ (at the center of the solid sphere).

23.79. Electric charge is distributed uniformly along a thin rod of length a, with total charge Q. Take the potential to be zero at infinity. Find the potential at the following points (Fig. 23.40): (a) point P, a distance x to the right of the rod, and (b) point R, a distance y above the right-hand end of the rod. (c) In parts (a) and (b), what does your result reduce to as x or y becomes much larger than a?

Figure 23.40 Problem 23.79.



23.80. (a) If a spherical raindrop of radius 0.650 mm carries a charge of -1.20 pC uniformly distributed over its volume, what is the potential at its surface? (Take the potential to be zero at an infinite distance from the raindrop.) (b) Two identical raindrops, each with radius and charge specified in part (a), collide and merge into one larger raindrop. What is the radius of this larger drop, and what is the potential at its surface, if its charge is uniformly distributed over its volume?

23.81. Two metal spheres of different sizes are charged such that the electric potential is the same at the surface of each. Sphere A has a radius three times that of sphere B. Let Q_A and Q_B be the charges on the two spheres, and let E_A and E_B be the electric-field magnitudes at the surfaces of the two spheres. What are (a) the ratio Q_B/Q_A and (b) the ratio E_B/E_A?

23.82. An alpha particle with kinetic energy 11.0 MeV makes a head-on collision with a lead nucleus at rest. What is the distance of closest approach of the two particles? (Assume that the lead nucleus remains stationary and that it may be treated as a point charge. The atomic number of lead is 82. The alpha particle is a helium nucleus, with atomic number 2.)

23.83. A metal sphere with radius R_1 has a charge Q_1. Take the electric potential to be zero at an infinite distance from the sphere. (a) What are the electric field and electric potential at the surface of the sphere? This sphere is now connected by a long, thin conducting wire to another sphere of radius R_2 that is several meters from the first sphere. Before the connection is made, this second sphere is uncharged. After electrostatic equilibrium has been reached, what are (b) the total charge on each sphere; (c) the electric potential at the surface of each sphere (d) the electric field at the surface of each sphere? Assume that the amount of charge on the wire is much less than the charge on each sphere.

23.84. Use the charge distribution and electric field calculated in Problem 22.57. (a) Show that for $r \geq R$ the potential is identical to that produced by a point charge Q. (Take the potential to be zero at infinity.) (b) Obtain an expression for the electric potential valid in the region $r \leq R$.

23.85. Nuclear Fusion in the Sun. The source of the sun's energy is a sequence of nuclear reactions that occur in its core. The first of these reactions involves the collision of two protons, which fuse together to form a heavier nucleus and release energy. For this process, called *nuclear fusion,* to occur, the two protons must first approach until their surfaces are essentially in contact. (a) Assume both protons are moving with the same speed and they collide head-on. If the radius of the proton is 1.2×10^{-15} m, what is the minimum speed that will allow fusion to occur? The charge distribution within a proton is spherically symmetric, so the electric field and potential outside a proton are the same as if it were a point charge. The mass of the proton is 1.67×10^{-27} kg. (b) Another nuclear fusion reaction that occurs in the sun's core involves a collision between two helium nuclei, each of which has 2.99 times the mass of the proton, charge $+2e$, and radius 1.7×10^{-15} m. Assuming the same collision geometry as in part (a), what minimum speed is required for this fusion reaction to take place if the nuclei must approach a center-to-center distance of about 3.5×10^{-15} m? As for the proton, the charge of the helium nucleus is uniformly distributed throughout its volume. (c) In Section 18.3 it was shown that the average translational kinetic energy of a particle with mass m in a gas at absolute temperature T is $\frac{3}{2}kT$, where k is the Boltzmann constant (given in Appendix F). For two protons with kinetic energy equal to this average value to be able to undergo the process described in part (a), what absolute temperature is required? What absolute temperature is required for two average helium nuclei to be able to undergo the process described in part (b)? (At these temperatures, atoms are completely ionized, so nuclei and electrons move separately.) (d) The temperature in the sun's core is about 1.5×10^7 K. How does this compare to the temperatures calculated in part (c)? How can the reactions described in parts (a) and (b) occur at all in the interior of the sun? (*Hint:* See the discussion of the distribution of molecular speeds in Section 18.5.)

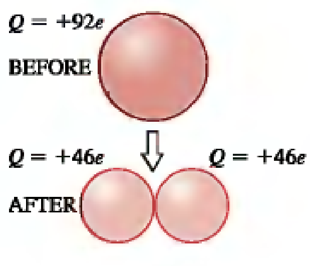
23.86. The electric potential V in a region of space is given by

$$V(x, y, z) = A(x^2 - 3y^2 + z^2)$$

where A is a constant. (a) Derive an expression for the electric field $\vec{E}$ at any point in this region. (b) The work done by the field when a 1.50-μC test charge moves from the point $(x, y, z) = (0, 0, 0.250\text{ m})$ to the origin is measured to be 6.00×10^{-5} J. Determine A. (c) Determine the electric field at the point (0, 0, 0.250 m). (d) Show that in every plane parallel to the xz-plane the equipotential contours are circles. (e) What is the radius of the equipotential contour corresponding to $V = 1280$ V and $y = 2.00$ m?

23.87. Nuclear Fission. The unstable nucleus of uranium-236 can be regarded as a uniformly charged sphere of charge $Q = +92e$ and radius $R = 7.4 \times 10^{-15}$ m. In nuclear fission, this can divide into two smaller nuclei, each with half the charge and half the volume of the original uranium-236 nucleus. This is one of the reactions that occurred in the nuclear weapon that exploded over Hiroshima, Japan, in August 1945. (a) Find the radii of the two "daughter" nuclei of charge $+46e$. (b) In a simple model for the fission process, immediately after the uranium-236 nucleus has undergone fission, the "daughter" nuclei are at rest and just touching, as shown in Fig. 23.41. Calculate the kinetic energy that each of the "daughter" nuclei will have when they are very far apart. (c) In this model the sum of the kinetic energies of the two "daughter" nuclei, calculated in part (b), is the energy released by the fission of one uranium-236 nucleus. Calculate the energy released by the fission of 10.0 kg of uranium-236. The atomic mass of uranium-236 is 236 u, where 1 u = 1 atomic mass unit = 1.66×10^{-24} kg. Express your answer both in joules and in kilotons of TNT (1 kiloton of TNT releases 4.18×10^{12} J when it explodes). (d) In terms of this model, discuss why an atomic bomb could just as well be called an "electric bomb."

Figure 23.41 Problem 23.87.



Challenge Problems

23.88. In a certain region, a charge distribution exists that is spherically symmetric but nonuniform. That is, the volume charge density $\rho(r)$ depends on the distance r from the center of the distribution but not on the spherical polar angles θ and ϕ. The electric potential $V(r)$ due to this charge distribution is

$$V(r) = \begin{cases} \dfrac{\rho_0 a^2}{18\epsilon_0}\left[1 - 3\left(\dfrac{r}{a}\right)^2 + 2\left(\dfrac{r}{a}\right)^3\right] & \text{for } r \le a \\ 0 & \text{for } r \ge a \end{cases}$$

where ρ_0 is a constant having units of C/m^3 and a is a constant having units of meters. (a) Derive expressions for $\vec{E}$ for the regions $r \le a$ and $r \ge a$. [*Hint:* Use Eq. (23.23).] Explain why $\vec{E}$ has only a radial component. (b) Derive an expression for $\rho(r)$ in each of the two regions $r \le a$ and $r \ge a$. [*Hint:* Use Gauss's law for two spherical shells, one of radius r and the other of radius $r + dr$. The charge contained in the infinitesimal spherical shell of radius dr is $dq = 4\pi r^2 \rho(r) dr$.] (c) Show that the net charge contained in the volume of a sphere of radius greater than or equal to a is zero. [*Hint:* Integrate the expressions derived in part (b) for $\rho(r)$ over a spherical volume of radius greater than or equal to a.] Is this result consistent with the electric field for $r > a$ that you calculated in part (a)?

23.89. In experiments in which atomic nuclei collide, head-on collisions like that described in Problem 23.82 do happen, but "near misses" are more common. Suppose the alpha particle in Problem 23.82 was not "aimed" at the center of the lead nucleus, but had an initial nonzero angular momentum (with respect to the stationary lead nucleus) of magnitude $L = p_0 b$, where p_0 is the magnitude of the initial momentum of the alpha particle and $b = 1.00 \times 10^{-12}$ m. What is the distance of closest approach? Repeat for $b = 1.00 \times 10^{-13}$ m and $b = 1.00 \times 10^{-14}$ m.

23.90. A hollow, thin-walled insulating cylinder of radius R and length L (like the cardboard tube in a roll of toilet paper) has charge Q uniformly distributed over its surface. (a) Calculate the electric potential at all points along the axis of the tube. Take the origin to be at the center of the tube, and take the potential to be zero at infinity. (b) Show that if $L \ll R$, the result of part (a) reduces to the potential on the axis of a ring of charge of radius R (See Example 23.11 in Section 23.3). (c) Use the result of part (a) to find the electric field at all points along the axis of the tube.

23.91. The Millikan Oil-Drop Experiment. The charge of an electron was first measured by the American physicist Robert Millikan during 1909–1913. In his experiment, oil is sprayed in very fine drops (around 10^{-4} mm in diameter) into the space between two parallel horizontal plates separated by a distance d. A potential difference V_{AB} is maintained between the parallel plates, causing a downward electric field between them. Some of the oil drops acquire a negative charge because of frictional effects or because of ionization of the surrounding air by x rays or radioactivity. The drops are observed through a microscope. (a) Show that an oil drop of radius r at rest between the plates will remain at rest if the magnitude of its charge is

$$q = \frac{4\pi}{3}\frac{\rho r^3 g d}{V_{AB}}$$

where ρ is the density of the oil. (Ignore the buoyant force of the air.) By adjusting V_{AB} to keep a given drop at rest, the charge on that drop can be determined, provided its radius is known. (b) Millikan's oil drops were much too small to measure their radii directly. Instead, Millikan determined r by cutting off the electric field and measuring the *terminal speed* v_t of the drop as it fell. (We discussed the concept of terminal speed in Section 5.3.) The viscous force F on a sphere of radius r moving with speed v through a fluid with viscosity η is given by Stokes's law: $F = 6\pi\eta r v$. When the drop is falling at v_t, the viscous force just balances the weight $w = mg$ of the drop. Show that the magnitude of the charge on the drop is

$$q = 18\pi\frac{d}{V_{AB}}\sqrt{\frac{\eta^3 v_t^3}{2\rho g}}$$

Within the limits of their experimental error, every one of the thousands of drops that Millikan and his coworkers measured had a charge equal to some small integer multiple of a basic charge e. That is, they found drops with charges of $\pm 2e$, $\pm 5e$, and so on, but none with values such as $0.76e$ or $2.49e$. A drop with charge $-e$ has acquired one extra electron; if its charge is $-2e$, it has acquired two extra electrons, and so on. (c) A charged oil drop in a Millikan oil-drop apparatus is observed to fall 1.00 mm at constant speed in 39.3 s if $V_{AB} = 0$. The same drop can be held at rest between two plates separated by 1.00 mm if $V_{AB} = 9.16$ V. How many excess electrons has the drop acquired, and what is the radius of the drop? The viscosity of air is 1.81×10^{-5} N · s/m^2, and the density of the oil is 824 kg/m^3.

23.92. Two point charges are moving to the right along the x-axis. Point charge 1 has charge $q_1 = 2.00\ \mu\text{C}$, mass $m_1 = 6.00 \times 10^{-5}$ kg, and speed v_1. Point charge 2 is to the right of q_1 and has charge $q_2 = -5.00\ \mu\text{C}$, mass $m_2 = 3.00 \times 10^{-5}$ kg, and speed v_2. At a particular instant, the charges are separated by a distance of 9.00 mm and have speeds $v_1 = 400$ m/s and $v_2 = 1300$ m/s. The only forces on the particles are the forces they exert on each other. (a) Determine the speed v_{cm} of the center of mass of the system. (b) The *relative energy* E_{rel} of the system is defined as the total energy minus the kinetic energy contributed by the motion of the center of mass:

$$E_{\text{rel}} = E - \frac{1}{2}(m_1 + m_2)v_{\text{cm}}^2$$

where $E = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + q_1q_2/4\pi\epsilon_0 r$ is the total energy of the system and r is the distance between the charges. Show that $E_{\text{rel}} = \frac{1}{2}\mu v^2 + q_1q_2/4\pi\epsilon_0 r$, where $\mu = m_1m_2/(m_1 + m_2)$ is called the *reduced mass* of the system and $v = v_2 - v_1$ is the relative speed of the moving particles. (c) For the numerical values given above, calculate the numerical value of E_{rel}. (d) Based on the result of part (c), for the conditions given above, will the particles escape from one another? Explain. (e) If the particles do escape, what will be their final relative speed when $r \to \infty$? If the particles do not escape, what will be their distance of maximum separation? That is, what will be the value of r when $v = 0$? (f) Repeat parts (c)–(e) for $v_1 = 400$ m/s and $v_2 = 1800$ m/s when the separation is 9.00 mm.

CAPACITANCE AND DIELECTRICS

24

? The energy used in a camera's flash unit is stored in a capacitor, which consists of two closely spaced conductors that carry opposite charges. If the amount of charge on the conductors is doubled, by what factor does the stored energy increase?

LEARNING GOALS

By studying this chapter, you will learn:

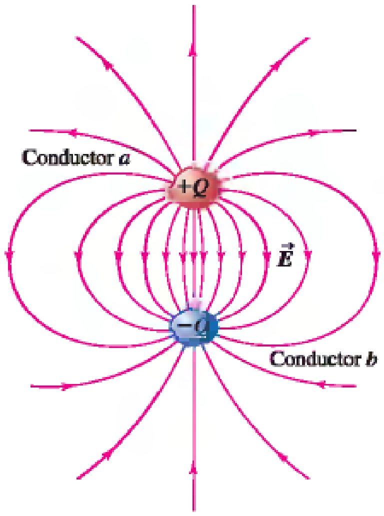
- The nature of capacitors, and how to calculate a quantity that measures their ability to store charge.
- How to analyze capacitors connected in a network.
- How to calculate the amount of energy stored in a capacitor.
- What dielectrics are, and how they make capacitors more effective.

When you set an old-fashioned spring mousetrap or pull back the string of an archer's bow, you are storing mechanical energy as elastic potential energy. A capacitor is a device that stores *electric* potential energy and electric charge. To make a capacitor, just insulate two conductors from each other. To store energy in this device, transfer charge from one conductor to the other so that one has a negative charge and the other has an equal amount of positive charge. Work must be done to move the charges through the resulting potential difference between the conductors, and the work done is stored as electric potential energy.

Capacitors have a tremendous number of practical applications in devices such as electronic flash units for photography, pulsed lasers, air bag sensors for cars, and radio and television receivers. We'll encounter many of these applications in later chapters (particularly Chapter 31, in which we'll see the crucial role played by capacitors in the alternating-current circuits that pervade our technological society). In this chapter, however, our emphasis is on the fundamental properties of capacitors. For a particular capacitor, the ratio of the charge on each conductor to the potential difference between the conductors is a constant, called the *capacitance*. The capacitance depends on the sizes and shapes of the conductors and on the insulating material (if any) between them. Compared to the case in which there is only vacuum between the conductors, the capacitance increases when an insulating material (a *dielectric*) is present. This happens because a redistribution of charge, called *polarization*, takes place within the insulating material. Studying polarization will give us added insight into the electrical properties of matter.

Capacitors also give us a new way to think about electric potential energy. The energy stored in a charged capacitor is related to the electric field in the space between the conductors. We will see that electric potential energy can be regarded as being stored *in the field itself*. The idea that the electric field is itself a storehouse of energy is at the heart of the theory of electromagnetic waves and our modern understanding of the nature of light, to be discussed in Chapter 32.

24.1 Any two conductors a and b insulated from each another form a capacitor.



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11.11.6 Electric Potential: Qualitative Introduction

11.12.1 and 11.12.3 Electric Potential, Field and, Force

24.1 Capacitors and Capacitance

Any two conductors separated by an insulator (or a vacuum) form a **capacitor** (Fig. 24.1). In most practical applications, each conductor initially has zero net charge and electrons are transferred from one conductor to the other; this is called *charging* the capacitor. Then the two conductors have charges with equal magnitude and opposite sign, and the *net* charge on the capacitor as a whole remains zero. We will assume throughout this chapter that this is the case. When we say that a capacitor has charge Q, or that a charge Q is *stored* on the capacitor, we mean that the conductor at higher potential has charge $+Q$ and the conductor at lower potential has charge $-Q$ (assuming that Q is positive). Keep this in mind in the following discussion and examples.

In circuit diagrams a capacitor is represented by either of these symbols:

In either symbol the vertical lines (straight or curved) represent the conductors and the horizontal lines represent wires connected to either conductor. One common way to charge a capacitor is to connect these two wires to opposite terminals of a battery. Once the charges Q and $-Q$ are established on the conductors, the battery is disconnected. This gives a fixed *potential difference* V_{ab} between the conductors (that is, the potential of the positively charged conductor a with respect to the negatively charged conductor b) that is just equal to the voltage of the battery.

The electric field at any point in the region between the conductors is proportional to the magnitude Q of charge on each conductor. It follows that the potential difference V_{ab} between the conductors is also proportional to Q. If we double the magnitude of charge on each conductor, the charge density at each point doubles, the electric field at each point doubles, and the potential difference between conductors doubles; however, the *ratio* of charge to potential difference does not change. This ratio is called the **capacitance** C of the capacitor:

$$C = \frac{Q}{V_{ab}} \qquad \text{(definition of capacitance)} \tag{24.1}$$

The SI unit of capacitance is called one **farad** (1 F), in honor of the 19th-century English physicist Michael Faraday. From Eq. (24.1), one farad is equal to one *coulomb per volt* (1 C/V):

$$1\ \text{F} = 1\ \text{farad} = 1\ \text{C/V} = 1\ \text{coulomb/volt}$$

CAUTION **Capacitance vs. coulombs** Don't confuse the symbol C for capacitance (which is always in italics) with the abbreviation C for coulombs (which is never italicized).

The greater the capacitance C of a capacitor, the greater the magnitude Q of charge on either conductor for a given potential difference V_{ab} and hence the greater the amount of stored energy. (Remember that potential is potential energy per unit charge.) Thus *capacitance is a measure of the ability of a capacitor to store energy.* We will see that the value of the capacitance depends only on the shapes and sizes of the conductors and on the nature of the insulating material between them. (The above remarks about capacitance being independent of Q and V_{ab} do not apply to certain special types of insulating materials. We won't discuss these materials in this book, however.)

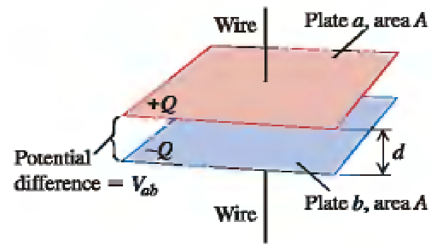
Calculating Capacitance: Capacitors in Vacuum

We can calculate the capacitance C of a given capacitor by finding the potential difference V_{ab} between the conductors for a given magnitude of charge Q and then using Eq. (24.1). For now we'll consider only *capacitors in vacuum;* that is, we'll assume that the conductors that make up the capacitor are separated by empty space.

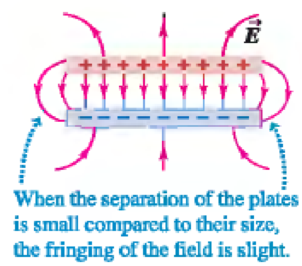
The simplest form of capacitor consists of two parallel conducting plates, each with area A, separated by a distance d that is small in comparison with their dimensions (Fig. 24.2a). When the plates are charged, the electric field is almost completely localized in the region between the plates (Fig. 24.2b). As we discussed in Example 22.8 (Section 22.4), the field between such plates is essentially *uniform,* and the charges on the plates are uniformly distributed over their opposing surfaces. We call this arrangement a **parallel-plate capacitor.**

24.2 A charged parallel-plate capacitor.

(a) Arrangement of the capacitor plates



(b) Side view of the electric field $\vec{E}$



We worked out the electric-field magnitude E for this arrangement in Example 21.13 (Section 21.5) using the principle of superposition of electric fields and again in Example 22.8 (Section 22.4) using Gauss's law. It would be a good idea to review those examples. We found that $E = \sigma/\epsilon_0$, where σ is the magnitude (absolute value) of the surface charge density on each plate. This is equal to the magnitude of the total charge Q on each plate divided by the area A of the plate, or $\sigma = Q/A$, so the field magnitude E can be expressed as

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

The field is uniform and the distance between the plates is d, so the potential difference (voltage) between the two plates is

$$V_{ab} = Ed = \frac{1}{\epsilon_0}\frac{Qd}{A}$$

From this we see that the capacitance C of a parallel-plate capacitor in vacuum is

$$C = \frac{Q}{V_{ab}} = \epsilon_0 \frac{A}{d} \qquad \text{(capacitance of a parallel-plate capacitor in vacuum)} \qquad (24.2)$$

The capacitance depends only on the geometry of the capacitor; it is directly proportional to the area A of each plate and inversely proportional to their separation d. The quantities A and d are constants for a given capacitor, and ϵ_0 is a universal constant. Thus in vacuum the capacitance C is a constant independent of the charge on the capacitor or the potential difference between the plates. If one of the capacitor plates is flexible, the capacitance C changes as the plate separation d changes. This is the operating principle of a condenser microphone (Fig. 24.3).

24.3 Inside a condenser microphone is a capacitor with one rigid plate and one flexible plate. The two plates are kept at a constant potential difference V_{ab}. Sound waves cause the flexible plate to move back and forth, varying the capacitance C and causing charge to flow to and from the capacitor in accordance with the relationship $C = Q/V_{ab}$. Thus a sound wave is converted to a charge flow that can be amplified and recorded digitally.

When matter is present between the plates, its properties affect the capacitance. We will return to this topic in Section 24.4. Meanwhile, we remark that if the space contains air at atmospheric pressure instead of vacuum, the capacitance differs from the prediction of Eq. (24.2) by less than 0.06%.

In Eq. (24.2), if A is in square meters and d in meters, C is in farads. The units of ϵ_0 are $C^2/N \cdot m^2$, so we see that

$$1\ \text{F} = 1\ \text{C}^2/\text{N}\cdot\text{m} = 1\ \text{C}^2/\text{J}$$

Because $1\ \text{V} = 1\ \text{J/C}$ (energy per unit charge), this is consistent with our definition $1\ \text{F} = 1\ \text{C/V}$. Finally, the units of ϵ_0 can be expressed as $1\ \text{C}^2/\text{N}\cdot\text{m}^2 = 1\ \text{F/m}$, so

$$\epsilon_0 = 8.85 \times 10^{-12}\ \text{F/m}$$

24.4 A commercial capacitor is labeled with the value of its capacitance. For these capacitors, $C = 2200\ \mu\text{F}$, $1000\ \mu\text{F}$, and $470\ \mu\text{F}$.

This relationship is useful in capacitance calculations, and it also helps us to verify that Eq. (24.2) is dimensionally consistent.

One farad is a very large capacitance, as the following example shows. In many applications the most convenient units of capacitance are the *microfarad* $(1\ \mu\text{F} = 10^{-6}\ \text{F})$ and the *picofarad* $(1\ \text{pF} = 10^{-12}\ \text{F})$. For example, the flash unit in a point-and-shoot camera uses a capacitor of a few hundred microfarads (Fig. 24.4), while capacitances in a radio tuning circuit are typically from 10 to 100 picofarads.

For *any* capacitor in vacuum, the capacitance C depends only on the shapes, dimensions, and separation of the conductors that make up the capacitor. If the conductor shapes are more complex than those of the parallel-plate capacitor, the expression for capacitance is more complicated than in Eq. (24.2). In the following examples we show how to calculate C for two other conductor geometries.

Example 24.1 Size of a 1-F capacitor

A parallel-plate capacitor has a capacitance of 1.0 F. If the plates are 1.0 mm apart, what is the area of the plates?

SOLUTION

IDENTIFY: This problem uses the relationship among the capacitance, plate separation, and plate area (our target variable) for a parallel-plate capacitor.

SET UP: We are given the values of C and d for a parallel-plate capacitor, so we use Eq. (24.2) and solve for the target variable A.

EXECUTE: From Eq. (24.2), the area A is

$$A = \frac{Cd}{\epsilon_0} = \frac{(1.0\ \text{F})(1.0 \times 10^{-3}\ \text{m})}{8.85 \times 10^{-12}\ \text{F/m}} = 1.1 \times 10^{8}\ \text{m}^2$$

EVALUATE: This corresponds to a square about 10 km (about 6 miles) on a side! This area is about a third larger than Manhattan Island. Clearly this is not a very practical design for a capacitor.

In fact, it's now possible to make 1-F capacitors a few centimeters on a side. The trick is to have an appropriate substance between the plates rather than a vacuum. We'll explore this further in Section 24.4.

Example 24.2 Properties of a parallel-plate capacitor

The plates of a parallel-plate capacitor in vacuum are 5.00 mm apart and $2.00\ \text{m}^2$ in area. A potential difference of 10,000 V (10.0 kV) is applied across the capacitor. Compute (a) the capacitance; (b) the charge on each plate; and (c) the magnitude of the electric field in the space between them.

SOLUTION

IDENTIFY: We are given the plate area A, the plate spacing d, and the potential difference V_{ab} for this parallel-plate capacitor. Our target variables are the capacitance C, charge Q, and electric-field magnitude E.

SET UP: We use Eq. (24.2) to calculate C and then find the charge Q on each plate using the given potential difference V_{ab} and Eq. (24.1). Once we have Q, we find the electric field between the plates using the relationship $E = Q/\epsilon_0 A$.

EXECUTE: (a) From Eq. (24.2),

$$C = \epsilon_0 \frac{A}{d} = \frac{(8.85 \times 10^{-12}\ \text{F/m})(2.00\ \text{m}^2)}{5.00 \times 10^{-3}\ \text{m}} = 3.54 \times 10^{-9}\ \text{F} = 0.00354\ \mu\text{F}$$

(b) The charge on the capacitor is

$$Q = CV_{ab} = (3.54 \times 10^{-9}\ \text{C/V})(1.00 \times 10^{4}\ \text{V}) = 3.54 \times 10^{-5}\ \text{C} = 35.4\ \mu\text{C}$$

The plate at higher potential has charge $+35.4\ \mu\text{C}$ and the other plate has charge $-35.4\ \mu\text{C}$.

(c) The electric-field magnitude is

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A} = \frac{3.54 \times 10^{-5}\ \text{C}}{(8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(2.00\ \text{m}^2)} = 2.00 \times 10^{6}\ \text{N/C}$$

EVALUATE: An alternative way to get the result in part (c) is to recall that the electric field is equal in magnitude to the potential gradient [Eq. (23.22)]. Since the field between the plates is uniform,

$$E = \frac{V_{ab}}{d} = \frac{1.00 \times 10^{4}\ \text{V}}{5.00 \times 10^{-3}\ \text{m}} = 2.00 \times 10^{6}\ \text{V/m}$$

(Remember that the newton per coulomb and the volt per meter are equivalent units.)

Example 24.3 A spherical capacitor

Two concentric spherical conducting shells are separated by vacuum. The inner shell has total charge $+Q$ and outer radius r_a, and the outer shell has charge $-Q$ and inner radius r_b (Fig. 24.5). (The inner shell is attached to the outer shell by thin insulating rods that have negligible effect on the capacitance.) Find the capacitance of this spherical capacitor.

SOLUTION

IDENTIFY: This isn't a parallel-plate capacitor, so we can't use the relationships developed for that particular geometry. Instead, we'll go back to the fundamental definition of capacitance: the magnitude of the charge on either conductor divided by the potential difference between the conductors.

SET UP: We use Gauss's law to find the electric field between the spherical conductors. From this value we determine the potential difference V_{ab} between the two conductors; we then use Eq. (24.1) to find the capacitance $C = Q/V_{ab}$.

EXECUTE: Using the same procedure as in Example 22.5 (Section 22.4), we take as our Gaussian surface a sphere with radius r between the two spheres and concentric with them. Gauss's law, Eq. (22.8), states that the electric flux through this surface is equal to the total charge enclosed within the surface, divided by ϵ_0:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl}}}{\epsilon_0}$$

By symmetry, $\vec{E}$ is constant in magnitude and parallel to $d\vec{A}$ at every point on this surface, so the integral in Gauss's law is equal to $(E)(4\pi r^2)$. The total charge enclosed is $Q_{\text{encl}} = Q$, so we have

$$(E)(4\pi r^2) = \frac{Q}{\epsilon_0}$$

$$E = \frac{Q}{4\pi\epsilon_0 r^2}$$

The electric field between the spheres is just that due to the charge on the inner sphere; the outer sphere has no effect. We found in Example 22.5 that the charge on a conducting sphere produces zero field *inside* the sphere, which also tells us that the outer conductor makes no contribution to the field between the conductors.

The above expression for E is the same as that for a point charge Q, so the expression for the potential can also be taken to be the same as for a point charge, $V = Q/4\pi\epsilon_0 r$. Hence the potential of the inner (positive) conductor at $r = r_a$ with respect to that of the outer (negative) conductor at $r = r_b$ is

$$V_{ab} = V_a - V_b = \frac{Q}{4\pi\epsilon_0 r_a} - \frac{Q}{4\pi\epsilon_0 r_b}$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) = \frac{Q}{4\pi\epsilon_0}\frac{r_b - r_a}{r_a r_b}$$

Finally, the capacitance is

$$C = \frac{Q}{V_{ab}} = 4\pi\epsilon_0 \frac{r_a r_b}{r_b - r_a}$$

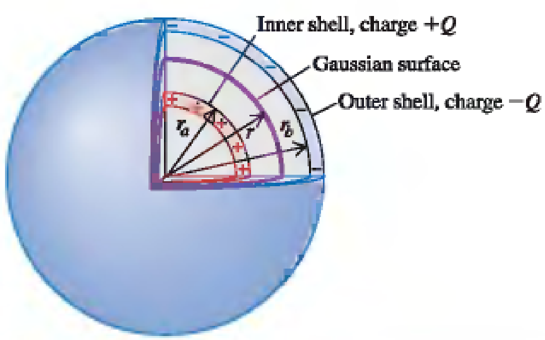
As an example, if $r_a = 9.5$ cm and $r_b = 10.5$ cm,

$$C = 4\pi(8.85 \times 10^{-12}\,\text{F/m})\frac{(0.095\,\text{m})(0.105\,\text{m})}{0.010\,\text{m}}$$

$$= 1.1 \times 10^{-10}\,\text{F} = 110\,\text{pF}$$

EVALUATE: We can relate this result to the capacitance of a parallel-plate capacitor. The quantity $4\pi r_a r_b$ is intermediate between the areas $4\pi r_a^2$ and $4\pi r_b^2$ of the two spheres; in fact, it's the *geometric mean* of these two areas, which we can denote by A_{gm}. The distance between spheres is $d = r_b - r_a$, so we can rewrite the above result as $C = \epsilon_0 A_{\text{gm}}/d$. This is exactly the same form as for parallel plates: $C = \epsilon_0 A/d$. The point is that if the distance between spheres is very small in comparison to their radii, they behave like parallel plates with the same area and spacing.

24.5 A spherical capacitor.



Example 24.4 A cylindrical capacitor

A long cylindrical conductor has a radius r_a and a linear charge density $+\lambda$. It is surrounded by a coaxial cylindrical conducting shell with inner radius r_b and linear charge density $-\lambda$ (Fig. 24.6). Calculate the capacitance per unit length for this capacitor, assuming that there is vacuum in the space between cylinders.

SOLUTION

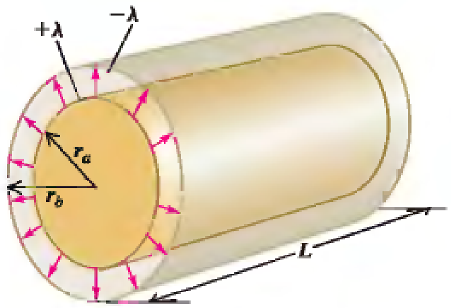
IDENTIFY: As in Example 24.3, we use the fundamental definition of capacitance.

SET UP: We first find expressions for the potential difference V_{ab} between the cylinders and the charge Q in a length L of the cylinders; we then find the capacitance of a length L using Eq. (24.1). Our target variable is this capacitance divided by L.

EXECUTE: To find the potential difference between the cylinders, we use a result that we worked out in Example 23.10 (Section 23.3). There we found that at a point outside a charged

Continued

24.6 A long cylindrical capacitor. The linear charge density λ is assumed to be positive in this figure. The magnitude of charge in a length L of either cylinder is λL.



cylinder a distance r from the axis, the potential due to the cylinder is

$$V = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r_0}{r}$$

where r_0 is the (arbitrary) radius at which $V = 0$. We can use this same result for the potential *between* the cylinders in the present problem because, according to Gauss's law, the charge on the outer cylinder doesn't contribute to the field between cylinders (see Example 24.3). In our case, we take the radius r_0 to be r_b, the radius of the inner surface of the outer cylinder, so that the outer conducting cylinder is at $V = 0$. Then the potential at the outer surface of the inner cylinder (where $r = r_a$) is just equal to the potential V_{ab} of the inner (positive) cylinder a with respect to the outer (negative) cylinder b, or

$$V_{ab} = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r_b}{r_a}$$

This potential difference is positive (assuming that λ is positive, as in Fig. 24.6) because the inner cylinder is at higher potential than the outer.

The total charge Q in a length L is $Q = \lambda L$, so from Eq. (24.1) the capacitance C of a length L is

$$C = \frac{Q}{V_{ab}} = \frac{\lambda L}{\dfrac{\lambda}{2\pi\epsilon_0} \ln \dfrac{r_b}{r_a}} = \frac{2\pi\epsilon_0 L}{\ln (r_b/r_a)}$$

The capacitance per unit length is

$$\frac{C}{L} = \frac{2\pi\epsilon_0}{\ln (r_b/r_a)}$$

Substituting $\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F/m} = 8.85\ \mathrm{pF/m}$, we get

$$\frac{C}{L} = \frac{55.6\ \mathrm{pF/m}}{\ln (r_b/r_a)}$$

EVALUATE: We see that the capacitance of the coaxial cylinders is determined entirely by the dimensions, just as for the parallel-plate case. Ordinary coaxial cables are made like this but with an insulating material instead of vacuum between the inner and outer conductors. A typical cable for TV antennas and VCR connections has a capacitance per unit length of 69 pF/m.

Test Your Understanding of Section 24.1 A capacitor has vacuum in the space between the conductors. If you double the amount of charge on each conductor, what happens to the capacitance? (i) It increases; (ii) it decreases; (iii) it remains the same; (iv) the answer depends on the size or shape of the conductors.

24.2 Capacitors in Series and Parallel

24.7 An assortment of commercially available capacitors.

Capacitors are manufactured with certain standard capacitances and working voltages (Fig. 24.7). However, these standard values may not be the ones you actually need in a particular application. You can obtain the values you need by combining capacitors; many combinations are possible, but the simplest combinations are a series connection and a parallel connection.

Capacitors in Series

Figure 24.8a is a schematic diagram of a **series connection.** Two capacitors are connected in series (one after the other) by conducting wires between points a and b. Both capacitors are initially uncharged. When a constant positive potential difference V_{ab} is applied between points a and b, the capacitors become charged; the figure shows that the charge on *all* conducting plates has the same magnitude. To see why, note first that the top plate of C_1 acquires a positive charge Q. The electric field of this positive charge pulls negative charge up to the bottom plate of C_1 until all of the field lines that begin on the top plate end on the bottom plate. This requires that the bottom plate have charge $-Q$. These negative charges had to come from the top plate of C_2, which becomes positively charged with charge $+Q$. This positive charge then pulls negative charge $-Q$ from the connection at point b onto the bottom plate of C_2. The total charge on the lower plate of C_1 and the upper plate of C_2 together must always be zero because these plates aren't connected to anything except each other. Thus *in a series connection the magnitude of charge on all plates is the same.*

Referring to Fig. 24.8a, we can write the potential differences between points a and c, c and b, and a and b as

$$V_{ac} = V_1 = \frac{Q}{C_1} \qquad V_{cb} = V_2 = \frac{Q}{C_2}$$

$$V_{ab} = V = V_1 + V_2 = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

and so

$$\frac{V}{Q} = \frac{1}{C_1} + \frac{1}{C_2} \tag{24.3}$$

Following a common convention, we use the symbols V_1, V_2, and V to denote the potential *differences* V_{ac} (across the first capacitor), V_{cb} (across the second capacitor), and V_{ab} (across the entire combination of capacitors), respectively.

The **equivalent capacitance** C_{eq} of the series combination is defined as the capacitance of a *single* capacitor for which the charge Q is the same as for the combination, when the potential difference V is the same. In other words, the combination can be replaced by an *equivalent capacitor* of capacitance C_{eq}. For such a capacitor, shown in Fig. 24.8b,

$$C_{eq} = \frac{Q}{V} \quad \text{or} \quad \frac{1}{C_{eq}} = \frac{V}{Q} \tag{24.4}$$

Combining Eqs. (24.3) and (24.4), we find

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

We can extend this analysis to any number of capacitors in series. We find the following result for the *reciprocal* of the equivalent capacitance:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots \quad \text{(capacitors in series)} \tag{24.5}$$

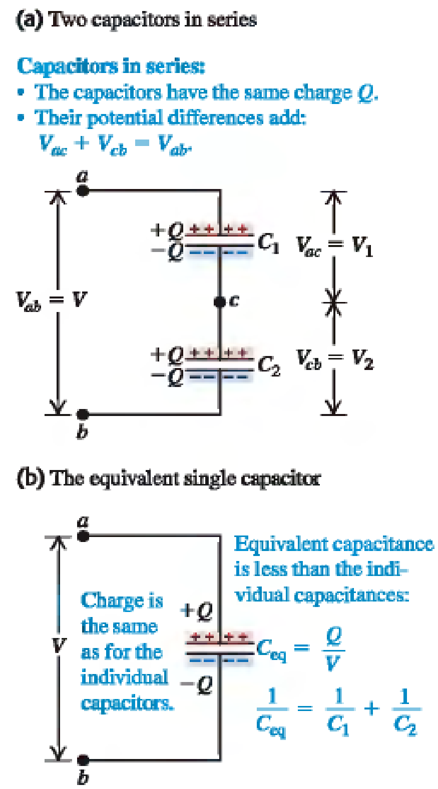
The reciprocal of the equivalent capacitance of a series combination equals the sum of the reciprocals of the individual capacitances. In a series connection the equivalent capacitance is always *less than* any individual capacitance.

CAUTION **Capacitors in series** The magnitude of charge is the same on all plates of all the capacitors in a series combination; however, the potential differences of the individual capacitors are *not* the same unless their individual capacitances are the same. The potential differences of the individual capacitors add to give the total potential difference across the series combination: $V_{total} = V_1 + V_2 + V_3 + \cdots$.

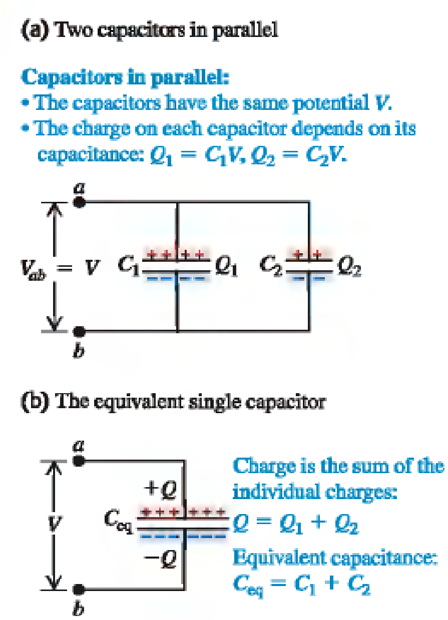
Capacitors in Parallel

The arrangement shown in Fig. 24.9a is called a **parallel connection.** Two capacitors are connected in parallel between points a and b. In this case the upper plates of the two capacitors are connected by conducting wires to form an equipotential surface, and the lower plates form another. Hence *in a parallel connection the potential difference for all individual capacitors is the same* and is equal to $V_{ab} = V$. The charges Q_1 and Q_2 are not necessarily equal, however,

24.8 A series connection of two capacitors.



24.9 A parallel connection of two capacitors.



since charges can reach each capacitor independently from the source (such as a battery) of the voltage V_{ab}. The charges are

$$Q_1 = C_1V \quad \text{and} \quad Q_2 = C_2V$$

The *total* charge Q of the combination, and thus the total charge on the equivalent capacitor, is

$$Q = Q_1 + Q_2 = (C_1 + C_2)V$$

so

$$\frac{Q}{V} = C_1 + C_2 \tag{24.6}$$

The parallel combination is equivalent to a single capacitor with the same total charge $Q = Q_1 + Q_2$ and potential difference V as the combination (Fig. 24.9b). The equivalent capacitance of the combination, C_{eq}, is the same as the capacitance Q/V of this single equivalent capacitor. So from Eq. (24.6),

$$C_{eq} = C_1 + C_2$$

In the same way we can show that for any number of capacitors in parallel,

$$C_{eq} = C_1 + C_2 + C_3 + \cdots \qquad \text{(capacitors in parallel)} \tag{24.7}$$

The equivalent capacitance of a parallel combination equals the *sum* of the individual capacitances. In a parallel connection the equivalent capacitance is always *greater than* any individual capacitance.

CAUTION Capacitors in parallel The potential differences are the same for all the capacitors in a parallel combination; however, the charges on individual capacitors are *not* the same unless their individual capacitances are the same. The charges on the individual capacitors add to give the total charge on the parallel combination: $Q_{total} = Q_1 + Q_2 + Q_3 + \cdots$. [Compare these statements to those in the "Caution" paragraph following Eq. (24.5).]

Problem-Solving Strategy 24.1 Equivalent Capacitance

MP

IDENTIFY *the relevant concepts:* The concept of equivalent capacitance is useful whenever two or more capacitors are connected.

SET UP *the problem* using the following steps:

1. Make a drawing of the capacitor arrangement.
2. Identify whether the capacitors are connected in series or in parallel. With more complicated combinations, you can sometimes identify parts that are simple series or parallel connections.
3. Keep in mind that when we say a capacitor has charge Q, we always mean that the plate at higher potential has charge $+Q$ and the other plate has charge $-Q$.

EXECUTE *the solution* as follows:

1. When capacitors are connected in series, as in Fig. 24.8a, they always have the same charge, assuming that they were uncharged before they were connected. The potential differences are *not* equal unless the capacitances are equal. The total potential difference across the combination is the sum of the individual potential differences.
2. When capacitors are connected in parallel, as in Fig. 24.9a, the potential difference V is always the same for all of the individual capacitors. The charges on the individual capacitors are *not* equal unless the capacitances are equal. The total charge on the combination is the sum of the individual charges.
3. For more complicated combinations, find the parts that are simple series or parallel connections and replace them with their equivalent capacitances, in a step-by-step reduction. If you then need to find the charge or potential difference for an individual capacitor, you may have to retrace your path to the original capacitors.

EVALUATE *your answer:* Check whether your result makes sense. If the capacitors are connected in series, the equivalent capacitance C_{eq} must be *smaller* than any of the individual capacitances. By contrast, if the capacitors are connected in parallel, C_{eq} must be *greater* than any of the individual capacitances.

Example 24.5 Capacitors in series and in parallel

In Figs. 24.8 and 24.9, let $C_1 = 6.0\ \mu\text{F}$, $C_2 = 3.0\ \mu\text{F}$, and $V_{ab} = 18$ V. Find the equivalent capacitance, and find the charge and potential difference for each capacitor when the two capacitors are connected (a) in series and (b) in parallel.

SOLUTION

IDENTIFY: This problem uses the ideas discussed in this section about capacitor connections.

SET UP: In both parts, one of the target variables is the equivalent capacitance C_{eq}. For the series combination in part (a), it is given by Eq. (24.5); for the parallel combination in part (b), C_{eq} is given by Eq. (24.6). In each part we find the charge and potential difference using the definition of capacitance, Eq. (24.1), and the rules outlined in the Problem-Solving Strategy 24.1.

EXECUTE: (a) Using Eq. (24.5) for the equivalent capacitance of the series combination (Fig. 24.8a), we find

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{6.0\ \mu\text{F}} + \frac{1}{3.0\ \mu\text{F}} \qquad C_{eq} = 2.0\ \mu\text{F}$$

The charge Q on each capacitor in series is the same as the charge on the equivalent capacitor:

$$Q = C_{eq}V = (2.0\ \mu\text{F})(18\ \text{V}) = 36\ \mu\text{C}$$

The potential difference across each capacitor is inversely proportional to its capacitance:

$$V_{ac} = V_1 = \frac{Q}{C_1} = \frac{36\ \mu\text{C}}{6.0\ \mu\text{F}} = 6.0\ \text{V}$$

$$V_{cb} = V_2 = \frac{Q}{C_2} = \frac{36\ \mu\text{C}}{3.0\ \mu\text{F}} = 12.0\ \text{V}$$

(b) To find the equivalent capacitance of the parallel combination (Fig. 24.9a), we use Eq. (24.6):

$$C_{eq} = C_1 + C_2 = 6.0\ \mu\text{F} + 3.0\ \mu\text{F} = 9.0\ \mu\text{F}$$

The potential difference across each of the two capacitors in parallel is the same as that across the equivalent capacitor, 18 V. The charges Q_1 and Q_2 are directly proportional to the capacitances C_1 and C_2, respectively:

$$Q_1 = C_1V = (6.0\ \mu\text{F})(18\ \text{V}) = 108\ \mu\text{C}$$

$$Q_2 = C_2V = (3.0\ \mu\text{F})(18\ \text{V}) = 54\ \mu\text{C}$$

EVALUATE: Note that the equivalent capacitance C_{eq} for the series combination in part (a) is indeed less than either C_1 or C_2, while for the parallel combination in part (b) the equivalent capacitance is indeed greater than either C_1 or C_2.

It's instructive to compare the potential differences and charges in each part of the example. For two capacitors in series, as in part (a), the charge is the same on either capacitor and the *larger* potential difference appears across the capacitor with the *smaller* capacitance. Furthermore, $V_{ac} + V_{cb} = V_{ab} = 18$ V, as it must. By contrast, for two capacitors in parallel, as in part (b), each capacitor has the same potential difference and the *larger* charge appears on the capacitor with the *larger* capacitance. Can you show that the total charge $Q_1 + Q_2$ on the parallel combination is equal to the charge $Q = C_{eq}V$ on the equivalent capacitor?

Example 24.6 A capacitor network

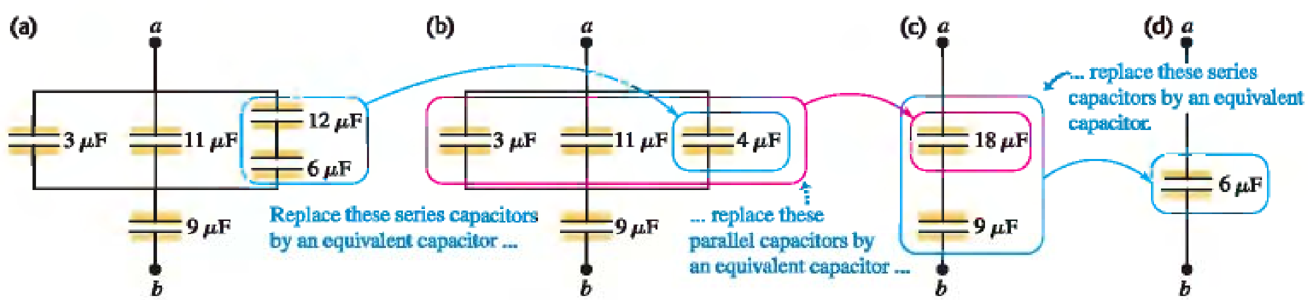
Find the equivalent capacitance of the combination shown in Fig. 24.10a.

SOLUTION

IDENTIFY: The five capacitors in Fig. 24.10a are neither all in series nor all in parallel. We can, however, identify portions of the arrangement that *are* either in series or parallel, which we combine to find the net equivalent capacitance.

SET UP: We use Eq. (24.5) to analyze portions of the network that are series connections and Eq. (24.7) to analyze portions that are parallel connections.

24.10 (a) A capacitor network between points a and b. (b) The 12-μF and 6-μF capacitors in series in (a) are replaced by an equivalent 4-μF capacitor. (c) The 3-μF, 11-μF, and 4-μF capacitors in parallel in (b) are replaced by an equivalent 18-μF capacitor. (d) Finally, the 18-μF and 9-μF capacitors in series in (c) are replaced by an equivalent 6-μF capacitor.



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EXECUTE: We first replace the 12-μF and 6-μF series combination by its equivalent capacitance; calling that C', we use Eq. (24.5):

$$\frac{1}{C'} = \frac{1}{12\,\mu\text{F}} + \frac{1}{6\,\mu\text{F}} \qquad C' = 4\,\mu\text{F}$$

This gives us the equivalent combination shown in Fig. 24.10b. Next we find the equivalent capacitance of the three capacitors in parallel, using Eq. (24.7). Calling their equivalent capacitance C'', we have

$$C'' = 3\,\mu\text{F} + 11\,\mu\text{F} + 4\,\mu\text{F} = 18\,\mu\text{F}$$

This gives us the simpler equivalent combination shown in Fig. 24.10c. Finally, we find the equivalent capacitance C_{eq} of these two capacitors in series (Fig. 24.10d):

$$\frac{1}{C_{\text{eq}}} = \frac{1}{18\,\mu\text{F}} + \frac{1}{9\,\mu\text{F}} \qquad C_{\text{eq}} = 6\,\mu\text{F}$$

EVALUATE: The equivalent capacitance of the network is 6 μF; that is, if a potential difference V_{ab} is applied across the terminals of the network, the net charge on the network is 6 μF times V_{ab}. How is this net charge related to the charges on the individual capacitors in Fig. 24.10a?

Test Your Understanding of Section 24.2 You want to connect a 4-μF capacitor and an 8-μF capacitor. (a) With which type of connection will the 4-μF capacitor have a greater *potential difference* across it than the 8-μF capacitor? (i) series; (ii) parallel; (iii) either series or parallel; (iv) neither series nor parallel. (b) With which type of connection will the 4-μF capacitor have a greater *charge* than the 8-μF capacitor? (i) series; (ii) parallel; (iii) either series or parallel; (iv) neither series nor parallel.

MP

24.3 Energy Storage in Capacitors and Electric-Field Energy

Many of the most important applications of capacitors depend on their ability to store energy. The electric potential energy stored in a charged capacitor is just equal to the amount of work required to charge it—that is, to separate opposite charges and place them on different conductors. When the capacitor is discharged, this stored energy is recovered as work done by electrical forces.

We can calculate the potential energy U of a charged capacitor by calculating the work W required to charge it. Suppose that when we are done charging the capacitor, the final charge is Q and the final potential difference is V. From Eq. (24.1) these quantities are related by

$$V = \frac{Q}{C}$$

Let q and v be the charge and potential difference, respectively, at an intermediate stage during the charging process; then $v = q/C$. At this stage the work dW required to transfer an additional element of charge dq is

$$dW = v\,dq = \frac{q\,dq}{C}$$

The total work W needed to increase the capacitor charge q from zero to a final value Q is

$$W = \int_0^W dW = \frac{1}{C}\int_0^Q q\,dq = \frac{Q^2}{2C} \qquad \text{(work to charge a capacitor)} \qquad (24.8)$$

This is also equal to the total work done by the electric field on the charge when the capacitor discharges. Then q *decreases* from an initial value Q to zero as the elements of charge dq "fall" through potential differences v that vary from V down to zero.

If we define the potential energy of an *uncharged* capacitor to be zero, then W in Eq. (24.8) is equal to the potential energy U of the charged capacitor. The final stored charge is $Q = CV$, so we can express U (which is equal to W) as

$$U = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV \qquad \text{(potential energy stored in a capacitor)} \qquad (24.9)$$

When Q is in coulombs, C in farads (coulombs per volt), and V in volts (joules per coulomb), U is in joules.

The last form of Eq. (24.9), $U = \frac{1}{2}QV$, shows that the total work W required to charge the capacitor is equal to the total charge Q multiplied by the *average* potential difference $\frac{1}{2}V$ during the charging process.

The expression $U = \frac{1}{2}(Q^2/C)$ in Eq. (24.9) shows that a charged capacitor is the electrical analog of a stretched spring with elastic potential energy $U = \frac{1}{2}kx^2$. The charge Q is analogous to the elongation x, and the *reciprocal* of the capacitance, $1/C$, is analogous to the force constant k. The energy supplied to a capacitor in the charging process is analogous to the work we do on a spring when we stretch it.

Equations (24.8) and (24.9) tell us that capacitance measures the ability of a capacitor to store both energy and charge. If a capacitor is charged by connecting it to a battery or other source that provides a fixed potential difference V, then increasing the value of C gives a greater charge $Q = CV$ and a greater amount of stored energy $U = \frac{1}{2}CV^2$. If instead the goal is to transfer a given quantity of charge Q from one conductor to another, Eq. (24.8) shows that the work W required is inversely proportional to C; the greater the capacitance, the easier it is to give a capacitor a fixed amount of charge.

Applications of Capacitors: Energy Storage

Most practical applications of capacitors take advantage of their ability to store and release energy. In electronic flash units used by photographers, the energy stored in a capacitor (see Fig. 24.4) is released by depressing the camera's shutter button. This provides a conducting path from one capacitor plate to the other through the flash tube. Once this path is established, the stored energy is rapidly converted into a brief but intense flash of light. An extreme example of the same principle is the Z machine at Sandia National Laboratories in New Mexico, which is used in experiments in controlled nuclear fusion (Fig. 24.11). A bank of charged capacitors releases more than a million joules of energy in just a few billionths of a second. For that brief space of time, the power output of the Z machine is 2.9×10^{14} W, or about 80 times the electric output of all the electric power plants on earth combined!

?

24.11 The Z machine uses a large number of capacitors in parallel to give a tremendous equivalent capacitance C (see Section 24.2). Hence a large amount of energy $U = \frac{1}{2}CV^2$ can be stored with even a modest potential difference V. The arcs shown here are produced when the capacitors discharge their energy into a target, which is no larger than a spool of thread. This heats the target to a temperature higher than 2×10^9 K.

In other applications, the energy is released more slowly. Springs in the suspension of an automobile, help smooth out the ride by absorbing the energy from sudden jolts and releasing that energy gradually; in an analogous way, a capacitor in an electronic circuit can smooth out unwanted variations in voltage due to power surges. And just as the presence of a spring gives a mechanical system a natural frequency at which it responds most strongly to an applied periodic force, so the presence of a capacitor gives an electric circuit a natural frequency for current oscillations. This idea is used in tuned circuits such as those in radio and television receivers, which respond to broadcast signals at one particular frequency and ignore signals at other frequencies. We'll discuss these circuits in detail in Chapter 31.

The energy-storage properties of capacitors also have some undesirable practical effects. Adjacent pins on the underside of a computer chip act like a capacitor, and the property that makes capacitors useful for smoothing out voltage variations acts to retard the rate at which the potentials of the chip's pins can be changed. This tendency limits how rapidly the chip can perform computations, an effect that becomes more important as computer chips become smaller and are pushed to operate at faster speeds.

Electric-Field Energy

We can charge a capacitor by moving electrons directly from one plate to another. This requires doing work against the electric field between the plates. Thus we can think of the energy as being stored *in the field* in the region between the plates. To develop this relationship, let's find the energy *per unit volume* in the space between the plates of a parallel-plate capacitor with plate area A and separation d. We call this the **energy density,** denoted by u. From Eq. (24.9) the total stored potential energy is $\frac{1}{2}CV^2$ and the volume between the plates is just Ad; hence the energy density is

$$u = \text{Energy density} = \frac{\frac{1}{2}CV^2}{Ad} \qquad (24.10)$$

From Eq. (24.2) the capacitance C is given by $C = \epsilon_0 A/d$. The potential difference V is related to the electric field magnitude E by $V = Ed$. If we use these expressions in Eq. (24.10), the geometric factors A and d cancel, and we find

$$u = \frac{1}{2}\epsilon_0 E^2 \quad \text{(electric energy density in a vacuum)} \qquad (24.11)$$

Although we have derived this relationship only for a parallel-plate capacitor, it turns out to be valid for any capacitor in vacuum and indeed *for any electric field configuration in vacuum.* This result has an interesting implication. We think of vacuum as space with no matter in it, but vacuum can nevertheless have electric fields and therefore energy. Thus "empty" space need not be truly empty after all. We will use this idea and Eq. (24.11) in Chapter 32 in connection with the energy transported by electromagnetic waves.

CAUTION **Electrical-field energy is electric potential energy** It's a common misconception that electric-field energy is a new kind of energy, different from the electric potential energy described before. This is *not* the case; it is simply a different way of interpreting electric potential energy. We can regard the energy of a given system of charges as being a shared property of all the charges, or we can think of the energy as being a property of the electric field that the charges create. Either interpretation leads to the same value of the potential energy.

Example 24.7 Transferring charge and energy between capacitors

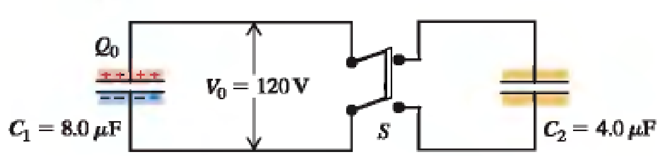
In Fig. 24.12 we charge a capacitor of capacitance $C_1 = 8.0\ \mu\text{F}$ by connecting it to a source of potential difference $V_0 = 120$ V (not shown in the figure). The switch S is initially open. Once C_1 is charged, the source of potential difference is disconnected. (a) What is the charge Q_0 on C_1 if switch S is left open? (b) What is the energy stored in C_1 if switch S is left open? (c) The capacitor of capacitance $C_2 = 4.0\ \mu\text{F}$ is initially uncharged. After we close switch S, what is the potential difference across each capacitor, and what is the charge on each capacitor? (d) What is the total energy of the system after we close switch S?

SOLUTION

IDENTIFY: Initially we have a single capacitor with a given potential difference between its plates. After the switch is closed, one wire connects the upper plates of the two capacitors and another wire connects the lower plates; in other words, the capacitors are connected in parallel.

SET UP: In parts (a) and (b) we find the charge and stored energy for capacitor C_1 using Eqs. (24.1) and (24.9), respectively. In part (c) we use the character of the parallel connection to determine how the charge Q_0 is shared between the two capacitors. In part (d) we again use Eq. (24.9) to find the energy stored in capacitors C_1 and C_2; the total energy is the sum of these values.

24.12 When the switch S is closed, the charged capacitor C_1 is connected to an uncharged capacitor C_2. The center part of the switch is an insulating handle; charge can flow only between the two upper terminals and between the two lower terminals.



EXECUTE: (a) The charge Q_0 on C_1 is

$$Q_0 = C_1V_0 = (8.0\ \mu\text{F})(120\ \text{V}) = 960\ \mu\text{C}$$

(b) The energy initially stored in the capacitor is

$$U_{\text{initial}} = \frac{1}{2}Q_0V_0 = \frac{1}{2}(960 \times 10^{-6}\ \text{C})(120\ \text{V}) = 0.058\ \text{J}$$

(c) When the switch is closed, the positive charge Q_0 becomes distributed over the upper plates of both capacitors and the negative charge $-Q_0$ is distributed over the lower plates of both capacitors. Let Q_1 and Q_2 be the magnitudes of the final charges on the two capacitors. From conservation of charge,

$$Q_1 + Q_2 = Q_0$$

In the final state, when the charges are no longer moving, both upper plates are at the same potential; they are connected by a conducting wire and so form a single equipotential surface. Both lower plates are also at the same potential, different from that of the upper plates. The final potential difference V between the plates is therefore the same for both capacitors, as we would expect for a parallel connection. The capacitor charges are

$$Q_1 = C_1V \qquad Q_2 = C_2V$$

When we combine these with the preceding equation for conservation of charge, we find

$$V = \frac{Q_0}{C_1 + C_2} = \frac{960\ \mu\text{C}}{8.0\ \mu\text{F} + 4.0\ \mu\text{F}} = 80\ \text{V}$$

$$Q_1 = 640\ \mu\text{C} \qquad Q_2 = 320\ \mu\text{C}$$

(d) The final energy of the system is the sum of the energies stored in each capacitor:

$$U_{\text{final}} = \frac{1}{2}Q_1V + \frac{1}{2}Q_2V = \frac{1}{2}Q_0V$$

$$= \frac{1}{2}(960 \times 10^{-6}\ \text{C})(80\ \text{V}) = 0.038\ \text{J}$$

EVALUATE: The final energy is less than the original energy $U_{\text{initial}} = 0.058$ J; the difference has been converted to energy of some other form. The conductors become a little warmer because of their resistance, and some energy is radiated as electromagnetic waves. We'll study the circuit behavior of capacitors in detail in Chapters 26 and 31.

Example 24.8 Electric-field energy

Suppose you want to store 1.00 J of electric potential energy in a volume of 1.00 m^3 in vacuum. (a) What is the magnitude of the required electric field? (b) If the field magnitude is 10 times larger, how much energy is stored per cubic meter?

SOLUTION

IDENTIFY: We use the relationship between the electric-field magnitude E and the energy density u, which equals the electric-field energy divided by the volume occupied by the field.

SET UP: In part (a) we use the given information to find u, then we use Eq. (24.11) to find the required value of E. This same equation gives us the relationship between changes in E and the corresponding changes in u.

EXECUTE: (a) The desired energy density is $u = 1.00\ \text{J/m}^3$. We solve Eq. (24.11) for E:

$$E = \sqrt{\frac{2u}{\epsilon_0}} = \sqrt{\frac{2(1.00\ \text{J/m}^3)}{8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2}}$$

$$= 4.75 \times 10^5\ \text{N/C} = 4.75 \times 10^5\ \text{V/m}$$

(b) Equation (24.11) shows that u is proportional to E^2. If E increases by a factor of 10, u increases by a factor of $10^2 = 100$, and the energy density is 100 J/m^3.

EVALUATE: The value of E found in part (a) is sizable, corresponding to a potential difference of nearly a half million volts over a distance of 1 meter. We will see in Section 24.4 that the field magnitudes in practical insulators can be as great as this or even larger.

Example 24.9 Two ways to calculate energy stored in a capacitor

The spherical capacitor described in Example 24.3 (Section 24.1) has charges $+Q$ and $-Q$ on its inner and outer conductors. Find the electric potential energy stored in the capacitor (a) by using the capacitance C found in Example 24.3 and (b) by integrating the electric-field energy density.

SOLUTION

IDENTIFY: This problem asks us to think about the energy stored in a capacitor, U, in two different ways: in terms of the work done to put the charges on the two conductors, $U = Q^2/2C$, and in terms of the energy in the electric field between the two conductors. Both descriptions are equivalent, so both must give us the same answer for U.

SET UP: In Example 24.3 we found the capacitance C and the field magnitude E between the conductors. We find the stored energy U in part (a) using the expression for C in Eq. (24.9). In part (b) we use the expression for E in Eq. (24.11) to find the electric-field energy density u between the conductors. The field magnitude depends on the distance r from the center of the capacitor, so u also depends on r. Hence we cannot find U by simply multiplying u by the volume between the conductors; instead, we must integrate u over this volume.

EXECUTE: (a) From Example 24.3, the spherical capacitor has capacitance

$$C = 4\pi\epsilon_0 \frac{r_a r_b}{r_b - r_a}$$

where r_a and r_b are the radii of the inner and outer conducting spheres. From Eq. (24.9) the energy stored in this capacitor is

$$U = \frac{Q^2}{2C} = \frac{Q^2}{8\pi\epsilon_0}\frac{r_b - r_a}{r_a r_b}$$

(b) The electric field in the volume between the two conducting spheres has magnitude $E = Q/4\pi\epsilon_0 r^2$. The electric field is zero inside the inner sphere and is also zero outside the inner surface of the outer sphere, because a Gaussian surface with radius $r < r_a$ or $r > r_b$ encloses zero net charge. Hence the energy density is nonzero only in the space between the spheres $(r_a < r < r_b)$. In this region,

$$u = \frac{1}{2}\epsilon_0 E^2 = \frac{1}{2}\epsilon_0\left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2 = \frac{Q^2}{32\pi^2\epsilon_0 r^4}$$

The energy density is *not* uniform; it decreases rapidly with increasing distance from the center of the capacitor. To find the

Continued

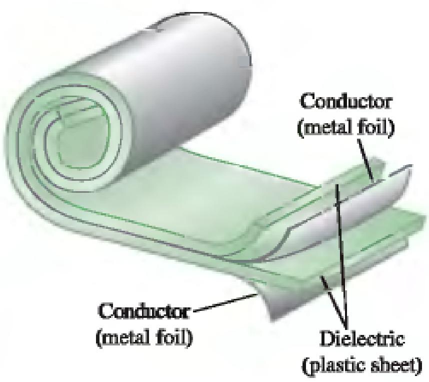
total electric-field energy, we integrate u (the energy per unit volume) over the volume between the inner and outer conducting spheres. Dividing this volume up into spherical shells of radius r, surface area $4\pi r^2$, thickness dr, and volume $dV = 4\pi r^2\,dr$, we have

$$U = \int u\,dV = \int_{r_a}^{r_b}\left(\frac{Q^2}{32\pi^2\epsilon_0 r^4}\right)4\pi r^2\,dr$$
$$= \frac{Q^2}{8\pi\epsilon_0}\int_{r_a}^{r_b}\frac{dr}{r^2} = \frac{Q^2}{8\pi\epsilon_0}\left(-\frac{1}{r_b}+\frac{1}{r_a}\right)$$
$$= \frac{Q^2}{8\pi\epsilon_0}\frac{r_b - r_a}{r_a r_b}$$

EVALUATE: We obtain the same result for U with either approach, as we must. We emphasize that electric potential energy can be regarded as being associated with either the *charges*, as in part (a), or the *field*, as in part (b); regardless of which viewpoint you choose, the amount of stored energy is the same.

Test Your Understanding of Section 24.3 You want to connect a 4-μF capacitor and an 8-μF capacitor. With which type of connection will the 4-μF capacitor have a greater amount of *stored energy* than the 8-μF capacitor? (i) series; (ii) parallel; (iii) either series or parallel; (iv) neither series nor parallel.

24.4 Dielectrics



24.13 A common type of capacitor uses dielectric sheets to separate the conductors.

Most capacitors have a nonconducting material, or **dielectric,** between their conducting plates. A common type of capacitor uses long strips of metal foil for the plates, separated by strips of plastic sheet such as Mylar. A sandwich of these materials is rolled up, forming a unit that can provide a capacitance of several microfarads in a compact package (Fig. 24.13).

Placing a solid dielectric between the plates of a capacitor serves three functions. First, it solves the mechanical problem of maintaining two large metal sheets at a very small separation without actual contact.

Second, using a dielectric increases the maximum possible potential difference between the capacitor plates. As we described in Section 23.3, any insulating material, when subjected to a sufficiently large electric field, experiences a partial ionization that permits conduction through it. This is called **dielectric breakdown.** Many dielectric materials can tolerate stronger electric fields without breakdown than can air. Thus using a dielectric allows a capacitor to sustain a higher potential difference V and so store greater amounts of charge and energy.

Third, the capacitance of a capacitor of given dimensions is *greater* when there is a dielectric material between the plates than when there is vacuum. We can demonstrate this effect with the aid of a sensitive *electrometer,* a device that measures the potential difference between two conductors without letting any appreciable charge flow from one to the other. Figure 24.14a shows an electrometer connected across a charged capacitor, with magnitude of charge Q on each plate and potential difference V_0. When we insert an uncharged sheet of dielectric, such as glass, paraffin, or polystyrene, between the plates, experiment shows that the potential difference *decreases* to a smaller value V (Fig. 24.14b). When we remove the dielectric, the potential difference returns to its original value V_0, showing that the original charges on the plates have not changed.

The original capacitance C_0 is given by $C_0 = Q/V_0$, and the capacitance C with the dielectric present is $C = Q/V$. The charge Q is the same in both cases, and V is less than V_0, so we conclude that the capacitance C with the dielectric present is *greater* than C_0. When the space between plates is completely filled by the dielectric, the ratio of C to C_0 (equal to the ratio of V_0 to V) is called the **dielectric constant** of the material, K:

$$K = \frac{C}{C_0} \qquad \text{(definition of dielectric constant)} \qquad (24.12)$$

When the charge is constant, $Q = C_0V_0 = CV$ and $C/C_0 = V_0/V$. In this case, Eq. (24.12) can be rewritten as

$$V = \frac{V_0}{K} \qquad \text{(when } Q \text{ is constant)} \tag{24.13}$$

With the dielectric present, the potential difference for a given charge Q is *reduced* by a factor K.

The dielectric constant K is a pure number. Because C is always greater than C_0, K is always greater than unity. Some representative values of K are given in Table 24.1. For vacuum, $K = 1$ by definition. For air at ordinary temperatures and pressures, K is about 1.0006; this is so nearly equal to 1 that for most purposes an air capacitor is equivalent to one in vacuum. Note that while water has a very large value of K, it is usually not a very practical dielectric for use in capacitors. The reason is that while pure water is a very poor conductor, it is also an excellent ionic solvent. Any ions that are dissolved in the water will cause charge to flow between the capacitor plates, so the capacitor discharges.

Table 24.1 Values of Dielectric Constant K at 20°C

Material	K	Material	K
Vacuum	1	Polyvinyl chloride	3.18
Air (1 atm)	1.00059	Plexiglas	3.40
Air (100 atm)	1.0548	Glass	5–10
Teflon	2.1	Neoprene	6.70
Polyethylene	2.25	Germanium	16
Benzene	2.28	Glycerin	42.5
Mica	3–6	Water	80.4
Mylar	3.1	Strontium titanate	310

No real dielectric is a perfect insulator. Hence there is always some *leakage current* between the charged plates of a capacitor with a dielectric. We tacitly ignored this effect in Section 24.2 when we derived expressions for the equivalent capacitances of capacitors in series, Eq. (24.5), and in parallel, Eq. (24.7). But if a leakage current flows for a long enough time to substantially change the charges from the values we used to derive Eqs. (24.5) and (24.7), those equations may no longer be accurate.

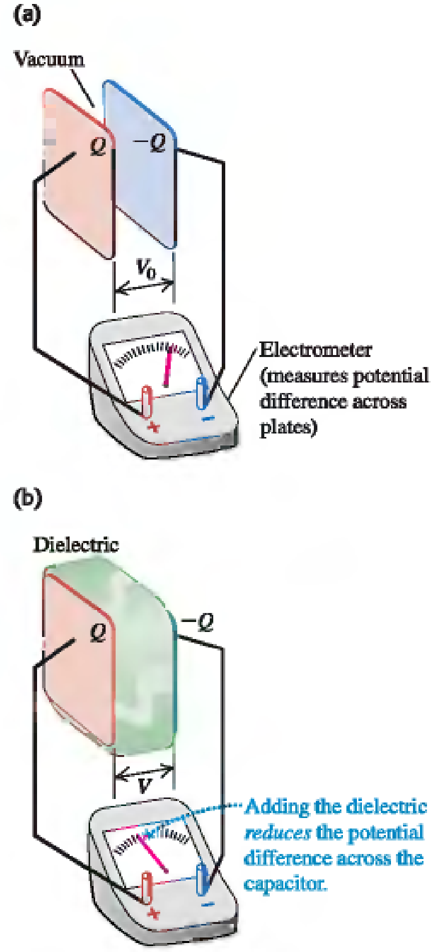
Induced Charge and Polarization

When a dielectric material is inserted between the plates while the charge is kept constant, the potential difference between the plates decreases by a factor K. Therefore the electric field between the plates must decrease by the same factor. If E_0 is the vacuum value and E is the value with the dielectric, then

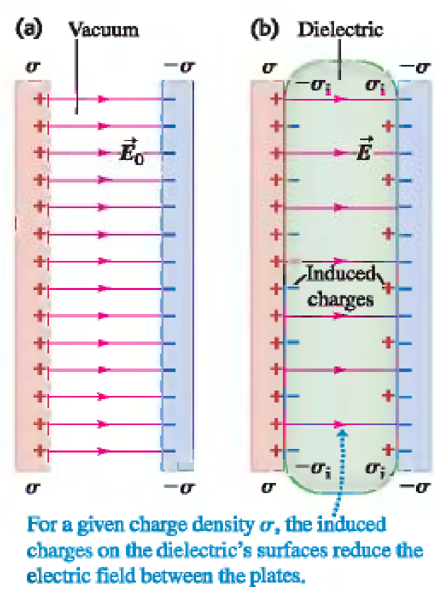
$$E = \frac{E_0}{K} \qquad \text{(when } Q \text{ is constant)} \tag{24.14}$$

Since the electric-field magnitude is smaller when the dielectric is present, the surface charge density (which causes the field) must be smaller as well. The surface charge on the conducting plates does not change, but an *induced* charge of the opposite sign appears on each surface of the dielectric (Fig. 24.15). The dielectric was originally electrically neutral and is still neutral; the induced surface charges arise as a result of *redistribution* of positive and negative charge within the dielectric material, a phenomenon called **polarization.** We first encountered polarization in Section 21.2, and we suggest that you reread the discussion of Fig. 21.8. We will assume that the induced surface charge is *directly proportional* to the electric-field magnitude E in the material; this is indeed the case for many common dielectrics. (This direct proportionality is analogous to

24.14 Effect of a dielectric between the plates of a parallel-plate capacitor. **(a)** With a given charge, the potential difference is V_0. **(b)** With the same charge but with a dielectric between the plates, the potential difference V is smaller than V_0.



24.15 Electric field lines with **(a)** vacuum between the plates and **(b)** dielectric between the plates.



Hooke's law for a spring.) In that case, K is a constant for any particular material. When the electric field is very strong or if the dielectric is made of certain crystalline materials, the relationship between induced charge and the electric field can be more complex; we won't consider such cases here.

We can derive a relationship between this induced surface charge and the charge on the plates. Let's denote the magnitude of the charge per unit area induced on the surfaces of the dielectric (the induced surface charge density) by σ_i. The magnitude of the surface charge density on the capacitor plates is σ, as usual. Then the *net* surface charge on each side of the capacitor has magnitude $(\sigma - \sigma_i)$, as shown in Fig. 24.15b. As we found in Example 21.13 (Section 21.5) and in Example 22.8 (Section 22.4), the field between the plates is related to the net surface charge density by $E = \sigma_{net}/\epsilon_0$. Without and with the dielectric, respectively, we have

$$E_0 = \frac{\sigma}{\epsilon_0} \qquad E = \frac{\sigma - \sigma_i}{\epsilon_0} \tag{24.15}$$

Using these expressions in Eq. (24.14) and rearranging the result, we find

$$\sigma_i = \sigma\left(1 - \frac{1}{K}\right) \qquad \text{(induced surface charge density)} \tag{24.16}$$

This equation shows that when K is very large, σ_i is nearly as large as σ. In this case, σ_i nearly cancels σ, and the field and potential difference are much smaller than their values in vacuum.

The product $K\epsilon_0$ is called the **permittivity** of the dielectric, denoted by ϵ:

$$\epsilon = K\epsilon_0 \qquad \text{(definition of permittivity)} \tag{24.17}$$

In terms of ϵ we can express the electric field within the dielectric as

$$E = \frac{\sigma}{\epsilon} \tag{24.18}$$

The capacitance when the dielectric is present is given by

$$C = KC_0 = K\epsilon_0\frac{A}{d} = \epsilon\frac{A}{d} \qquad \text{(parallel-plate capacitor, dielectric between plates)} \tag{24.19}$$

We can repeat the derivation of Eq. (24.11) for the energy density u in an electric field for the case in which a dielectric is present. The result is

$$u = \frac{1}{2}K\epsilon_0 E^2 = \frac{1}{2}\epsilon E^2 \qquad \text{(electric energy density in a dielectric)} \tag{24.20}$$

In empty space, where $K = 1$, $\epsilon = \epsilon_0$ and Eqs. (24.19) and (24.20) reduce to Eqs. (24.2) and (24.11), respectively, for a parallel-plate capacitor in vacuum. For this reason, ϵ_0 is sometimes called the "permittivity of free space" or the "permittivity of vacuum." Because K is a pure number, ϵ and ϵ_0 have the same units, $C^2/N \cdot m^2$ or F/m.

Equation (24.19) shows that extremely high capacitances can be obtained with plates that have a large surface area A and are separated by a small distance d by a dielectric with a large value of K. In an *electrolytic double-layer capacitor*, tiny carbon granules adhere to each plate: The value of A is the combined surface area of the granules, which can be tremendous. The plates with granules attached are separated by a very thin dielectric sheet. A capacitor of this kind can have a capacitance of 5000 farads yet fit in the palm of your hand (compare Example 24.1 in Section 24.1).

Several practical devices make use of the way in which a capacitor responds to a change in dielectric constant. One example is an electric stud finder, used by home repair workers to locate metal studs hidden behind a wall's surface. It consists of a metal plate with associated circuitry. The plate acts as one half of a capacitor, with the wall acting as the other half. If the stud finder moves over a metal stud, the effective dielectric constant for the capacitor changes, changing the capacitance and triggering a signal.

Problem-Solving Strategy 24.2 Dielectrics

MP

IDENTIFY *the relevant concepts:* The relationships in this section are useful whenever there is an electric field in a dielectric, such as a dielectric between charged capacitor plates. Typically you will be asked to relate the potential difference between the plates, the electric field in the capacitor, the charge density on the capacitor plates, and the induced charge density on the surfaces of the capacitor.

SET UP *the problem* using the following steps:

1. Make a drawing of the situation.
2. Identify the target variables, and choose which of the key equations of this section will help you find those variables.

EXECUTE *the solution* as follows:

1. In problems such as the next example, it is easy to get lost in a blizzard of formulas. Ask yourself at each step what kind of quantity each symbol represents. For example, distinguish clearly between charges and charge densities, and between electric fields and electric potential differences.
2. As you calculate, continually check for consistency of units. This effort is a bit more complex with electrical quantities than it was in mechanics. Distances must always be in meters. Remember that a microfarad is 10^{-6} farad, and so on. Don't confuse the numerical value of ϵ_0 with the value of $1/4\pi\epsilon_0$. There are several alternative sets of units for electric-field magnitude, including N/C and V/m. The units of ϵ_0 are $C^2/N \cdot m^2$ or F/m.

EVALUATE *your answer:* When you check numerical values, remember that with a dielectric present, (a) the capacitance is always greater than without a dielectric; (b) for a given amount of charge on the capacitor, the electric field and potential difference are less than without a dielectric; and (c) the induced surface charge density σ_i on the dielectric is always less in magnitude than the charge density σ on the capacitor plates.

Example 24.10 A capacitor with and without a dielectric

Suppose the parallel plates in Fig. 24.15 each have an area of $2000\ \text{cm}^2$ $(2.00 \times 10^{-1}\ \text{m}^2)$ and are $1.00\ \text{cm}$ $(1.00 \times 10^{-2}\ \text{m})$ apart. The capacitor is connected to a power supply and charged to a potential difference $V_0 = 3000\ \text{V} = 3.00\ \text{kV}$. It is then disconnected from the power supply, and a sheet of insulating plastic material is inserted between the plates, completely filling the space between them. We find that the potential difference decreases to 1000 V while the charge on each capacitor plate remains constant. Compute (a) the original capacitance C_0; (b) the magnitude of charge Q on each plate; (c) the capacitance C after the dielectric is inserted; (d) the dielectric constant K of the dielectric; (e) the permittivity ϵ of the dielectric; (f) the magnitude of the induced charge Q_i on each face of the dielectric; (g) the original electric field E_0 between the plates; and (h) the electric field E after the dielectric is inserted.

SOLUTION

IDENTIFY: This problem uses most of the relationships we have discussed for capacitors and dielectrics.

SET UP: Most of the target variables can be obtained in several different ways. The methods used below are a representative sample; we encourage you to think of others and compare your results.

EXECUTE: (a) With vacuum between the plates, we use Eq. (24.19) with $K = 1$:

$$C_0 = \epsilon_0 \frac{A}{d} = (8.85 \times 10^{-12}\ \text{F/m})\frac{2.00 \times 10^{-1}\ \text{m}^2}{1.00 \times 10^{-2}\ \text{m}}$$
$$= 1.77 \times 10^{-10}\ \text{F} = 177\ \text{pF}$$

(b) Using the definition of capacitance, Eq. (24.1),

$$Q = C_0 V_0 = (1.77 \times 10^{-10}\ \text{F})(3.00 \times 10^3\ \text{V})$$
$$= 5.31 \times 10^{-7}\ \text{C} = 0.531\ \mu\text{C}$$

(c) When the dielectric is inserted, the charge remains the same but the potential decreases to $V = 1000\ \text{V}$. Hence from Eq. (24.1), the new capacitance is

$$C = \frac{Q}{V} = \frac{5.31 \times 10^{-7}\ \text{C}}{1.00 \times 10^3\ \text{V}} = 5.31 \times 10^{-10}\ \text{F} = 531\ \text{pF}$$

(d) From Eq. (24.12), the dielectric constant is

$$K = \frac{C}{C_0} = \frac{5.31 \times 10^{-10}\ \text{F}}{1.77 \times 10^{-10}\ \text{F}} = \frac{531\ \text{pF}}{177\ \text{pF}} = 3.00$$

Alternatively, from Eq. (24.13),

$$K = \frac{V_0}{V} = \frac{3000\ \text{V}}{1000\ \text{V}} = 3.00$$

(e) Using K from part (d) in Eq. (24.17), the permittivity is

$$\epsilon = K\epsilon_0 = (3.00)(8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)$$
$$= 2.66 \times 10^{-11}\ \text{C}^2/\text{N}\cdot\text{m}^2$$

(f) Multiplying Eq. (24.15) by the area of each plate gives the induced charge $Q_i = \sigma_i A$ in terms of the charge $Q = \sigma A$ on each plate:

$$Q_i = Q\left(1 - \frac{1}{K}\right) = (5.31 \times 10^{-7}\ \text{C})\left(1 - \frac{1}{3.00}\right)$$
$$= 3.54 \times 10^{-7}\ \text{C}$$

Continued

(g) Since the electric field between the plates is uniform, its magnitude is the potential difference divided by the plate separation:

$$E_0 = \frac{V_0}{d} = \frac{3000\ \text{V}}{1.00 \times 10^{-2}\ \text{m}} = 3.00 \times 10^5\ \text{V/m}$$

(h) With the new potential difference after the dielectric is inserted,

$$E = \frac{V}{d} = \frac{1000\ \text{V}}{1.00 \times 10^{-2}\ \text{m}} = 1.00 \times 10^5\ \text{V/m}$$

or, from Eq. (24.17),

$$E = \frac{\sigma}{\epsilon} = \frac{Q}{\epsilon A} = \frac{5.31 \times 10^{-7}\ \text{C}}{(2.66 \times 10^{-11}\ \text{C}^2/\text{N}\cdot\text{m}^2)(2.00 \times 10^{-1}\ \text{m}^2)}$$
$$= 1.00 \times 10^5\ \text{V/m}$$

or, from Eq. (24.15),

$$E = \frac{\sigma - \sigma_i}{\epsilon_0} = \frac{Q - Q_i}{\epsilon_0 A}$$
$$= \frac{(5.31 - 3.54) \times 10^{-7}\ \text{C}}{(8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(2.00 \times 10^{-1}\ \text{m}^2)}$$
$$= 1.00 \times 10^5\ \text{V/m}$$

or, from Eq. (24.14),

$$E = \frac{E_0}{K} = \frac{3.00 \times 10^5\ \text{V/m}}{3.00} = 1.00 \times 10^5\ \text{V/m}$$

EVALUATE: It's always useful to check the results by finding them in more than one way, as we did in parts (d) and (h). Our results show that inserting the dielectric increased the capacitance by a factor of $K = 3.00$ and reduced the electric field between the plates by a factor of $1/K = 1/3.00$. It did so by developing induced charges on the faces of the dielectric of magnitude $Q(1 - 1/K) = Q(1 - 1/3.00) = 0.667Q$.

Example 24.11 Energy storage with and without a dielectric

Find the total energy stored in the electric field of the capacitor in Example 24.10 and the energy density, both before and after the dielectric sheet is inserted.

SOLUTION

IDENTIFY: In this problem we have to extend the analysis of Example 24.10 to include the ideas of energy stored in a capacitor and electric-field energy.

SET UP: We use Eq. (24.9) to find the stored energy before and after the dielectric is inserted, and Eq. (24.20) to find the energy density.

EXECUTE: Let the original energy be U_0 and let the energy with the dielectric in place be U. From Eq. (24.9),

$$U_0 = \frac{1}{2}C_0V_0^2 = \frac{1}{2}(1.77 \times 10^{-10}\ \text{F})(3000\ \text{V})^2 = 7.97 \times 10^{-4}\ \text{J}$$

$$U = \frac{1}{2}CV^2 = \frac{1}{2}(5.31 \times 10^{-10}\ \text{F})(1000\ \text{V})^2 = 2.66 \times 10^{-4}\ \text{J}$$

The final energy is one-third of the original energy.

The energy density without the dielectric is given by Eq. (24.20) with $K = 1$:

$$u_0 = \frac{1}{2}\epsilon_0 E_0^2 = \frac{1}{2}(8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(3.0 \times 10^5\ \text{N/C})^2$$
$$= 0.398\ \text{J/m}^3$$

With the dielectric in place,

$$u = \frac{1}{2}\epsilon E^2 = \frac{1}{2}(2.66 \times 10^{-11}\ \text{C}^2/\text{N}\cdot\text{m}^2)(1.00 \times 10^5\ \text{N/C})^2$$
$$= 0.133\ \text{J/m}^3$$

The energy density with the dielectric is one-third of the original energy density.

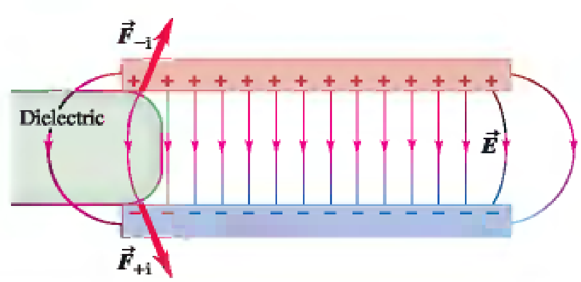
EVALUATE: We can check our answer for u_0 by noting that the volume between the plates is $V = (0.200\ \text{m})^2(0.0100\ \text{m}) = 0.00200\ \text{m}^3$. Since the electric field is uniform between the plates, u_0 is uniform as well and the energy density is just the stored energy divided by the volume:

$$u_0 = \frac{U_0}{V} = \frac{7.97 \times 10^{-4}\ \text{J}}{0.00200\ \text{m}^3} = 0.398\ \text{J/m}^3$$

which agrees with our earlier answer. You should use the same approach to check the value for U, the energy density with the dielectric.

We can generalize the results of this example. When a dielectric is inserted into a capacitor while the charge on each plate remains the same, the permittivity ϵ increases by a factor of K (the dielectric constant), the electric field decreases by a factor of $1/K$, and the energy density $u = \frac{1}{2}\epsilon E^2$ decreases by a factor of $1/K$. Where did the energy go? The answer lies in the fringing field at the edges of a real parallel-plate capacitor. As Fig. 24.16 shows, that field tends to pull the dielectric into the space between the plates, doing work on it as it does so. We could attach a spring to the left end of the dielectric in Fig. 24.16 and use this force to stretch the spring. Because work is done by the field, the field energy density decreases.

24.16 The fringing field at the edges of the capacitor exerts forces $\vec{F}_{-i}$ and $\vec{F}_{+i}$ on the negative and positive induced surface charges of a dielectric, pulling the dielectric into the capacitor.



Dielectric Breakdown

We mentioned earlier that when any dielectric material is subjected to a sufficiently strong electric field, *dielectric breakdown* takes place and the dielectric becomes a conductor (Fig. 24.17). This occurs when the electric field is so strong that electrons are ripped loose from their molecules and crash into other molecules, liberating even more electrons. This avalanche of moving charge, forming a spark or arc discharge, often starts quite suddenly.

Because of dielectric breakdown, capacitors always have maximum voltage ratings. When a capacitor is subjected to excessive voltage, an arc may form through a layer of dielectric, burning or melting a hole in it. This arc creates a conducting path (a short circuit) between the conductors. If a conducting path remains after the arc is extinguished, the device is rendered permanently useless as a capacitor.

The maximum electric-field magnitude that a material can withstand without the occurrence of breakdown is called its **dielectric strength.** This quantity is affected significantly by temperature, trace impurities, small irregularities in the metal electrodes, and other factors that are difficult to control. For this reason we can give only approximate figures for dielectric strengths. The dielectric strength of dry air is about 3×10^6 V/m. Values of dielectric strength for a few common insulating materials are shown in Table 24.2. Note that the values are all substantially greater than the value for air. For example, a layer of polycarbonate 0.01 mm thick (about the smallest practical thickness) has 10 times the dielectric strength of air and can withstand a maximum voltage of about $(3 \times 10^7 \text{ V/m})(1 \times 10^{-5} \text{ m}) = 300 \text{ V}$.

24.17 A very strong electric field caused dielectric breakdown in a block of Plexiglas. The resulting flow of charge etched this pattern into the block.

Table 24.2 Dielectric Constant and Dielectric Strength of Some Insulating Materials

Material	Dielectric Constant, K	Dielectric Strength, E_m (V/m)
Polycarbonate	2.8	3×10^7
Polyester	3.3	6×10^7
Polypropylene	2.2	7×10^7
Polystyrene	2.6	2×10^7
Pyrex glass	4.7	1×10^7

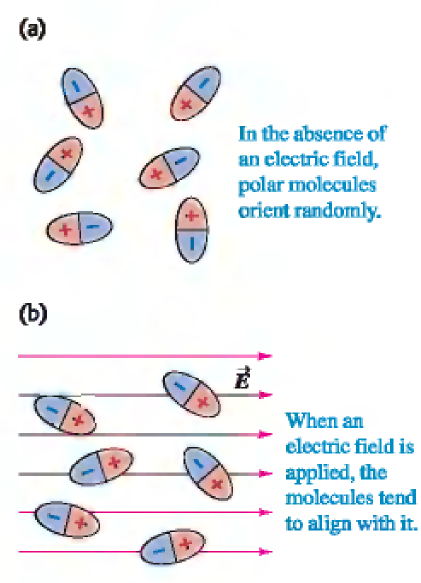
Test Your Understanding of Section 24.4 The space between the plates of an isolated parallel-plate capacitor is filled by a slab of dielectric with dielectric constant K. The two plates of the capacitor have charges Q and $-Q$. You pull out the dielectric slab. If the charges do not change, how does the energy in the capacitor change when you remove the slab? (i) It increases; (ii) it decreases; (iii) it remains the same.

*24.5 Molecular Model of Induced Charge

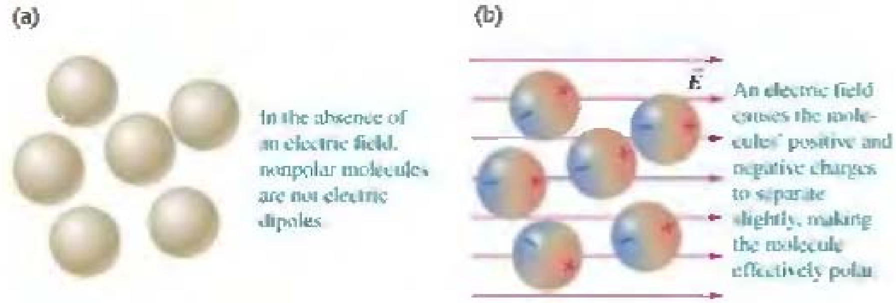
In Section 24.4 we discussed induced surface charges on a dielectric in an electric field. Now let's look at how these surface charges can arise. If the material were a *conductor,* the answer would be simple. Conductors contain charge that is free to move, and when an electric field is present, some of the charge redistributes itself on the surface so that there is no electric field inside the conductor. But an ideal dielectric has *no* charges that are free to move, so how can a surface charge occur?

To understand this, we have to look again at rearrangement of charge at the *molecular* level. Some molecules, such as H_2O and N_2O, have equal amounts of positive and negative charges but a lopsided distribution, with excess positive charge concentrated on one side of the molecule and negative charge on the other. As we described in Section 21.7, such an arrangement is called an *electric dipole,* and the molecule is called a *polar molecule.* When no electric field is present in a gas or liquid with polar molecules, the molecules are oriented randomly (Fig. 24.18a). When they are placed in an electric field, however, they tend

24.18 Polar molecules (a) without and (b) with an applied electric field $\vec{E}$.



24.19 Nonpolar molecules **(a)** without and **(b)** with an applied electric field $\vec{E}$.



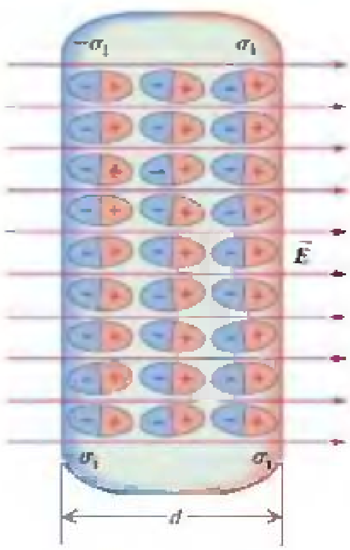
to orient themselves as in Fig. 24.18b, as a result of the electric-field torques described in Section 21.7. Because of thermal agitation, the alignment of the molecules with $\vec{E}$ is not perfect.

Even a molecule that is *not* ordinarily polar *becomes* a dipole when it is placed in an electric field because the field pushes the positive charges in the molecules in the direction of the field and pushes the negative charges in the opposite direction. This causes a redistribution of charge within the molecule (Fig. 24.19). Such dipoles are called *induced* dipoles.

With either polar or nonpolar molecules, the redistribution of charge caused by the field leads to the formation of a layer of charge on each surface of the dielectric material (Fig. 24.20). These layers are the surface charges described in Section 24.4; their surface charge density is denoted by σ_i. The charges are *not* free to move indefinitely, as they would be in a conductor, because each charge is bound to a molecule. They are in fact called **bound charges** to distinguish them from the **free charges** that are added to and removed from the conducting capacitor plates. In the interior of the material the net charge per unit volume remains zero. As we have seen, this redistribution of charge is called *polarization,* and we say that the material is *polarized.*

The four parts of Fig. 24.21 show the behavior of a slab of dielectric when it is inserted in the field between a pair of oppositely charged capacitor plates. Figure 24.21a shows the original field. Figure 24.21b is the situation after the dielectric has been inserted but before any rearrangement of charges has occurred.

24.20 Polarization of a dielectric in an electric field $\vec{E}$ gives rise to thin layers of bound charges on the surfaces, creating surface charge densities σ_i and $-\sigma_i$. The sizes of the molecules are greatly exaggerated for clarity.



24.21 **(a)** Electric field of magnitude E_0 between two charged plates. **(b)** Introduction of a dielectric of dielectric constant K. **(c)** The induced surface charges and their field. **(d)** Resultant field of magnitude E_0/K.

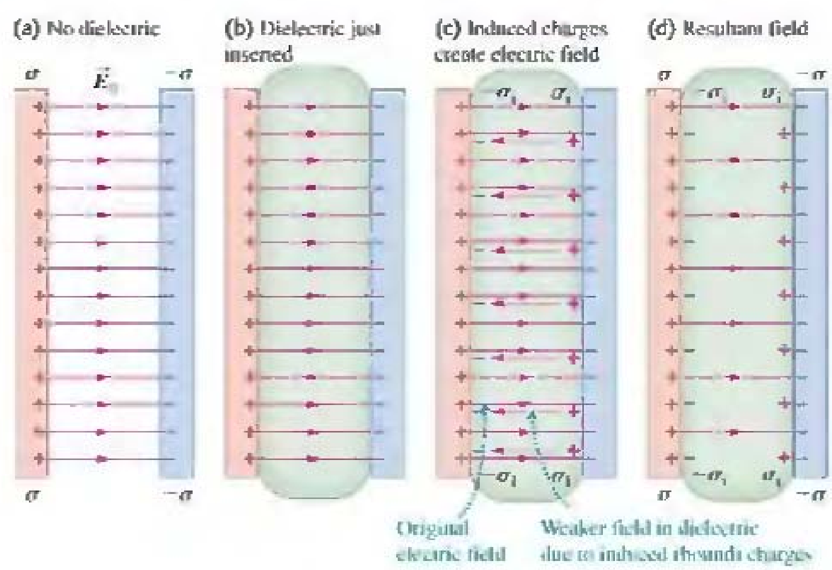
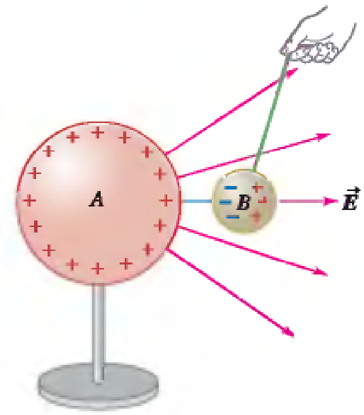


Figure 24.21c shows by thinner arrows the additional field set up in the dielectric by its induced surface charges. This field is *opposite* to the original field, but it is not great enough to cancel the original field completely because the charges in the dielectric are not free to move indefinitely. The resultant field in the dielectric, shown in Fig. 24.21d, is therefore decreased in magnitude. In the field-line representation, some of the field lines leaving the positive plate go through the dielectric, while others terminate on the induced charges on the faces of the dielectric.

As we discussed in Section 21.2, polarization is also the reason a charged body, such as an electrified plastic rod, can exert a force on an *uncharged* body such as a bit of paper or a pith ball. Figure 24.22 shows an uncharged dielectric sphere B in the radial field of a positively charged body A. The induced positive charges on B experience a force toward the right, while the force on the induced negative charges is toward the left. The negative charges are closer to A, and thus are in a stronger field, than are the positive charges. The force toward the left is stronger than that toward the right, and B is attracted toward A, even though its net charge is zero. The attraction occurs whether the sign of A's charge is positive or negative (see Fig. 21.8). Furthermore, the effect is not limited to dielectrics; an uncharged conducting body would be attracted in the same way.

24.22 A neutral sphere B in the radial electric field of a positively charged sphere A is attracted to the charge because of polarization.



Test Your Understanding of Section 24.5 A parallel-plate capacitor has charges Q and $-Q$ on its two plates. A dielectric slab with $K = 3$ is then inserted into the space between the plates as shown in Fig. 24.21. Rank the following electric-field magnitudes in order from largest to smallest. (i) the field before the slab is inserted; (ii) the resultant field after the slab is inserted; (iii) the field due to the bound charges.

*24.6 Gauss's Law in Dielectrics

We can extend the analysis of Section 24.4 to reformulate Gauss's law in a form that is particularly useful for dielectrics. Figure 24.23 is a close-up view of the left capacitor plate and left surface of the dielectric in Fig. 24.15b. Let's apply Gauss's law to the rectangular box shown in cross section by the purple line; the surface area of the left and right sides is A. The left side is embedded in the conductor that forms the left capacitor plate, and so the electric field everywhere on that surface is zero. The right side is embedded in the dielectric, where the electric field has magnitude E, and $E_\perp = 0$ everywhere on the other four sides. The total charge enclosed, including both the charge on the capacitor plate and the induced charge on the dielectric surface, is $Q_{encl} = (\sigma - \sigma_i)A$, so Gauss's law gives

$$EA = \frac{(\sigma - \sigma_i)A}{\epsilon_0} \tag{24.21}$$

This equation is not very illuminating as it stands because it relates two unknown quantities: E inside the dielectric and the induced surface charge density σ_i. But now we can use Eq. (24.16), developed for this same situation, to simplify this equation by eliminating σ_i. Equation (24.16) is

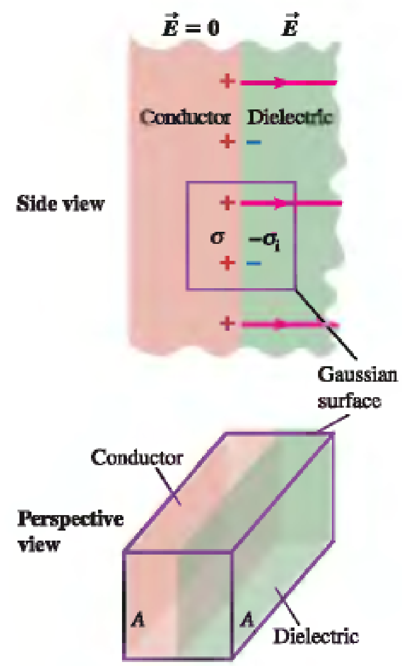
$$\sigma_i = \sigma\left(1 - \frac{1}{K}\right) \quad \text{or} \quad \sigma - \sigma_i = \frac{\sigma}{K}$$

Combining this with Eq. (24.21), we get

$$EA = \frac{\sigma A}{K\epsilon_0} \quad \text{or} \quad KEA = \frac{\sigma A}{\epsilon_0} \tag{24.22}$$

Equation (24.22) says that the flux of $K\vec{E}$, not $\vec{E}$, through the Ganssian surface in Fig. 24.23 is equal to the enclosed *free* charge σA divided by ϵ_0. It turns out

24.23 Gauss's law with a dielectric. This figure shows a close-up of the left-hand capacitor plate in Fig. 24.15b. The Gaussian surface is a rectangular box that lies half in the conductor and half in the dielectric.



that for *any* Gaussian surface, whenever the induced charge is proportional to the electric field in the material, we can rewrite Gauss's law as

$$\oint K\vec{E}\cdot d\vec{A} = \frac{Q_{\text{encl-free}}}{\epsilon_0} \qquad \text{(Gauss's law in a dielectric)} \qquad (24.23)$$

where $Q_{\text{encl-free}}$ is the total *free* charge (not bound charge) enclosed by the Gaussian surface. The significance of these results is that the right sides contain only the *free* charge on the conductor, not the bound (induced) charge. In fact, although we have not proved it, Eq. (24.23) remains valid even when different parts of the Gaussian surface are embedded in dielectrics having different values of K, provided that the value of K in each dielectric is independent of the electric field (usually the case for electric fields that are not too strong) and that we use the appropriate value of K for each point on the Gaussian surface.

Example 24.12 A spherical capacitor with dielectric

In the spherical capacitor of Example 24.3 (Section 24.1), the volume between the concentric spherical conducting shells is filled with an insulating oil with dielectric constant K. Use Gauss's law to find the capacitance.

SOLUTION

IDENTIFY: This is essentially the same problem as Example 24.3. The only difference is the presence of the dielectric.

SET UP: As we did in Example 24.3, we use a spherical Gaussian surface of radius r between the two spheres. Since a dielectric is present, we use Gauss's law in the form of Eq. (24.23).

EXECUTE: The spherical symmetry of the problem is not changed by the presence of the dielectric, so we have

$$\oint K\vec{E}\cdot d\vec{A} = \oint KE\,dA = KE\oint dA = (KE)\,(4\pi r^2) = \frac{Q}{\epsilon_0}$$

$$E = \frac{Q}{4\pi K\epsilon_0 r^2} = \frac{Q}{4\pi\epsilon r^2}$$

where $\epsilon = K\epsilon_0$ is the permittivity of the dielectric (introduced in Section 24.4). Compared to the case in which there is vacuum between the conducting shells, the electric field is reduced by a factor of $1/K$. The potential difference V_{ab} between the shells is likewise reduced by a factor of $1/K$, and so the capacitance $C = Q/V_{ab}$ is *increased* by a factor of K, just as for a parallel-plate capacitor when a dielectric is inserted. Using the result for the vacuum case in Example 24.3, we find that the capacitance with the dielectric is

$$C = \frac{4\pi K\epsilon_0 r_a r_b}{r_b - r_a} = \frac{4\pi\epsilon r_a r_b}{r_b - r_a}$$

EVALUATE: In this case the dielectric completely fills the volume between the two conductors, so the capacitance is just K times the value with no dielectric. The result is more complicated if the dielectric only partially fills this volume (see Challenge Problem 24.76).

Test Your Understanding of Section 24.6 A single point charge q is imbedded in a dielectric of dielectric constant K. At a point inside the dielectric a distance r from the point charge, what is the magnitude of the electric field? (i) $q/4\pi\epsilon_0 r^2$; (ii) $Kq/4\pi\epsilon_0 r^2$; (iii) $q/4\pi K\epsilon_0 r^2$; (iv) none of these.

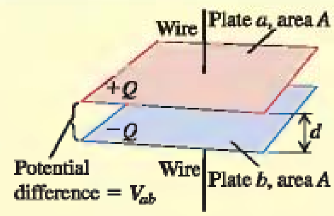
CHAPTER 24 SUMMARY

Capacitors and capacitance: A capacitor is any pair of conductors separated by an insulating material. When the capacitor is charged, there are charges of equal magnitude Q and opposite sign on the two conductors, and the potential V_{ab} of the positively charged conductor with respect to the negatively charged conductor is proportional to Q. The capacitance C is defined as the ratio of Q to V_{ab}. The SI unit of capacitance is the farad (F): $1\ \text{F} = 1\ \text{C/V}$.

A parallel-plate capacitor consists of two parallel conducting plates, each with area A, separated by a distance d. If they are separated by vacuum, the capacitance depends only on A and d. For other geometries, the capacitance can be found by using the definition $C = Q/V_{ab}$. (See Examples 24.1–24.4.)

$$C = \frac{Q}{V_{ab}} \tag{24.1}$$

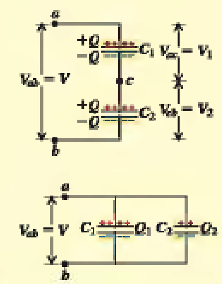
$$C = \frac{Q}{V_{ab}} = \epsilon_0 \frac{A}{d} \tag{24.2}$$



Capacitors in series and parallel: When capacitors with capacitances $C_1, C_2, C_3, \ldots$ are connected in series, the reciprocal of the equivalent capacitance C_{eq} equals the sum of the reciprocals of the individual capacitances. When capacitors are connected in parallel, the equivalent capacitance C_{eq} equals the sum of the individual capacitances. (See Examples 24.5 and 24.6.)

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots \tag{24.5}$$
(capacitors in series)

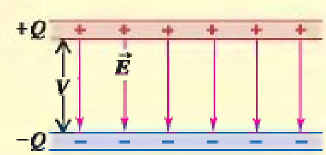
$$C_{eq} = C_1 + C_2 + C_3 + \cdots \tag{24.7}$$
(capacitors in parallel)



Energy in a capacitor: The energy U required to charge a capacitor C to a potential difference V and a charge Q is equal to the energy stored in the capacitor. This energy can be thought of as residing in the electric field between the conductors; the energy density u (energy per unit volume) is proportional to the square of the electric-field magnitude. (See Examples 24.7–24.9.)

$$U = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV \tag{24.9}$$

$$u = \frac{1}{2}\epsilon_0 E^2 \tag{24.11}$$



Dielectrics: When the space between the conductors is filled with a dielectric material, the capacitance increases by a factor K, called the dielectric constant of the material. The quantity $\epsilon = K\epsilon_0$ is called the permittivity of the dielectric. For a fixed amount of charge on the capacitor plates, induced charges on the surface of the dielectric decrease the electric field and potential difference between the plates by the same factor K. The surface charge results from polarization, a microscopic rearrangement of charge in the dielectric. (See Example 24.10.)

Under sufficiently strong fields, dielectrics become conductors, a situation called dielectric breakdown. The maximum field that a material can withstand without breakdown is called its dielectric strength.

In a dielectric, the expression for the energy density is the same as in vacuum but with ϵ_0 replaced by $\epsilon = K\epsilon$. (See Example 24.11.)

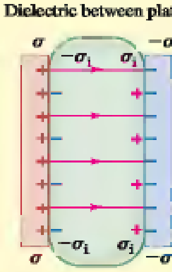
Gauss's law in a dielectric has almost the same form as in vacuum, with two key differences: $\vec{E}$ is replaced by $K\vec{E}$ and Q_{encl} is replaced by $Q_{encl\text{-}free}$, which includes only the free charge (not bound charge) enclosed by the Gaussian surface. (See Example 24.12.)

$$C = KC_0 = K\epsilon_0 \frac{A}{d} = \epsilon \frac{A}{d} \tag{24.19}$$
(parallel-plate capacitor filled with dielectric)

$$u = \frac{1}{2}K\epsilon_0 E^2 = \frac{1}{2}\epsilon E^2 \tag{24.20}$$

$$\oint K\vec{E} \cdot d\vec{A} = \frac{Q_{encl\text{-}free}}{\epsilon_0} \tag{24.23}$$

Dielectric between plates



Key Terms

capacitor, *816*
capacitance, *816*
farad, *816*
parallel-plate capacitor, *817*
series connection, *820*
equivalent capacitance, *821*
parallel connection, *821*
energy density, *826*
dielectric, *828*
dielectric breakdown, *828*
dielectric constant, *828*
polarization, *829*
permittivity, *830*
dielectric strength, *833*
bound charge, *834*
free charge, *834*

Answer to Chapter Opening Question ?

Equation (24.9) shows that the energy stored in a capacitor with capacitance C and charge Q is $U = Q^2/2C$. If the charge Q is doubled, the stored energy increases by a factor of $2^2 = 4$. Note that if the value of Q is too great, the electric-field magnitude inside the capacitor will exceed the dielectric strength of the material between the plates and dielectric breakdown will occur (see Section 24.4). This puts a practical limit on the amount of energy that can be stored.

Answers to Test Your Understanding Questions

24.1 Answer: (iii) The capacitance does not depend on the value of the charge Q. Doubling the value of Q causes the potential difference V_{ab} to double, so the capacitance $C = Q/V_{ab}$ remains the same. These statements are true no matter what the geometry of the capacitor.

24.2 Answers: (a) (i), (b) (iv) In a series connection the two capacitors carry the same charge Q but have different potential differences $V_{ab} = Q/C$; the capacitor with the smaller capacitance C has the greater potential difference. In a parallel connection the two capacitors have the same potential difference V_{ab} but carry different charges $Q = CV_{ab}$; the capacitor with the larger capacitance C has the greater charge. Hence a 4-μF capacitor will have a greater potential difference than an 8-μF capacitor if the two are connected in series. The 4-μF capacitor cannot carry more charge than the 8-μF capacitor no matter how they are connected: In a series connection they will carry the same charge, and in a parallel connection the 8-μF capacitor will carry more charge.

24.3 Answer: (i) Capacitors connected in series carry the same charge Q. To compare the amount of energy stored, we use the expression $U = Q^2/2C$ from Eq. (24.9); it shows that the capacitor with the *smaller* capacitance $(C = 4\ \mu\text{F})$ has more stored energy in a series combination. By contrast, capacitors in parallel have the same potential difference V, so to compare them we use $U = \frac{1}{2}CV^2$ from Eq. (24.9). It shows that in a parallel combination, the capacitor with the *larger* capacitance $(C = 8\ \mu\text{F})$ has more stored energy. (If we had instead used $U = \frac{1}{2}CV^2$ to analyze the series combination, we would have to account for the different potential differences across the two capacitors. Likewise, using $U = Q^2/2C$ to study the parallel combination would require us to account for the different charges on the capacitors.)

24.4 Answers: (i) Here Q remains the same, so we use $U = Q^2/2C$ from Eq. (24.9) for the stored energy. Removing the dielectric lowers the capacitance by a factor of $1/K$; since U is inversely proportional to C, the stored energy *increases* by a factor of K. It takes work to pull the dielectric slab out of the capacitor because the fringing field tries to pull the slab back in (Fig. 24.16). The work that you do goes into the energy stored in the capacitor.

24.5 Answer: (i), (iii), (ii) Equation (24.14) says that if E_0 is the initial electric-field magnitude (before the dielectric slab is inserted), then the resultant field magnitude after the slab is inserted is $E_0/K = E_0/3$. The magnitude of the resultant field equals the difference between the initial field magnitude and the magnitude E_i of the field due to the bound charges (see Fig. 24.21). Hence $E_0 - E_i = E_0/3$ and $E_i = 2E_0/3$.

24.6 Answer: (iii) Equation (24.23) shows that this situation is the same as an isolated point charge in vacuum but with $\vec{E}$ replaced by $K\vec{E}$. Hence KE at the point of interest is equal to $q/4\pi\epsilon_0 r^2$, and so $E = q/4\pi K\epsilon_0 r^2$. As in Example 24.12, filling the space with a dielectric reduces the electric field by a factor of $1/K$.

PROBLEMS

For instructor-assigned homework, go to www.masteringphysics.com

Discussion Questions

Q24.1. Equation (24.2) shows that the capacitance of a parallel-plate capacitor becomes larger as the plate separation d decreases. However, there is a practical limit to how small d can be made, which places limits on how large C can be. Explain what sets the limit on d. (*Hint:* What happens to the magnitude of the electric field as $d \to 0$?)

Q24.2. Suppose several different parallel-plate capacitors are charged up by a constant-voltage source. Thinking of the actual movement and position of the charges on an atomic level, why does it make sense that the capacitances are proportional to the surface areas of the plates? Why does it make sense that the capacitances are *inversely* proportional to the distance between the plates?

Q24.3. Suppose the two plates of a capacitor have different areas. When the capacitor is charged by connecting it to a battery, do the charges on the two plates have equal magnitude, or may they be different? Explain your reasoning.

Q24.4. At the Fermi National Accelerator Laboratory (Fermilab) in Illinois, protons are accelerated around a ring 2 km in radius to speeds that approach that of light. The energy for this is stored in capacitors the size of a house. When these capacitors are being charged, they make a very loud creaking sound. What is the origin of this sound?

Q24.5. In the parallel-plate capacitor of Fig. 24.2, suppose the plates are pulled apart so that the separation d is much larger than the size of the plates. (a) Is it still accurate to say that the electric field between the plates is uniform? Why or why not? (b) In the situation shown in Fig. 24.2, the potential difference between the plates is $V_{ab} = Qd/\epsilon_0 A$. If the plates are pulled apart as described above, is V_{ab} more or less than this formula would indicate? Explain your reasoning. (c) With the plates pulled apart as described above, is the capacitance more than, less than, or the same as that given by Eq. (24.2)? Explain your reasoning.

Q24.6. A parallel-plate capacitor is charged by being connected to a battery and is kept connected to the battery. The separation between the plates is then doubled. How does the electric field change? The charge on the plates? The total energy? Explain your reasoning.

Q24.7. A parallel-plate capacitor is charged by being connected to a battery and is then disconnected from the battery. The separation between the plates is then doubled. How does the electric field change? The potential difference? The total energy? Explain your reasoning.

Q24.6. Two parallel-plate capacitors, identical except that one has twice the plate separation of the other, are charged by the same voltage source. Which capacitor has a stronger electric field between the plates? Which capacitor has a greater charge? Which has greater energy density? Explain your reasoning.

Q24.9. The charged plates of a capacitor attract each other, so to pull the plates farther apart requires work by some external force. What becomes of the energy added by this work? Explain your reasoning.

Q24.18. The two plates of a capacitor are given charges $\pm Q$. The capacitor is then disconnected from the charging device so that the charges on the plates can't change, and the capacitor is immersed in a tank of oil. Does the electric field between the plates increase, decrease, or stay the same? Explain your reasoning. How can this field be measured?

Q24.11. As shown in Table 24.1, water has a very large dielectric constant $K = 80.4$. Why do you think water is not commonly used as a dielectric in capacitors?

Q24.12. Is dielectric strength the same thing as dielectric constant? Explain any differences between the two quantities. Is there a simple relationship between dielectric strength and dielectric constant (see Table 24.2)?

Q24.13. A capacitor made of aluminum foil strips separated by Mylar film was subjected to excessive voltage, and the resulting dielectric breakdown melted holes in the Mylar. After this, the capacitance was found to be about the same as before, but the breakdown voltage was much less. Why?

Q24.14. Suppose you bring a slab of dielectric close to the gap between the plates of a charged capacitor, preparing to slide it between the plates. What force will you feel? What does this force tell you about the energy stored between the plates once the dielectric is in place, compared to before the dielectric is in place?

Q24.15. The freshness of fish can be measured by placing a fish between the plates of a capacitor and measuring the capacitance. How does this work? (*Hint:* As time passes, the fish dries out. See Table 24.1.)

Q24.18. *Electrolytic* capacitors use as their dielectric an extremely thin layer of nonconducting oxide between a metal plate and a conducting solution. Discuss the advantage of such a capacitor over one constructed using a solid dielectric between the metal plates.

Q24.17. In terms of the dielectric constant K, what happens to the electric flux through the Gaussian surface shown in Fig. 24.23 when the dielectric is inserted into the previously empty space between the plates? Explain.

Q24.18. A parallel-plate capacitor is connected to a power supply that maintains a fixed potential difference between the plates. (a) If a sheet of dielectric is then slid between the plates, what happens to (i) the electric field between the plates, (ii) the magnitude of charge on each plate, and (iii) the energy stored in the capacitor? (b) Now suppose that before the dielectric is inserted, the charged capacitor is disconnected from the power supply. In this case, what happens to (i) the electric field between the plates, (ii) the magnitude of charge on each plate, (iii) the energy stored in the capacitor? Explain any differences between the two situations.

Q24.19. Liquid dielectrics that have polar molecules (such as water) always have dielectric constants that decrease with increasing temperature. Why?

Q24.20. A conductor is an extreme case of a dielectric, since if an electric field is applied to a conductor, charges are free to move within the conductor to set up "induced charges." What is the dielectric constant of a perfect conductor? Is it $K = 0$, $K \to \infty$, or something in between? Explain your reasoning.

Exercises

Section 24.1 Capacitors and Capacitance

24.1. A capacitor has a capacitance of 7.28 μF. What amount of charge must be placed on each of its plates to make the potential difference between its plates equal to 25.0 V?

24.2. The plates of a parallel-plate capacitor are 3.28 mm apart, and each has an area of 12.2 cm^2. Each plate carries a charge of magnitude 4.35×10^{-8} C. The plates are in vacuum. (a) What is the capacitance? (b) What is the potential difference between the plates? (c) What is the magnitude of the electric field between the plates?

24.3. A parallel-plate air capacitor of capacitance 245 pF has a charge of magnitude 0.148 μC on each plate. The plates are 0.328 mm apart. (a) What is the potential difference between the plates? (b) What is the area of each plate? (c) What is the electric-field magnitude between the plates? (d) What is the surface charge density on each plate?

24.4. Capacitance of an Oscilloscope. Oscilloscopes have parallel metal plates inside them to deflect the electron beam. These plates are called the *deflecting plates.* Typically, they are squares 3.0 cm on a side and separated by 5.0 mm, with vacuum in between. What is the capacitance of these deflecting plates and hence of the oscilloscope? (*Note:* This capacitance can sometimes have an effect on the circuit you are trying to study and must be taken into consideration in your calculations.)

24.5. A 10.0-μF parallel-plate capacitor with circular plates is connected to a 12.0-V battery. (a) What is the charge on each plate? (b) How much charge would be on the plates if their separation were doubled while the capacitor remained connected to the battery? (c) How much charge would be on the plates if the capacitor were connected to the 12.0-V battery after the radius of each plate was doubled without changing their separation?

24.6. A 10.0-μF parallel-plate capacitor is connected to a 12.0-V battery. After the capacitor is fully charged, the battery is disconnected without loss of any of the charge on the plates. (a) A voltmeter is connected across the two plates without discharging them. What does it read? (b) What would the voltmeter read if (i) the plate separation were doubled; (ii) the radius of each plate were doubled and, but their separation was unchanged?

24.7. How far apart would parallel pennies have to be to make a 1.00-pF capacitor? Does your answer suggest that you are justified in treating these pennies as infinite sheets? Explain.

24.8. A 5.00-pF, parallel-plate, air-filled capacitor with circular plates is to be used in a circuit in which it will be subjected to potentials of up to 1.00×10^2 V. The electric field between the plates is to be no greater than 1.00×10^4 N/C. As a budding electrical engineer for Live-Wire Electronics, your tasks are to (a) design the capacitor by finding what its physical dimensions and separation must be; (b) find the maximum charge these plates can hold.

24.9. A capacitor is made from two hollow, coaxial, iron cylinders, one inside the other. The inner cylinder is negatively charged and the outer is positively charged; the magnitude of the charge on each is 10.0 pC. The inner cylinder has radius 0.50 mm, the outer one has radius 5.00 mm, and the length of each cylinder is 18.0 cm. (a) What is the capacitance? (b) What applied potential difference is necessary to produce these charges on the cylinders?

24.10. A cylindrical capacitor consists of a solid inner conducting core with radius 0.250 cm, surrounded by an outer hollow conducting tube. The two conductors are separated by air, and the length of the cylinder is 12.0 cm. The capacitance is 36.7 pF. (a) Calculate the inner radius of the hollow tube. (b) When the capacitor is charged to 125 V, what is the charge per unit length λ on the capacitor?

24.11. A cylindrical capacitor has an inner conductor of radius 1.5 mm and an outer conductor of radius 3.5 mm. The two conductors are separated by vacuum, and the entire capacitor is 2.8 m long. (a) What is the capacitance per unit length? (b) The potential of the inner conductor is 350 mV higher than that of the outer conductor. Find the charge (magnitude and sign) on both conductors.

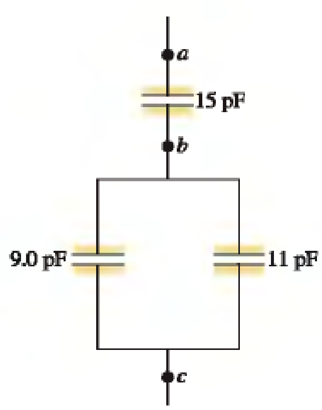
24.12. A spherical capacitor is formed from two concentric, spherical, conducting shells separated by vacuum. The inner sphere has radius 15.0 cm and the capacitance is 116 pF. (a) What is the radius of the outer sphere? (b) If the potential difference between the two spheres is 220 V, what is the magnitude of charge on each sphere?

24.13. A spherical capacitor contains a charge of 3.30 nC when connected to a potential difference of 220 V. If its plates are separated by vacuum and the inner radius of the outer shell is 4.00 cm, calculate: (a) the capacitance; (b) the radius of the inner sphere; (c) the electric field just outside the surface of the inner sphere.

Section 24.2 Capacitors in Series and Parallel

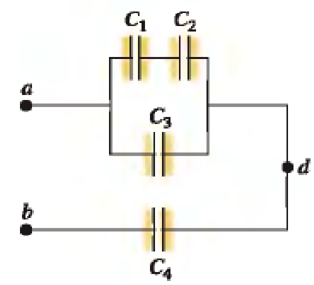
24.14. For the system of capacitors shown in Fig. 24.24, find the equivalent capacitance (a) between b and c, and (b) between a and c.

Figure 24.24 Exercise 24.14.



24.15. In Fig. 24.25, each capacitor has $C = 4.00\ \mu F$ and $V_{ab} = +28.0$ V. Calculate (a) the charge on each capacitor; (b) the potential difference across each capacitor; (c) the potential difference between points a and d.

Figure 24.25 Exercise 24.15.

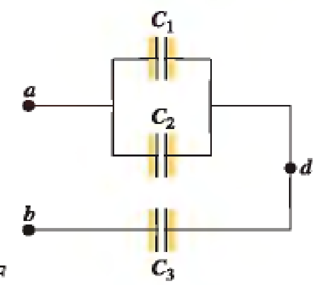


24.16. In Fig. 24.8a, let $C_1 = 3.00\ \mu F$, $C_2 = 5.00\ \mu F$, and $V_{ab} = +52.0$ V. Calculate (a) the charge on each capacitor and (b) the potential difference across each capacitor.

24.17. In Fig. 24.9a, let $C_1 = 3.00\ \mu F$, $C_2 = 5.00\ \mu F$, and $V_{ab} = +52.0$ V. Calculate (a) the charge on each capacitor and (b) the potential difference across each capacitor.

24.19. In Fig. 24.26, $C_1 = 6.00\ \mu F$, $C_2 = 3.00\ \mu F$, and $C_3 = 5.00\ \mu F$. The capacitor network is connected to an applied potential V_{ab}. After the charges on the capacitors have reached their final values, the charge on C_2 is $40.0\ \mu C$. (a) What are the charges on capacitors C_1 and C_3? (b) What is the applied voltage V_{ab}?

Figure 24.26 Exercises 24.18 and 24.19.



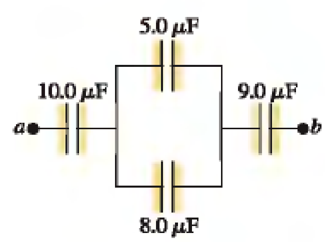
24.19. In Fig. 24.26, $C_1 = 3.00\ \mu F$ and $V_{ab} = 120$ V. The charge on capacitor C_1 is $150\ \mu C$. Calculate the voltage across the other two capacitors.

24.20. Two parallel-plate vacuum capacitors have plate spacings d_1 and d_2 and equal plate areas A. Show that when the capacitors are connected in series, the eqnivalent capacitance is the same as for a single capacitor with plate area A and spacing $d_1 + d_2$.

24.21. Two parallel-plate vacuum capacitors have areas A_1 and A_2 and equal plate spacings d. Show that when the capacitors are connected in parallel, the equivalent capacitance is the same as for a single capacitor with plate area $A_1 + A_2$ and spacing d.

24.22. Figure 24.27 shows a system of four capacitors, where the potential difference across ab is 50.0 V. (a) Find the eqnivalent capacitance of this system between a and b. (b) How much charge is stored by this combination of capacitors? (c) How much charge is stored in each of the 10.0-μF and the 9.0-μF capacitors?

Figure 24.27 Exercise 24.22.



24.23. Suppose the 3-μF capacitor in Fig. 24.10a were removed and replaced by a different one, and that this changed the equivalent capacitance between points a and b to 8 μF. What would be the capacitance of the replacement capacitor?

Section 24.3 Energy Storage in Capacitors and Electric-Field Energy

24.24. A parallel-plate air capacitor has a capacitance of 920 pF. The charge on each plate is $2.55\ \mu C$. (a) What is the potential difference between the plates? (b) If the charge is kept constant, what will be the potential difference between the plates if the separation is doubled? (c) How much work is required to double the separation?

24.25. A 5.80-μF, parallel-plate, air capacitor has a plate separation of 5.00 mm and is charged to a potential difference of 400 V. Calculate the energy density in the region between the plates, in units of J/m^3.

24.26. An air capacitor is made from two flat parallel plates 1.50 mm apart. The magnitude of charge on each plate is 0.0180 μC when the potential difference is 200 V. (a) What is the capacitance? (b) What is the area of each plate? (c) What maximum voltage can be applied without dielectric breakdown? (Dielectric breakdown for air occurs at an electric-field strength of 3.0×10^6 V/m.) (d) When the charge is 0.0180 μC, what total energy is stored?

24.27. A 450-μF capacitor is charged to 295 V. Then a wire is connected between the plates. How many joules of thermal energy are produced as the capacitor discharges if all of the energy that was stored goes into heating the wire?

24.28. A capacitor of capacitance C is charged to a potential difference V_0. The terminals of the charged capacitor are then connected to those of an uncharged capacitor of capacitance $C/2$. Compute (a) the original charge of the system; (b) the final potential difference across each capacitor; (c) the final energy of the system; (d) the decrease in energy when the capacitors are connected. (e) Where did the "lost" energy go?

24.28. A parallel-plate vacuum capacitor with plate area A and separation x has charges $+Q$ and $-Q$ on its plates. The capacitor is disconnected from the source of charge, so the charge on each plate remains fixed. (a) What is the total energy stored in the capacitor? (b) The plates are pulled apart an additional distance dx. What is the change in the stored energy? (c) If F is the force with which the plates attract each other, then the change in the stored energy must equal the work $dW = Fdx$ done in pulling the plates apart. Find an expression for F. (d) Explain why F is *not* equal to QE, where E is the electric field between the plates.

24.30. A parallel-plate vacuum capacitor has 8.38 J of energy stored in it. The separation between the plates is 2.30 mm. If the separation is decreased to 1.15 mm, what is the energy stored (a) if the capacitor is disconnected from the potential source so the charge on the plates remains constant, and (b) if the capacitor remains connected to the potential source so the potential difference between the plates remains constant?

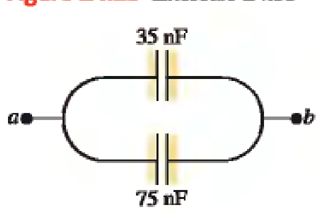
24.31. (a) How much charge does a battery have to supply to a 5.0-μF capacitor to create a potential difference of 1.5 V across its plates? How much energy is stored in the capacitor in this case? (b) How much charge would the battery have to supply to store 1.0 J of energy in the capacitor? What would be the potential across the capacitor in that case?

24.32. For the capacitor network shown in Fig. 24.28, the potential difference across ab is 36 V. Find (a) the total charge stored in this network; (b) the charge on each capacitor; (c) the total energy stored in the network; (d) the energy stored in each capacitor; (e) the potential differences across each capacitor.

Figure 24.28 Exercise 24.32.

24.33. For the capacitor network shown in Fig. 24.29, the potential difference across ab is 220 V. Find (a) the total charge stored in this network; (b) the charge on each capacitor; (c) the total energy stored in the network; (d) the energy stored in each capacitor; (e) the potential difference across each capacitor.

Figure 24.29 Exercise 24.33



24.34. A 0.350-m-long cylindrical capacitor consists of a solid conducting core with a radius of 1.20 mm and an outer hollow conducting tube with an inner radius of 2.00 mm. The two conductors are separated by air and charged to a potential difference of 6.00 V. Calculate (a) the charge per length for the capacitor; (b) the total charge on the capacitor; (c) the capacitance; (d) the energy stored in the capacitor when fully charged.

24.35. A cylindrical air capacitor of length 15.0 m stores 3.20×10^{-9} J of energy when the potential difference between the two conductors is 4.00 V. (a) Calculate the magnitude of the charge on each conductor. (b) Calculate the ratio of the radii of the inner and outer conductors.

24.36. A capacitor is formed from two concentric spherical conducting shells separated by vacuum. The inner sphere has radius 12.5 cm, and the outer sphere has radius 14.8 cm. A potential difference of 120 V is applied to the capacitor. (a) What is the energy density at $r = 12.6$ cm, just outside the inner sphere? (b) What is the energy density at $r = 14.7$ cm, just inside the outer sphere? (c) For a parallel-plate capacitor the energy density is uniform in the region between the plates, except near the edges of the plates. Is this also true for a spherical capacitor?

24.37. You have two identical capacitors and an external potential source. (a) Compare the total energy stored in the capacitors when they are connected to the applied potential in series and in parallel. (b) Compare the maximum amount of charge stored in each case. (c) Energy storage in a capacitor can be limited by the maximum electric field between the plates. What is the ratio of the electric field for the series and parallel combinations?

Section 24.4 Dielectrics

24.38. A parallel-plate capacitor has capacitance $C_0 = 5.00$ pF when there is air between the plates. The separation between the plates is 1.50 mm. (a) What is the maximum magnitude of charge Q that can be placed on each plate if the electric field in the region between the plates is not to exceed 3.00×10^4 V/m? (b) A dielectric with $K = 2.70$ is inserted between the plates of the capacitor, completely filling the volume between the plates. Now what is the maximum magnitude of charge on each plate if the electric field between the plates is not to exceed 3.00×10^4 V/m?

24.38. Two parallel plates have equal and opposite charges. When the space between the plates is evacuated, the electric field is $E = 3.20 \times 10^5$ V/m. When the space is filled with dielectric, the electric field is $E = 2.50 \times 10^5$ V/m. (a) What is the charge density on each surface of the dielectric? (b) What is the dielectric constant?

24.40. A budding electronics hobbyist wants to make a simple 1.0-nF capacitor for tuning her crystal radio, using two sheets of aluminum foil as plates, with a few sheets of paper between them as a dielectric. The paper has a dielectric constant of 3.0, and the thickness of one sheet of it is 0.20 mm. (a) If the sheets of paper measure 22×28 cm and she cuts the aluminum foil to the same dimensions, how many sheets of paper should she use between her plates to get the proper capacitance? (b) Suppose for convenience she wants to use a single sheet of posterboard, with the same dielectric constant but a thickness of 12.0 mm, instead of the paper. What area of aluminum foil will she need for her plates to get her 1.0 nF of capacitance? (c) Suppose she goes high-tech and finds a sheet of Teflon of the same thickness as the posterboard to use as a dielectric. Will she need a larger or smaller area of Teflon than of posterboard? Explain.

24.41. The dielectric to be used in a parallel-plate capacitor has a dielectric constant of 3.60 and a dielectric strength of 1.60×10^7 V/m. The capacitor is to have a capacitance of 1.25×10^{-9} F and must be able to withstand a maximum potential difference of 5500 V. What is the minimum area the plates of the capacitor may have?

24.42. Show that Eq. (24.20) holds for a parallel-plate capacitor with a dielectric material between the plates. Use a derivation analogous to that used for Eq. (24.11).

24.43. A capacitor has parallel plates of area 12 cm^2 separated by 2.0 mm. The space between the plates is filled with polystyrene (see Table 24.2). (a) Find the permittivity of polystyrene. (b) Find the maximum permissible voltage across the capacitor to avoid dielectric breakdown. (c) When the voltage equals the value found in part (b), find the surface charge density on each plate and the induced surface-charge density on the surface of the dielectric.

24.44. A constant potential difference of 12 V is maintained between the terminals of a 0.25-μF, parallel-plate, air capacitor. (a) A sheet of Mylar is inserted between the plates of the capacitor, completely filling the space between the plates. When this is done, how much additional charge flows onto the positive plate of the capacitor (see Table 24.1)? (b) What is the total induced charge on either face of the Mylar sheet? (c) What effect does the Mylar sheet have on the electric field between the plates? Explain how you can reconcile this with the increase in charge on the plates, which acts to *increase* the electric field.

24.45. When a 360-nF air capacitor $(1\ \text{nF} = 10^{-9}\ \text{F})$ is connected to a power supply, the energy stored in the capacitor is 1.85×10^{-5} J. While the capacitor is kept connected to the power supply, a slab of dielectric is inserted that completely fills the space between the plates. This increases the stored energy by 2.32×10^{-5} J. (a) What is the potential difference between the capacitor plates? (b) What is the dielectric constant of the slab?

24.46. A parallel-plate capacitor has capacitance $C = 12.5$ pF when the volume between the plates is filled with air. The plates are circular, with radius 3.00 cm. The capacitor is connected to a battery and a charge of magnitude 25.0 pC goes onto each plate. With the capacitor still connected to the battery, a slab of dielectric is inserted between the plates, completely filling the space between the plates. After the dielectric has been inserted, the charge on each plate has magnitude 45.0 pC. (a) What is the dielectric constant K of the dielectric? (b) What is the potential difference between the plates before and after the dielectric has been inserted? (c) What is the electric field at a point midway between the plates before and after the dielectric has been inserted?

24.47. A 12.5-μF capacitor is connected to a power supply that keeps a constant potential difference of 24.0 V across the plates. A piece of material having a dielectric constant of 3.75 is placed between the plates, completely filling the space between them. (a) How much energy is stored in the capacitor before and after the dielectric is inserted? (b) By how much did the energy change during the insertion? Did it increase or decrease?

*Section 24.6 Gauss's Law in Dielectrics

***24.48.** A parallel-plate capacitor has plates with area 0.0225 m^2 separated by 1.00 mm of Teflon. (a) Calculate the charge on the plates when they are charged to a potential difference of 12.0 V. (b) Use Gauss's law (Eq. 24.23) to calculate the electric field inside the Teflon. (c) Use Gauss's law to calculate the electric field if the voltage source is disconnected and the Teflon is removed.

***24.48.** A parallel-plate capacitor has the volume between its plates filled with plastic with dielectric constant K. The magnitude of the charge on each plate is Q. Each plate has area A, and the distance between the plates is d. (a) Use Gauss's law as stated in Eq. (24.23) to calculate the magnitude of the electric field in the dielectric. (b) Use the electric field determined in part (a) to calculate the potential difference between the two plates. (c) Use the result of part (b) to determine the capacitance of the capacitor. Compare your result to Eq. (24.12).

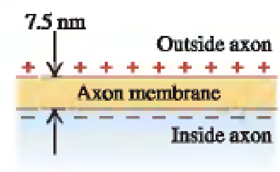
Problems

24.50. A parallel-plate air capacitor is made by using two plates 16 cm square, spaced 4.7 mm apart. It is connected to a 12-V battery. (a) What is the capacitance? (b) What is the charge on each plate? (c) What is the electric field between the plates? (d) What is the energy stored in the capacitor? (e) If the battery is disconnected and then the plates are pulled apart to a separation of 9.4 mm, what are the answers to parts (a)–(d)?

24.51. Suppose the battery in Problem 24.50 remains connected while the plates are pulled apart. What are the answers then to parts (a)–(d) after the plates have been pulled apart?

24.52. Cell Membranes. Cell membranes (the walled enclosure around a cell) are typically about 7.5 nm thick. They are partially permeable to allow charged material to pass in and out, as needed. Equal but opposite charge densities build up on the inside and outside faces of such a membrane, and these charges prevent additional charges from passing through the cell wall. We can model a cell membrane as a parallel-plate capacitor, with the membrane itself containing proteins embedded in an organic material to give the membrane a dielectric constant of about 10. (See Fig. 24.30.) (a) What is the capacitance per square centimeter of such a cell wall? (b) In its normal resting state, a cell has a potential difference of 85 mV across its membrane. What is the electric field inside this membrane?

Figure 24.30 Problem 24.52.



24.53. Electronic flash units for cameras contain a capacitor for storing the energy used to produce the flash. In one such unit, the flash lasts for $\frac{1}{675}$ s with an average light power output of 2.70×10^5 W. (a) If the conversion of electrical energy to light is 95% efficient (the rest of the energy goes to thermal energy), how much energy must be stored in the capacitor for one flash? (b) The capacitor has a potential difference between its plates of 125 V when the stored energy equals the value calculated in part (a). What is the capacitance?

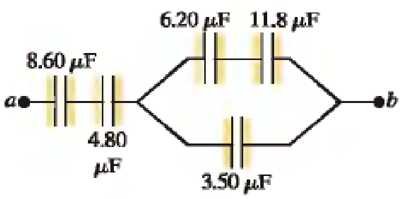
24.54. In one type of computer keyboard, each key holds a small metal plate that serves as one plate of a parallel-plate, air-filled capacitor. When the key is depressed, the plate separation decreases and the capacitance increases. Electronic circuitry detects the change in capacitance and thus detects that the key has been pressed. In one particular keyboard, the area of each metal plate is 42.0 mm^2, and the separation between the plates is 0.700 mm before the key is depressed. (a) Calculate the capacitance before the key is depressed. (b) If the circuitry can detect a change in capacitance of 0.250 pF, how far must the key be depressed before the circuitry detects its depression?

24.55. Consider a cylindrical capacitor like that shown in Fig. 24.6. Let $d = r_b - r_a$ be the spacing between the inner and outer conductors. (a) Let the radii of the two conductors be only slightly different, so that $d \ll r_a$. Show that the result derived in Example 24.4 (Section 24.1) for the capacitance of a cylindrical capacitor then reduces to Eq. (24.2), the equation for the capacitance of a parallel-plate capacitor, with A being the surface area of each cylinder. Use the result that $\ln(1+z) \cong z$ for $|z| \ll 1$. (b) Even though the earth is essentially spherical, its surface appears flat to us because its radius is so large. Use this idea to explain why the result of part (a) makes sense from a purely geometrical standpoint.

24.56. In Fig. 24.9a, let $C_1 = 9.0\,\mu\text{F}$, $C_2 = 4.0\,\mu\text{F}$, and $V_{ab} = 28$ V. Suppose the charged capacitors are disconnected from the source and from each other, and then reconnected to each other with plates of *opposite* sign together. By how much does the energy of the system decrease?

24.57. For the capacitor network shown in Fig. 24.31, the potential difference across ab is 12.0 V. Find (a) the total energy stored in this network and (b) the energy stored in the 4.80-μF capacitor.

Figure 24.31 Problem 24.57.



24.58. Several 0.25-μF capacitors are available. The voltage across each is not to exceed 600 V. You need to make a capacitor with capacitance 0.25 μF to be connected across a potential difference of 960 V. (a) Show in a diagram how an equivalent capacitor with the desired properties can be obtained. (b) No dielectric is a perfect insulator that would not permit the flow of any charge through its volume. Suppose that the dielectric in one of the capacitors in your diagram is a moderately good conductor. What will happen in this case when your combination of capacitors is connected across the 960-V potential difference?

24.58. In Fig. 24.32, $C_1 = C_5 = 8.4\,\mu\text{F}$ and $C_2 = C_3 = C_4 = 4.2\,\mu\text{F}$. The applied potential is $V_{ab} = 220$ V. (a) What is the equivalent capacitance of the network between points a and b? (b) Calculate the charge on each capacitor and the potential difference across each capacitor.

Figure 24.32 Problem 24.59.

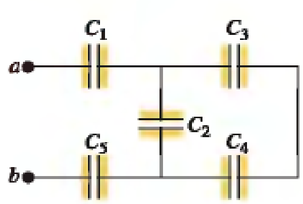
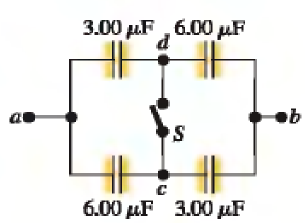


Figure 24.33 Problem 24.60.



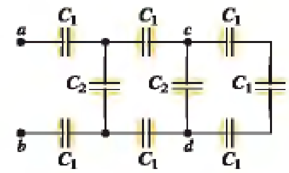
24.60. The capacitors in Fig. 24.33 are initially uncharged and are connected, as in the diagram, with switch S open. The applied potential difference is $V_{ab} = +210$ V. (a) What is the potential difference V_{cd}? (b) What is the potential difference across each capacitor after switch S is closed? (c) How much charge flowed through the switch when it was closed?

24.61. Three capacitors having capacitances of 8.4, 8.4, and 4.2 μF are connected in series across a 36-V potential difference. (a) What is the charge on the 4.2-μF capacitor? (b) What is the total energy stored in all three capacitors? (c) The capacitors are disconnected from the potential difference without allowing them to discharge. They are then reconnected in parallel with each other, with the positively charged plates connected together. What is the voltage across each capacitor in the parallel combination? (d) What is the total energy now stored in the capacitors?

24.62. Capacitance of a Thundercloud. The charge center of a thundercloud, drifting 3.0 km above the earth's surface, contains 20 C of negative charge. Assuming the charge center has a radius of 1.0 km, and modeling the charge center and the earth's surface as parallel plates, calculate: (a) the capacitance of the system; (b) the potential difference between charge center and ground; (c) the average strength of the electric field between cloud and ground; (d) the electrical energy stored in the system.

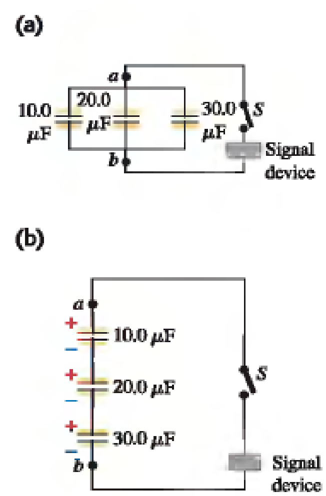
24.63. In Fig. 24.34, each capacitance C_1 is 6.9 μF, and each capacitance C_2 is 4.6 μF. (a) Compute the equivalent capacitance of the network between points a and b. (b) Compute the charge on each of the three capacitors nearest a and b when $V_{ab} = 420$ V. (c) With 420 V across a and b, compute V_{cd}.

Figure 24.34 Problem 24.63.



24.64. Each combination of capacitors between points a and b in Fig. 24.35 is first connected across a 120-V battery, charging the combination to 120 V. These combinations are then connected to make the circuits shown. When the switch S is thrown, a surge of charge for the discharging capacitors flows to trigger the signal device. How much charge flows through the signal device?

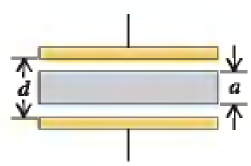
Figure 24.35 Problem 24.64.



24.65. A parallel-plate capacitor with only air between the plates is charged by connecting it to a battery. The capacitor is then disconnected from the battery, without any of the charge leaving the plates. (a) A voltmeter reads 45.0 V when placed across the capacitor. When a dielectric is inserted between the plates, completely filling the space, the voltmeter reads 11.5 V. What is the dielectric constant of this material? (b) What will the voltmeter read if the dielectric is now pulled partway out so it fills only one-third of the space between the plates?

24.66. An air capacitor is made by using two flat plates, each with area A, separated by a distance d. Then a metal slab having thickness a (less than d) and the same shape and size as the plates is inserted between them, parallel to the plates and not touching either plate (Fig. 24.36). (a) What is the capacitance of this arrangement? (b) Express the capacitance as a multiple of the capacitance C_0 when the metal slab is not present. (c) Discuss what happens to the capacitance in the limits $a \to 0$ and $a \to d$.

Figure 24.36 Problem 24.66.



24.67. Capacitance of the Earth. (a) Discuss how the concept of capacitance can also be applied to a *single* conductor. (*Hint:* In the relationship $C = Q/V_{ab}$, think of the second conductor as being located at infinity.) (b) Use Eq. (24.1) to show that $C = 4\pi\epsilon_0 R$ for a solid conducting sphere of radius R. (c) Use your result in part (b) to calculate the capacitance of the earth, which is a good conductor of radius 6380 km. Compare to typical capacitors used in electronic circuits that have capacitances ranging from 10 pF to 100 μF.

24.68. A solid conducting sphere of radius R carries a charge Q. Calculate the electric-field energy density at a point a distance r from the center of the sphere for (a) $r < R$ and (b) $r > R$. (c) Calculate the total electric-field energy associated with the charged sphere. (*Hint:* Consider a spherical shell of radius r and thickness dr that has volume $dV = 4\pi r^2\,dr$, and find the energy stored in this volume. Then integrate from $r = 0$ to $r \to \infty$.) (d) Explain why the result of part (c) can be interpreted as the amount of work required to assemble the charge Q on the sphere. (e) By using Eq. (24.9) and the result of part (c), show that the capacitance of the sphere is as given in Problem 24.67.

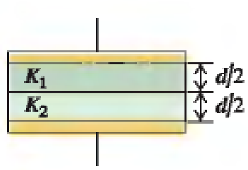
24.69. Earth-Ionosphere Capacitance. The earth can be considered as a single-conductor capacitor (see Problem 24.67). It can also be considered in combination with a charged layer of the atmosphere, the ionosphere, as a spherical capacitor with two plates, the surface of the earth being the negative plate. The ionosphere is at a level of about 70 km, and the potential difference between earth and ionosphere is about 350,000 V. Calculate: (a) the capacitance of this system; (b) the total charge on the capacitor; (c) the energy stored in the system.

24.70. The inner cylinder of a long, cylindrical capacitor has radius r_a and linear charge density $+\lambda$. It is surrounded by a coaxial cylindrical conducting shell with inner radius r_b and linear charge density $-\lambda$ (see Fig. 24.6). (a) What is the energy density in the region between the conductors at a distance r from the axis? (b) Integrate the energy density calculated in part (a) over the volume between the conductors in a length L of the capacitor to obtain the total electric-field energy per unit length. (c) Use Eq. (24.9) and the capacitance per unit length calculated in Example 24.4 (Section 24.1) to calculate U/L. Does your result agree with that obtained in part (b)?

24.71. A parallel-plate capacitor has the space between the plates filled with two slabs of dielectric, one with constant K_1 and one with constant K_2 (Fig. 24.37). Each slab has thickness $d/2$, where d is the plate separation. Show that the capacitance is

$$C = \frac{2\epsilon_0 A}{d}\left(\frac{K_1 K_2}{K_1 + K_2}\right)$$

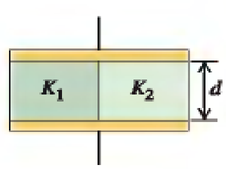
Figure 24.37 Problem 24.71.



24.72. A parallel-plate capacitor has the space between the plates filled with two slabs of dielectric, one with constant K_1 and one with constant K_2 (Fig. 24.38). The thickness of each slab is the same as the plate separation d, and each slab fills half of the volume between the plates. Show that the capacitance is

$$C = \frac{\epsilon_0 A(K_1 + K_2)}{2d}$$

Figure 24.38 Problem 24.72.



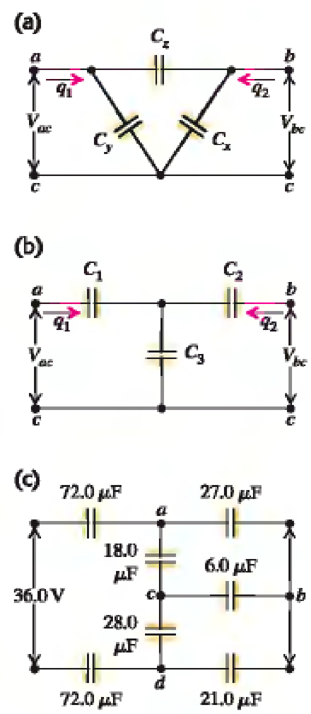
Challenge Problems

24.73. Capacitors in networks cannot always be grouped into simple series or parallel combinations. As an example, Fig. 24.39a shows three capacitors C_x, C_y, and C_z in a *delta network*, so called because of its triangular shape. This network has *three* terminals a, b, and c and hence cannot be transformed into a single equivalent capacitor. It can be shown that as far as any effect on the external circuit is concerned, a delta network is equivalent to what is called a *Y network*. For example, the delta network of Fig. 24.39a can be replaced by the Y network of Fig. 24.39b. (The name "Y network" also refers to the shape of the network.) (a) Show that the transformation equations that give C_1, C_2, and C_3 in terms of C_x, and C_y, and C_z are

$$C_1 = (C_xC_y + C_yC_z + C_zC_x)/C_x$$
$$C_2 = (C_xC_y + C_yC_z + C_zC_x)/C_y$$
$$C_3 = (C_xC_y + C_yC_z + C_zC_x)/C_z$$

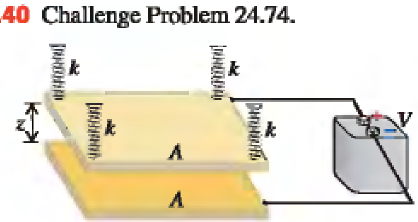
(*Hint:* The potential difference V_{ac} must be the same in both circuits, as V_{bc} must be. Also, the charge q_1 that flows from point a along the wire as indicated must be the same in both circuits, as must q_2. Obtain a relationship for V_{ac} as a function of q_1 and q_2 and the capacitances for each network, and obtain a separate relationship for V_{bc} as a function of the charges for each network. The coefficients of corresponding charges in corresponding equations must be the same for both networks.) (b) For the network shown in Fig. 24.39c, determine the equivalent capacitance between the terminals at the left end of the network. (*Hint:* Use the delta-Y transformation derived in part (a). Use points a, b, and c to form the delta, and transform the delta into a Y. The capacitors can then be combined using the relationships for series and parallel combinations of capacitors.) (c) Determine the charges of, and the potential differences across, each capacitor in Fig. 24.39c.

Figure 24.39 Challenge Problem 24.73.



24.74. The parallel-plate air capacitor in Fig. 24.40 consists of two horizontal conducting plates of equal area A. The bottom plate rests on a fixed support, and the top plate is suspended by four

Figure 24.40 Challenge Problem 24.74.



springs with spring constant k, positioned at each of the four corners of the top plate as shown in the figure. When uncharged, the plates are separated by a distance z_0. A battery is connected to the plates and produces a potential difference V between them. This causes the plate separation to decrease to z. Neglect any fringing effects. (a) Show that the electrostatic force between the charged plates has a magnitude $\epsilon_0 AV^2/2z^2$. (*Hint:* See Exercise 24.29.) (b) Obtain an expression that relates the plate separation z to the potential difference V. The resulting equation will be cubic in z. (c) Given the values $A = 0.300\ \text{m}^2$, $z_0 = 1.20\ \text{mm}$, $k = 25.0\ \text{N/m}$, and $V = 120\ \text{V}$, find the two values of z for which the top plate will be in equilibrium. (*Hint:* You can solve the cubic equation by plugging a trial value of z into the equation and then adjusting your guess until the equation is satisfied to three significant figures. Locating the roots of the cubic equation graphically can help you pick starting values of z for this trial-and-error procedure. One root of the cubic equation has a nonphysical negative value.) (d) For each of the two values of z found in part (c), is the equilibrium stable or unstable? For stable equilibrium a small displacement of the object will give rise to a net force tending to return the object to the equilibrium position. For unstable equilibrium a small displacement gives rise to a net force that takes the object farther away from equilibrium.

24.75. Two square conducting plates with sides of length L are separated by a distance D. A dielectric slab with constant K with dimensions $L \times L \times D$ is inserted a distance x into the space between the plates, as shown in Fig. 24.41. (a) Find the capacitance C of this system (see Problem 24.72). (b) Suppose that the capacitor is connected to a battery that maintains a constant potential difference V between the plates. If the dielectric slab is inserted an additional distance dx into the space between the plates, show that the change in stored energy is

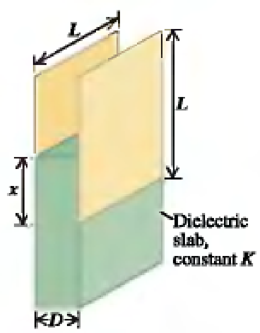


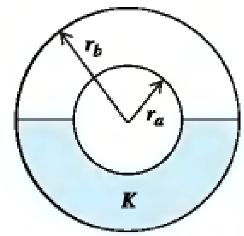
Figure 24.41 Challenge Problem 24.75.

$$dU = +\frac{(K-1)\epsilon_0 V^2 L}{2D}dx$$

(c) Suppose that before the slab is moved by dx, the plates are disconnected from the battery, so that the charges on the plates remain constant. Determine the magnitude of the charge on each plate, and then show that when the slab is moved dx farther into the space between the plates, the stored energy changes by an amount that is the *negative* of the expression for dU given in part (b). (d) If F is the force exerted on the slab by the charges on the plates, then dU should equal the work done *against* this force to move the slab a distance dx. Thus $dU = -F\,dx$. Show that applying this expression to the result of part (b) suggests that the electric force on the slab pushes it *out* of the capacitor, while the result of part (c) suggests that the force pulls the slab *into* the capacitor. (e) Figure 24.16 shows that the force in fact pulls the slab into the capacitor. Explain why the result of part (b) gives an incorrect answer for the direction of this force, and calculate the magnitude of the force. (This method does not require knowledge of the nature of the fringing field.)

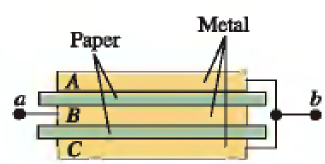
24.76. An isolated spherical capacitor has charge $+Q$ on its inner conductor (radius r_a) and charge $-Q$ on its outer conductor (radius r_b). Half of the volume between the two conductors is then filled with a liquid dielectric of constant K, as shown in cross section in Fig. 24.42. (a) Find the capacitance of the half-filled capacitor. (b) Find the magnitude of $\vec{E}$ in the volume between the two conductors as a function of the distance r from the center of the capacitor. Give answers for both the upper and lower halves of this volume. (c) Find the surface density of free charge on the upper and lower halves of the inner and outer conductors. (d) Find the surface density of bound charge on the inner $(r = r_a)$ and outer $(r = r_b)$ surfaces of the dielectric. (e) What is the surface density of bound charge on the flat surface of the dielectric? Explain.

Figure 24.42 Challenge Problem 24.76.



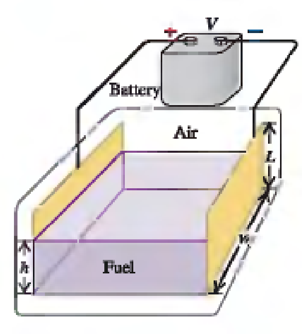
24.77. Three square metal plates A, B, and C, each 12.0 cm on a side and 1.50 mm thick, are arranged as in Fig. 24.43. The plates are separated by sheets of paper 0.45 mm thick and with dielectric constant 4.2. The outer plates are connected together and connected to point b. The inner plate is connected to point a. (a) Copy the diagram and show by plus and minus signs the charge distribution on the plates when point a is maintained at a positive potential relative to point b. (b) What is the capacitance between points a and b?

Figure 24.43 Challenge Problem 24.77.



24.78. A fuel gauge uses a capacitor to determine the height of the fuel in a tank. The effective dielectric constant K_{eff} changes from a value of 1 when the tank is empty to a value of K, the dielectric constant of the fuel, when the tank is full. The appropriate electronic circuitry can determine the effective dielectric constant of the combined air and fuel between the capacitor plates. Each of the two rectangular plates has a width w and a length L (Fig. 24.44). The height of the fuel between the plates is h. You can ignore any fringing effects. (a) Derive an expression for K_{eff} as a function of h. (b) What is the effective dielectric constant for a tank $\frac{1}{4}$ full, $\frac{1}{2}$ full, and $\frac{3}{4}$ full if the fuel is gasoline $(K = 1.95)$? (c) Repeat part (b) for methanol $(K = 33.0)$. (d) For which fuel is this fuel gauge more practical?

Figure 24.44 Challenge Problem 24.78.



25 CURRENT, RESISTANCE, AND ELECTROMOTIVE FORCE

LEARNING GOALS

By studying this chapter, you will learn:

- The meaning of electric current, and how charges move in a conductor.
- What is meant by the resistivity and conductivity of a substance.
- How to calculate the resistance of a conductor from its dimensions and its resistivity.
- How an electromotive force (emf) makes it possible for current to flow in a circuit.
- How to do calculations involving energy and power in circuits.

? In a flashlight, is the amount of current that flows out of the bulb less than, greater than, or equal to the amount of current that flows into the bulb?

In the past four chapters we studied the interactions of electric charges *at rest;* now we're ready to study charges *in motion.* An *electric current* consists of charges in motion from one region to another. When this motion takes place within a conducting path that forms a closed loop, the path is called an *electric circuit.*

Fundamentally, electric circuits are a means for conveying *energy* from one place to another. As charged particles move within a circuit, electric potential energy is transferred from a source (such as a battery or generator) to a device in which that energy is either stored or converted to another form: into sound in a stereo system or into heat and light in a toaster or light bulb. From a technological standpoint, electric circuits are useful because they allow energy to be transported without any moving parts (other than the moving charged particles themselves). Electric circuits are at the heart of flashlights, CD players, computers, radio and television transmitters and receivers, and household and industrial power distribution systems. The nervous systems of animals and humans are specialized electric circuits that carry vital signals from one part of the body to another.

In Chapter 26 we will see how to analyze electric circuits and will examine some practical applications of circuits. Before we can do so, however, you must understand the basic properties of electric currents. These properties are the subject of this chapter. We'll begin by describing the nature of electric conductors and considering how they are affected by temperature. We'll learn why a short, fat, cold copper wire is a better conductor than a long, skinny, hot steel wire. We'll study the properties of batteries and see how they cause current and energy transfer in a circuit. In this analysis we will use the concepts of current, potential difference (or voltage), resistance, and electromotive force. Finally, we'll look at electric current in a material from a microscopic viewpoint.

25.1 Current

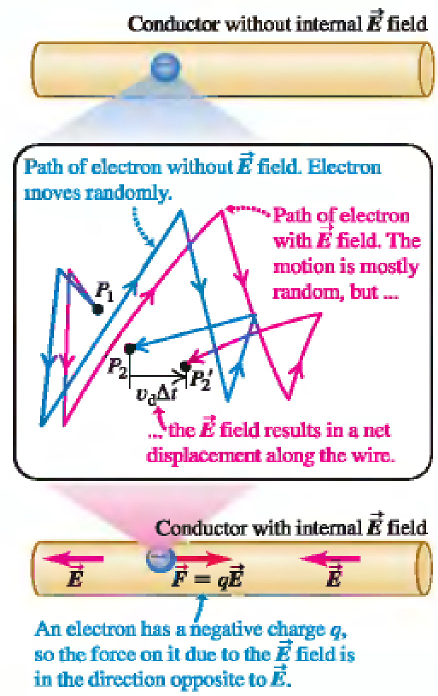
A **current** is any motion of charge from one region to another. In this section we'll discuss currents in conducting materials. The vast majority of technological applications of charges in motion involve currents of this kind.

In electrostatic situations (discussed in Chapters 21 through 24) the electric field is zero everywhere within the conductor, and there is *no* current. However, this does not mean that all charges within the conductor are at rest. In an ordinary metal such as copper or alumium, some of the electrons are free to move within the conducting material. These free electrons move randomly in all directions, somewhat like the molecules of a gas but with much greater speeds, of the order of 10^6 m/s. The electrons nonetheless do not escape from the conducting material, because they are attracted to the positive ions of the material. The motion of the electrons is random, so there is no *net* flow of charge in any direction and hence no current.

Now consider what happens if a constant, steady electric field $\vec{E}$ is established inside a conductor. (We'll see later how this can be done.) A charged particle (such as a free electron) inside the conducting material is then subjected to a steady force $\vec{F} = q\vec{E}$. If the charged particle were moving in *vacuum,* this steady force would cause a steady acceleration in the direction of $\vec{F}$, and after a time the charged particle would be moving in that direction at high speed. But a charged particle moving in a *conductor* undergoes frequent collisions with the massive, nearly stationary ions of the material. In each such collision the particle's direction of motion undergoes a random change. The net effect of the electric field $\vec{E}$ is that in addition to the random motion of the charged particles within the conductor, there is also a very slow net motion or *drift* of the moving charged particles as a group in the direction of the electric force $\vec{F} = q\vec{E}$ (Fig. 25.1). This motion is described in terms of the **drift velocity** $\vec{v}_d$ of the particles. As a result, there is a net current in the conductor.

While the random motion of the electrons has a very fast average speed of about 10^6 m/s, the drift speed is very slow, often on the order of 10^{-4} m/s. Given that the electrons move so slowly, you may wonder why the light comes on immediately when you turn on the switch of a flashlight. The reason is that the electric field is set up in the wire with a speed approaching the speed of light, and electrons start to move all along the wire at very nearly the same time. The time that it takes any individual electron to get from the switch to the light bulb isn't really relevant. A good analogy is a group of soldiers standing at attention when the sergeant orders them to start marching; the order reaches the soldiers' ears at the speed of sound, which is much faster than their marching speed, so all the soldiers start to march essentially in unison.

25.1 If there is no electric field inside a conductor, an electron moves randomly from point P_1 to point P_2 in a time Δt. If an electric field $\vec{E}$ is present, the electric force $\vec{F} = q\vec{E}$ imposes a small drift (greatly exaggerated here) that takes the electron to point P'_2, a distance $v_d \Delta t$ from P_2 in the direction of the force.



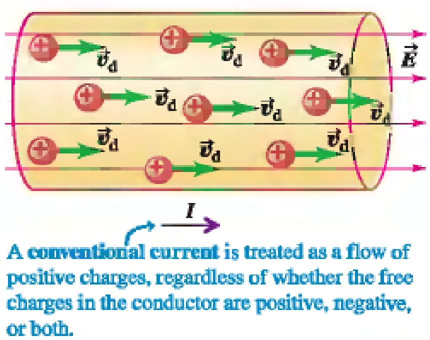
The Direction of Current Flow

The drift of moving charges through a conductor can be interpreted in terms of work and energy. The electric field $\vec{E}$ does work on the moving charges. The resulting kinetic energy is transferred to the material of the conductor by means of collisions with the ions, which vibrate about their equilibrium positions in the crystalline structure of the conductor. This energy transfer increases the average vibrational energy of the ions and therefore the temperature of the material. Thus much of the work done by the electric field goes into heating the conductor, *not* into making the moving charges move ever faster and faster. This heating is sometimes useful, as in an electric toaster, but in many situations is simply an unavoidable by-product of current flow.

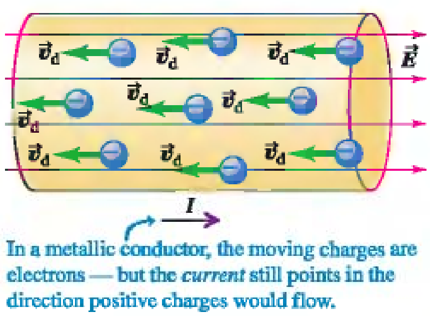
In different current-carrying materials, the charges of the moving particles may be positive or negative. In metals the moving charges are always (negative) electrons, while in an ionized gas (plasma) or an ionic solution the moving

25.2 The same current can be produced by **(a)** positive charges moving in the direction of the electric field $\vec{E}$ or **(b)** the same number of negative charges moving at the same speed in the direction opposite to $\vec{E}$.

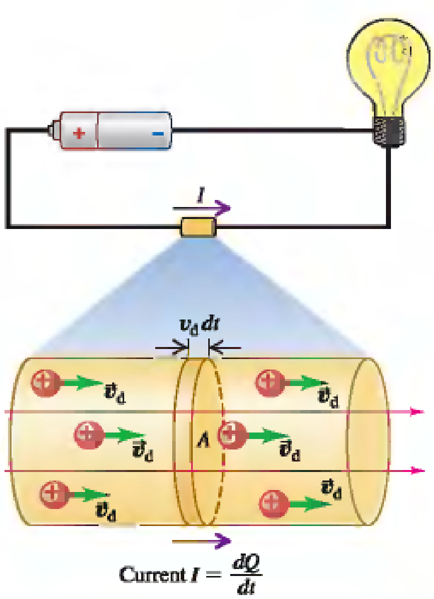
(a)



(b)



25.3 The current I is the time rate of charge transfer through the cross-sectional area A. The random component of each moving charged particle's motion averages to zero, and the current is in the same direction as $\vec{E}$ whether the moving charges are positive (as shown here) or negative (see Fig. 25.2b).



charges may include both electrons and positively charged ions. In a semiconductor material such as germanium or silicon, conduction is partly by electrons and partly by motion of *vacancies,* also known as *holes;* these are sites of missing electrons and act like positive charges.

Fig. 25.2 shows segments of two different current-carrying materials. In Fig. 25.2a the moving charges are positive, the electric force is in the same direction as $\vec{E}$, and the drift velocity $\vec{v}_d$ is from left to right. In Fig. 25.2b the charges are negative, the electric force is opposite to $\vec{E}$, and the drift velocity $\vec{v}_d$ is from right to left. In both cases there is a net flow of positive charge from left to right, and positive charges end up to the right of negative ones. We *define* the current, denoted by I, to be in the direction in which there is a flow of *positive* charge. Thus we describe currents as though they consisted entirely of positive charge flow, even in cases in which we know that the actual current is due to electrons. Hence the current is to the right in both Figs. 25.2a and 25.2b. This choice or convention for the direction of current flow is called **conventional current.** While the direction of the conventional current is *not* necessarily the same as the direction in which charged particles are actually moving, we'll find that the sign of the moving charges is of little importance in analyzing electric circuits.

Fig. 25.3 shows a segment of a conductor in which a current is flowing. We consider the moving charges to be *positive,* so they are moving in the same direction as the current. We define the current through the cross-sectional area A to be *the net charge flowing through the area per unit time.* Thus, if a net charge dQ flows through an area in a time dt, the current I through the area is

$$I = \frac{dQ}{dt} \qquad \text{(definition of current)} \qquad (25.1)$$

CAUTION **Current is not a vector** Although we refer to the *direction* of a current, current as defined by Eq. (25.1) is *not* a vector quantity. In a current-carrying wire, the current is always along the length of the wire, regardless of whether the wire is straight or curved. No single vector could describe motion along a curved path, which is why current is not a vector. We'll usually describe the direction of current either in words (as in "the current flows clockwise around the circuit") or by choosing a current to be positive if it flows in one direction along a conductor and negative if it flows in the other direction.

The SI unit of current is the **ampere;** one ampere is defined to be *one coulomb per second* $(1\text{ A} = 1\text{ C/s})$. This unit is named in honor of the French scientist André Marie Ampère (1775–1836). When an ordinary flashlight (D-cell size) is turned on, the current in the flashlight is about 0.5 to 1 A; the current in the wires of a car engine's starter motor is around 200 A. Currents in radio and television circuits are usually expressed in *milliamperes* $(1\text{ mA} = 10^{-3}\text{ A})$ or *microamperes* $(1\,\mu\text{A} = 10^{-6}\text{ A})$, and currents in computer circuits are expressed in *nanoamperes* $(1\text{ nA} = 10^{-9}\text{ A})$ or *picoamperes* $(1\text{ pA} = 10^{-12}\text{ A})$.

Current, Drift Velocity, and Current Density

We can express current in terms of the drift velocity of the moving charges. Let's consider again the situation of Fig. 25.3, a conductor with cross-sectional area A and an electric field $\vec{E}$ directed from left to right. To begin with, we'll assume that the free charges in the conductor are positive; then the drift velocity is in the same direction as the field.

Suppose there are n moving charged particles per unit volume. We call n the **concentration** of particles; its SI unit is m^{-3}. Assume that all the particles move with the same drift velocity with magnitude v_d. In a time interval dt, each particle moves a distance $v_d\,dt$. The particles that flow out of the right end of the shaded cylinder with length $v_d\,dt$ during dt are the particles that were within this cylinder at the beginning of the interval dt. The volume of the cylinder is $Av_d\,dt$, and the number of particles within it is $nAv_d\,dt$. If each particle has a charge q, the charge dQ that flows out of the end of the cylinder during time dt is

$$dQ = q(nAv_d\,dt) = nqv_dA\,dt$$

and the current is

$$I = \frac{dQ}{dt} = nqv_dA$$

The current *per unit cross-sectional area* is called the **current density** J:

$$J = \frac{I}{A} = nqv_d$$

The units of current density are amperes per square meter $(\mathrm{A/m^2})$.

If the moving charges are negative rather than positive, as in Fig. 25.2b, the drift velocity is opposite to $\vec{E}$. But the *current* is still in the same direction as $\vec{E}$ at each point in the conductor. Hence the current I and current density J don't depend on the sign of the charge, and so in the above expressions for I and J we replace the charge q by its absolute value $|q|$:

$$I = \frac{dQ}{dt} = n|q|v_dA \qquad \text{(general expression for current)} \qquad (25.2)$$

$$J = \frac{I}{A} = n|q|v_d \qquad \text{(general expression for current density)} \qquad (25.3)$$

The current in a conductor is the product of the concentration of moving charged particles, the magnitude of charge of each such particle, the magnitude of the drift velocity, and the cross-sectional area of the conductor.

We can also define a *vector* current density $\vec{J}$ that includes the direction of the drift velocity:

$$\vec{J} = nq\vec{v}_d \qquad \text{(vector current density)} \qquad (25.4)$$

There are *no* absolute value signs in Eq. (25.4). If q is positive, $\vec{v}_d$ is in the same direction as $\vec{E}$; if q is negative, $\vec{v}_d$ is opposite to $\vec{E}$. In either case, $\vec{J}$ is in the same direction as $\vec{E}$. Equation (25.3) gives the *magnitude* J of the vector current density $\vec{J}$.

CAUTION **Current density vs. current** Note that current density $\vec{J}$ is a vector, but current I is not. The difference is that the current density $\vec{J}$ describes how charges flow at a certain point, and the vector's direction tells you about the direction of the flow at that point. By contrast, the current I describes how charges flow through an extended object such as a wire. For example, I has the same value at all points in the circuit of Fig. 25.3, but $\vec{J}$ does not: the current density is directed downward in the left-hand side of the loop and upward in the right-hand side. The magnitude of $\vec{J}$ can also vary around a circuit. In Fig. 25.3 the current density magnitude $J = I/A$ is less in the battery (which has a large cross-sectional area A) than in the wires (which have a small cross-sectional area).

In general, a conductor may contain several different kinds of moving charged particles having charges $q_1, q_2, \ldots$, concentrations $n_1, n_2, \ldots$, and drift velocities with magnitudes $v_{d1}, v_{d2}, \ldots$. An example is current flow in an ionic solution (Fig. 25.4). In a sodium chloride solution, current can be carried by both positive sodium ions and negative chlorine ions; the total current I is found by adding up the currents due to each kind of charged particle, using Eq. (25.2). Likewise, the total vector current density $\vec{J}$ is found by using Eq. (25.4) for each kind of charged particle and adding the results.

We will see in Section 25.4 that it is possible to have a current that is *steady* (that is, one that is constant in time) only if the conducting material forms a

25.4 Part of the electric circuit that includes this light bulb passes through a beaker with a solution of sodium chloride. The current in the solution is carried by both positive charges (Na^+ ions) and negative charges (Cl^- ions).

closed loop, called a *complete circuit.* In such a steady situation, the total charge in every segment of the conductor is constant. Hence the rate of flow of charge *out* at one end of a segment at any instant equals the rate of flow of charge *in* at the other end of the segment, and *the current is the same at all cross sections of the circuit.* We'll make use of this observation when we analyze electric circuits later in this chapter.

In many simple circuits, such as flashlights or cordless electric drills, the direction of the current is always the same; this is called *direct current.* But home appliances such as toasters, refrigerators, and televisions use *alternating current,* in which the current continuously changes direction. In this chapter we'll consider direct current only. Alternating current has many special features worthy of detailed study, which we'll examine in Chapter 31.

Example 25.1 Current density and drift velocity in a wire

An 18-gauge copper wire (the size usually used for lamp cords) has a nominal diameter of 1.02 mm. This wire carries a constant current of 1.67 A to a 200-watt lamp. The density of free electrons is 8.5×10^{28} electrons per cubic meter. Find the magnitudes of (a) the current density and (b) the drift velocity.

SOLUTION

IDENTIFY: This problem uses the relationships among current, current density, and drift velocity.

SET UP: We are given the current and the dimensions of the wire, so we use Eq. (25.3) to find the magnitude J of the current density. We then use Eq. (25.3) again to find the drift speed v_d from J and the concentration of electrons.

EXECUTE: (a) The cross-sectional area is

$$A = \frac{\pi d^2}{4} = \frac{\pi(1.02 \times 10^{-3}\,\text{m})^2}{4} = 8.17 \times 10^{-7}\,\text{m}^2$$

The magnitude of the current density is

$$J = \frac{I}{A} = \frac{1.67\,\text{A}}{8.17 \times 10^{-7}\,\text{m}^2} = 2.04 \times 10^6\,\text{A/m}^2$$

(b) Solving Eq. (25.3) for the drift velocity magnitude v_d, we find

$$v_d = \frac{J}{n|q|} = \frac{2.04 \times 10^6\,\text{A/m}^2}{(8.5 \times 10^{28}\,\text{m}^{-3})|-1.60 \times 10^{-19}\,\text{C}|}$$
$$= 1.5 \times 10^{-4}\,\text{m/s} = 0.15\,\text{mm/s}$$

EVALUATE: At this speed an electron would require 6700 s, or about 1 hr 50 min, to travel the length of a wire 1 m long. The speeds of random motion of the electrons are of the order of 10^6 m/s. So in this example the drift speed is around 10^{10} times slower than the speed of random motion. Picture the electrons as bouncing around frantically, with a very slow and sluggish drift!

Test Your Understanding of Section 25.1 Suppose we replaced the wire in Example 25.1 with 12-gauge copper wire, which has twice the diameter of 18-gauge wire. If the current remains the same, what effect would this have on the magnitude of the drift velocity v_d? (i) none—v_d would be unchanged; (ii) v_d would be twice as great; (iii) v_d would be four times greater; (iv) v_d would be half as great; (v) v_d would be one-fourth as great.

25.2 Resistivity

The current density $\vec{J}$ in a conductor depends on the electric field $\vec{E}$ and on the properties of the material. In general, this dependence can be quite complex. But for some materials, especially metals, at a given temperature, $\vec{J}$ is nearly *directly proportional* to $\vec{E}$, and the ratio of the magnitudes of E and J is constant. This relationship, called Ohm's law, was discovered in 1826 by the German physicist Georg Simon Ohm (1787–1854). The word "law" should actually be in quotation marks, since **Ohm's law,** like the ideal-gas equation and Hooke's law, is an *idealized model* that describes the behavior of some materials quite well but is not a general description of *all* matter. In the following discussion we'll assume that Ohm's law is valid, even though there are many situations in which it is not. The situation is comparable to our representation of the behavior of the static and kinetic friction forces; we treated these friction forces as being directly proportional to the normal force, even though we knew that this was at best an approximate description.

Table 25.1 Resistivities at Room Temperature $(20\ ^\circ\text{C})$

		Substance	$\rho(\Omega\cdot\text{m})$		Substance	$\rho(\Omega\cdot\text{m})$
Conductors				**Semiconductors**		
Metals		Silver	1.47×10^{-8}		Pure carbon (graphite)	3.5×10^{-5}
		Copper	1.72×10^{-8}		Pure germanium	0.60
		Gold	2.44×10^{-8}		Pure silicon	2300
		Aluminum	2.75×10^{-8}	**Insulators**		
		Tungsten	5.25×10^{-8}		Amber	5×10^{14}
		Steel	20×10^{-8}		Glass	10^{10}–10^{14}
		Lead	22×10^{-8}		Lucite	$>10^{13}$
		Mercury	95×10^{-8}		Mica	10^{11}–10^{15}
Alloys		Manganin (Cu 84%, Mn 12%, Ni 4%)	44×10^{-8}		Quartz (fused)	75×10^{16}
		Constantan (Cu 60%, Ni 40%)	49×10^{-8}		Sulfur	10^{15}
		Nichrome	100×10^{-8}		Teflon	$>10^{13}$
					Wood	10^{8}–10^{11}

We define the **resistivity** ρ of a material as the ratio of the magnitudes of electric field and current density:

$$\rho = \frac{E}{J} \qquad \text{(definition of resistivity)} \qquad (25.5)$$

The greater the resistivity, the greater the field needed to cause a given current density, or the smaller the current density caused by a given field. From Eq. (25.5) the units of ρ are $(\text{V/m})/(\text{A/m}^2) = \text{V}\cdot\text{m/A}$. As we will discuss in the next section, 1 V/A is called one *ohm* ($1\ \Omega$; we use the Greek letter Ω, or omega, which is alliterative with "ohm"). So the SI units for ρ are $\Omega\cdot\text{m}$ (ohm-meters). Table 25.1 lists some representative values of resistivity. A perfect conductor would have zero resistivity, and a perfect insulator would have an infinite resistivity. Metals and alloys have the smallest resistivities and are the best conductors. The resistivities of insulators are greater than those of the metals by an enormous factor, on the order of 10^{22}.

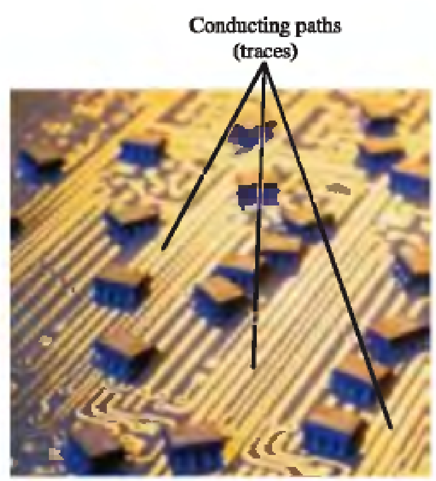
The reciprocal of resistivity is **conductivity.** Its units are $(\Omega\cdot\text{m})^{-1}$. Good conductors of electricity have larger conductivity than insulators. Conductivity is the direct electrical analog of thermal conductivity. Comparing Table 25.1 with Table 17.5 (Thermal Conductivities), we note that good electrical conductors, such as metals, are usually also good conductors of heat. Poor electrical conductors, such as ceramic and plastic materials, are also poor thermal conductors. In a metal the free electrons that carry charge in electrical conduction also provide the principal mechanism for heat conduction, so we should expect a correlation between electrical and thermal conductivity. Because of the enormous difference in conductivity between electrical conductors and insulators, it is easy to confine electric currents to well-defined paths or circuits (Fig. 25.5). The variation in *thermal* conductivity is much less, only a factor of 10^3 or so, and it is usually impossible to confine heat currents to that extent.

Semiconductors have resistivities intermediate between those of metals and those of insulators. These materials are important because of the way their resistivities are affected by temperature and by small amounts of impurities.

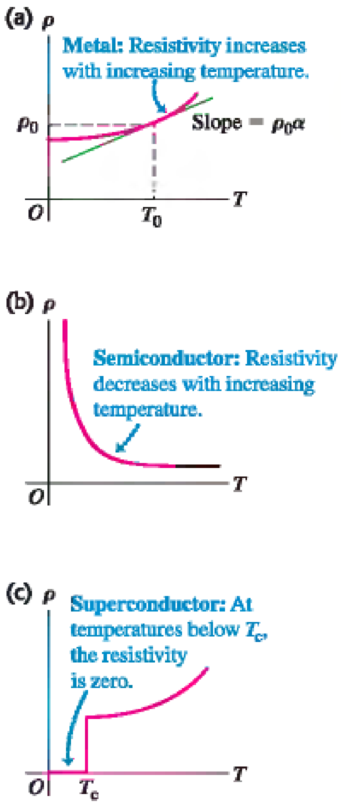
A material that obeys Ohm's law reasonably well is called an *ohmic* conductor or a *linear* conductor. For such materials, at a given temperature, ρ is a *constant* that does not depend on the value of E. Many materials show substantial departures from Ohm's-law behavior; they are *nonohmic,* or *nonlinear.* In these materials, J depends on E in a more complicated manner.

Analogies with fluid flow can be a big help in developing intuition about electric current and circuits. For example, in the making of wine or maple syrup, the product is sometimes filtered to remove sediments. A pump forces the fluid through the filter under pressure; if the flow rate (analogous to J) is proportional to the pressure difference between the upstream and downstream sides (analogous to E), the behavior is analogous to Ohm's law.

25.5 The copper "wires," or traces, on this circuit board are printed directly onto the surface of the dark-colored insulating board. Even though the traces are very close to each other (only about a millimeter apart), the board has such a high resistivity (and low conductivity) compared to the copper that no current can flow between the traces.



25.6 Variation of resistivity ρ with absolute temperature T for **(a)** a normal metal, **(b)** a semiconductor, and **(c)** a superconductor. In (a) the linear approximation to ρ as a function of T is shown as a green line; the approximation agrees exactly at $T = T_0$, where $\rho = \rho_0$.



Resistivity and Temperature

The resistivity of a *metallic* conductor nearly always increases with increasing temperature, as shown in Fig. 25.6a. As temperature increases, the ions of the conductor vibrate with greater amplitude, making it more likely that a moving electron will collide with an ion as in Fig. 25.1; this impedes the drift of electrons through the conductor and hence reduces the current. Over a small temperature range (up to 100 C° or so), the resistivity of a metal can be represented approximately by the equation

$$\rho(T) = \rho_0[1 + \alpha(T - T_0)] \quad \text{(temperature dependence of resistivity)} \tag{25.6}$$

where ρ_0 is the resistivity at a reference temperature T_0 (often taken as 0°C or 20°C) and $\rho(T)$ is the resistivity at temperature T, which may be higher or lower than T_0. The factor α is called the **temperature coefficient of resistivity.** Some representative values are given in Table 25.2. The resistivity of the alloy manganin is practically independent of temperature.

Table 25.2 Temperature Coefficients of Resistivity (Approximate Values Near Room Temperature)

Material	$\alpha\,[(°\text{C})^{-1}]$	Material	$\alpha\,[(°\text{C})^{-1}]$
Aluminum	0.0039	Lead	0.0043
Brass	0.0020	Manganin	0.00000
Carbon (graphite)	−0.0005	Mercury	0.00088
Constantan	0.00001	Nichrome	0.0004
Copper	0.00393	Silver	0.0038
Iron	0.0050	Tungsten	0.0045

The resistivity of graphite (a nonmetal) *decreases* with increasing temperature, since at higher temperatures, more electrons are "shaken loose" from the atoms and become mobile; hence the temperature coefficient of resistivity of graphite is negative. This same behavior occurs for semiconductors (Fig. 25.6b). Measuring the resistivity of a small semiconductor crystal is therefore a sensitive measure of temperature; this is the principle of a type of thermometer called a *thermistor.*

Some materials, including several metallic alloys and oxides, show a phenomenon called *superconductivity.* As the temperature decreases, the resistivity at first decreases smoothly, like that of any metal. But then at a certain critical temperature T_c a phase transition occurs and the resistivity suddenly drops to zero, as shown in Fig. 25.6c. Once a current has been established in a superconducting ring, it continues indefinitely without the presence of any driving field.

Superconductivity was discovered in 1911 by the Dutch physicist Heike Kamerlingh Onnes (1853–1926). He discovered that at very low temperatures, below 4.2 K, the resistivity of mercury suddenly dropped to zero. For the next 75 years, the highest T_c attained was about 20 K. This meant that superconductivity occurred only when the material was cooled using expensive liquid helium, with a boiling-point temperature of 4.2 K, or explosive liquid hydrogen, with a boiling point of 20.3 K. But in 1986 Karl Müller and Johannes Bednorz discovered an oxide of barium, lanthanum, and copper with a T_c of nearly 40 K, and the race was on to develop "high-temperature" superconducting materials.

By 1987 a complex oxide of yttrium, copper, and barium had been found that has a value of T_c well above the 77 K boiling temperature of liquid nitrogen, a refrigerant that is both inexpensive and safe. The current (2006) record for T_c at atmospheric pressure is 138 K, and materials that are superconductors at room temperature may become a reality. The implications of these discoveries for power-distribution systems, computer design, and transportation are enormous. Meanwhile, superconducting electromagnets cooled by liquid helium are used in particle accelerators and some experimental magnetic-levitation railroads. Superconductors have other exotic properties that require an understanding of magnetism to explore; we will discuss these further in Chapter 29.

Test Your Understanding of Section 25.2 You maintain a constant electric field inside a piece of semiconductor while lowering the semiconductor's temperature. What happens to the current density in the semiconductor? (i) It increases; (ii) it decreases; (iii) it remains the same.

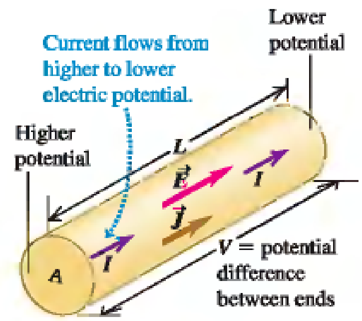
25.3 Resistance

For a conductor with resistivity ρ, the current density $\vec{J}$ at a point where the electric field is $\vec{E}$ is given by Eq. (25.5), which we can write as

$$\vec{E} = \rho\vec{J} \tag{25.7}$$

When Ohm's law is obeyed, ρ is constant and independent of the magnitude of the electric field, so $\vec{E}$ is directly proportional to $\vec{J}$. Often, however, we are more interested in the total current in a conductor than in $\vec{J}$ and more interested in the potential difference between the ends of the conductor than in $\vec{E}$. This is so largely because current and potential difference are much easier to measure than are $\vec{J}$ and $\vec{E}$.

Suppose our conductor is a wire with uniform cross-sectional area A and length L, as shown in Fig. 25.7. Let V be the potential difference between the higher-potential and lower-potential ends of the conductor, so that V is positive. The *direction* of the current is always from the higher-potential end to the lower-potential end. That's because current in a conductor flows in the direction of $\vec{E}$, no matter what the sign of the moving charges (Fig. 25.2), and because $\vec{E}$ points in the direction of *decreasing* electric potential (see Section 23.2). As the current flows through the potential difference, electric potential energy is lost; this energy is transferred to the ions of the conducting material during collisions.



25.7 A conductor with uniform cross section. The current density is uniform over any cross section, and the electric field is constant along the length.

We can also relate the *value* of the current I to the potential difference between the ends of the conductor. If the magnitudes of the current density $\vec{J}$ and the electric field $\vec{E}$ are uniform throughout the conductor, the total current I is given by $I = JA$, and the potential difference V between the ends is $V = EL$. When we solve these equations for J and E, respectively, and substitute the results in Eq. (25.7), we obtain

$$\frac{V}{L} = \frac{\rho I}{A} \quad \text{or} \quad V = \frac{\rho L}{A}I \tag{25.8}$$

This shows that when ρ is constant, the total current I is proportional to the potential difference V.

The ratio of V to I for a particular conductor is called its **resistance** R:

$$R = \frac{V}{I} \tag{25.9}$$

Comparing this definition of R to Eq. (25.8), we see that the resistance R of a particular conductor is related to the resistivity ρ of its material by

$$R = \frac{\rho L}{A} \quad \text{(relationship between resistance and resistivity)} \tag{25.10}$$

If ρ is constant, as is the case for ohmic materials, then so is R.

The equation

$$V = IR \quad \text{(relationship among voltage, current, and resistance)} \tag{25.11}$$

is often called Ohm's law, but it is important to understand that the real content of Ohm's law is the direct proportionality (for some materials) of V to I or of J to E. Equation (25.9) or (25.11) *defines* resistance R for *any* conductor, whether or not it obeys Ohm's law, but only when R is constant can we correctly call this relationship Ohm's law.

Interpreting Resistance

Equation (25.10) shows that the resistance of a wire or other conductor of uniform cross section is directly proportional to its length and inversely proportional to its cross-sectional area. It is also proportional to the resistivity of the material of which the conductor is made.

The flowing-fluid analogy is again useful. In analogy to Eq. (25.10), a narrow water hose offers more resistance to flow than a fat one, and a long hose has more resistance than a short one (Fig. 25.8). We can increase the resistance to flow by stuffing the hose with cotton or sand; this corresponds to increasing the resistivity. The flow rate is approximately proportional to the pressure difference between the ends. Flow rate is analogous to current, and pressure difference is analogous to potential difference ("voltage"). Let's not stretch this analogy too far, though; the water flow rate in a pipe is usually *not* proportional to its cross-sectional area (see Section 14.6).

25.8 A long fire hose offers substantial resistance to water flow. To make water pass through the hose rapidly, the upstream end of the hose must be at much higher pressure than the end where the water emerges. In an analogous way, there must be a large potential difference between the ends of a long wire in order to cause a substantial electric current through the wire.

The SI unit of resistance is the **ohm,** equal to one volt per ampere $(1\ \Omega = 1\ \text{V/A})$. The *kilohm* $(1\ \text{k}\Omega = 10^3\ \Omega)$ and the *megohm* $(1\ \text{M}\Omega = 10^6\ \Omega)$ are also in common use. A 100-m length of 12-gauge copper wire, the size usually used in household wiring, has a resistance at room temperature of about 0.5 Ω. A 100-W, 120-V light bulb has a resistance (at operating temperature) of 140 Ω. If the same current I flows in both the copper wire and the light bulb, the potential difference $V = IR$ is much greater across the light bulb, and much more potential energy is lost per charge in the light bulb. This lost energy is converted by the light bulb filament into light and heat. You don't want your household wiring to glow white-hot, so its resistance is kept low by using wire of low resistivity and large cross-sectional area.

Because the resistivity of a material varies with temperature, the resistance of a specific conductor also varies with temperature. For temperature ranges that are not too great, this variation is approximately a linear relationship, analogous to Eq. (25.6):

$$R(T) = R_0[1 + \alpha(T - T_0)] \tag{25.12}$$

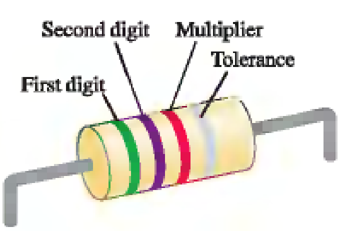
In this equation, $R(T)$ is the resistance at temperature T and R_0 is the resistance at temperature T_0, often taken to be 0°C or 20°C. The *temperature coefficient of resistance* α is the same constant that appears in Eq. (25.6) if the dimensions L and A in Eq. (25.10) do not change appreciably with temperature; this is indeed the case for most conducting materials (see Problem 25.67). Within the limits of validity of Eq. (25.12), the *change* in resistance resulting from a temperature change $T - T_0$ is given by $R_0\alpha(T - T_0)$.

Table 25.3 Color Codes for Resistors

Color	Value as Digit	Value as Multiplier
Black	0	1
Brown	1	10
Red	2	10^2
Orange	3	10^3
Yellow	4	10^4
Green	5	10^5
Blue	6	10^6
Violet	7	10^7
Gray	8	10^8
White	9	10^9

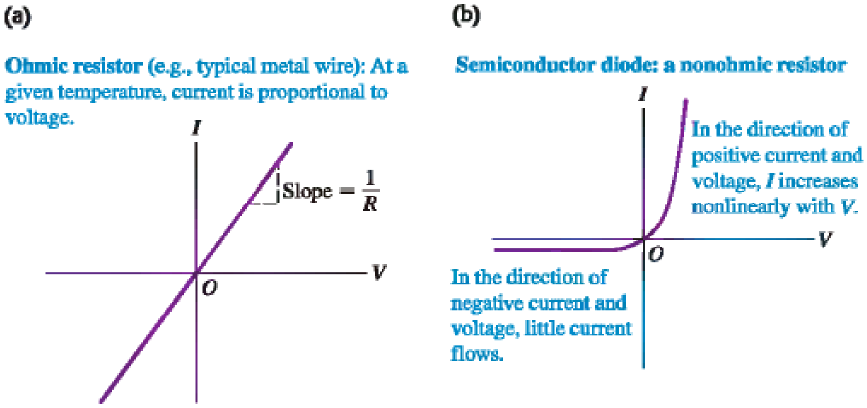
A circuit device made to have a specific value of resistance between its ends is called a **resistor.** Resistors in the range 0.01 to $10^7\ \Omega$ can be bought off the shelf. Individual resistors used in electronic circuitry are often cylindrical, a few millimeters in diameter and length, with wires coming out of the ends. The resistance may be marked with a standard code using three or four color bands near one end (Fig. 25.9), according to the scheme shown in Table 25.3. The first two bands (starting with the band nearest an end) are digits, and the third is a power-of-10 multiplier, as shown in Fig. 25.9. For example, green–violet–red means $57 \times 10^2\ \Omega$, or 5.7 kΩ. The fourth band, if present, indicates the precision (tolerance) of the value; no band means ±20%, a silver band ±10%, and a gold band ±5%. Another important characteristic of a resistor is the maximum *power* it can dissipate without damage. We'll return to this point in Section 25.5.

25.9 This resistor has a resistance of 5.7 kΩ with a precision (tolerance) of ±10%.



For a resistor that obeys Ohm's law, a graph of current as a function of potential difference (voltage) is a straight line (Fig. 25.10a). The slope of the line is $1/R$. If the sign of the potential difference changes, so does the sign of the current produced; in Fig. 25.7 this corresponds to interchanging the higher- and lower-potential ends of the conductor, so the electric field, current density, and current

25.10 Current–voltage relationships for two devices. Only for a resistor that obeys Ohm's law as in (a) is current I proportional to voltage V.



all reverse direction. In devices that do not obey Ohm's law, the relationship of voltage to current may not be a direct proportion, and it may be different for the two directions of current. Figure 25.10b shows the behavior of a semiconductor *diode*, a device used to convert alternating current to direct current and to perform a wide variety of logic functions in computer circuitry. For positive potentials V of the anode (one of two terminals of the diode) with respect to the cathode (the other terminal), I increases exponentially with increasing V; for negative potentials the current is extremely small. Thus a positive potential difference V causes a current to flow in the positive direction, but a potential difference of the other sign causes little or no current. Hence a diode acts like a one-way valve in a circuit.

Example 25.2 Electric field, potential difference, and resistance in a wire

The 18-gauge copper wire in Example 25.1 (Section 25.1) has a diameter of 1.02 mm and a cross-sectional area of $8.20 \times 10^{-7}\ \text{m}^2$. It carries a current of 1.67 A. Find (a) the electric-field magnitude in the wire; (b) the potential difference between two points in the wire 50.0 m apart; (c) the resistance of a 50.0-m length of this wire.

SOLUTION

IDENTIFY: We are given the values of cross-sectional area A and current I. Our target variables are the electric-field magnitude E, potential difference V, and resistance R.

SET UP: The magnitude of the current density is $J = I/A$ and the resistivity ρ is given in Table 25.1. We find the electric-field magnitude by using Eq. (25.5), $E = \rho J$. Once we have found E, the potential difference is simply the product of E and the length of the wire. We find the resistance by using Eq. (25.11).

EXECUTE: (a) From Table 25.1, the resistivity of copper is $1.72 \times 10^{-8}\ \Omega \cdot \text{m}$. Hence, using Eq. (25.5),

$$E = \rho J = \frac{\rho I}{A} = \frac{(1.72 \times 10^{-8}\ \Omega \cdot \text{m})(1.67\ \text{A})}{8.20 \times 10^{-7}\ \text{m}^2} = 0.0350\ \text{V/m}$$

(b) The potential difference is given by

$$V = EL = (0.0350\ \text{V/m})(50.0\ \text{m}) = 1.75\ \text{V}$$

(c) From Eq. (25.11) the resistance of a 50.0-m length of this wire is

$$R = \frac{V}{I} = \frac{1.75\ \text{V}}{1.67\ \text{A}} = 1.05\ \Omega$$

EVALUATE: To check our result in part (c), we calculate the resistance using Eq. (25.10):

$$R = \frac{\rho L}{A} = \frac{(1.72 \times 10^{-8}\ \Omega \cdot \text{m})(50.0\ \text{m})}{8.20 \times 10^{-7}\ \text{m}^2} = 1.05\ \Omega$$

We emphasize that the resistance of the wire is *defined* to be the ratio of voltage to current. If the wire is made of nonohmic material, then R is different for different values of V but is always given by $R = V/I$. Resistance is also always given by $R = \rho L/A$; if the material is nonohmic, ρ is not constant but depends on E (or, equivalently, on $V = EL$).

Example 25.3 Temperature dependence of resistance

Suppose the resistance of the wire in Example 25.2 is $1.05\ \Omega$ at a temperature of 20°C. Find the resistance at 0°C and at 100°C.

SOLUTION

IDENTIFY: This example concerns how resistance (the target variable) depends on temperature. As Table 25.2 shows, this temperature dependence differs for different substances.

SET UP: Our target variables are the values of the wire resistance R at two temperatures, $T = 0°\text{C}$ and $T = 100°\text{C}$. To find these values we use Eq. (25.12). Note that we are given the resistance $R_0 = 1.05\ \Omega$ at a reference temperature $T_0 = 20°\text{C}$, and we know from Example 25.2 that the wire is made of copper.

EXECUTE: From Table 25.2 the temperature coefficient of resistivity of copper is $\alpha = 0.00393\ (\text{C}°)^{-1}$. From Eq. (25.12), the resistance at $T = 0°\text{C}$ is

$$\begin{aligned} R &= R_0[1 + \alpha(T - T_0)] \\ &= (1.05\ \Omega)\{1 + [0.00393\ (\text{C}°)^{-1}][0°\text{C} - 20°\text{C}]\} \\ &= 0.97\ \Omega \end{aligned}$$

At $T = 100°\text{C}$,

$$\begin{aligned} R &= (1.05\ \Omega)\{1 + [0.00393\ (\text{C}°)^{-1}][100°\text{C} - 20°\text{C}]\} \\ &= 1.38\ \Omega \end{aligned}$$

EVALUATE: The resistance at 100°C is greater than that at 0°C by a factor of $(1.38\ \Omega)/(0.97\ \Omega) = 1.42$. In other words, raising the temperature of ordinary copper wire from 0°C to 100°C increases its resistance by 42%. From Eq. (25.11), $V = IR$, this means that 42% more voltage V is required to produce the same current I at 100°C than at 0°C. This is a substantial effect that must be taken into account in designing electric circuits that are to operate over a wide range of temperatures.

Example 25.4 Calculating resistance

The hollow cylinder shown in Fig. 25.11 has length L and inner and outer radii a and b. It is made of a material with resistivity ρ. A potential difference is set up between the inner and outer surfaces of the cylinder (each of which is an equipotential surface) so that current flows radially through the cylinder. What is the resistance to this radial current flow?

SOLUTION

IDENTIFY: Figure 25.11 shows that the current flows radially from the inside of the conductor toward the outside, *not* along the length of the conductor as in Fig. 25.7. Hence we must use the ideas of this section to derive a new formula for resistance (our target variable) appropriate for radial current flow.

SET UP: We can't use Eq. (25.10) directly because the cross section through which the charge travels is *not* constant; it varies from $2\pi aL$ at the inner surface to $2\pi bL$ at the outer surface. Instead, we calculate the resistance to radial current flow through a thin cylindrical shell of inner radius r and thickness dr. We then combine the resistances for all such shells between the inner and outer radii of the cylinder.

EXECUTE: The area A for the shell is $2\pi rL$, the surface area that the current encounters as it flows outward. The length of the current path through the shell is dr. The resistance dR of this shell, between inner and outer surfaces, is that of a conductor with length dr and area $2\pi rL$:

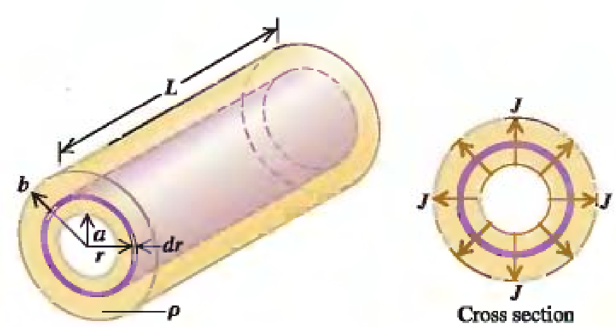
$$dR = \frac{\rho\, dr}{2\pi rL}$$

The current has to pass successively through all such shells between the inner and outer radii a and b. From Eq. (25.11) the potential difference across one shell is $dV = I\, dR$, and the total potential difference between the inner and outer surfaces is the sum of the potential differences for all shells. The total current is the same through each shell, so the total resistance is the sum of the resistances of all the shells. If the area $2\pi rL$ were constant, we could just integrate dr from $r = a$ to $r = b$ to get the total length of the current path. But the area increases as the current passes through shells of greater radius, so we have to integrate the above expression for dR. The total resistance is thus given by

$$R = \int dR = \frac{\rho}{2\pi L}\int_a^b \frac{dr}{r} = \frac{\rho}{2\pi L}\ln\frac{b}{a}$$

EVALUATE: The conductor geometry shown in Fig. 25.11 plays an important role in your body's nervous system. Each neuron, or nerve cell, has a long extension called a nerve fiber or *axon*. An axon has a cylindrical membrane shaped much like the resistor in Fig. 25.11, with one conducting fluid inside the membrane and another outside it. Ordinarily all of the inner fluid is at the same potential, so no current tends to flow along the length of the axon. If the axon is stimulated at a certain point along its length, however, charged ions flow radially across the cylindrical membrane at that point, as in Fig. 25.11. This flow causes a potential difference between that point and other points along the length of the axon, which makes a nerve signal flow along that length.

25.11 Finding the resistance for radial current flow.



Test Your Understanding of Section 25.3 Suppose you increase the voltage across the copper wire in Examples 25.2 and 25.3. The increased voltage causes more current to flow, which makes the temperature of the wire increase. (The same thing happens to the coils of an electric oven or a toaster when a voltage is applied to them. We'll explore this issue in more depth in Section 25.5.) If you double the voltage across the wire, the current in the wire increases. By what factor does it increase? (i) 2; (ii) greater than 2; (iii) less than 2. ▮

25.4 Electromotive Force and Circuits

For a conductor to have a steady current, it must be part of a path that forms a closed loop or **complete circuit.** Here's why. If you establish an electric field $\vec{E}_1$ inside an isolated conductor with resistivity ρ that is *not* part of a complete circuit, a current begins to flow with current density $\vec{J} = \vec{E}_1/\rho$ (Fig. 25.12a). As a result a net positive charge quickly accumulates at one end of the conductor and a net negative charge accumulates at the other end (Fig. 25.12b). These charges themselves produce an electric field $\vec{E}_2$ in the direction opposite to $\vec{E}_1$, causing the total electric field and hence the current to decrease. Within a very small fraction of a second, enough charge builds up on the conductor ends that the total electric field $\vec{E} = \vec{E}_1 + \vec{E}_2 = \mathbf{0}$ inside the conductor. Then $\vec{J} = \mathbf{0}$ as well, and the current stops altogether (Fig. 25.12c). So there can be no steady motion of charge in such an *incomplete* circuit.

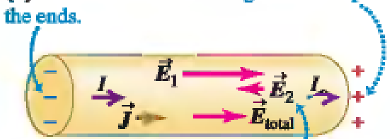
To see how to maintain a steady current in a *complete* circuit, we recall a basic fact about electric potential energy: If a charge q goes around a complete circuit and returns to its starting point, the potential energy must be the same at the end of the round trip as at the beginning. As described in Section 25.3, there is always a *decrease* in potential energy when charges move through an ordinary conducting material with resistance. So there must be some part of the circuit in which the potential energy *increases.*

The problem is analogous to an ornamental water fountain that recycles its water. The water pours out of openings at the top, cascades down over the terraces and spouts (moving in the direction of decreasing gravitational potential energy), and collects in a basin in the bottom. A pump then lifts it back to the top (increasing the potential energy) for another trip. Without the pump, the water would just fall to the bottom and stay there.

25.12 If an electric field is produced inside a conductor that is *not* part of a complete circuit, current flows for only a very short time.

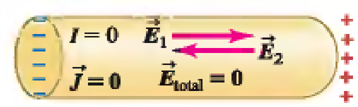
(a) An electric field $\vec{E}_1$ produced inside an isolated conductor causes a current.

(b) The current causes charge to build up at the ends.



The charge buildup produces an opposing field $\vec{E}_2$, thus reducing the current.

(c) After a very short time $\vec{E}_2$ has the same magnitude as $\vec{E}_1$; then the total field is $\vec{E}_{total} = \mathbf{0}$ and the current stops completely.



Electromotive Force

In an electric circuit there must be a device somewhere in the loop that acts like the water pump in a water fountain (Fig. 25.13). In this device a charge travels "uphill," from lower to higher potential energy, even though the electrostatic force is trying to push it from higher to lower potential energy. The direction of current in such a device is from lower to higher potential, just the opposite of what happens in an ordinary conductor. The influence that makes current flow from lower to higher potential is called **electromotive force** (abbreviated **emf** and pronounced "ee-em-eff"). This is a poor term because emf is *not* a force but an energy-per-unit-charge quantity, like potential. The SI unit of emf is the same as that for potential, the volt $(1\ \mathrm{V} = 1\ \mathrm{J/C})$. A typical flashlight battery has an emf of 1.5 V; this means that the battery does 1.5 J of work on every coulomb of charge that passes through it. We'll use the symbol $\mathcal{E}$ (a script capital E) for emf.

Every complete circuit with a steady current must include some device that provides emf. Such a device is called a **source of emf.** Batteries, electric generators, solar cells, thermocouples, and fuel cells are all examples of sources of emf. All such devices convert energy of some form (mechanical, chemical, thermal, and so on) into electric potential energy and transfer it into the circuit to which the device is connected. An *ideal* source of emf maintains a constant potential

25.13 Just as a water fountain requires a pump, an electric circuit requires a source of electromotive force to sustain a steady current.

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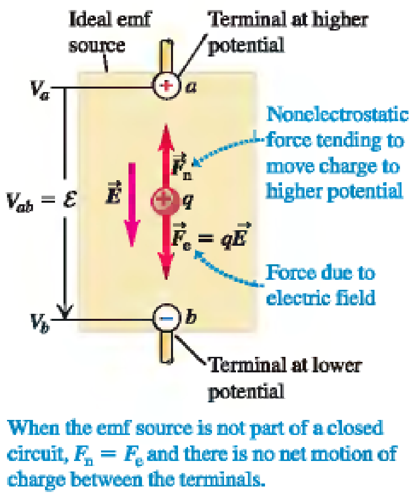
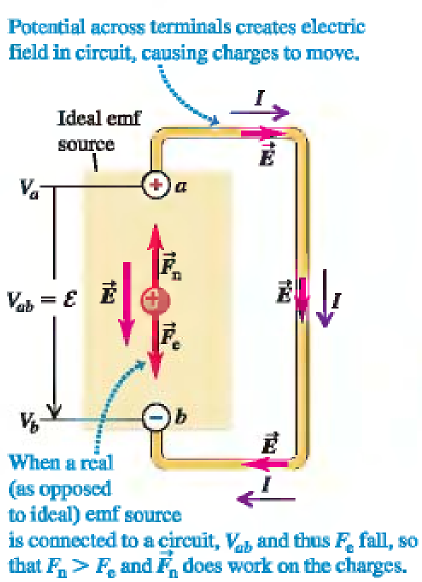
12.1 DC Series Circuits (Qualitative)

25.14 Schematic diagram of a source of emf in an "open-circuit" situation. The electric-field force $\vec{F}_e = q\vec{E}$ and the nonelectrostatic force $\vec{F}_n$ are shown for a positive charge q.



25.15 Schematic diagram of an ideal source of emf in a complete circuit. The electric-field force $\vec{F}_e = q\vec{E}$ and the nonelectrostatic force $\vec{F}_n$ are shown for a positive charge q The current is in the direction from a to b in the external circuit and from b to a within the source.



difference between its terminals, independent of the current through it. We define electromotive force quantitatively as the magnitude of this potential difference. As we will see, such an ideal source is a mythical beast, like the frictionless plane and the massless rope. We will discuss later how real-life sources of emf differ in their behavior from this idealized model.

Fig. 25.14 is a schematic diagram of an ideal source of emf that maintains a potential difference between conductors a and b, called the *terminals* of the device. Terminal a, marked +, is maintained at *higher* potential than terminal b, marked −. Associated with this potential difference is an electric field $\vec{E}$ in the region around the terminals, both inside and outside the source. The electric field inside the device is directed from a to b, as shown. A charge q within the source experiences an electric force $\vec{F}_e = q\vec{E}$. But the source also provides an additional influence, which we represent as a nonelectrostatic force $\vec{F}_n$. This force, operating inside the device, pushes charge from b to a in an "uphill" direction against the electric force $\vec{F}_e$. Thus $\vec{F}_n$ maintains the potential difference between the terminals. If $\vec{F}_n$ were not present, charge would flow between the terminals until the potential difference was zero. The origin of the additional influence $\vec{F}_n$ depends on the kind of source. In a generator it results from magnetic-field forces on moving charges. In a battery or fuel cell it is associated with diffusion processes and varying electrolyte concentrations resulting from chemical reactions. In an electrostatic machine such as a Van de Graaff generator (see Fig. 22.27), an actual mechanical force is applied by a moving belt or wheel.

If a positive charge q is moved from b to a inside the source, the nonelectrostatic force $\vec{F}_n$ does a positive amount of work $W_n = q\mathcal{E}$ on the charge. This displacement is *opposite* to the electrostatic force $\vec{F}_e$, so the potential energy associated with the charge *increases* by an amount equal to qV_{ab}, where $V_{ab} = V_a - V_b$ is the (positive) potential of point a with respect to point b. For the ideal source of emf that we've described, $\vec{F}_e$ and $\vec{F}_n$ are equal in magnitude but opposite in direction, so the total work done on the charge q is zero; there is an increase in potential energy but *no* change in the kinetic energy of the charge. It's like lifting a book from the floor to a high shelf at constant speed. The increase in potential energy is just equal to the non-electrostatic work W_n, so $q\mathcal{E} = qV_{ab}$, or

$$V_{ab} = \mathcal{E} \qquad \text{(ideal source of emf)} \tag{25.13}$$

Now let's make a complete circuit by connecting a wire with resistance R to the terminals of a source (Fig. 25.15). The potential difference between terminals a and b sets up an electric field within the wire; this causes current to flow around the loop from a toward b, from higher to lower potential. Where the wire bends, equal amounts of positive and negative charge persist on the "inside" and "outside" of the bend. These charges exert the forces that cause the current to follow the bends in the wire.

From Eq. (25.11) the potential difference between the ends of the wire in Fig. 25.15 is given by $V_{ab} = IR$. Combining with Eq. (25.13), we have

$$\mathcal{E} = V_{ab} = IR \qquad \text{(ideal source of emf)} \tag{25.14}$$

That is, when a positive charge q flows around the circuit, the potential *rise* $\mathcal{E}$ as it passes through the ideal source is numerically equal to the potential *drop* $V_{ab} = IR$ as it passes through the remainder of the circuit. Once $\mathcal{E}$ and R are known, this relationship determines the current in the circuit.

CAUTION **Current is not "used up" in a circuit** It's a common misconception that in a closed circuit, current is something that squirts out of the positive terminal of a battery and is consumed or "used up" by the time it reaches the negative terminal. In fact the current is the *same* at every point in a simple loop circuit like that in Fig. 25.15, even if the thickness of the wires is different at different points in the circuit. This happens because charge is conserved (that is, it can be neither created nor destroyed) and because charge cannot accumulate in the circuit devices we have described. If charge did accumulate, the potential differences would change with time. It's like the flow of water in an ornamental fountain; water flows out of the top of the fountain at the same rate at which it reaches the bottom, no matter what the dimensions of the fountain. None of the water is "used up" along the way!

Internal Resistance

Real sources of emf in a circuit don't behave in exactly the way we have described; the potential difference across a real source in a circuit is *not* equal to the emf as in Eq. (25.14). The reason is that charge moving through the material of any real source encounters *resistance*. We call this the **internal resistance** of the source, denoted by r. If this resistance behaves according to Ohm's law, r is constant and independent of the current I. As the current moves through r, it experiences an associated drop in potential equal to Ir. Thus, when a current is flowing through a source from the negative terminal b to the positive terminal a, the potential difference V_{ab} between the terminals is

$$V_{ab} = \mathcal{E} - Ir \quad \text{(terminal voltage, source with internal resistance)} \tag{25.15}$$

The potential V_{ab}, called the **terminal voltage,** is less than the emf $\mathcal{E}$ because of the term Ir representing the potential drop across the internal resistance r. Expressed another way, the increase in potential energy qV_{ab} as a charge q moves from b to a within the source is now less than the work $q\mathcal{E}$ done by the nonelectrostatic force $\vec{F}_n$, since some potential energy is lost in traversing the internal resistance.

A 1.5-V battery has an emf of 1.5 V, but the terminal voltage V_{ab} of the battery is equal to 1.5 V only if no current is flowing through it so that $I = 0$ in Eq. (25.15). If the battery is part of a complete circuit through which current is flowing, the terminal voltage will be less than 1.5 V. *For a real source of emf, the terminal voltage equals the emf only if no current is flowing through the source* (Fig. 25.16). Thus we can describe the behavior of a source in terms of two properties: an emf $\mathcal{E}$, which supplies a constant potential difference independent of current, in series with an internal resistance r.

The current in the external circuit connected to the source terminals a and b is still determined by $V_{ab} = IR$. Combining this with Eq. (25.15), we find

$$\mathcal{E} - Ir = IR \quad \text{or} \quad I = \frac{\mathcal{E}}{R + r} \quad \text{(current, source with internal resistance)} \tag{25.16}$$

That is, the current equals the source emf divided by the *total* circuit resistance $(R + r)$.

CAUTION **A battery is not a "current source"** You might have thought that a battery or other source of emf always produces the same current, no matter what circuit it's used in. But as Eq. (25.16) shows, the current that a source of emf produces in a given circuit depends on the resistance R of the external circuit (as well as on the internal resistance r of the source). The greater the resistance, the less current the source will produce. It's analogous to pushing an object through a thick, viscous liquid such as oil or molasses; if you exert a certain steady push (emf), you can move a small object at high speed (small R, large I) or a large object at low speed (large R, small I).

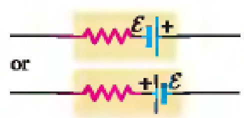
25.16 The emf of this battery—that is, the terminal voltage when it's not connected to anything—is 12 V. But because the battery has internal resistance, the terminal voltage of the battery is less than 12 V when it is supplying current to a light bulb.

Symbols for Circuit Diagrams

An important part of analyzing any electric circuit is drawing a schematic *circuit diagram.* Table 25.4 shows the usual symbols used in circuit diagrams. We will use these symbols extensively in this chapter and the next. We usually assume that the wires that connect the various elements of the circuit have negligible resistance; from Eq. (25.11), $V = IR$, the potential difference between the ends of such a wire is zero.

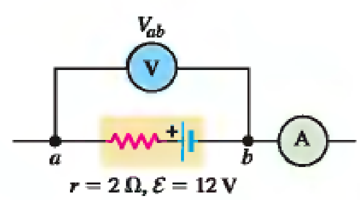
Table 25.4 includes two *meters* that are used to measure the properties of circuits. Idealized meters do not disturb the circuit in which they are connected. A **voltmeter,** introduced in Section 23.2, measures the potential difference between its terminals; an idealized voltmeter has infinitely large resistance and measures potential difference without having any current diverted through it. An ammeter measures the current passing through it; an idealized **ammeter** has zero resistance and has no potential difference between its terminals. Because meters act as part of the circuit in which they are connected, these properties are important to remember.

Table 25.4 Symbols for Circuit Diagrams

	Conductor with negligible resistance
	Resistor
	Source of emf (longer vertical line always represents the positive terminal, usually the terminal with higher potential)
	Source of emf with internal resistance r (r can be placed on either side)
	Voltmeter (measures potential difference between its terminals)
	Ammeter (measures current through it)

Conceptual Example 25.5 A source in an open circuit

Fig. 25.17 shows a source (a battery) with an emf $\mathcal{E}$ of 12 V and an internal resistance r of 2 Ω. (For comparison, the internal resistance of a commercial 12-V lead storage battery is only a few thousandths of an ohm.) The wires to the left of a and to the right of the ammeter A are not connected to anything. Determine the readings of the idealized voltmeter V and the idealized ammeter A.



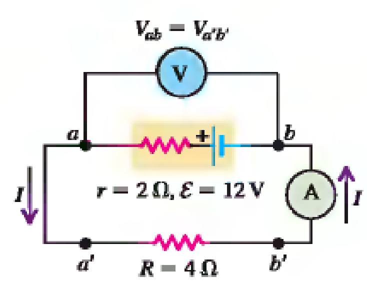
25.17 A source of emf in an open circuit.

SOLUTION

There is no current because there is no complete circuit. (There is no current through our idealized voltmeter, with its infinitely large resistance.) Hence the ammeter A reads $I = 0$. Because there is no current through the battery, there is no potential difference across its internal resistance. From Eq. (25.15) with $I = 0$, the potential difference V_{ab} across the battery terminals is equal to the emf. So the voltmeter reads $V_{ab} = \mathcal{E} = 12$ V. The terminal voltage of a real, nonideal source equals the emf *only* if there is no current flowing through the source, as in this example.

Example 25.6 A source in a complete circuit

Using the battery in Conceptual Example 25.5, we add a 4-Ω resistor to form the complete circuit shown in Fig. 25.18. What are the voltmeter and ammeter readings now?



25.18 A source of emf in a complete circuit.

SOLUTION

IDENTIFY Our first target variable is the current I through the circuit $aa'b'b$ (equal to the ammeter reading). The second is the potential difference V_{ab} (equal to the voltmeter reading).

SET UP: We find I using Eq. (25.16). To find V_{ab}, we note that we can regard this either as the potential difference across the source or as the potential difference around the circuit through the external resistor.

EXECUTE: The ideal ammeter has zero resistance, so the resistance external to the source is $R = 4\ \Omega$. From Eq. (25.16), the current through the circuit $aa'b'b$ is

$$I = \frac{\mathcal{E}}{R + r} = \frac{12\ \text{V}}{4\ \Omega + 2\Omega} = 2\ \text{A}$$

The ammeter A reads $I = 2$ A.

Our idealized conducting wires have zero resistance, and the idealized ammeter A also has zero resistance. So there is no potential difference between points a and a' or between points b and b'; that is, $V_{ab} = V_{a'b'}$. We can find V_{ab} by considering a and b either as the terminals of the resistor or as the terminals of the source. Considering them as terminals of the resistor, we use Ohm's law ($V = IR$):

$$V_{a'b'} = IR = (2\ \text{A})(4\ \Omega) = 8\ \text{V}$$

Considering them as the terminals of the source, we have

$$V_{ab} = \mathcal{E} - Ir = 12\ \text{V} - (2\ A)(2\ \Omega) = 8\ \text{V}$$

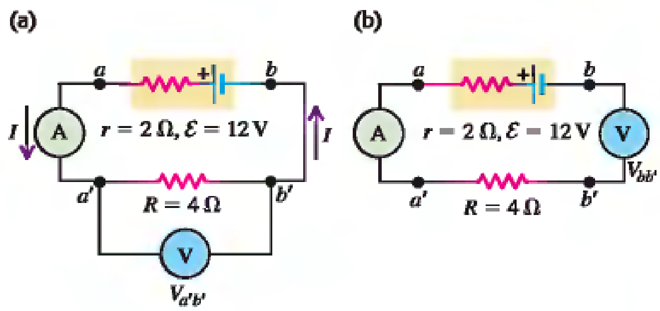
Either way, we conclude that the voltmeter reads $V_{ab} = 8$ V.

EVALUATE: With a current flowing through the source, the terminal voltage V_{ab} is less than the emf. The smaller the internal resistance r, the less the difference between V_{ab} and $\mathcal{E}$.

Conceptual Example 25.7 Using voltmeters and ammeters

The voltmeter and ammeter in Example 25.6 are moved to different positions in the circuit. What are the voltmeter and ammeter readings in the situations shown in (a) Fig. 25.19a and (b) Fig. 25.19b?

25.19 Different placements of a voltmeter and an ammeter in a complete circuit.



SOLUTION

(a) The voltmeter now measures the potential difference between points a' and b'. But as mentioned in Example 25.6, $V_{ab} = V_{a'b'}$, so the voltmeter reads the same as in Example 25.6: $V_{a'b'} = 8$ V.

CAUTION **Current in a simple loop** You might be tempted to conclude that the ammeter in Fig. 25.19a, which is located "upstream" of the resistor, would have a higher reading than the one located "downstream" of the resistor in Fig. 25.18. But this conclusion is based on the misconception that current is somehow "used up" as it moves through a resistor. As charges move through a resistor, there is a decrease in electric potential energy, but there is *no* change in the current. *The current in a simple loop is the same at every point.* An ammeter placed as in Fig. 25.19a reads the same as one placed as in Fig. 25.18: $I = 2$ A.

(b) There is no current through the voltmeter because it has infinitely large resistance. Since the voltmeter is now part of the circuit, there is no current at all in the circuit, and the ammeter reads $I = 0$.

The voltmeter measures the potential difference $V_{bb'}$ between points b and b'. Since $I = 0$, the potential difference across the resistor is $V_{a'b'} = IR = 0$, and the potential difference between the ends a and a' of the idealized ammeter is also zero. So $V_{bb'}$ is equal to V_{ab}, the terminal voltage of the source. As in Conceptual Example 25.5, there is no current flowing, so the terminal voltage equals the emf, and the voltmeter reading is $V_{ab} = \mathcal{E} = 12$ V.

This example shows that ammeters and voltmeters are circuit elements, too. Moving the voltmeter from the position in Fig. 25.19a to that in Fig. 25.19b changes the current and potential differences in the circuit—in this case rather dramatically. If you want to measure the potential difference between two points in a circuit without disturbing the circuit, use a voltmeter as in Figs. 25.18 or 25.19a, *not* as in Fig. 25.19b.

Example 25.8 A source with a short circuit

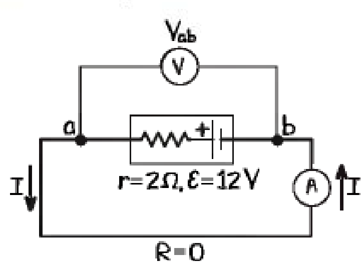
Using the same battery as in the preceding three examples, we now replace the 4-Ω resistor with a zero-resistance conductor. What are the meter readings now?

SOLUTION

IDENTIFY: Our target variables are I and V_{ab}, the same as in Example 25.6. The only difference from that example is that the external resistance is now $R = 0$.

SET UP: Figure 25.20 shows the new circuit. There is now a zero-resistance path between points a and b (through the lower loop in Fig. 25.20). Hence the potential difference between these points must be zero, which we can use to help solve the problem.

25.20 Our sketch for this problem.



Continued

EXECUTE: We must have $V_{ab} = IR = I(0) = 0$, no matter what the current. Knowing this, we can find the current I from Eq. (25.15):

$$V_{ab} = \mathcal{E} - Ir = 0$$

$$I = \frac{\mathcal{E}}{r} = \frac{12\text{ V}}{2\ \Omega} = 6\text{ A}$$

The ammeter reads $I = 6$ A and the voltmeter reads $V_{ab} = 0$.

EVALUATE: The current has a different value than in Example 25.6, even though the same battery is used. A source does *not* deliver the same current in all situations; the amount of current depends on the internal resistance r and on the resistance of the external circuit.

The situation in this example is called a *short circuit.* The terminals of the battery are connected directly to each other, with no external resistance. The short-circuit current is equal to the emf $\mathcal{E}$ divided by the internal resistance r. *Warning:* A short circuit can be an extremely dangerous situation. An automobile battery or a household power line has very small internal resistance (much less than in these examples), and the short-circuit current can be great enough to melt a small wire or cause a storage battery to explode. Don't try it!

Potential Changes Around a Circuit

The net change in potential energy for a charge q making a round trip around a complete circuit must be zero. Hence the net change in *potential* around the circuit must also be zero; in other words, the algebraic sum of the potential differences and emfs around the loop is zero. We can see this by rewriting Eq. (25.16) in the form

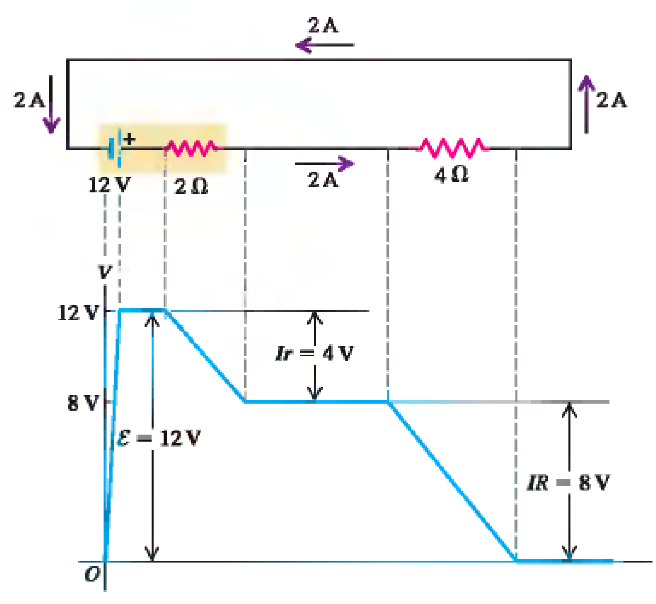
$$\mathcal{E} - Ir - IR = 0$$

A potential gain of $\mathcal{E}$ is associated with the emf, and potential drops of Ir and IR are associated with the internal resistance of the source and the external circuit, respectively. Fig. 25.21 is a graph showing how the potential varies as we go around the complete circuit of Fig. 25.18. The horizontal axis doesn't necessarily represent actual distances, but rather various points in the loop. If we take the potential to be zero at the negative terminal of the battery, then we have a rise $\mathcal{E}$ and a drop Ir in the battery and an additional drop IR in the external resistor, and as we finish our trip around the loop, the potential is back where it started.

In this section we have considered only situations in which the resistances are ohmic. If the circuit includes a nonlinear device such as a diode (see Fig. 25.10b), Eq. (25.16) is still valid but cannot be solved algebraically because R is not a constant. In such a situation, the current I can be found by using numerical techniques (see Challenge Problem 25.84).

Finally, we remark that Eq. (25.15) is not always an adequate representation of the behavior of a source. The emf may not be constant, and what we have

25.21 Potential rises and drops in a circuit.

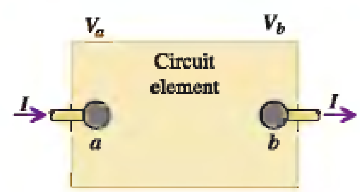


described as an internal resistance may actually be a more complex voltage–current relationship that doesn't obey Ohm's law. Nevertheless, the concept of internal resistance frequently provides an adequate description of batteries, generators, and other energy converters. The principal difference between a fresh flashlight battery and an old one is not in the emf, which decreases only slightly with use, but in the internal resistance, which may increase from less than an ohm when the battery is fresh to as much as 1000 Ω or more after long use. Similarly, a car battery can deliver less current to the starter motor on a cold morning than when the battery is warm, not because the emf is appreciably less but because the internal resistance increases with decreasing temperature. Cold-climate dwellers take a number of measures to avoid this loss, from using special battery warmers to soaking the battery in warm water on very cold mornings.

Test Your Understanding of Section 25.4 Rank the following circuits in order from highest to lowest current. (i) a 1.4-Ω resistor connected to a 1.5-V battery that has an internal resistance of 0.10 Ω; (ii) a 1.8-Ω resistor connected to a 4.0-V battery that has a terminal voltage of 3.6 V but an unknown internal resistance; (iii) an unknown resistor connected to a 12.0-V battery that has an internal resistance of 0.20 Ω and a terminal voltage of 11.0 V.

25.5 Energy and Power in Electric Circuits

Let's now look at some energy and power relationships in electric circuits. The box in Fig. 25.22 represents a circuit element with potential difference $V_a - V_b = V_{ab}$ between its terminals and current I passing through it in the direction from a toward b. This element might be a resistor, a battery, or something else; the details don't matter. As charge passes through the circuit element, the electric field does work on the charge. In a source of emf, additional work is done by the force $\vec{F}_n$ that we mentioned in Section 25.4.



25.22 The power input to the circuit element between a and b is $P = (V_a - V_b)I = V_{ab}I$.

As an amount of charge q passes through the circuit element, there is a change in potential energy equal to qV_{ab}. For example, if $q > 0$ and $V_{ab} = V_a - V_b$ is positive, potential energy decreases as the charge "falls" from potential V_a to lower potential V_b. The moving charges don't gain *kinetic* energy, because the rate of charge flow (that is, the current) out of the circuit element must be the same as the rate of charge flow into the element. Instead, the quantity qV_{ab} represents electrical energy transferred into the circuit element. This situation occurs in the coils of a toaster or electric oven, in which electrical energy is converted to thermal energy.

It may happen that the potential at b is higher than that at a. In this case V_{ab} is negative, and a net transfer of energy *out* of the circuit element occurs. The element then acts as a source, delivering electrical energy into the circuit to which it is attached. This is the usual situation for a battery, which converts chemical energy into electrical energy and delivers it to the external circuit. Thus qV_{ab} can denote either a quantity of energy delivered to a circuit element or a quantity of energy extracted from that element.

In electric circuits we are most often interested in the *rate* at which energy is either delivered to or extracted from a circuit element. If the current through the element is I, then in a time interval dt an amount of charge $dQ = I\,dt$ passes through the element. The potential energy change for this amount of charge is $V_{ab}\,dQ = V_{ab}I\,dt$. Dividing this expression by dt, we obtain the *rate* at which energy is transferred either into or out of the circuit element. The time rate of energy transfer is *power*, denoted by P, so we write

$$P = V_{ab}I \quad \text{(rate at which energy is delivered to or extracted from a circuit element)} \tag{25.17}$$

The unit of V_{ab} is one volt, or one joule per coulomb, and the unit of I is one ampere, or one coulomb per second. Hence the unit of $P = V_{ab}I$ is one watt, as it should be:

$$(1\ \text{J/C})(1\ \text{C/s}) = 1\ \text{J/s} = 1\ \text{W}$$

Let's consider a few special cases.

Power Inout to a Pure Resistance

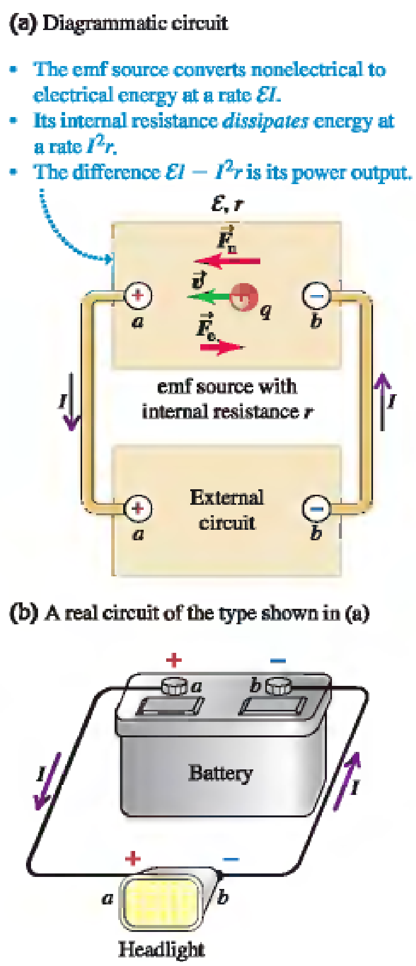
If the circuit element in Fig. 25.22 is a resistor, the potential difference is $V_{ab} = IR$. From Eq. (25.17) the electrical power delivered to the resistor by the circuit is

$$P = V_{ab}I = I^2R = \frac{V_{ab}{}^2}{R} \qquad \text{(power delivered to a resistor)} \qquad (25.18)$$

In this case the potential at a (where the current enters the resistor) is always higher than that at b (where the current exits). Current enters the higher-potential terminal of the device, and Eq. (25.18) represents the rate of transfer of electric potential energy *into* the circuit element.

What becomes of this energy? The moving charges collide with atoms in the resistor and transfer some of their energy to these atoms, increasing the internal energy of the material. Either the temperature of the resistor increases or there is a flow of heat out of it, or both. In any of these cases we say that energy is *dissipated* in the resistor at a rate I^2R. Every resistor has a *power rating,* the maximum power the device can dissipate without becoming overheated and damaged. In practical applications the power rating of a resistor is often just as important a characteristic as its resistance value. Of course, some devices, such as electric heaters, are designed to get hot and transfer heat to their surroundings. But if the power rating is exceeded, even such a device may melt or even explode.

25.23 Energy conversion in a simple circuit.



Power Output of a Source

The upper rectangle in Fig. 25.23a represents a source with emf $\mathcal{E}$ and internal resistance r, connected by ideal (resistanceless) conductors to an external circuit represented by the lower box. This could describe a car battery connected to one of the car's headlights (Fig. 25.23b). Point a is at higher potential than point b, so $V_a > V_b$ and V_{ab} is positive. Note that the current I is *leaving* the source at the higher-potential terminal (rather than entering there). Energy is being delivered to the external circuit, and the rate of its delivery to the circuit is given by Eq. (25.17):

$$P = V_{ab}I$$

For a source that can be described by an emf $\mathcal{E}$ and an internal resistance r, we may use Eq. (25.15):

$$V_{ab} = \mathcal{E} - Ir$$

Multiplying this equation by I, we find

$$P = V_{ab}I = \mathcal{E}I - I^2r \qquad (25.19)$$

What do the terms $\mathcal{E}I$ and I^2r mean? In Section 25.4 we defined the emf $\mathcal{E}$ as the work per unit charge performed on the charges by the nonelectrostatic force as the charges are pushed "uphill" from b to a in the source. In a time dt, a charge $dQ = I\,dt$ flows through the source; the work done on it by this nonelectrostatic force is $\mathcal{E}\,dQ = \mathcal{E}I\,dt$. Thus $\mathcal{E}I$ is the *rate* at which work is done on the circulating charges by whatever agency causes the nonelectrostatic force in the source. This term represents the rate of conversion of nonelectrical energy to electrical energy within the source. The term I^2r is the rate at which electrical energy is *dissipated* in the internal resistance of the source. The difference $\mathcal{E}I - I^2r$ is the *net* electrical power output of the source—that is, the rate at which the source delivers electrical energy to the remainder of the circuit.

Power Input to a Source

Suppose that the lower rectangle in Fig. 25.23a is itself a source, with an emf *larger* than that of the upper source and with its emf opposite to that of the upper source. Fig. 25.24 shows a practical example, an antomobile battery (the upper circuit element) being charged by the car's alternator (the lower element). The current I in the circuit is then *opposite* to that shown in Fig. 25.23; the lower source is pushing current backward through the upper source. Because of this reversal of current, instead of Eq. (25.15) we have for the upper source

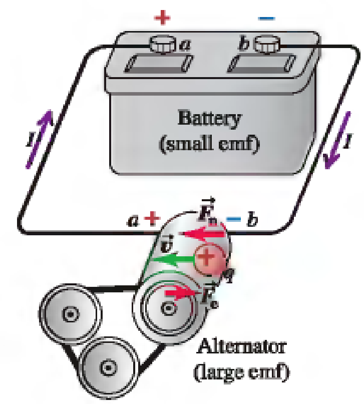
$$V_{ab} = \mathcal{E} + Ir$$

and instead of Eq. (25.19), we have

$$P = V_{ab}I = \mathcal{E}I + I^2R \qquad (25.20)$$

Work is being done *on*, rather than *by*, the agent that causes the nonelectrostatic force in the upper source. There is a conversion of electrical energy into nonelectrical energy in the upper source at a rate $\mathcal{E}I$. The term I^2r in Eq. (25.20) is again the rate of dissipation of energy in the internal resistance of the upper source, and the sum $\mathcal{E}I + I^2r$ is the total electrical power *input* to the upper source. This is what happens when a rechargeable battery (a storage battery) is connected to a charger. The charger supplies electrical energy to the battery; part of it is converted to chemical energy, to be reconverted later, and the remainder is dissipated (wasted) in the battery's internal resistance, warming the battery and causing a heat flow out of it. If you have a power tool or laptop computer with a rechargeable battery, you may have noticed that it gets warm while it is charging.

25.24 When two sources are connected in a simple loop circuit, the source with the larger emf delivers energy to the other source.



Problem-Solving Strategy 25.1 Power and Energy in Circuits

IDENTIFY *the relevant concepts:*
The ideas of electric power input and output can be applied to any electric circuit. In most cases you'll know when these concepts are needed because the problem will ask you explicitly to consider power or energy.

SET UP *the problem* using the following steps:
1. Make a drawing of the circuit.
2. Identify the circuit elements, including sources of emf and resistors. In later chapters we will add other kinds of circuit elements, including capacitors and inductors (described in Chapter 30).
3. Determine the target variables. Typically they will be the power input or output for each circuit element, or the total amount of energy put into or taken out of a circuit element in a given time.

EXECUTE *the solution* as follows:
1. A source of emf $\mathcal{E}$ delivers power $\mathcal{E}I$ into a circuit when the current I runs through the source from − to +. The energy is converted from chemical energy in a battery, from mechanical energy in a generator, or whatever. In this case the source has a *positive* power output to the circuit or, equivalently, a *negative* power input to the source.
2. A source of emf takes power $\mathcal{E}I$ from a circuit—that is, it has a negative power output or, equivalently, a positive power input—when current passes through the source in the direction from + to −. This occurs in charging a storage battery, when electrical energy is converted back to chemical energy. In this case the source has a *negative* power output to the circuit or, equivalently, a *positive* power input to the source.
3. No matter what the direction of the current through a resistor, there is always a *positive* power input to the resistor. It removes energy from a circuit at a rate given by $VI = I^2R = V^2/R$, where V is the potential difference across the resistor.
4. There is also a *positive* power input to the internal resistance r of a source, irrespective of the direction of the current. The internal resistance always removes energy from the circuit, converting it into heat at a rate I^2r.
5. You may need to calculate the total energy delivered to or extracted from a circuit element in a given amount of time. If the power into or out of a circuit element is constant, this integral is just the product of power and elapsed time. (In Chapter 26 we will encounter situations in which the power is not constant. In such cases, calculating the total energy requires an integral.)

EVALUATE *your answer:* Check your results, including a check that energy is conserved. This conservation can be expressed in either of two forms: "net power input = net power output" or "the algebraic sum of the power inputs to the circuit elements is zero."

Example 25.9 Power input and output in a complete circuit

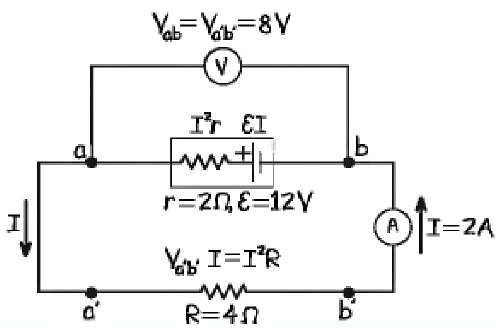
For the situation that we analyzed in Example 25.6, find the rate of energy conversion (chemical to electrical) and the rate of dissipation of energy in the battery and the net power output of the battery.

SOLUTION

IDENTIFY: Our target variables are the power output of the source of emf, the power input to the internal resistance, and the net power output of the source.

SET UP: Fig. 25.25 shows the circuit. We use Eq. (25.17) to find the power input or output of a circuit element and Eq. (25.19) to find the source's net power output.

25.25 Our sketch for this problem.



EXECUTE: From Example 25.6 the current in the circuit is $I = 2$ A. The rate of energy conversion in the battery is

$$\mathcal{E}I = (12\text{ V})(2\text{ A}) = 24\text{ W}$$

The rate of dissipation of energy in the battery is

$$I^2r = (2\text{ A})^2(2\,\Omega) = 8\text{ W}$$

The electrical power *output* of the source is the difference between these: $\mathcal{E}I - I^2r = 16$ W.

EVALUATE: The power output is also given by the terminal voltage $V_{ab} = 8$ V (calculated in Example 25.6) multiplied by the current:

$$V_{ab}I = (8\text{ V})(2\text{ A}) = 16\text{ W}$$

The electrical power input to the resistor is

$$V_{a'b'}I = (8\text{ V})(2\text{ A}) = 16\text{ W}$$

This equals the rate of dissipation of electrical energy in the resistor:

$$I^2R = (2\text{ A})^2(4\,\Omega) = 16\text{ W}$$

Note that our results agree with Eq. (25.19), which states that $V_{ab}I = \mathcal{E}I - I^2R$; the left side of this equation equals 16 W, and the right side equals 24 W − 8 W = 16 W. This verifies the consistency of the various power quantities.

Example 25.10 Increasing the resistance

Suppose the 4-Ω resistor in Fig. 25.25 is replaced by an 8-Ω resistor. How does this affect the electrical power dissipated in the resistor?

SOLUTION

IDENTIFY: Our target variable is the power dissipated in the resistor to which the source of emf is connected.

SET UP: The situation is the same as that in Example 25.9, but with a different value of the external resistance R.

EXECUTE: According to Eq. (25.18), the power dissipated in the resistor is given by $P = I^2R$. If you were in a hurry, you might conclude that since R now has twice the value that it had in Example 25.9, the power should also be twice as great, or $2(16\text{ W}) = 32$ W. Or you might instead try to use the formula $P = V_{ab}^2/R$; this formula would lead you to conclude that the power should be one-half as great as in the preceding example, or $(16\text{ W})/2 = 8$ W. Which answer is correct?

In fact, *both* of these conclusions are *incorrect*. The first is incorrect because changing the resistance R also changes the current in the circuit (remember, a source of emf does *not* generate the same current in all situations). The second conclusion is also incorrect because the potential difference V_{ab} across the resistor changes when the current changes. To get the correct answer, we first use the same technique as in Example 25.6 to find the current:

$$I = \frac{\mathcal{E}}{R + r} = \frac{12\text{ V}}{8\,\Omega + 2\,\Omega} = 1.2\text{ A}$$

The greater resistance causes the current to decrease. The potential difference across the resistor is

$$V_{ab} = IR = (1.2\text{ A})(8\,\Omega) = 9.6\text{ V}$$

which is greater than that with the 4-Ω resistor. We can then find the power dissipated in the resistor in either of two ways:

$$P = I^2R = (1.2\text{ A})^2(8\,\Omega) = 12\text{ W} \quad \text{or}$$

$$P = \frac{V_{ab}^2}{R} = \frac{(9.6\text{ V})^2}{8\,\Omega} = 12\text{ W}$$

EVALUATE: Increasing the resistance R causes a *reduction* in the power input to the resistor. In the expression $P = I^2R$ the decrease in current is more important than the increase in resistance; in the expression $P = V_{ab}^2/R$ the increase in resistance is more important than the increase in V_{ab}. This same principle applies to ordinary light bulbs; a 50-W light bulb has a greater resistance than does a 100-W light bulb.

Can you show that replacing the 4-Ω resistor with an 8-Ω resistor decreases both the rate of energy conversion (chemical to electrical) in the battery and the rate of energy dissipation in the battery?

Example 25.11 Power in a short circuit

For the circuit that we analyzed in Example 25.8, find the rates of energy conversion and energy dissipation in the battery and the net power output of the battery.

SOLUTION

IDENTIFY: Our target variables are again the power inputs and outputs associated with the battery.

SET UP: Fig. 25.26 shows the circuit. This is once again the same situation as in Example 25.9, but now the external resistance R is zero.

EXECUTE: We found in Example 25.8 that the current in this situation is $I = 6\ \text{A}$. The rate of energy conversion (chemical to electrical) in the battery is

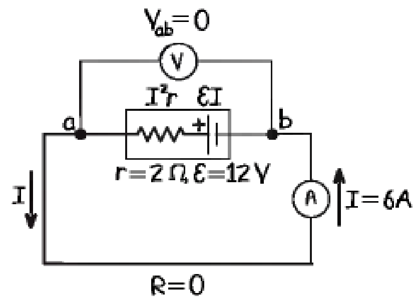
$$\mathcal{E}I = (12\ \text{V})(6\ \text{A}) = 72\ \text{W}$$

The rate of dissipation of energy in the battery is

$$I^2r = (6\ \text{A})^2(2\ \Omega) = 72\ \text{W}$$

The net power output of the source, given by $V_{ab}I$, is zero because the terminal voltage V_{ab} is zero.

25.26 Our sketch for this problem.



EVALUATE: With ideal wires and an ideal ammeter so that $R = 0$, *all* of the converted energy is dissipated within the source. This is why a short-circuited battery is quickly ruined and in some cases may even explode.

Test Your Understanding of Section 25.5 Rank the following circuits in order from highest to lowest values of the net power output of the battery. (i) a 1.4-Ω resistor connected to a 1.5-V battery that has an internal resistance of 0.10 Ω; (ii) a 1.8-Ω; resistor connected to a 4.0-V battery that has a terminal voltage of 3.6 V but an unknown internal resistance; (iii) an unknown resistor connected to a 12.0-V battery that has an internal resistance of 0.20 Ω; and a terminal voltage of 11.0 V.

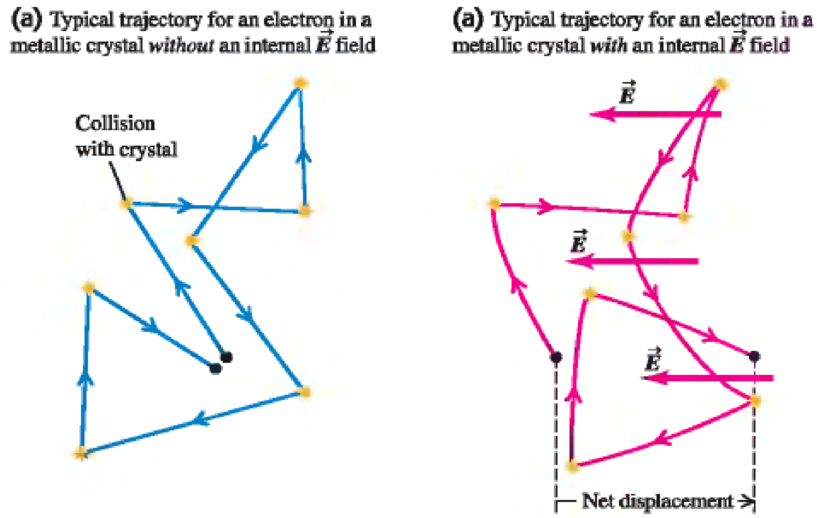
MP

*25.6 Theory of Metallic Conduction

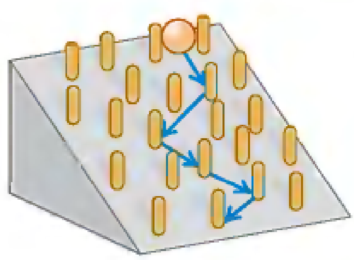
We can gain additional insight into electrical conduction by looking at the microscopic origin of conductivity. We'll consider a very simple model that treats the electrons as classical particles and ignores their quantum-mechanical, wavelike behavior in solids. Using this model, we'll derive an expression for the resistivity of a metal. Even though this model is not entirely correct conceptually, it will still help you to develop an intuitive idea of the microscopic basis of conduction.

In the simplest microscopic model of conduction in a metal, each atom in the metallic crystal gives up one or more of its outer electrons. These electrons are then free to move through the crystal, colliding at intervals with the stationary positive ions. The motion of the electrons is analogous to the motion of molecules of a gas moving through a porous bed of sand, and they are often referred to as an "electron gas."

If there is no electric field, the electrons move in straight lines between collisions, the directions of their velocities are random, and on average they never get anywhere (Fig. 25.27a). But if an electric field is present, the paths curve slightly because of the acceleration caused by electric-field forces. Figure 25.27b shows a few paths of an electron in an electric field directed from right to left. As we mentioned in Section 25.1, the average speed of random motion is of the order of 10^6 m/s, while the average drift speed is *much* slower, of the order of 10^{-4} m/s.



25.27 Random motions of an electron in a metallic crystal (a) with zero electric field and (b) with an electric field that causes drift. The curvatures of the paths are greatly exaggerated.



25.28 The motion of a ball rolling down an inclined plane and bouncing off pegs in its path is analogous to the motion of an electron in a metallic conductor with an electric field present.

The average time between collisions is called the **mean free time,** denoted by τ. Figure 25.28 shows a mechanical analog of this electron motion.

We would like to derive from this model an expression for the resistivity ρ of a material, defined by Eq. (25.5):

$$\rho = \frac{E}{J} \tag{25.21}$$

where E and J are the magnitudes of electric field and current density. The current density $\vec{J}$ is in turn given by Eq. (25.4):

$$\vec{J} = nq\vec{v}_d \tag{25.22}$$

where n is the number of free electrons per unit volume, q is the charge of each, and $\vec{v}_d$ is their average drift velocity. (We also know that $q = -e$ in an ordinary metal; we'll use that fact later.)

We need to relate the drift velocity $\vec{v}_d$ to the electric field $\vec{E}$. The value of $\vec{v}_d$ is determined by a steady-state condition in which, on average, the velocity *gains* of the charges due to the force of the $\vec{E}$ field are just balanced by the velocity *losses* due to collisions.

To clarify this process, let's imagine turning on the two effects one at a time. Suppose that before time $t = 0$ there is no field. The electron motion is then completely random. A typical electron has velocity $\vec{v}_0$ at time $t = 0$, and the value of $\vec{v}_0$ averaged over many electrons (that is, the initial velocity of an average electron) is zero: $(\vec{v}_0)_{av} = \mathbf{0}$. Then at time $t = 0$ we turn on a constant electric field $\vec{E}$. The field exerts a force $\vec{F} = q\vec{E}$ on each charge, and this causes an acceleration $\vec{a}$ in the direction of the force, given by

$$\vec{a} = \frac{\vec{F}}{m} = \frac{q\vec{E}}{m}$$

where m is the electron mass. *Every* electron has this acceleration.

We wait for a time τ, the average time between collisions, and then "turn on" the collisions. An electron that has velocity $\vec{v}_0$ at time $t = 0$ has a velocity at time $t = \tau$ equal to

$$\vec{v} = \vec{v}_0 + \vec{a}\tau$$

The velocity $\vec{v}_{av}$ of an *average* electron at this time is the sum of the averages of the two terms on the right. As we have pointed out, the initial velocity $\vec{v}_0$ is zero for an average electron, so

$$\vec{v}_{av} = \vec{a}\tau = \frac{q\tau}{m}\vec{E} \tag{25.23}$$

After time $t = \tau$, the tendency of the collisions to decrease the velocity of an average electron (by means of randomizing collisions) just balances the tendency of the $\vec{E}$ field to increase this velocity. Thus the velocity of an average electron, given by Eq. (25.23), is maintained over time and is equal to the drift velocity $\vec{v}_d$:

$$\vec{v}_d = \frac{q\tau}{m}\vec{E}$$

Now we substitute this equation for the drift velocity $\vec{v}_d$ into Eq. (25.22):

$$\vec{J} = nq\vec{v}_d = \frac{nq^2\tau}{m}\vec{E}$$

Comparing this with Eq. (25.21), which we can rewrite as $\vec{J} = \vec{E}/\rho$, and substituting $q = -e$, we see that the resistivity ρ is given by

$$\rho = \frac{m}{ne^2\tau} \tag{25.24}$$

If n and τ are independent of $\vec{E}$, then the resistivity is independent of $\vec{E}$ and the conducting material obeys Ohm's law.

Turning the interactions on one at a time may seem artificial. But the derivation would come out the same if each electron had its own clock and the $t = 0$ times were different for different electrons. If τ is the average time between collisions, then $\vec{v}_d$ is still the average electron drift velocity, even though the motions of the various electrons aren't actually correlated in the way we postulated.

What about the temperature dependence of resistivity? In a perfect crystal with no atoms out of place, a correct quantum-mechanical analysis would let the free electrons move through the crystal with no collisions at all. But the atoms vibrate about their equilibrium positions. As the temperature increases, the amplitudes of these vibrations increase, collisions become more frequent, and the mean free time τ decreases. So this theory predicts that the resistivity of a metal increases with temperature. In a superconductor, roughly speaking, there are no inelastic collisions, τ is infinite, and the resistivity ρ is zero.

In a pure semiconductor such as silicon or germanium, the number of charge carriers per unit volume, n, is not constant but increases very rapidly with increasing temperature. This increase in n far outweighs the decrease in the mean free time, and in a semiconductor the resistivity always decreases rapidly with increasing temperature. At low temperatures, n is very small, and the resistivity becomes so large that the material can be considered an insulator.

Electrons gain energy between collisions through the work done on them by the electric field. During collisions they transfer some of this energy to the atoms of the material of the conductor. This leads to an increase in the material's internal energy and temperature; that's why wires carrying current get warm. If the electric field in the material is large enough, an electron can gain enough energy between collisions to knock off electrons that are normally bound to atoms in the material. These can then knock off more electrons, and so on, possibly leading to an avalanche of current. This is the microscopic basis of dielectric breakdown in insulators.

Example 25.12 Mean free time in copper

Calculate the mean free time between collisions in copper at room temperature.

SOLUTION

IDENTIFY: This problem uses the ideas developed in this section.

SET UP: We can find an expression for mean free time τ in terms of n, ρ, e, and m by rearranging Eq. (25.24). From Example 25.1 and Table 25.1, for copper $n = 8.5 \times 10^{28}\ \text{m}^{-3}$ and $\rho = 1.72 \times 10^{-8}\ \Omega \cdot \text{m}$. Also, $e = 1.60 \times 10^{-19}\ \text{C}$ and $m = 9.11 \times 10^{-31}\ \text{kg}$ for electrons.

EXECUTE: From Eq. (25.24), we get

$$\tau = \frac{m}{ne^2\rho}$$
$$= \frac{9.11 \times 10^{-31}\ \text{kg}}{(8.5 \times 10^{28}\ \text{m}^{-3})(1.60 \times 10^{-19}\ \text{C})^2(1.72 \times 10^{-8}\ \Omega \cdot \text{m})}$$
$$= 2.4 \times 10^{-14}\ \text{s}$$

EVALUATE: Taking the reciprocal of this time, we find that each electron averages about 4×10^{13} collisions every second!

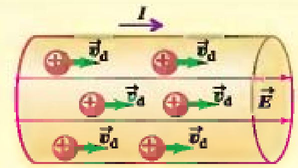
Test Your Understanding of Section 25.6 Which of the following factors will, if increased, make it more difficult to produce a certain amount of current in a conductor? (There may be more than one correct answer.) (i) the mass of the moving charged particles in the conductor; (ii) the number of moving charged particles per cubic meter; (iii) the amount of charge on each moving particle; (iv) the average time between collisions for a typical moving charged particle.

CHAPTER 25 SUMMARY

Current and current density: Current is the amount of charge flowing through a specified area, per unit time. The SI unit of current is the ampere, equal to one coulomb per second $(1\ \text{A} = 1\ \text{C/s})$. The current I through an area A depends on the concentration n and charge q of the charge carriers, as well as on the magnitude of their drift velocity $\vec{v}_d$. The current density is current per unit cross-sectional area. Current is conventionally described in terms of a flow of positive charge, even when the actual charge carriers are negative or of both signs. (See Example 25.1.)

$$I = \frac{dQ}{dt} = n|q|v_d A \qquad (25.2)$$

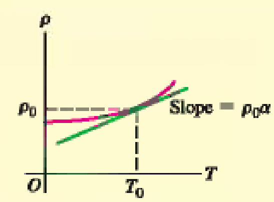
$$\vec{J} = nq\vec{v}_d \qquad (25.4)$$



Resistivity: The resistivity ρ of a material is the ratio of the magnitudes of electric field and current density. Good conductors have small resistivity; good insulators have large resistivity. Ohm's law, obeyed approximately by many materials, states that ρ is a constant independent of the value of E. Resistivity usually increases with temperature; for small temperature changes this variation is represented approximately by Eq. (25.6), where α is the temperature coefficient of resistivity.

$$\rho = \frac{E}{J} \qquad (25.5)$$

$$\rho(T) = \rho_0[1 + \alpha(T - T_0)] \qquad (25.6)$$

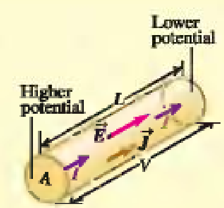


Metal: ρ increases with increasing T.

Resistors: For materials obeying Ohm's law, the potential difference V across a particular sample of material is proportional to the current I through the material. The ratio $V/I = R$ is the resistance of the sample. The SI unit of resistance is the ohm $(1\ \Omega = 1\ \text{V/A})$. The resistance of a cylindrical conductor is related to its resistivity ρ, length L, and cross-sectional area A. (See Examples 25.2–25.4.)

$$V = IR \qquad (25.11)$$

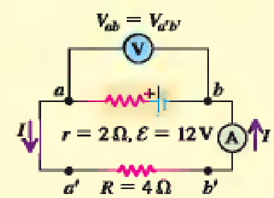
$$R = \frac{\rho L}{A} \qquad (25.10)$$



Circuits and emf: A complete circuit has a continuous current-carrying path. A complete circuit carrying a steady current must contain a source of electromotive force (emf) $\mathcal{E}$. The SI unit of electromotive force is the volt (1 V). An ideal source of emf maintains a constant potential difference, independent of current through the device, but every real source of emf has some internal resistance r. The terminal potential difference V_{ab} then depends on current. (See Examples 25.5–25.8.)

$$V_{ab} = \mathcal{E} - Ir \qquad (25.15)$$

(source with internal resistance)



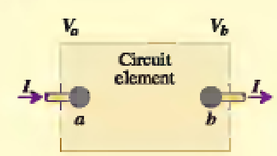
Energy and power in circuits: A circuit element with a potential difference $V_a - V_b = V_{ab}$ and a current I puts energy into a circuit if the current direction is from lower to higher potential in the device, and it takes energy out of the circuit if the current is opposite. The power P (rate of energy transfer) is equal to the product of the potential difference and the current. A resistor always takes electrical energy out of a circuit. (See Examples 25.9–25.11.)

$$P = V_{ab}I \qquad (25.17)$$

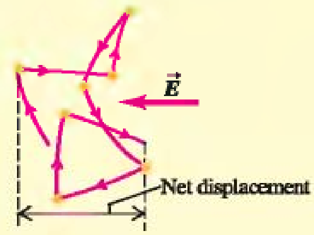
(general circuit element)

$$P = V_{ab}I = I^2R = \frac{V_{ab}^2}{R} \qquad (25.18)$$

(power into a resistor)



Conduction in metals: The microscopic basis of conduction in metals is the motion of electrons that move freely through the metallic crystal, bumping into ion cores in the crystal. In a crude classical model of this motion, the resistivity of the material can be related to the electron mass, charge, speed of random motion, density, and mean free time between collisions. (See Example 25.12.)



Key Terms

current, *847*
drift velocity, *847*
conventional current, *848*
ampere, *848*
concentration, *848*
current density, *849*
Ohm's law, *850*
resistivity, *851*
conductivity, *851*
temperature coefficient of resistivity, *852*
resistance, *853*
ohm, *854*
resistor, *854*
complete circuit, *857*
electromotive force (emf), *857*
source of emf, *857*
internal resistance, *859*
terminal voltage, *859*
voltmeter, *860*
ammeter, *860*
mean free time, *868*

Answer to Chapter Opening Question ?

The current out equals the current in. In other words, charge must enter the bulb at the same rate as it exits the bulb. It is not "used up" or consumed as it flows through the bulb.

Answers to Test Your Understanding Questions

25.1 Answer: (v) Doubling the diameter increases the cross-sectional area A by a factor of 4. Hence the current density magnitude $J = I/A$ is reduced to $\frac{1}{4}$ of the value in Example 25.1, and the magnitude of the drift velocity $v_d = J/n|q|$ is reduced by the same factor. The new magnitude is $v_d = (0.15\text{ mm/s})/4 = 0.038\text{ mm/s}$. This behavior is the same as that of an incompressible fluid, which slows down when it moves from a narrow pipe to a broader one (see Section 14.4).

25.2 Answer (ii) Figure 25.6b shows that the resistivity ρ of a semiconductor increases as the temperature decreases. From Eq. (25.5), the magnitude of the current density is $J = E/\rho$, so the current density decreases as the temperature drops and the resistivity increases.

25.3 Answer (iii) Solving Eq. (25.11) for the current shows that $I = V/R$. If the resistance R of the wire remained the same, doubling the voltage V would make the current I double as well. However, we saw in Example 25.3 that the resistance is *not* constant: As the current increases and the temperature increases, R increases as well. Thus doubling the voltage produces a current that is *less* than double the original current. An ohmic conductor is one for which $R = V/I$ has the same value no matter what the voltage, so the wire is *nonohmic*. (In many practical problems the temperature change of the wire is so small that it can be ignored, so we can safely regard the wire as being ohmic. We do so in almost all examples in this book.)

25.4 Answer: (iii), (ii), (i) For circuit (i), we find the current from Eq. (25.16): $I = \mathcal{E}/(R + r) = (1.5\text{ V})/(1.4\ \Omega + 0.10\ \Omega) = 1.0\text{ A}$. For circuit (ii), we note that the terminal voltage $V_{ab} = 3.6\text{ V}$ equals the voltage IR across the 1.8-Ω resistor: $V_{ab} = IR$, so $I = V_{ab}/R = (3.6\text{ V})/(1.8\ \Omega) = 2.0\text{ A}$. For circuit (iii), we use Eq. (25.15) for the terminal voltage: $V_{ab} = \mathcal{E} - Ir$, so $I = (\mathcal{E} - V_{ab})/r = (12.0\text{ V} - 11.0\text{ V})/(0.20\ \Omega) = 5.0\text{ A}$.

25.5 Answer: (iii), (ii), (i) These are the same circuits that we analyzed in Test Your Understanding of Section 25.4. In each case the net power output of the battery is $P = V_{ab}I$, where V_{ab} is the battery terminal voltage. For circuit (i), we found that $I = 1.0\text{ A}$, so $V_{ab} = \mathcal{E} - Ir = 1.5\text{ V} - (1.0\text{ A})(0.10\ \Omega) = 1.4\text{ V}$, so $P = (1.4\text{ V})(1.0\text{ A}) = 1.4\text{ W}$. For circuit (ii), we have $V_{ab} = 3.6\text{ V}$ and found that $I = 2.0\text{ A}$, so $P = (3.6\text{ V})(2.0\text{ A}) = 7.2\text{ W}$. For circuit (iii), we have $V_{ab} = 11.0\text{ V}$ and found that $I = 5.0\text{ A}$, so $P = (11.0\text{ V})(5.0\text{ A}) = 55\text{ A}$.

25.6 Answer: (i) The difficulty of producing a certain amount of current increases as the resistivity ρ increases. From Eq. (25.24), $\rho = m/ne^2\tau$, so increasing the mass m will increase the resistivity. That's because a more massive charged particle will respond more sluggishly to an applied electric field and hence drift more slowly. To produce the same current, a greater electric field would be needed. (Increasing n, e, or τ; would decrease the resistivity and make it easier to produce a given current.)

PROBLEMS

For instructor-assigned homework, go to **www.masteringphysics.com** MP

Discussion Questions

Q25.1. The definition of resistivity $(\rho = E/J)$ implies that an electric field exists inside a conductor. Yet we saw in Chapter 21 that there can be no electric field inside a conductor. Is there a contradiction here? Explain.

Q25.2. A cylindrical rod has resistance R. If we triple its length and diameter, what is its resistance, in terms of R?

Q25.3. A cylindrical rod has resistivity ρ. If we triple its length and diameter, what is its resistivity, in terms of ρ?

Q25.4. Two copper wires with different diameters are joined end to end. If a current flows in the wire combination, what happens to electrons when they move from the larger-diameter wire into the smaller-diameter wire? Does their drift speed increase, decrease, or stay the same? If the drift speed changes, what is the force that causes the change? Explain your reasoning.

Q25.5. When is a 1.5-V AAA battery *not* actually a 1.5-V battery? That is, when do its terminals provide a potential difference of less than 1.5 V?

Q25.6. Can the potential difference between the terminals of a battery ever be opposite in direction to the emf? If it can, give an example. If it cannot, explain why not.

Q25.7. A rule of thumb used to determine the internal resistance of a source is that it is the open-circuit voltage divided by the short-circuit current. Is this correct? Why or why not?

Q25.8. Batteries are always labeled with their emf; for instance, an AA flashlight battery is labeled "1.5 volts." Would it also be appropriate to put a label on batteries stating how much current they provide? Why or why not?

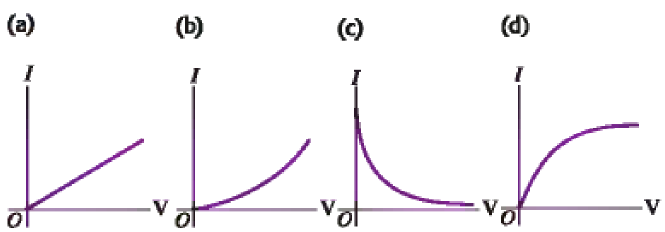
Q25.9. We have seen that a coulomb is an enormous amount of charge; it is virtually impossible to place a charge of 1 C on an object. Yet, a current of 10 A, 10 C/s, is quite reasonable. Explain this apparent discrepancy.

Q25.10. Electrons in an electric circuit pass through a resistor. The wire on either side of the resistor has the same diameter. (a) How does the drift speed of the electrons before entering the resistor compare to the speed after leaving the resistor? Explain your reasoning. (b) How does the potential energy for an electron before entering the resistor compare to the potential energy after leaving the resistor? Explain your reasoning.

Q25.11. Current causes the temperature of a real resistor to increase. Why? What effect does this heating have on the resistance? Explain.

Q25.12. Which of the graphs in Fig. 25.29 best illustrates the current I in a real resistor as a function of the potential difference V across it? Explain. (*Hint:* See Discussion Question Q25.11.)

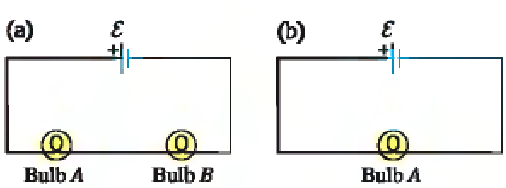
Figure 25.29 Question Q25.12.



Q25.13. Why does an electric light bulb nearly always burn out just as you turn on the light, almost never while the light is shining?

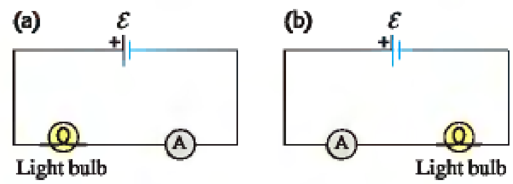
Q25.14. A light bulb glows because it has resistance. The brightness of a light bulb increases with the electrical power dissipated in the bulb. (a) In the circuit shown in Fig. 25.30a, the two bulbs A and B are identical. Compared to bulb A, does bulb B glow more brightly, just as brightly, or less brightly? Explain your reasoning. (b) Bulb B is removed from the circuit and the circuit is completed as shown in Fig. 25.30b. Compared to the brightness of bulb A in Fig. 25.30a, does bulb A now glow more brightly, just as brightly, or less brightly? Explain your reasoning.

Figure 25.30 Question Q25.14.



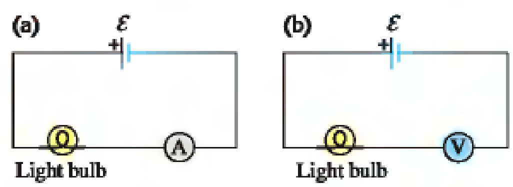
Q25.15. (See Discussion Question Q25.14.) An ideal ammeter A is placed in a circuit with a battery and a light bulb as shown in Fig. 25.31a, and the ammeter reading is noted. The circuit is then reconnected as in Fig. 25.31b, so that the positions of the ammeter and light bulb are reversed. (a) How does the ammeter reading in the situation shown in Fig. 25.31a compare to the reading in the situation shown in Fig. 25.31b? Explain your reasoning. (b) In which situation does the light bulb glow more brightly? Explain your reasoning.

Figure 25.31 Question Q25.15.



Q25.10. (See Discussion Question Q25.14.) Will a light bulb glow more brightly when it is connected to a battery as shown in Fig. 25.32a, in which an ideal ammeter A is placed in the circuit, or when it is connected as shown in Fig. 25.32b, in which an ideal voltmeter V is placed in the circuit? Explain your reasoning.

Figure 25.32 Question Q25.16.



Q25.17. The energy that can be extracted from a storage battery is always less than the energy that goes into it while it is being charged. Why?

Q25.10. Eight flashlight batteries in series have an emf of about 12 V, similar to that of a car battery. Could they be used to start a car with a dead battery? Why or why not?

Q25.19. Small aircraft often have 24-V electrical systems rather than the 12-V systems in automobiles, even though the electrical power requirements are roughly the same in both applications. The explanation given by aircraft designers is that a 24-V system weighs less than a 12-V system because thinner wires can be used. Explain why this is so.

Q25.20. Long-distance, electric-power, transmission lines always operate at very high voltage, sometimes as much as 750 kV. What are the advantages of such high voltages? What are the disadvantages?

Q25.21. Ordinary household electric lines in North America usually operate at 120 V. Why is this a desirable voltage, rather than a value considerably larger or smaller? On the other hand, automobiles usually have 12-V electrical systems. Why is this a desirable voltage?

Q25.22. A fuse is a device designed to break a circuit, usually by melting when the current exceeds a certain value. What characteristics should the material of the fuse have?

Q25.23. High-voltage power supplies are sometimes designed intentionally to have rather large internal resistance as a safety precaution. Why is such a power supply with a large internal resistance safer than a supply with the same voltage but lower internal resistance?

Q25.24. The text states that good thermal conductors are also good electrical conductors. If so, why don't the cords used to connect toasters, irons, and similar heat-producing appliances get hot by conduction of heat from the heating element?

Exercises

Section 25.1 Current

25.1. A current of 3.6 A flows through an automobile headlight. How many coulombs of charge flow through the headlight in 3.0 h?

25.2. A silver wire 2.6 mm in diameter transfers a charge of 420 C in 80 min. Silver contains 5.8×10^{28} free electrons per cubic meter. (a) What is the current in the wire? (b) What is the magnitude of the drift velocity of the electrons in the wire?

25.3. A 5.00-A current runs through a 12-gauge copper wire (diameter 2.05 mm) and through a light bulb. Copper has 8.5×10^{28} free electrons per cubic meter. (a) How many electrons pass through the light bulb each second? (b) What is the current density in the wire? (c) At what speed does a typical electron pass by any given point in the wire? (d) If you were to use wire of twice the diameter, which of the above answers would change? Would they increase or decrease?

25.4. An 18-gauge wire (diameter 1.02 mm) carries a current with a current density of $1.50 \times 10^6 \text{ A/m}^2$. Calculate (a) the current in the wire and (b) the drift velocity of electrons in the wire.

25.5. Copper has 8.5×10^{28} free electrons per cubic meter. A 71.0-cm length of 12-gauge copper wire that is 2.05 mm in diameter carries 4.85 A of current. (a) How much time does it take for an electron to travel the length of the wire? (b) Repeat part (a) for 6-gauge copper wire (diameter 4.12 mm) of the same length that carries the same current. (c) Generally speaking, how does changing the diameter of a wire that carries a given amount of current affect the drift velocity of the electrons in the wire?

25.6. Consider the 18-gauge wire in Example 25.1. How many atoms are in 1.00 m^3 of copper? With the density of free electrons given in the example, how many free electrons are there per copper atom?

25.7. The current in a wire varies with time according to the relationship $I = 55 \text{ A} - (0.65 \text{ A/s}^2)t^2$. (a) How many coulombs of charge pass a cross section of the wire in the time interval between $t = 0$ and $t = 8.0$ s? (b) What constant current would transport the same charge in the same time interval?

25.8. Current passes through a solution of sodium chloride. In 1.00 s, $2.68 \times 10^{16}\ Na^+$ ions arrive at the negative electrode and $3.92 \times 10^{16}\ Cl^-$ ions arrive at the positive electrode. (a) What is the current passing between the electrodes? (b) What is the direction of the current?

25.9. Assume that in silver metal there is one free electron per silver atom. Compute the free electron density for silver, and compare it to the value given in Exercise 25.2.

Section 25.2 Resistivity and Section 25.3 Resistance

25.10. (a) At room temperature what is the strength of the electric field in a 12-gauge copper wire (diameter 2.05 mm) that is needed to cause a 2.75-A current to flow? (b) What field would be needed if the wire were made of silver instead?

25.11. A 1.50-m cylindrical rod of diameter 0.500 cm is connected to a power supply that maintains a constant potential difference of 15.0 V across its ends, while an ammeter measures the current through it. You observe that at room temperature (20.0°C) the ammeter reads 18.5 A, while at 92.0°C it reads 17.2 A. You can ignore any thermal expansion of the rod. Find (a) the resistivity and (b) the temperature coefficient of resistivity at 20°C for the material of the rod.

25.12. A copper wire has a square cross section 2.3 mm on a side. The wire is 4.0 m long and carries a current of 3.6 A. The density of free electrons is $8.5 \times 10^{28}/\text{m}^3$. Find the magnitudes of (a) the current density in the wire and (b) the electric field in the wire. (c) How much time is required for an electron to travel the length of the wire?

25.13. In an experiment conducted at room temperature, a current of 0.820 A flows through a wire 3.26 mm in diameter. Find the magnitude of the electric field in the wire if the wire is made of (a) tungsten; and (b) aluminum.

25.14. A wire 6.50 m long with diameter of 2.05 mm has a resistance of 0.0290 Ω. What material is the wire most likely made of?

25.15. A cylindrical tungsten filament 15.0 cm long with a diameter of 1.00 mm is to be used in a machine for which the temperature will range from room temperature (20°C) up to 120°C. It will carry a current of 12.5 A at all temperatures (consult Tables 25.1 and 25.2). (a) What will be the maximum electric field in this filament, and (b) what will be its resistance with that field? (c) What will be the maximum potential drop over the full length of the filament?

25.16. What length of copper wire, 0.462 mm in diameter, has a resistance of 1.00 Ω?

25.17. In household wiring, copper wire 2.05 mm in diameter is often used. Find the resistance of a 24.0-m length of this wire.

25.18. What diameter must a copper wire have if its resistance is to be the same as that of an equal length of aluminum wire with diameter 3.26 mm?

25.19. You need to produce a set of cylindrical copper wires 3.50 m long that will have a resistance of 0.125 Ω each. What will be the mass of each of these wires?

25.20. A tightly coiled spring having 75 coils, each 3.50 cm in diameter, is made of insulated metal wire 3.25 mm in diameter. An ohmmeter connected across its opposite ends reads 1.74 Ω. What is the resistivity of the metal?

25.21. An aluminum cube has sides of length of 1.80 m. What is the resistance between two opposite faces of the cube?

25.22. A battery-powered light bulb has a tungsten filament. When the switch connecting the bulb to the battery is first turned on and the temperature of the bulb is 20°C, the current in the bulb is 0.860 A. After the bulb has been on for 30 s, the current is 0.220 A. What is then the temperature of the filament?

25.23. A rectangular solid of pure germanium measures 12 cm × 12 cm × 25 cm. Assuming that each of its faces is an equipotential surface, what is the resistance between opposite faces that are (a) farthest apart and (b) closest together?

25.24. You apply a potential difference of 4.50 V between the ends of a wire that is 2.50 m in length and 0.654 mm in radius. The resulting current through the wire is 17.6 A. What is the resistivity of the wire?

25.25. A current-carrying gold wire has diameter 0.84 mm. The electric field in the wire is 0.49 V/m. What are (a) the current carried by the wire; (b) the potential difference between two points in the wire 6.4 m apart; (c) the resistance of a 6.4-m length of this wire?

25.26. The potential difference between points in a wire 75.0 cm apart is 0.938 V when the current density is 4.40×10^7 A/m^2. What are (a) the magnitude of $\vec{E}$ in the wire and (b) the resistivity of the material of which the wire is made?

25.27. (a) What is the resistance of a Nichrome wire at 0.0°C if its resistance is 100.00 Ω at 11.5°C? (b) What is the resistance of a carbon rod at 25.8°C if its resistance is 0.0160 Ω at 0.0°C?

25.26. A carbon resistor is to be used as a thermometer. On a winter day when the temperature is 4.0°C, the resistance of the carbon resistor is 217.3 Ω. What is the temperature on a spring day when the resistance is 215.8 Ω? (Take the reference temperature T_0 to be 4.0°C.)

25.29. A strand of wire has resistance 5.60 μΩ. Find the net resistance of 120 such strands if they are (a) placed side by side to form a cable of the same length as a single strand, and (b) connected end to end to form a wire 120 times as long as a single strand.

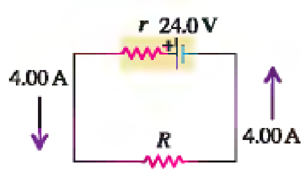
25.30. A hollow aluminum cylinder is 2.50 m long and has an inner radius of 3.20 cm and an outer radius of 4.60 cm. Treat each surface (inner, outer, and the two end faces) as an equipotential surface. At room temperature, what will an ohmmeter read if it is connected between (a) the opposite faces and (b) the inner and outer surfaces?

Section 25.4 Electromotive Force and Circuits

25.31. A copper transmission cable 100 km long and 10.0 cm in diameter carries a current of 125 A. (a) What is the potential drop across the cable? (b) How much electrical energy is dissipated as thermal energy every hour?

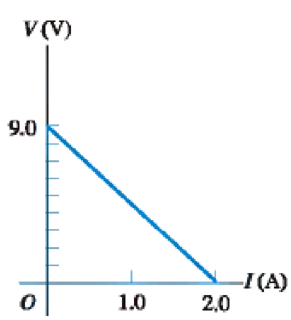
25.32. Consider the circuit shown in Fig. 25.33. The terminal voltage of the 24.0-V battery is 21.2 V. What are (a) the internal resistance r of the battery and (b) the resistance R of the circuit resistor?

Figure 25.33 Exercise 25.32.



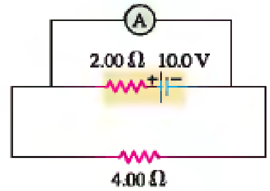
25.33. An idealized voltmeter is connected across the terminals of a battery while the current is varied. Figure 25.34 shows a graph of the voltmeter reading V as a function of the current I through the battery. Find (a) the emf $\mathcal{E}$ and (b) the internal resistance of the battery.

Figure 25.34 Exercise 25.33.



25.34. An idealized ammeter is connected to a battery as shown in Fig. 25.35. Find (a) the reading of the ammeter, (b) the current through the 4.00-Ω resistor, (c) the terminal voltage of the battery.

Figure 25.35 Exercise 25.34.

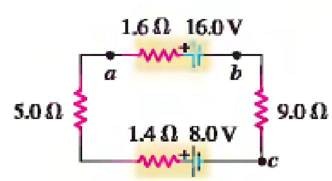


25.35. An ideal voltmeter V is connected to a 2.0-Ω resistor and a battery with emf 5.0 V and internal resistance 0.5 Ω as shown in Fig. 25.36. (a) What is the current in the 2.0-Ω resistor? (b) What is the terminal voltage of the battery? (c) What is the reading on the voltmeter? Explain your answers.

Figure 25.36 Exercise 25.35.

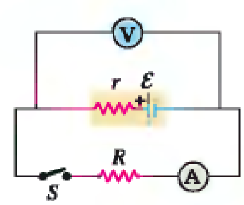
25.36. The circuit shown in Fig. 25.37 contains two batteries, each with an emf and an internal resistance, and two resistors. Find (a) the current in the circuit (magnitude *and* direction); (b) the terminal voltage V_{ab} of the 16.0-V battery; (c) the potential difference V_{ac} of point a with respect to point c. (d) Using Fig. 25.21 as a model, graph the potential rises and drops in this circuit.

Figure 25.37 Exercises 25.36, 25.38, 25.39, and 25.48.



25.37. When switch S in Fig. 25.38 is open, the voltmeter V of the battery reads 3.08 V. When the switch is closed, the voltmeter reading drops to 2.97 V, and the ammeter A reads 1.65 A. Find the emf, the internal resistance of the battery, and the circuit resistance R. Assume that the two meters are ideal, so they don't affect the circuit.

Figure 25.38 Exercise 25.37.



25.38. In the circuit of Fig. 25.37, the 5.0-Ω resistor is removed and replaced by a resistor of unknown resistance R. When this is done, an ideal voltmeter connected across the points b and c reads 1.9 V. Find (a) the current in the circuit and (b) the resistance R. (c) Graph the potential rises and drops in this circuit (see Fig. 25.21).

25.38. In the circuit shown in Fig. 25.37, the 16.0-V battery is removed and reinserted with the opposite polarity, so that its negative terminal is now next to point a. Find (a) the current in the circuit (magnitude *and* direction); (b) the terminal voltage V_{ba} of the 16.0-V battery; (c) the potential difference V_{ac} of point a with respect to point c. (d) Graph the potential rises and drops in this circuit (see Fig. 25.21).

25.40. The following measurements were made on a Thyrite resistor:

I(A)	0.50	1.00	2.00	4.00
V_{ab}(V)	2.55	3.11	3.77	4.58

(a) Graph V_{ab} as a function of I. (b) Does Thyrite obey Ohm's law? How can you tell? (c) Graph the resistance $R = V_{ab}/I$ as a function of I.

25.41. The following measurements of current and potential difference were made on a resistor constructed of Nichrome wire:

I(A)	0.50	1.00	2.00	4.00
V_{ab}(V)	1.94	3.88	7.76	15.52

(a) Graph V_{ab} as a function of I. (b) Does Nichrome obey Ohm's law? How can you tell? (c) What is the resistance of the resistor in ohms?

Section 25.5 Energy and Power in Electric Circuits

25.42. A resistor with a 15.0-V potential difference across its ends develops thermal energy at a rate of 327 W. (a) What is its resistance? (b) What is the current in the resistor?

25.43. Light Bulbs. The power rating of a light bulb (such as a 100-W bulb) is the power it dissipates when connected across a 120-V potential difference. What is the resistance of (a) a 100-W bulb and (b) a 60-W bulb? (c) How much current does each bulb draw in normal use?

25.44. If a "75-W" bulb (see Problem 25.43) is connected across a 220-V potential difference (as is used in Europe), how much power does it dissipate?

25.45. European Light Bulb. In Europe the standard voltage in homes is 220 V instead of the 120 V used in the United States. Therefore a "100-W" European bulb would be intended for use with a 220-V potential difference (see Problem 25.44). (a) If you bring a "100-W" European bulb home to the United States, what should be its U.S. power rating? (b) How much current will the 100-W European bulb draw in normal use in the United States?

25.49. A battery-powered global positioning system (GPS) receiver operating on 9.0 V draws a current of 0.13 A. How much electrical energy does it consume during 1.5 h?

25.47. Consider a resistor with length L, uniform cross-sectional area A, and uniform resistivity ρ that is carrying a current with uniform current density J. Use Eq. (25.18) to find the electrical power dissipated per unit volume, p. Express your result in terms of (a) E and J; (b) J and ρ; (c) E and ρ.

25.49. Consider the circuit of Fig. 25.37. (a) What is the total rate at which electrical energy is dissipated in the 5.00-Ω and 9.00-Ω resistors? (b) What is the power output of the 16.0-V battery? (c) At what rate is electrical energy being converted to other forms in the 8.0-V battery? (d) Show that the power output of the 16.0-V battery equals the overall rate of dissipation of electrical energy in the rest of the circuit.

25.49. The capacity of a storage battery, such as those used in automobile electrical systems, is rated in ampere-hours $(A \cdot h)$. A 50-A · h battery can supply a current of 50 A for 1.0 h, or 25 A for 2.0 h, and so on. (a) What total energy can be supplied by a 12-V, 60-A · h battery if its internal resistance is negligible? (b) What volume (in liters) of gasoline has a total heat of combustion equal to the energy obtained in part (a)? (See Section 17.6; the density of gasoline is 900 kg/m^3.) (c) If a generator with an average electrical power output of 0.45 kW is connected to the battery, how much time will be required for it to charge the battery fully?

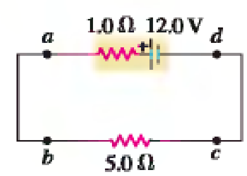
25.50. In the circuit analyzed in Example 25.9 the 4.0-Ω resistor is replaced by a 8.0-Ω resistor, as in Example 25.10. (a) Calculate the rate of conversion of chemical energy to electrical energy in the battery. How does your answer compare to the result calculated in Example 25.9? (b) Calculate the rate of electrical energy dissipation in the internal resistance of the battery. How does your answer compare to the result calculated in Example 25.9? (c) Use the results of parts (a) and (b) to calculate the net power output of the battery. How does your result compare to the electrical power dissipated in the 8.0-Ω resistor as calculated for this circuit in Example 25.10?

25.51. A 25.0-Ω bulb is connected across the terminals of a 12.0-V battery having 3.50 Ω of internal resistance. What percentage of the power of the battery is dissipated across the internal resistance and hence is not available to the bulb?

25.52. An idealized voltmeter is connected across the terminals of a 15.0-V battery, and a 75.0-Ω appliance is also connected across its terminals. If the voltmeter reads 11.3 V: (a) how much power is being dissipated by the appliance, and (b) what is the internal resistance of the battery?

25.53. In the circuit in Fig. 25.39, find (a) the rate of conversion of internal (chemical) energy to electrical energy within the battery; (b) the rate of dissipation of electrical energy in the battery; (c) the rate of dissipation of electrical energy in the external resistor.

Figure 25.39 Exercise 25.53.



25.54. A typical small flashlight contains two batteries, each having an emf of 1.5 V, connected in series with a bulb having resistance 17 Ω. (a) If the internal resistance of the batteries is negligible, what power is delivered to the bulb? (b) If the batteries last for 5.0 h, what is the total energy delivered to the bulb? (c) The resistance of real batteries increases as they run down. If the initial internal resistance is negligible, what is the combined internal resistance of both batteries when the power to the bulb has decreased to half its initial value? (Assume that the resistance of the bulb is constant. Actually, it will change somewhat when the current through the filament changes, because this changes the temperature of the filament and hence the resistivity of the filament wire.)

25.55. A "540-W" electric heater is designed to operate from 120-V lines. (a) What is its resistance? (b) What current does it draw? (c) If the line voltage drops to 110 V, what power does the heater take? (Assume that the resistance is constant. Actually, it will change because of the change in temperature.) (d) The heater coils are metallic, so that the resistance of the heater decreases with decreasing temperature. If the change of resistance with temperature is taken into account, will the electrical power consumed by the heater be larger or smaller than what you calculated in part (c)? Explain.

*Section 25.6 Theory of Metallic Conduction

***25.56.** Pure silicon contains approximately 1.0×10^{16} free electrons per cubic meter. (a) Referring to Table 25.1, calculate the mean free time τ for silicon at room temperature. (b) Your answer in part (a) is much greater than the mean free time for copper given in Example 25.12. Why, then, does pure silicon have such a high resistivity compared to copper?

Problems

25.57. An electrical conductor designed to carry large currents has a circular cross section 2.50 mm in diameter and is 14.0 m long. The resistance between its ends is 0.104 Ω. (a) What is the resistivity of the material? (b) If the electric-field magnitude in the conductor is 1.28 V/m, what is the total current? (c) If the material has 8.5×10^{28} free electrons per cubic meter, find the average drift speed under the conditions of part (b).

25.58. A plastic tube 25.0 m long and 4.00 cm in diameter is dipped into a silver solution, depositing a layer of silver 0.100 mm thick uniformly over the outer surface of the tube. If this coated tube is then connected across a 12.0-V battery, what will be the current?

25.59. On your first day at work as an electrical technician, you are asked to determine the resistance per meter of a long piece of wire. The company you work for is poorly equipped. You find a battery, a voltmeter, and an ammeter, but no meter for directly measuring resistance (an ohmmeter). You put the leads from the voltmeter across the terminals of the battery, and the meter reads 12.6 V. You cut off a 20.0-m length of wire and connect it to the battery, with an ammeter in series with it to measure the current in the wire. The ammeter reads 7.00 A. You then cut off a 40.0-m length of wire and connect it to the battery, again with the ammeter in series to measure the current. The ammeter reads 4.20 A. Even though the equipment you have available to you is limited, your boss assures you of its high quality: The ammeter has very small resistance, and the voltmeter has very large resistance. What is the resistance of 1 meter of wire?

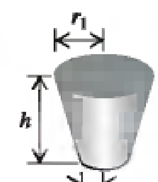
25.60. A 2.0-mm length of wire is made by welding the end of a 120-cm-long silver wire to the end of an 80-cm-long copper wire. Each piece of wire is 0.60 mm in diameter. The wire is at room temperature, so the resistivities are as given in Table 25.1. A potential difference of 5.0 V is maintained between the ends of the 2.0-m composite wire. (a) What is the current in the copper section? (b) What is the current in the silver section? (c) What is the magnitude of $\vec{E}$ in the copper? (d) What is the magnitude of $\vec{E}$ in the silver? (e) What is the potential difference between the ends of the silver section of wire?

25.61. A 3.00-m length of copper wire at 20°C has a 1.20-m-long section with diameter 1.60 mm and a 1.80-m-long section with diameter 0.80 mm. There is a current of 2.5 mA in the 1.60-mm-diameter section. (a) What is the current in the 0.80-mm-diameter section? (b) What is the magnitude of $\vec{E}$ in the 1.60-mm-diameter section? (c) What is the magnitude of $\vec{E}$ in the 0.80-mm-diameter section? (d) What is the potential difference between the ends of the 3.00-m length of wire?

25.62. Critical Current Density in Superconductors. One problem with some of the newer high-temperature superconductors is getting a large enough current density for practical use without causing the resistance to reappear. The maximum current density for which the material will remain a superconductor is called the critical current density of the material. In 1987, IBM research labs had produced thin films with critical current densities of 1.0×10^5 A/cm^2. (a) How much current could an 18-gauge wire (see Example 25.1 in Section 25.1) of this material carry and still remain superconducting? (b) Researchers are trying to develop superconductors with critical current densities of 1.0×10^6 A/cm^2. What diameter cylindrical wire of such a material would be needed to carry 1000 A without losing its superconductivity?

25.63. A material of resistivity ρ is formed into a solid, truncated cone of height h and radii r_1 and r_2 at either end (Fig. 25.40). (a) Calculate the resistance of the cone between the two flat end faces. (*Hint:* Imagine slicing the cone into very many thin disks, and calculate the resistance of one such disk.) (b) Show that your result agrees with Eq. (25.10) when $r_1 = r_2$.

Figure 25.40 Problem 25.63.



25.64. The region between two concentric conducting spheres with radii a and b is filled with a conducting material with resistivity ρ. (a) Show that the resistance between the spheres is given by

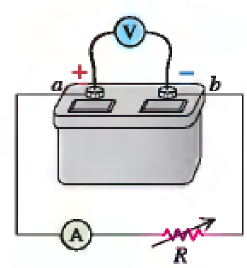
$$R = \frac{\rho}{4\pi}\left(\frac{1}{a} - \frac{1}{b}\right)$$

(b) Derive an expression for the current density as a function of radius, in terms of the potential difference V_{ab} between the spheres. (c) Show that the result in part (a) reduces to Eq. (25.10) when the separation $L = b - a$ between the spheres is small.

25.65. Leakage in a Dielectric. Two parallel plates of a capacitor have equal and opposite charges Q. The dielectric has a dielectric constant K and a resistivity ρ. Show that the "leakage" current I carried by the dielectric is given by $I = Q/K\epsilon_0\rho$.

25.66. In the circuit shown in Fig. 25.41, R is a variable resistor whose value can range from 0 to ∞, and a and b are the terminals of a battery having an emf $\mathcal{E} = 15.0$ V and an internal resistance of 4.00 Ω. The ammeter and voltmeter are both idealized meters. As R varies over its full range of values, what will be the largest and smallest readings of (a) the voltmeter and (b) the ammeter? (c) Sketch qualitative graphs of the readings of both meters as functions of R, as R ranges from 0 to ∞.

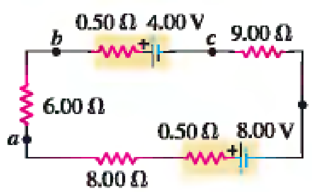
Figure 25.41 Problem 25.66.



25.67. The temperature coefficient of resistance α in Eq. (25.12) equals the temperature coefficient of resistivity α in Eq. (25.6) only if the coefficient of thermal expansion is small. A cylindrical column of mercury is in a vertical glass tube. At 20°C, the length of the mercury column is 12.0 cm. The diameter of the mercury column is 1.6 mm and doesn't change with temperature because glass has a small coefficient of thermal expansion. The coefficient of volume expansion of the mercury is given in Table 17.2, its resistivity at 20°C is given in Table 25.1, and its temperature coefficient of resistivity is given in Table 25.2. (a) At 20°C, what is the resistance between the ends of the mercury column? (b) The mercury column is heated to 60°C. What is the change in its resistivity? (c) What is the change in its length? Explain why the coefficient of volume expansion, rather than the coefficient of linear expansion, determines the change in length. (d) What is the change in its resistance? (*Hint:* Since the percentage changes in ρ and L are small, you may find it helpful to derive from Eq. (25.10) an equation for ΔR in terms of $\Delta\rho$ and ΔL.) (e) What is the temperature coefficient of resistance α for the mercury column, as defined in Eq. (25.12)? How does this value compare with the temperature coefficient of resistivity? Is the effect of the change in length important?

25.68. (a) What is the potential difference V_{ad} in the circuit of Fig. 25.42? (b) What is the terminal voltage of the 4.00-V battery? (c) A battery with emf 10.30z V and internal resistance 0.50 Ω is inserted in the circuit at d, with its negative terminal connected to the negative terminal of the 8.00-V battery. What is the difference of potential V_{bc} between the terminals of the 4.00-V battery now?

Figure 25.42 Problem 25.68.



25.68. The potential difference across the terminals of a battery is 8.4 V when there is a current of 1.50 A in the battery from the negative to the positive terminal. When the current is 3.50 A in the reverse direction, the potential difference becomes 9.4 V. (a) What is the internal resistance of the battery? (b) What is the emf of the battery?

25.70. A person with body resistance between his hands of 10 kΩ accidentally grasps the terminals of a 14-kV power supply. (a) If the internal resistance of the power supply is 2000 Ω, what is the current through the person's body? (b) What is the power dissipated in his body? (c) If the power supply is to be made safe by increasing its internal resistance, what should the internal resistance be for the maximum current in the above situation to be 1.00 mA or less?

25.71. The average bulk resistivity of the human body (apart from surface resistance of the skin) is about 5.0 Ω · m. The conducting path between the hands can be represented approximately as a cylinder 1.6 m long and 0.10 m in diameter. The skin resistance can be made negligible by soaking the hands in salt water. (a) What is the resistance between the hands if the skin resistance is negligible? (b) What potential difference between the hands is needed for a lethal shock current of 100 mA? (Note that your result shows that small potential differences produce dangerous currents when the skin is damp.) (c) With the current in part (b), what power is dissipated in the body?

25.72. A typical cost for electric power is 12.0¢ per kilowatt-hour. (a) Some people leave their porch light on all the time. What is the yearly cost to keep a 75-W bulb burning day and night? (b) Suppose your refrigerator uses 400 W of power when it's running, and it runs 8 hours a day. What is the yearly cost of operating your refrigerator?

25.73. A 12.6-V car battery with negligible internal resistance is connected to a series combination of a 3.2-Ω resistor that obeys Ohm's law and a thermistor that does not obey Ohm's law but instead has a current–voltage relationship $V = \alpha I + \beta I^2$, with $\alpha = 3.8\ \Omega$ and $\beta = 1.3\ \Omega/\text{A}$. What is the current through the 3.2-Ω resistor?

25.74. A cylindrical copper cable 1.50 km long is connected across a 220.0-V potential difference. (a) What should be its diameter so that it produces heat at a rate of 50.0 W? (b) What is the electric field inside the cable under these conditions?

25.75. A Nonideal Ammeter. Unlike the idealized ammeter described in Section 25.4, any real ammeter has a nonzero resistance. (a) An ammeter with resistance R_A is connected in series with a resistor R and a battery of emf $\mathcal{E}$ and internal resistance r. The current measured by the ammeter is I_A. Find the current through the circuit if the ammeter is removed so that the battery and the resistor form a complete circuit. Express your answer in terms of I_A, r, R_A, and R. The more "ideal" the ammeter, the smaller the difference between this current and the current I_A. (b) If $R = 3.80\ \Omega$, $\mathcal{E} = 7.50$ V, and $r = 0.45\ \Omega$, find the maximum value of the ammeter resistance R_A so that I_A is within 1.0% of the current in the circuit when the ammeter is absent. (c) Explain why your answer in part (b) represents a *maximum* value.

25.76. A 1.50-m cylinder of radius 1.10 cm is made of a complicated mixture of materials. Its resistivity depends on the distance x from the left end and obeys the formula $\rho(x) = a + bx^2$, where a and b are constants. At the left end, the resistivity is $2.25 \times 10^{-8}\ \Omega \cdot \text{m}$, while at the right end it is $8.50 \times 10^{-8}\ \Omega \cdot \text{m}$. (a) What is the resistance of this rod? (b) What is the electric field at its midpoint if it carries a 1.75-A current? (c) If we cut the rod into two 75.0-cm halves, what is the resistance of each half?

25.77. According to the U.S. National Electrical Code, copper wire used for interior wiring of houses, hotels, office buildings, and industrial plants is permitted to carry no more than a specified maximum amount of current. The table below shows the maximum current I_{max} for several common sizes of wire with varnished cambric insulation. The "wire gauge" is a standard used to describe the diameter of wires. Note that the larger the diameter of the wire, the *smaller* the wire gauge.

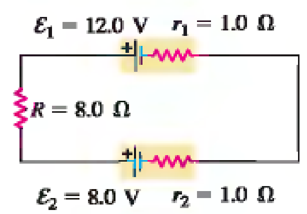
Wire gauge	Diameter (cm)	I_{max} (A)
14	0.163	18
12	0.205	25
10	0.259	30
8	0.326	40
6	0.412	60
5	0.462	65
4	0.519	85

(a) What considerations determine the maximum current-carrying capacity of household wiring? (b) A total of 4200 W of power is to be supplied through the wires of a house to the household electrical appliances. If the potential difference across the group of appliances is 120 V, determine the gauge of the thinnest permissible wire that can be used. (c) Suppose the wire used in this house is of the gauge found in part (b) and has total length 42.0 m. At what rate is energy dissipated in the wires? (d) The house is built in a community where the consumer cost of electric energy is $0.11 per kilowatt-hour. If the house were built with wire of the next larger diameter than that found in part (b), what would be the savings in electricity costs in one year? Assume that the appliances are kept on for an average of 12 hours a day.

25.78. A toaster using a Nichrome heating element operates on 120 V. When it is switched on at 20°C, the heating element carries an initial current of 1.35 A. A few seconds later the current reaches the steady value of 1.23 A. (a) What is the final temperature of the element? The average value of the temperature coefficient of resistivity for Nichrome over the temperature range is $4.5 \times 10^{-4}\ (\mathrm{C}°)^{-1}$. (b) What is the power dissipated in the heating element initially and when the current reaches a steady value?

25.79. In the circuit of Fig. 25.43, find (a) the current through the 8.0-Ω resistor and (b) the total rate of dissipation of electrical energy in the 8.0-Ω resistor and in the internal resistance of the batteries. (c) In one of the batteries, chemical energy is being converted into electrical energy. In which one is this happening, and at what rate? (d) In one of the batteries, electrical energy is being converted into chemical energy. In which one is this happening, and at what rate? (e) Show that the overall rate of production of electrical energy equals the overall rate of consumption of electrical energy in the circuit.

Figure 25.43 Problem 25.79.



25.80. A lightning bolt strikes one end of a steel lightning rod, producing a 15,000-A current burst that lasts for 65 μs. The rod is 2.0 m long and 1.8 cm in diameter, and its other end is connected to the ground by 35 m of 8.0-mm-diameter copper wire. (a) Find the potential difference between the top of the steel rod and the lower end of the copper wire during the current burst. (b) Find the total energy deposited in the rod and wire by the current burst.

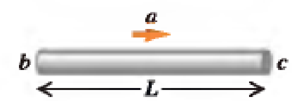
25.81. A 12.0-V battery has an internal resistance of 0.24 Ω and a capacity of 50.0 A · h (see Exercise 25.49). The battery is charged by passing a 10-A current through it for 5.0 h. (a) What is the terminal voltage during charging? (b) What total electrical energy is supplied to the battery during charging? (c) What electrical energy is dissipated in the internal resistance during charging? (d) The battery is now completely discharged through a resistor, again with a constant current of 10 A. What is the external circuit resistance? (e) What total electrical energy is supplied to the external resistor? (f) What total electrical energy is dissipated in the internal resistance? (g) Why are the answers to parts (b) and (e) not the same?

25.82. Repeat Problem 25.81 with charge and discharge currents of 30 A. The charging and discharging times will now be 1.7 h rather than 5.0 h. What differences in performance do you see?

Challenge Problems

25.83. The *Tolman-Stewart experiment* in 1916 demonstrated that the free charges in a metal have negative charge and provided a quantitative measurement of their charge-to-mass ratio, $|q|/m$. The experiment consisted of abruptly stopping a rapidly rotating spool of wire and measuring the potential difference that this produced between the ends of the wire. In a simplified model of this experiment, consider a metal rod of length L that is given a uniform acceleration $\vec{a}$ to the right. Initially the free charges in the metal lag behind the rod's motion, thus setting up an electric field $\vec{E}$ in the rod. In the steady state this field exerts a force on the free charges that makes them accelerate along with the rod. (a) Apply $\Sigma\vec{F} = m\vec{a}$ to the free charges to obtain an expression for $|q|/m$ in terms of the magnitudes of the induced electric field $\vec{E}$ and the acceleration $\vec{a}$. (b) If all the free charges in the metal rod have the same acceleration, the electric field $\vec{E}$ is the same at all points in the rod. Use this fact to rewrite the expression for $|q|/m$ in terms of the potential V_{bc} between the ends of the rod (Fig. 25.44). (c) If the free charges have negative charge, which end of the rod, b or c, is at higher potential? (d) If the rod is 0.50 m long and the free charges are electrons (charge $q = -1.60 \times 10^{-19}$ C, mass 9.11×10^{-31} kg), what magnitude of acceleration is required to produce a potential difference of 1.0 mV between the ends of the rod? (e) Discuss why the actual experiment used a rotating spool of thin wire rather than a moving bar as in our simplified analysis.

Figure 25.44 Challenge Problem 25.83.

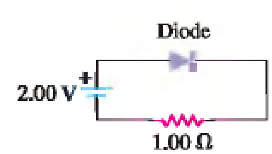


25.84. The current–voltage relationship of a semiconductor diode is given by

$$I = I_S\left[\exp\left(\frac{eV}{kT}\right) - 1\right]$$

where I and V are the current through and the voltage across the diode, respectively. I_S is a constant characteristic of the device, e is the magnitude of the electron charge, k is the Boltzmann constant, and T is the Kelvin temperature. Such a diode is connected in series with a resistor with $R = 1.00\ \Omega$ and a battery with $\mathcal{E} = 2.00$ V. The polarity of the battery is such that the current through the diode is in the forward direction (Fig. 25.45). The battery has negligible internal resistance. (a) Obtain an equation for V. Note that you cannot solve for V algebraically. (b) The value of V must be obtained by using a numerical method. One approach is to try a value of V, see how the left- and right-hand sides of the equation compare for this V, and use this to refine your guess for V. Using $I_S = 1.50$ mA and $T = 293$ K, obtain a solution (accurate to three significant figures) for the voltage drop V across the diode and the current I through it.

Figure 25.45 Challenge Problem 25.84.



25.85. The resistivity of a semiconductor can be modified by adding different amounts of impurities. A rod of semiconducting material of length L and cross-sectional area A lies along the x-axis between $x = 0$ and $x = L$. The material obeys Ohm's law, and its resistivity varies along the rod according to $\rho(x) = \rho_0 \exp(-x/L)$. The end of the rod at $x = 0$ is at a potential V_0 greater than the end at $x = L$. (a) Find the total resistance of the rod and the current in the rod. (b) Find the electric-field magnitude

$E(x)$ in the rod as a function of x. (c) Find the electric potential $V(x)$ in the rod as a function of x. (d) Graph the functions $\rho(x)$, $E(x)$, and $V(x)$ for values of x between $x = 0$ and $x = L$.

25.86. A source with emf $\mathcal{E}$ and internal resistance r is connected to an external circuit. (a) Show that the power output of the source is maximum when the current in the circuit is one-half the short-circuit current of the source. (b) If the external circuit consists of a resistance R, show that the power output is maximum when $R = r$ and that the maximum power is $\mathcal{E}^2/4r$.

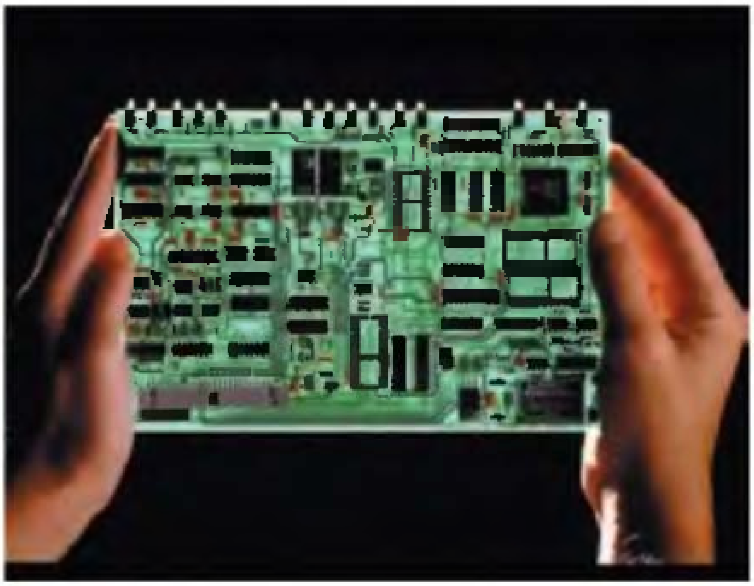
25.87. The temperature coefficient of resistivity α is given by

$$\alpha = \frac{1}{\rho}\frac{d\rho}{dT}$$

where ρ is the resistivity at the temperature T. Equation (25.6) then follows if α is assumed constant and much smaller than $(T - T_0)^{-1}$. (a) If α is not constant but is given by $\alpha = -n/T$, where T is the Kelvin temperature and n is a constant, show that the resistivity is given by $\rho = a/T^n$, where a is a constant. (b) From Fig. 25.10, you can see that such a relationship might be used as a rough approximation for a semiconductor. Using the values of ρ and α for carbon from Tables 25.1 and 25.2, determine a and n. (In Table 25.1, assume that "room temperature" means 293 K). (c) Using your result from part (b), determine the resistivity of carbon at $-196°C$ and $300°C$. (Remember to express T in kelvins.)

DIRECT-CURRENT CIRCUITS

26



? In a complex circuit like the one on this circuit board, is it possible to connect several resistors with different resistances so that they all have the same potential difference? If so, will the current be the same through all of the resistors?

LEARNING GOALS

By studying this chapter, you will learn:

- How to analyze circuits with multiple resistors in series or parallel.
- Rules that you can apply to any circuit with more than one loop.
- How to use an ammeter, voltmeter, ohmmeter, or potentiometer in a circuit.
- How to analyze circuits that include both a resistor and a capacitor.
- How electric power is distributed in the home.

If you look inside your TV, your computer, or your stereo receiver or under the hood of a car, you will find circuits of much greater complexity than the simple circuits we studied in Chapter 25. Whether connected by wires or integrated in a semiconductor chip, these circuits often include several sources, resistors, and other circuit elements, such as capacitors, transformers, and motors, interconnected in a *network.*

In this chapter we study general methods for analyzing such networks, including how to find unknown voltages, currents, and properties of circuit elements. We'll learn how to determine the equivalent resistance for several resistors connected in series or in parallel. For more general networks we need two rules called *Kirchhoff's rules.* One is based on the principle of conservation of charge applied to a junction; the other is derived from energy conservation for a charge moving around a closed loop. We'll discuss instruments for measuring various electrical quantities. We also look at a circuit containing resistance and capacitance, in which the current varies with time.

Our principal concern in this chapter is with **direct-current** (dc) circuits, in which the direction of the current does not change with time. Flashlights and automobile wiring systems are examples of direct-current circuits. Household electrical power is supplied in the form of **alternating current** (ac), in which the current oscillates back and forth. The same principles for analyzing networks apply to both kinds of circuits, and we conclude this chapter with a look at household wiring systems. We'll discuss alternating-current circuits in detail in Chapter 31.

26.1 Resistors in Series and Parallel

Resistors turn up in all kinds of circuits, ranging from hair dryers and space heaters to circuits that limit or divide current or reduce or divide a voltage. Such circuits often contain several resistors, so it's appropriate to consider *combinations* of resistors. A simple example is a string of light bulbs used for holiday decorations;

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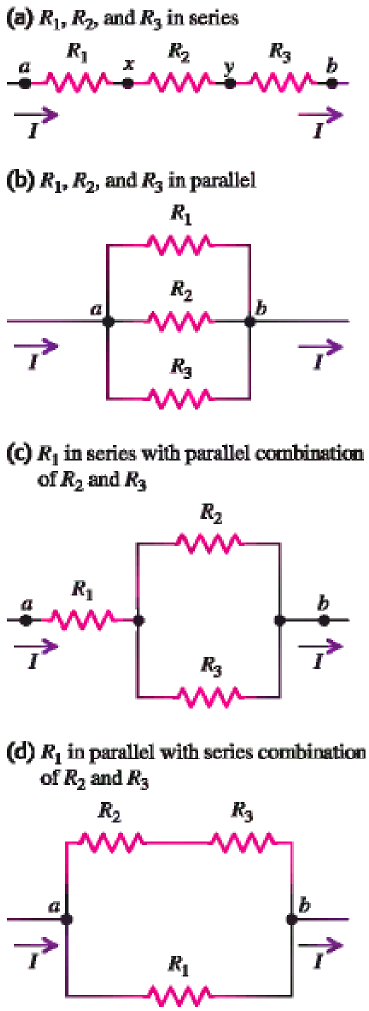
12.1 DC Series Circuits (Qualitative)

26.1 Four different ways of connecting three resistors.



each bulb acts as a resistor, and from a circuit-analysis perspective the string of bulbs is simply a combination of resistors.

Suppose we have three resistors with resistances R_1, R_2, and R_3. Figure 26.1 shows four different ways in which they might be connected between points a and b. When several circuit elements such as resistors, batteries, and motors are connected in sequence as in Fig. 26.1a, with only a single current path between the points, we say that they are connected in **series.** We studied *capacitors* in series in Section 24.2; we found that, because of conservation of charge, capacitors in series all have the same charge if they are initially uncharged. In circuits we're often more interested in the *current,* which is charge flow per unit time.

The resistors in Fig. 26.1b are said to be connected in **parallel** between points a and b. Each resistor provides an alternative path between the points. For circuit elements that are connected in parallel, the *potential difference* is the same across each element. We studied capacitors in parallel in Section 24.2.

In Fig. 26.1c, resistors R_2 and R_3 are in parallel, and this combination is in series with R_1. In Fig. 26.1d, R_2 and R_3 are in series, and this combination is in parallel with R_1.

For any combination of resistors we can always find a *single* resistor that could replace the combination and result in the same total current and potential difference. For example, a string of holiday light bulbs could be replaced by a single, appropriately chosen light bulb that would draw the same current and have the same potential difference between its terminals as the original string of bulbs. The resistance of this single resistor is called the **equivalent resistance** of the combination. If any one of the networks in Fig. 26.1 were replaced by its equivalent resistance R_{eq}, we could write

$$V_{ab} = IR_{eq} \quad \text{or} \quad R_{eq} = \frac{V_{ab}}{I}$$

where V_{ab} is the potential difference between terminals a and b of the network and I is the current at point a or b. To compute an equivalent resistance, we assume a potential difference V_{ab} across the actual network, compute the corresponding current I, and take the ratio V_{ab}/I.

Resistors in Series

We can derive general equations for the equivalent resistance of a series or parallel combination of resistors. If the resistors are in *series,* as in Fig. 26.1a, the current I must be the same in all of them. (As we discussed in Section 25.4, current is *not* "used up" as it passes through a circuit.) Applying $V = IR$ to each resistor, we have

$$V_{ax} = IR_1 \qquad V_{xy} = IR_2 \qquad V_{yb} = IR_3$$

The potential differences across each resistor need not be the same (except for the special case in which all three resistances are equal). The potential difference V_{ab} across the entire combination is the sum of these individual potential differences:

$$V_{ab} = V_{ax} + V_{xy} + V_{yb} = I(R_1 + R_2 + R_3)$$

and so

$$\frac{V_{ab}}{I} = R_1 + R_2 + R_3$$

The ratio V_{ab}/I is, by definition, the equivalent resistance R_{eq}. Therefore

$$R_{eq} = R_1 + R_2 + R_3$$

It is easy to generalize this to any number of resistors:

$$R_{eq} = R_1 + R_2 + R_3 + \cdots \qquad \text{(resistors in series)} \qquad (26.1)$$

The equivalent resistance of *any number* of resistors in series equals the sum of their individual resistances.

The equivalent resistance is *greater than* any individual resistance.

Let's compare this result with Eq. (24.5) for *capacitors* in series. Resistors in series add directly because the voltage across each is directly proportional to its resistance and to the common current. Capacitors in series add reciprocally because the voltage across each is directly proportional to the common charge but *inversely* proportional to the individual capacitance.

Resistors in Parallel

If the resistors are in *parallel*, as in Fig. 26.1b, the current through each resistor need not be the same. But the potential difference between the terminals of each resistor must be the same and equal to V_{ab} (Fig. 26.2). (Remember that the potential difference between any two points does not depend on the path taken between the points.) Let's call the currents in the three resistors I_1, I_2, and I_3. Then from $I = V/R$,

$$I_1 = \frac{V_{ab}}{R_1} \qquad I_2 = \frac{V_{ab}}{R_2} \qquad I_3 = \frac{V_{ab}}{R_3}$$

In general, the current is different through each resistor. Because charge is not accumulating or draining out of point a, the total current I must equal the sum of the three currents in the resistors:

$$I = I_1 + I_2 + I_3 = V_{ab}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) \quad \text{or}$$

$$\frac{I}{V_{ab}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

But by the definition of the equivalent resistance R_{eq}, $I/V_{ab} = 1/R_{eq}$, so

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Again it is easy to generalize to *any number* of resistors in parallel:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots \qquad \text{(resistors in parallel)} \qquad (26.2)$$

For *any number* of resistors in parallel, the *reciprocal* of the equivalent resistance equals the *sum of the reciprocals* of their individual resistances.

The equivalent resistance is always *less than* any individual resistance.

We can compare this result with Eq. (24.7) for *capacitors* in parallel. Resistors in parallel add reciprocally because the current in each is proportional to the common voltage across them and *inversely* proportional to the resistance of each. Capacitors in parallel add directly because the charge on each is proportional to the common voltage across them and *directly* proportional to the capacitance of each.

For the special case of *two* resistors in parallel,

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1 + R_2}{R_1 R_2} \quad \text{and}$$

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2} \qquad \text{(two resistors in parallel)} \qquad (26.3)$$

26.2 A car's headlights are connected in parallel. Hence each headlight is exposed to the full potential difference supplied by the car's electrical system, giving maximum brightness. Another advantage is that if one headlight burns out, the other one keeps shining (see Example 26.2).

ActivPhysics ONLINE 12.2 DC Parallel Circuits

Because $V_{ab} = I_1R_1 = I_2R_2$, it follows that

$$\frac{I_1}{I_2} = \frac{R_2}{R_1} \quad \text{(two resistors in parallel)} \tag{26.4}$$

This shows that the currents carried by two resistors in parallel are *inversely proportional* to their resistances. More current goes through the path of least resistance.

Problem-Solving Strategy 26.1 Resistors in Series and Parallel

MP

IDENTIFY *the relevant concepts:* Many resistor networks are made up of resistors in series, in parallel, or a combination of the two. The key concept is that such a network can be replaced by a single equivalent resistor.

SET UP *the problem* using the following steps:

1. Make a drawing of the resistor network.
2. Determine whether the resistors are connected in series or parallel. Note that you can often consider networks such as those in Figs. 26.1c and 26.1d as combinations of series and parallel arrangements.
3. Determine what the target variables are. They could include the equivalent resistance of the network, the potential difference across each resistor, or the current through each resistor.

EXECUTE *the solution* as follows:

1. Use Eq. (26.1) or (26.2) to find the equivalent resistance for a series or a parallel combination, respectively.
2. If the network is more complex, try reducing it to series and parallel combinations. For example, in Fig. 26.1c we first replace the parallel combination of R_2 and R_3 with its equivalent resistance; this then forms a series combination with R_1. In Fig. 26.1d, the combination of R_2 and R_3 in series forms a parallel combination with R_1.
3. When calculating potential differences, remember that when resistors are connected in series, the total potential difference across the combination equals the sum of the individual potential differences. When resistors are connected in parallel, the potential difference is the same for every resistor and equals the potential difference across the parallel combination.
4. Keep in mind the analogous statements for current. When resistors are connected in series, the current is the same through every resistor and equals the current through the series combination. When resistors are connected in parallel, the total current through the combination equals the sum of the currents through the individual resistors.

EVALUATE *your answer:* Check whether your results are consistent. If resistors are connected in series, the equivalent resistance should be greater than that of any individual resistor; if they are connected in parallel, the equivalent resistance should be less than that of any individual resistor.

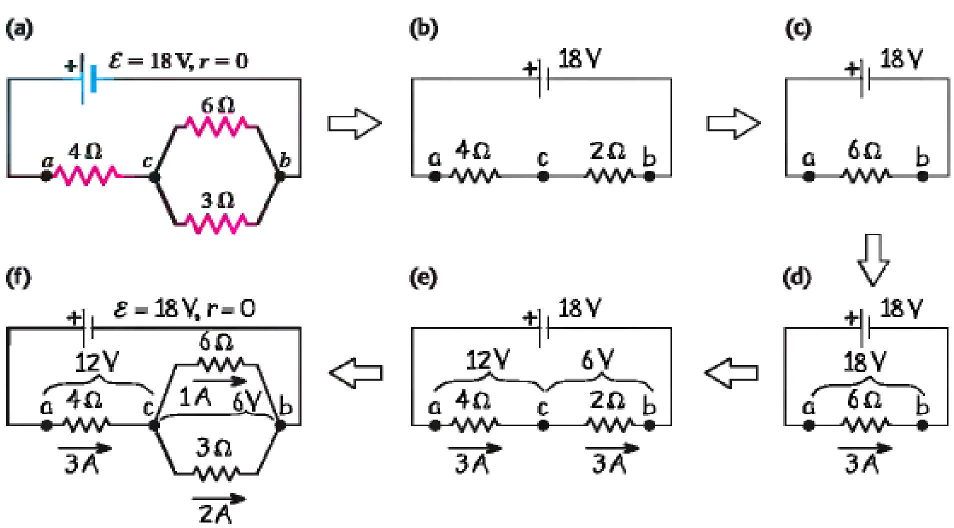
Example 26.1 Equivalent resistance

Compute the equivalent resistance of the network in Fig. 26.3a, and find the current in each resistor. The source of emf has negligible internal resistance.

SOLUTION

IDENTIFY: This network of three resistors is a *combination* of series and parallel resistances, just as in Fig. 26.1c. The 6-Ω and

26.3 Steps in reducing a combination of resistors to a single equivalent resistor and finding the current in each resistor.



3-Ω resistors are in parallel, and their combination is in series with the 4-Ω resistor.

SET UP: We first determine the equivalent resistance R_{eq} of this network as a whole. Given this value, we find the current in the emf, which is the same as the current in the 4-Ω resistor. This same current is split between the 6-Ω and 3-Ω resistors; we determine how much goes into each resistor by using the principle that the potential difference must be the same across these two resistors (because they are connected in parallel).

EXECUTE: Figures 26.3b and 26.3c show successive steps in reducing the network to a single equivalent resistance. From Eq.(26.2) the 6-Ω and 3-Ω resistors in parallel in Fig. 26.3a are equivalent to the single 2-Ω resistor in Fig. 26.3b:

$$\frac{1}{R_{eq}} = \frac{1}{6\,\Omega} + \frac{1}{3\,\Omega} = \frac{1}{2\,\Omega}$$

[You can find the same result using Eq. (26.3).] From Eq. (26.1) the series combination of this 2-Ω resistor with the 4-Ω resistor is equivalent to the single 6-Ω resistor in Fig. 26.3c.

To find the current in each resistor of the original network, we reverse the steps by which we reduced the network. In the circuit shown in Fig. 26.3d (identical to Fig. 26.3c), the current is $I = V_{ab}/R = (18\text{ V})/(6\,\Omega) = 3\text{ A}$. So the current in the 4-Ω and 2-Ω resistors in Fig. 26.3e (identical to Fig. 26.3b) is also 3 A. The potential difference V_{cb} across the 2-Ω resistor is therefore $V_{cb} = IR = (3\text{ A})(2\,\Omega) = 6\text{ V}$. This potential difference must also be 6 V in Fig. 26.3f (identical to Fig. 26.3a). Using $I = V_{cb}/R$, the currents in the 6-Ω and 3-Ω resistors in Fig. 26.3f are $(6\text{ V})/(6\,\Omega) = 1\text{ A}$ and $(6\text{ V})/(3\,\Omega) = 2\text{ A}$, respectively.

EVALUATE: Note that for the two resistors in parallel between points c and b in Fig. 26.3f, there is twice as much current through the 3-Ω resistor as through the 6-Ω resistor; more current goes through the path of least resistance, in accordance with Eq. (26.4). Note also that the total current through these two resistors is 3 A, the same as it is through the 4-Ω resistor between points a and c.

Example 26.2 Series versus parallel combinations

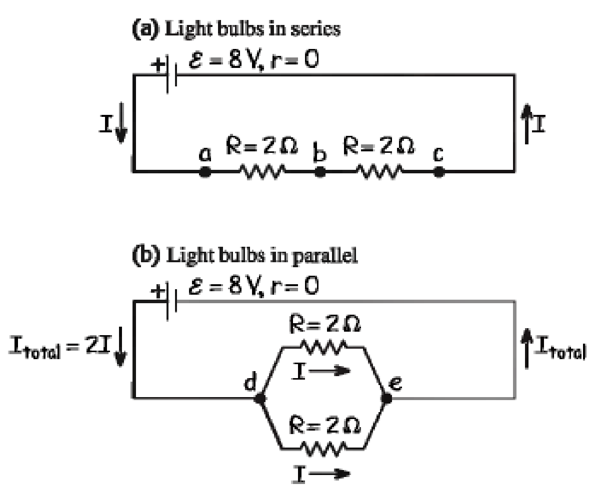
Two identical light bulbs are to be connected to a source with $\mathcal{E} = 8\text{ V}$ and negligible internal resistance. Each light bulb has a resistance $R = 2\,\Omega$. Find the current through each bulb, the potential difference across each bulb, and the power delivered to each bulb and to the entire network if the bulbs are connected (a) in series and (b) in parallel. (c) Suppose one of the bulbs burns out; that is, its filament breaks and current can no longer flow through it. What happens to the other bulb in the series case? In the parallel case?

SOLUTION

IDENTIFY: The light bulbs are just resistors in simple series and parallel connections.

SET UP: Figures 26.4a and 26.4b show our sketches of the series and parallel circuits, respectively. Once we have found the current through each light bulb, we can find the power delivered to each bulb using Eq. (25.18), $P = I^2R = V^2/R$.

26.4 Our sketches for this problem.



EXECUTE: (a) From Eq. (26.1) the equivalent resistance of the two bulbs between points a and c in Fig. 26.4a is the sum of their individual resistances:

$$R_{eq} = 2R = 2(2\,\Omega) = 4\,\Omega$$

The current is the same through either light bulb in series:

$$I = \frac{V_{ac}}{R_{eq}} = \frac{8\text{ V}}{4\,\Omega} = 2\text{ A}$$

Since the bulbs have the same resistance, the potential difference is the same across each bulb:

$$V_{ab} = V_{bc} = IR = (2\text{ A})(2\,\Omega) = 4\text{ V}$$

This is one-half of the 8-V terminal voltage of the source. From Eq.(25.18), the power delivered to each light bulb is

$$P = I^2R = (2\text{ A})^2(2\,\Omega) = 8\text{ W} \quad \text{or}$$

$$P = \frac{V_{ab}^2}{R} = \frac{V_{bc}^2}{R} = \frac{(4\text{ V})^2}{2\,\Omega} = 8\text{ W}$$

The total power delivered to both bulbs is $P_{total} = 2P = 16\text{ W}$. Alternatively, we can find the total power by using the equivalent resistance $R_{eq} = 4\,\Omega$, through which the current is $I = 2\text{ A}$ and across which the potential difference is $V_{ac} = 8\text{ V}$:

$$P_{total} = I^2R_{eq} = (2\text{ A})^2(4\,\Omega) = 16\text{ W} \quad \text{or}$$

$$P_{total} = \frac{V_{ac}^2}{R_{eq}} = \frac{(8\text{ V})^2}{4\,\Omega} = 16\text{ W}$$

(b) If the light bulbs are in parallel, as in Fig. 26.4b, the potential difference V_{de} across each bulb is the same and equal to 8 V,

Continued

the terminal voltage of the source. Hence the current through each light bulb is

$$I = \frac{V_{de}}{R} = \frac{8\ \text{V}}{2\ \Omega} = 4\ \text{A}$$

and the power delivered to each bulb is

$$P = I^2R = (4\ \text{A})^2(2\ \Omega) = 32\ \text{W} \qquad \text{or}$$
$$P = \frac{V_{de}^2}{R} = \frac{(8\ \text{V})^2}{2\ \Omega} = 32\ \text{W}$$

Both the potential difference across each bulb and the current through each bulb are twice as great as in the series case. Hence the power delivered to each bulb is *four* times greater, and each bulb glows more brightly than in the series case. If the goal is to produce the maximum amount of light from each bulb, a parallel arrangement is superior to a series arrangement.

The total power delivered to the parallel network is $P_{\text{total}} = 2P = 64\ \text{W}$, four times greater than in the series case. The increased power compared to the series case isn't obtained "for free"; energy is extracted from the source four times more rapidly in the parallel case than in the series case. If the source is a battery, it will be used up four times as fast.

We can also find the total power by using the equivalent resistance R_{eq}, given by Eq. (26.2):

$$\frac{1}{R_{\text{eq}}} = 2\left(\frac{1}{2\ \Omega}\right) = 1\ \Omega^{-1} \qquad \text{or} \qquad R_{\text{eq}} = 1\ \Omega$$

The total current through the equivalent resistor is $I_{\text{total}} = 2I = 2(4\ \text{A}) = 8\ \text{A}$, and the potential difference across the equivalent resistor is 8 V. Hence the total power is

$$P_{\text{total}} = I^2R_{\text{eq}} = (8\ \text{A})^2(1\ \Omega) = 64\ \text{W} \qquad \text{or}$$
$$P_{\text{total}} = \frac{V_{de}^2}{R} = \frac{(8\ \text{V})^2}{1\ \Omega} = 64\ \text{W}$$

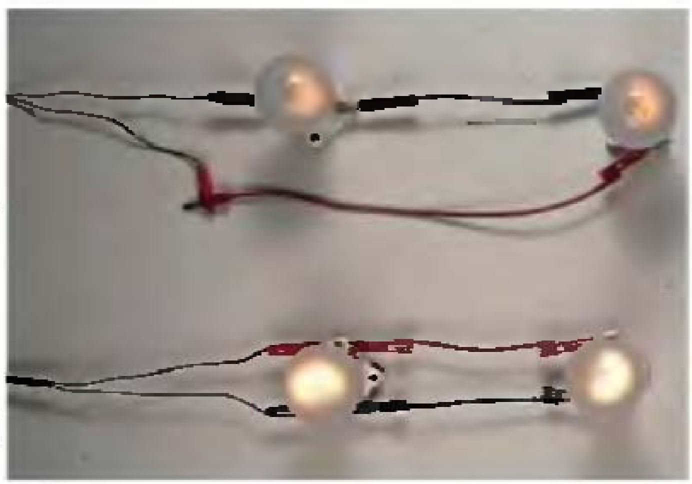
The potential difference across the equivalent resistance is the same for both the series and parallel cases, but for the parallel case the value of R_{eq} is less, and so $P_{\text{total}} = V^2/R_{\text{eq}}$ is greater.

(c) In the series case the same current flows through both bulbs. If one of the bulbs burns out, there will be no current at all in the circuit, and neither bulb will glow.

In the parallel case the potential difference across either bulb remains equal to 8 V even if one of the bulbs burns out. Hence the current through the functional bulb remains equal to 4 A, and the power delivered to that bulb remains equal to 32 W, the same as before the other bulb burned out. This is another of the merits of a parallel arrangement of light bulbs: If one fails, the other bulbs are unaffected. This principle is used in household wiring systems, which we'll discuss in Section 26.5.

EVALUATE: Our calculation isn't completely accurate, because the resistance $R = V/I$ of real light bulbs is *not* a constant independent of the potential difference V across the bulb. (The resistance of the filament increases with increasing operating temperature and hence with increasing V.) But it is indeed true that light bulbs connected in series across a source glow less brightly than when connected in parallel across the same source (Fig. 26.5).

26.5 When connected to the same source, two light bulbs in series (shown at top) draw less power and glow less brightly than when they are in parallel (shown at bottom).



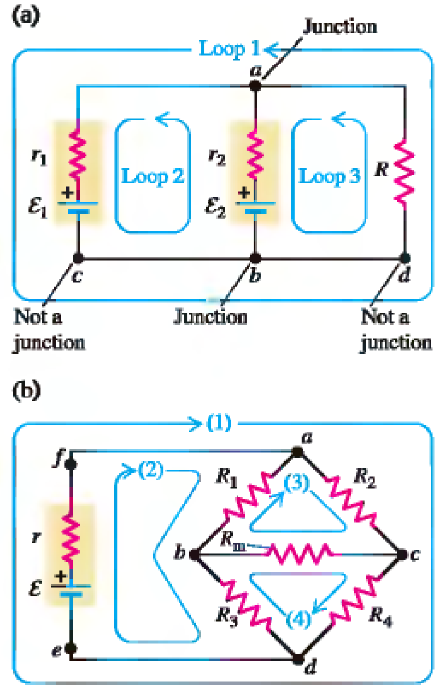
Test Your Understanding of Section 26.1 Suppose all three of the resistors shown in Fig. 26.1 have the same resistance, so $R_1 = R_2 = R_3 = R$. Rank the four arrangements shown in parts (a)–(d) of Fig. 26.1 in order of their equivalent resistance, from highest to lowest.

26.2 Kirchhoff's Rules

Many practical resistor networks cannot be reduced to simple series-parallel combinations. Figure 26.6a shows a dc power supply with emf $\mathcal{E}_1$ charging a battery with a smaller emf $\mathcal{E}_2$ and feeding current to a light bulb with resistance R. Figure 26.6b is a "bridge" circuit, used in many different types of measurement and control systems. (One important application of a "bridge" circuit is described in Problem 26.79.) We don't need any new principles to compute the currents in these networks, but there are some techniques that help us handle such problems systematically. We will describe the techniques developed by the German physicist Gustav Robert Kirchhoff (1824–1887).

First, here are two terms that we will use often. A **junction** in a circuit is a point where three or more conductors meet. Junctions are also called *nodes* or *branch points*. A **loop** is any closed conducting path. In Fig. 26.6a points a and b are junctions, but points c and d are not; in Fig. 26.6b the points a, b, c, and d are junctions, but points e and f are not. The blue lines in Figs. 26.6a and 26.6b show some possible loops in these circuits.

26.6 Two networks that cannot be reduced to simple series-parallel combinations of resistors.



Kirchhoff's rules are the following two statements:

Kirchhoff's junction rule: *The algebraic sum of the currents into any junction is zero.* That is,

$$\sum I = 0 \qquad \text{(junction rule, valid at any junction)} \qquad (26.5)$$

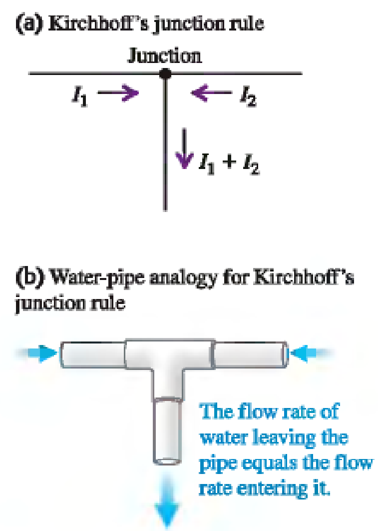
Kirchhoff's loop rule: *The algebraic sum of the potential differences in any loop,* including those associated with emfs and those of resistive elements, *must equal zero.* That is,

$$\sum V = 0 \qquad \text{(loop rule, valid for any closed loop)} \qquad (26.6)$$

The junction rule is based on *conservation of electric charge.* No charge can accumulate at a junction, so the total charge entering the junction per unit time must equal the total charge leaving per unit time (Fig. 26.7a). Charge per unit time is current, so if we consider the currents entering a junction to be positive and those leaving to be negative, the algebraic sum of currents into a junction must be zero. It's like a T branch in a water pipe (Fig. 26.7b); if you have 1 liter per minute coming in one pipe, you can't have 3 liters per minute going out the other two pipes. We may as well confess that we used the junction rule (without saying so) in Section 26.1 in the derivation of Eq. (26.2) for resistors in parallel.

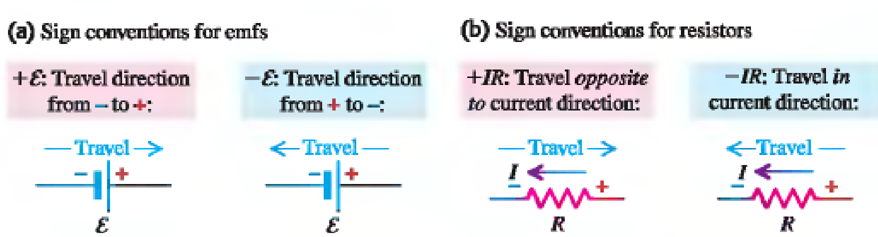
The loop rule is a statement that the electrostatic force is *conservative.* Suppose we go around a loop, measuring potential differences across successive circuit elements as we go. When we return to the starting point, we must find that the *algebraic sum* of these differences is zero; otherwise, we could not say that the potential at this point has a definite value.

26.7 (a) Kirchhoff's junction rule states that as much current flows into a junction as flows out of it. (b) A water-pipe analogy.



Sign Conventions for the Loop Rule

In applying the loop rule, we need some sign conventions. Problem-Solving Strategy 26.2 describes in detail how to use these, but here's a quick overview. We first assume a direction for the current in each branch of the circuit and mark it on a diagram of the circuit. Then, starting at any point in the circuit, we imagine traveling around a loop, adding emfs and IR terms as we come to them. When we travel through a source in the direction from $-$ to $+$, the emf is considered to be *positive;* when we travel from $+$ to $-$, the emf is considered to be *negative* (Fig. 26.8a). When we travel through a resistor in the *same* direction as the assumed current, the IR term is *negative* because the current goes in the direction of decreasing potential. When we travel through a resistor in the direction *opposite* to the assumed current, the IR term is *positive* because this represents a rise of potential (Fig. 26.8b).



26.8 Use these sign conventions when you apply Kirchhoff's loop rule. In each part of the figure "Travel" is the direction that we imagine going around the loop, which is not necessarily the direction of the current.

Kirchhoff's two rules are all we need to solve a wide variety of network problems. Usually, some of the emfs, currents, and resistances are known, and others are unknown. We must always obtain from Kirchhoff's rules a number of independent equations equal to the number of unknowns so that we can solve the equations simultaneously. Often the hardest part of the solution is not understanding the basic principles but keeping track of algebraic signs!

Problem-Solving Strategy 26.2 Kirchhoff's Rules

IDENTIFY *the relevant concepts:* Kirchhoff's rules are important tools for analyzing any circuit more complicated than a single loop.

SET UP *the problem* using the following steps:

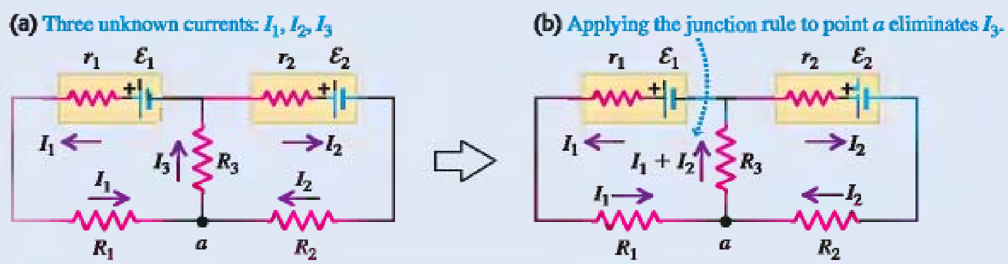
1. Draw a *large* circuit diagram so you have plenty of room for labels. Label all quantities, known and unknown, including an assumed direction for each unknown current and emf. Often you will not know in advance the actual direction of an unknown current or emf, but this doesn't matter. If the actual direction of a particular quantity is opposite to your assumption, the result will come out with a negative sign. If you use Kirchhoff's rules correctly, they will give you the directions as well as the magnitudes of unknown currents and emfs.
2. When you label currents, it is usually best to use the junction rule immediately to express the currents in terms of as few quantities as possible. For example, Fig. 26.9a shows a circuit correctly labeled; Fig. 26.9b shows the same circuit, relabeled by applying the junction rule to point a to eliminate I_3.
3. Determine which quantities are the target variables.

EXECUTE *the solution* as follows:

1. Choose any closed loop in the network and designate a direction (clockwise or counterclockwise) to travel around the loop when applying the loop rule. The direction doesn't have to be the same as any assumed current direction.
2. Travel around the loop in the designated direction, adding potential differences as you cross them. Remember that a positive potential difference corresponds to an increase in potential and a negative potential difference corresponds to a decrease in potential. An emf is counted as positive when you traverse it from $(-)$ to $(+)$, and negative when you go from $(+)$ to $(-)$. An IR term is negative if you travel through the resistor in the same direction as the assumed current and positive if you pass through it in the opposite direction. Figure 26.8 summarizes these sign conventions.
3. Equate the sum in Step 2 to zero.
4. If necessary, choose another loop to get a different relationship among the unknowns, and continue until you have as many independent equations as unknowns or until every circuit element has been included in at least one of the chosen loops.
5. Solve the equations simultaneously to determine the unknowns. This step involves algebra, not physics, but it can be fairly complex. Be careful with algebraic manipulations; one sign error will prove fatal to the entire solution.
6. You can use this same bookkeeping system to find the potential V_{ab} of any point a with respect to any other point b. Start at b and add the potential changes you encounter in going from b to a, using the same sign rules as in Step 2. The algebraic sum of these changes is $V_{ab} = V_a - V_b$.

EVALUATE *your answer:* Check all the steps in your algebra. A useful strategy is to consider a loop other than the ones you used to solve the problem; if the sum of potential drops around this loop isn't zero, you made an error somewhere in your calculations. As always, ask yourself whether the answers make sense.

26.9 Applying the junction rule to point a reduces the number of unknown currents from three to two.



Example 26.3 A single-loop circuit

The circuit shown in Fig. 26.10a contains two batteries, each with an emf and an internal resistance, and two resistors. Find (a) the current in the circuit, (b) the potential difference V_{ab}, and (c) the power output of the emf of each battery.

SOLUTION

IDENTIFY: This single-loop circuit has no junctions, so we don't need Kirchhoff's junction rule to solve for the target variables.

SET UP: To apply the loop rule to the single loop, we first assume a direction for the current; let's assume a counterclockwise direction, as shown in Fig. 26.10a.

EXECUTE: (a) Starting at a and going counterclockwise, we add potential increases and decreases and equate the sum to zero, as in Eq. (26.6). The resulting equation is

$$-I(4\,\Omega) - 4\text{ V} - I(7\,\Omega) + 12\text{ V} - I(2\,\Omega) - I(3\,\Omega) = 0$$

Collecting terms containing I and solving for I, we find

$$8\text{ V} = I(16\ \Omega) \qquad \text{and} \qquad I = 0.5\text{ A}$$

The result for I is positive, showing that our assumed current direction is correct. For an exercise, try assuming the opposite direction for I; you should then get $I = -0.5$ A, indicating that the actual current is opposite to this assumption.

(b) To find V_{ab}, the potential at a with respect to b, we start at b and add potential changes as we go toward a. There are two possible paths from b to a; taking the lower one first, we find

$$V_{ab} = (0.5\text{ A})(7\ \Omega) + 4\text{ V} + (0.5\text{ A})(4\ \Omega) = 9.5\text{ V}$$

Point a is at 9.5 V higher potential than b. All the terms in this sum, including the IR terms, are positive because each represents an *increase* in potential as we go from b toward a. If we use the upper path instead, the resulting equation is

$$V_{ab} = 12\text{ V} - (0.5\text{ A})(2\ \Omega) - (0.5\text{ A})(3\ \Omega) = 9.5\text{ V}$$

Here the IR terms are negative because our path goes in the direction of the current, with potential decreases through the resistors. The result is the same as for the lower path, as it must be in order for the total potential change around the complete loop to be zero. In each case, potential rises are taken to be positive and drops are taken to be negative.

(c) The power output of the emf of the 12-V battery is

$$P = \mathcal{E}I = (12\text{ V})(0.5\text{ A}) = 6\text{ W}$$

and the power output of the emf of the 4-V battery is

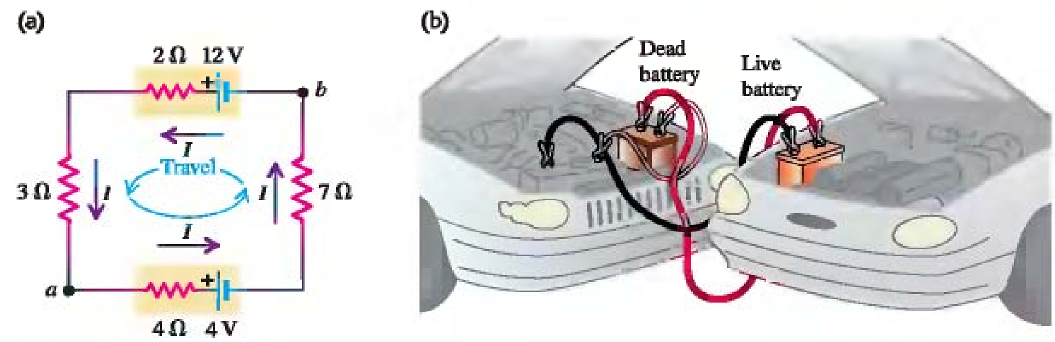
$$P = \mathcal{E}I = (-4\text{ V})(0.5\text{ A}) = -2\text{ W}$$

The negative sign in $\mathcal{E}$ for the 4-V battery appears because the current actually runs from the higher-potential side of the battery to the lower-potential side. The negative value of P means that we are *storing* energy in that battery, and it is being *recharged* by the 12-V battery.

EVALUATE: By applying the expression $P = I^2R$ to each of the four resistors in Fig. 26.10a, you should be able to show that the total power dissipated in all four resistors is 4 W. Of the 6 W provided by the emf of the 12-V battery, 2 W goes into storing energy in the 4-V battery and 4 W is dissipated in the resistances.

The circuit shown in Fig. 26.10a is very much like that used when a 12-V automobile battery is used to recharge a run-down battery in another automobile (Fig. 26.10b). The 3-Ω and 7-Ω resistors in Fig. 26.10a represent the resistances of the jumper cables and of the conducting path through the automobile with the run-down battery. (The values of the resistances in actual automobiles and jumper cables are different from those used in this example.)

26.10 (a) In this example we travel around the loop in the same direction as the assumed current, so all the IR terms are negative. The potential decreases as we travel from + to − through the bottom emf but increases as we travel from − to + through the top emf. (b) A real-life example of a circuit of this kind.



Example 26.4 Charging a battery

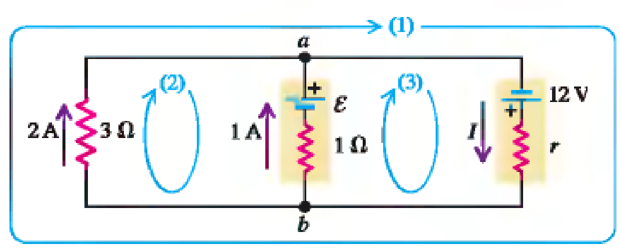
In the circuit shown in Fig. 26.11, a 12-V power supply with unknown internal resistance r is connected to a run-down rechargeable battery with unknown emf $\mathcal{E}$ and internal resistance 1 Ω and to an indicator light bulb of resistance 3 Ω carrying a current of 2 A. The current through the run-down battery is 1 A in the direction shown. Find the unknown current I, the internal resistance r, and the emf $\mathcal{E}$.

SOLUTION

IDENTIFY: This circuit has more than one loop, so we must apply both the junction rule and the loop rule.

SET UP: We assume the direction of the current through the 12-V power supply to be as shown. There are three target variables, so we need three equations.

26.11 In this circuit a power supply charges a run-down battery and lights a bulb. An assumption has been made about the polarity of the emf $\mathcal{E}$ of the run-down battery. Is this assumption correct?



EXECUTE: First we apply the junction rule, Eq. (26.5), to point a. We find

$$-I + 1\text{ A} + 2\text{ A} = 0 \qquad \text{so} \qquad I = 3\text{ A}$$

Continued

To determine r, we apply the loop rule, Eq. (26.6), to the outer loop labeled (1); we find

$$12\text{ V} - (3\text{ A})r - (2\text{ A})(3\ \Omega) = 0 \quad \text{so} \quad r = 2\ \Omega$$

The terms containing the resistances r and $3\ \Omega$ are negative because our loop traverses those elements in the same direction as the current and hence finds potential *drops*. If we had chosen to traverse loop (1) in the opposite direction, every term would have had the opposite sign, and the result for r would have been the same.

To determine $\mathcal{E}$, we apply the loop rule to loop (2):

$$-\mathcal{E} + (1\text{ A})(1\ \Omega) - (2\text{ A})(3\ \Omega) = 0 \quad \text{so} \quad \mathcal{E} = -5\text{ V}$$

The term for the 1-Ω resistor is positive because in traversing it in the direction opposite to the current, we find a potential *rise*. The negative value for $\mathcal{E}$ shows that the actual polarity of this emf is opposite to the assumption made in Fig. 26.11; the positive terminal of this source is really on the right side. As in Example 26.3, the battery is being recharged.

EVALUATE: We can check our result for $\mathcal{E}$ by using loop (3), obtaining the equation

$$12\text{ V} - (3\text{ A})(2\ \Omega) - (1\text{ A})(1\ \Omega) + \mathcal{E} = 0$$

from which we again find $\mathcal{E} = -5\text{ V}$.

As an additional consistency check, we note that $V_{ba} = V_b - V_a$ equals the voltage across the 3-Ω resistance, which is $(2\text{ A})(3\ \Omega) = 6\text{ V}$. Going from a to b by the top branch, we encounter potential differences $+12\text{ V} - (3\text{ A})(2\ \Omega) = +6\text{ V}$, and going by the middle branch we find $-(-5\text{ V}) + (1\text{ A})(1\ \Omega) = +6\text{ V}$. The three ways of getting V_{ba} give the same results. Make sure that you understand all the signs in these calculations.

Example 26.5 Power in a battery-charging circuit

In the circuit of Example 26.4 (shown in Fig. 26.11), find the power delivered by the 12-V power supply and by the battery being recharged, and find the power dissipated in each resistor.

SOLUTION

IDENTIFY: We use the results of Section 25.5, in which we found that the power delivered *from* an emf to a circuit is $\mathcal{E}I$ and the power delivered *to* a resistor from a circuit is $V_{ab}I = I^2R$.

SET UP: We know the values of each emf, each current, and each resistance from Example 26.4.

EXECUTE: The power output from the emf of the power supply is

$$P_{\text{supply}} = \mathcal{E}_{\text{supply}} I_{\text{supply}} = (12\text{ V})(3\text{ A}) = 36\text{ W}$$

The power dissipated in the power supply's internal resistance r is

$$P_{r\text{-supply}} = I_{\text{supply}}{}^2 r_{\text{supply}} = (3\text{ A})^2(2\ \Omega) = 18\text{ W}$$

so the power supply's *net* power output is $P_{\text{net}} = 36\text{ W} - 18\text{ W} = 18\text{ W}$. Alternatively, from Example 26.4 the terminal voltage of the battery is $V_{ba} = 6\text{ V}$, so the net power output is

$$P_{\text{net}} = V_{ba} I_{\text{supply}} = (6\text{ V})(3\text{ A}) = 18\text{ W}$$

The power output of the emf $\mathcal{E}$ of the battery being charged is

$$P_{\text{emf}} = \mathcal{E} I_{\text{battery}} = (-5\text{ V})(1\text{ A}) = -5\text{ W}$$

This is negative because the 1-A current runs through the battery from the higher-potential side to the lower-potential side. (As we mentioned in Example 26.4, the polarity assumed for this battery in Fig. 26.11 was wrong.) We are storing energy in the battery as we charge it. Additional power is dissipated in the battery's internal resistance; this power is

$$P_{r\text{-battery}} = I_{\text{battery}}{}^2 r_{\text{battery}} = (1\text{ A})^2(1\ \Omega) = 1\text{ W}$$

The total power input to the battery is thus $1\text{ W} + |-5\text{ W}| = 6\text{ W}$. Of this, 5 W represents useful energy stored in the battery; the remainder is wasted in its internal resistance.

The power dissipated in the light bulb is

$$P_{\text{bulb}} = I_{\text{bulb}}{}^2 R_{\text{bulb}} = (2\text{ A})^2(3\ \Omega) = 12\text{ W}$$

EVALUATE: As a check, note that all of the power from the supply is accounted for. Of the 18 W of net power from the power supply, 5 W goes to recharge the battery, 1 W is dissipated in the battery's internal resistance, and 12 W is dissipated in the light bulb.

Example 26.6 A complex network

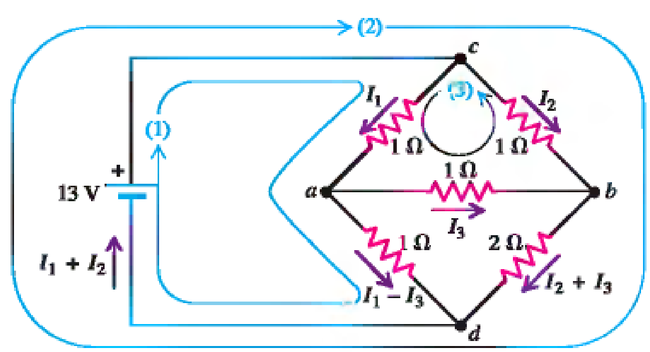
Figure 26.12 shows a "bridge" circuit of the type described at the beginning of this section (see Fig. 26.6b). Find the current in each resistor and the equivalent resistance of the network of five resistors.

SOLUTION

IDENTIFY: This network cannot be represented in terms of series and parallel combinations. Hence we must use Kirchhoff's rules to find the values of the target variables.

SET UP: There are five different currents to determine, but by applying the junction rule to junctions a and b, we can represent them in terms of three unknown currents, as shown in the figure. The current in the battery is $I_1 + I_2$.

26.12 A network circuit with several resistors.



EXECUTE: We apply the loop rule to the three loops shown, obtaining the following three equations:

$$13\text{ V} - I_1(1\,\Omega) - (I_1 - I_3)(1\,\Omega) = 0 \qquad (1)$$
$$-I_2(1\,\Omega) - (I_2 + I_3)(2\,\Omega) + 13\text{ V} = 0 \qquad (2)$$
$$-I_1(1\,\Omega) - I_3(1\,\Omega) + I_2(1\,\Omega) = 0 \qquad (3)$$

This is a set of three simultaneous equations for the three unknown currents. They may be solved by various methods; one straightforward procedure is to solve the third equation for I_2, obtaining $I_2 = I_1 + I_3$, and then substitute this expression into the second equation to eliminate I_2. When this is done, we are left with the two equations

$$13\text{ V} = I_1(2\,\Omega) - I_3(1\,\Omega) \qquad (1')$$
$$13\text{ V} = I_1(3\,\Omega) + I_3(5\,\Omega) \qquad (2')$$

Now we can eliminate I_3 by multiplying Eq. (1′) by 5 and adding the two equations. We obtain

$$78\text{ V} = I_1(13\,\Omega) \qquad I_1 = 6\text{ A}$$

We substitute this result back into Eq. (1′) to obtain $I_3 = -1$ A, and finally, from Eq. (3) we find $I_2 = 5$ A. The negative value of I_3 tells us that its direction is opposite to our initial assumption.

The total current through the network is $I_1 + I_2 = 11$ A, and the potential drop across it is equal to the battery emf—namely, 13 V. The equivalent resistance of the network is

$$R_{eq} = \frac{13\text{ V}}{11\text{ A}} = 1.2\,\Omega$$

EVALUATE: You can check the results $I_1 = 6$ A, $I_2 = 5$ A, and $I_3 = -1$ A by substituting these values into the three equations (1), (2), and (3). What do you find?

Example 26.7 A potential difference within a complex network

In the circuit of Example 26.6 (Fig. 26.12), find the potential difference V_{ab}.

SOLUTION

IDENTIFY: Our target variable is $V_{ab} = V_a - V_b$, which is the potential at point *a* with respect to point *b*.

SET UP: To find V_{ab}, we start at point *b* and follow a path to point *a*, adding potential rises and drops as we go. We can follow any of several paths from *b* to *a*; the value of V_{ab} must be independent of which path we choose, which gives ns a natural way to check our result.

EXECUTE: The simplest path to follow is through the center 1-Ω resistor. We have found $I_3 = -1$ A, showing that the actual current direction in this branch is from right to left. Thus, as we go from *b* to *a*, there is a *drop* of potential with magnitude $IR = (1\text{ A})(1\,\Omega) = 1$ V, and $V_{ab} = -1$ V. That is, the potential at point *a* is 1 V less than that at point *b*.

EVALUATE: To test our result, let's try a path from *b* to *a* that goes through the lower two resistors. The currents through these are

$$I_2 + I_3 = 5\text{ A} + (-1\text{ A}) = 4\text{ A} \quad \text{and}$$
$$I_1 - I_3 = 6\text{ A} - (-1\text{ A}) = 7\text{ A}$$

and so

$$V_{ab} = -(4\text{ A})(2\,\Omega) + (7\text{ A})(1\,\Omega) = -1\text{ V}$$

We suggest that you try some other paths from *b* to *a* to verify that they also give this result.

Test Your Understanding of Section 26.2 Subtract Eq. (1) from Eq. (2) in Example 26.6. To which loop in Fig. 26.12 does this equation correspond? Would this equation have simplified the solution of Example 26.6?

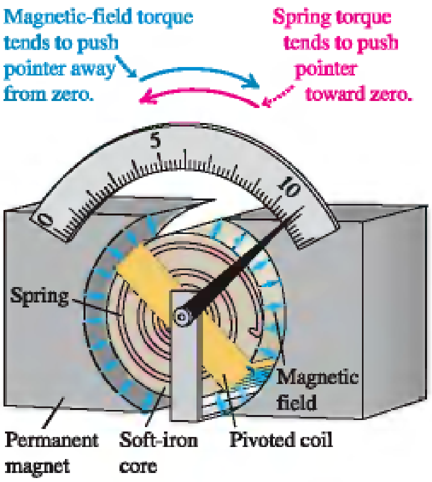
26.3 Electrical Measuring Instruments

26.13 This ammeter (top) and voltmeter (bottom) are both d'Arsonval galvanometers. The difference has to do with their internal connections (see Fig. 26.15).

We've been talking about potential difference, current, and resistance for two chapters, so it's about time we said something about how to *measure* these quantities. Many common devices, including car instrument panels, battery chargers, and inexpensive electrical instruments, measure potential difference (voltage), current, or resistance using a **d'Arsonval galvanometer** (Fig. 26.13). In the following discussion we'll often call it just a *meter*. A pivoted coil of fine wire is placed in the magnetic field of a permanent magnet (Fig. 26.14). Attached to the coil is a spring, similar to the hairspring on the balance wheel of a watch. In the equilibrium position, with no current in the coil, the pointer is at zero. When there is a current in the coil, the magnetic field exerts a torque on the coil that is proportional to the current. (We'll discuss this magnetic interaction in detail in Chapter 27.) As the coil turns, the spring exerts a restoring torque that is proportional to the angular displacement.

Thus the angular deflection of the coil and pointer is directly proportional to the coil current, and the device can be calibrated to measure current. The maximum deflection, typically 90° or so, is called *full-scale deflection*. The essential electrical characteristics of the meter are the current I_{fs} required for

26.14 A d'Arsonval galvanometer, showing a pivoted coil with attached pointer, a permanent magnet supplying a magnetic field that is uniform in magnitude, and a spring to provide restoring torque, which opposes magnetic-field torque.



ActivPhysics ONLINE
12.4 Using Ammeters and Voltmeters

full-scale deflection (typically on the order of 10 μA to 10 mA) and the resistance R_c of the coil (typically on the order of 10 to 1000 Ω).

The meter deflection is proportional to the *current* in the coil. If the coil obeys Ohm's law, the current is proportional to the *potential difference* between the terminals of the coil, and the deflection is also proportional to this potential difference. For example, consider a meter whose coil has a resistance $R_c = 20.0\ \Omega$ and that deflects full scale when the current in its coil is $I_{fs} = 1.00$ mA. The corresponding potential difference for full-scale deflection is

$$V = I_{fs}R_c = (1.00 \times 10^{-3}\ \text{A})(20.0\ \Omega) = 0.0200\ \text{V}$$

Ammeters

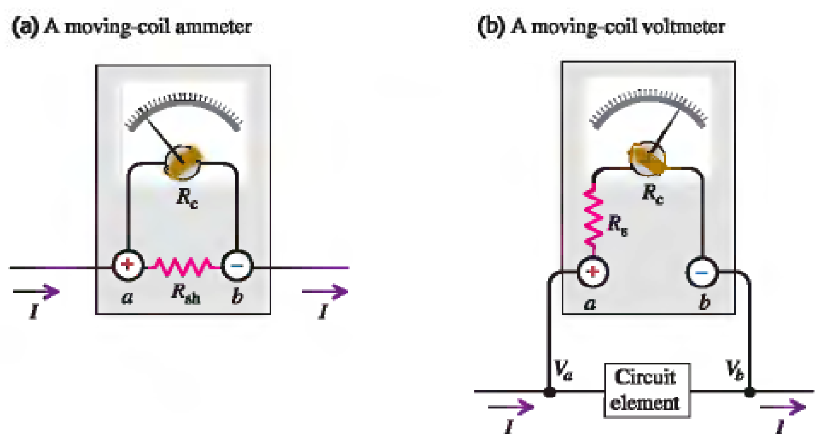
A current-measuring instrument is usually called an **ammeter** (or milliammeter, microammeter, and so forth, depending on the range). *An ammeter always measures the current passing through it.* An *ideal* ammeter, discussed in Section 25.4, would have *zero* resistance, so including it in a branch of a circuit would not affect the current in that branch. Real ammeters always have some finite resistance, but it is always desirable for an ammeter to have as little resistance as possible.

We can adapt any meter to measure currents that are larger than its full-scale reading by connecting a resistor in parallel with it (Fig. 26.15a) so that some of the current bypasses the meter coil. The parallel resistor is called a **shunt resistor** or simply a *shunt,* denoted as R_{sh}.

Suppose we want to make a meter with full-scale current I_{fs} and coil resistance R_c into an ammeter with full-scale reading I_a. To determine the shunt resistance R_{sh} needed, note that at full-scale deflection the total current through the parallel combination is I_a, the current through the coil of the meter is I_{fs}, and the current through the shunt is the difference $I_a - I_{fs}$. The potential difference V_{ab} is the same for both paths, so

$$I_{fs}R_c = (I_a - I_{fs})R_{sh} \qquad \text{(for an ammeter)} \tag{26.7}$$

26.15 Using the same meter to measure (a) current and (b) voltage.



Example 26.8 Designing an ammeter

What shunt resistance is required to make the 1.00-mA, 20.0-Ω meter described above into an ammeter with a range of 0 to 50.0 mA?

SOLUTION

IDENTIFY: Since the meter is being used as an ammeter, its internal connections are as shown in Fig. 26.15a. Our target variable is the shunt resistance R_{sh}.

SET UP: We want the ammeter to be able to handle a maximum current $I_a = 50.0\text{ mA} = 50.0 \times 10^{-3}$ A. The resistance of the coil is $R_c = 20.0\ \Omega$, and the meter shows full-scale deflection when the current through the coil is $I_{fs} = 1.00 \times 10^{-3}$ A. We find the shunt resistance R_{sh} using Eq. (26.7).

EXECUTE: Solving Eq. (26.7) for R_{sh}, we find

$$R_{sh} = \frac{I_{fs}R_c}{I_a - I_{fs}} = \frac{(1.00 \times 10^{-3}\ \text{A})(20.0\ \Omega)}{50.0 \times 10^{-3}\ \text{A} - 1.00 \times 10^{-3}\ \text{A}}$$
$$= 0.408\ \Omega$$

EVALUATE: It's useful to consider the equivalent resistance R_{eq} of the ammeter as a whole. From Eq. (26.2),

$$\frac{1}{R_{eq}} = \frac{1}{R_c} + \frac{1}{R_{sh}} = \frac{1}{20.0\ \Omega} + \frac{1}{0.408\ \Omega}$$
$$R_{eq} = 0.400\ \Omega$$

The shunt resistance is so small in comparison to the meter resistance that the equivalent resistance is very nearly equal to the shunt resistance. The result is a low-resistance instrument with the desired range of 0 to 50.0 mA. At full-scale deflection, $I = I_a = 50.0$ mA, the current through the galvanometer is 1.00 mA, the current through the shunt resistor is 49.0 mA, and $V_{ab} = 0.0200$ V. If the current I is *less* than 50.0 mA, the coil current and the deflection are proportionally less, but the resistance R_{eq} is still 0.400 Ω.

Voltmeters

This same basic meter may also be used to measure potential difference or *voltage*. A voltage-measuring device is called a **voltmeter** (or millivoltmeter, and so forth, depending on the range). A voltmeter always measures the potential difference between two points, and its terminals must be connected to these points. (Example 25.7 in Section 25.4 described what can happen if a voltmeter is connected incorrectly.) As we discussed in Section 25.4, an ideal voltmeter would have *infinite* resistance, so connecting it between two points in a circuit would not alter any of the currents. Real voltmeters always have finite resistance, but a voltmeter should have large enough resistance that connecting it in a circuit does not change the other currents appreciably.

For the meter described in Example 26.8 the voltage across the meter coil at full-scale deflection is only $I_{fs}R_c = (1.00 \times 10^{-3}\ \text{A})(20.0\ \Omega) = 0.0200$ V. We can extend this range by connecting a resistor R_s in *series* with the coil (Fig. 26.15b). Then only a fraction of the total potential difference appears across the coil itself, and the remainder appears across R_s. For a voltmeter with full-scale reading V_V, we need a series resistor R_s in Fig. 26.15b such that

$$V_V = I_{fs}(R_c + R_s) \qquad \text{(for a voltmeter)} \qquad (26.8)$$

Example 26.9 Designing a voltmeter

How can we make a galvanometer with $R_c = 20.0\ \Omega$ and $I_{fs} = 1.00$ mA into a voltmeter with a maximum range of 10.0 V?

SOLUTION

IDENTIFY: Since this meter is being used as a voltmeter, its internal connections are as shown in Fig. 26.15b. Our target variable is the series resistance R_s.

SET UP: The maximum allowable voltage across the voltmeter is $V_V = 10.0$ V. We want this to occur when the current through the coil (of resistance $R_c = 20.0\ \Omega$) is $I_{fs} = 1.00 \times 10^{-3}$ A. We find the series resistance R_s with Eq.(26.8).

EXECUTE: From Eq. (26.8),

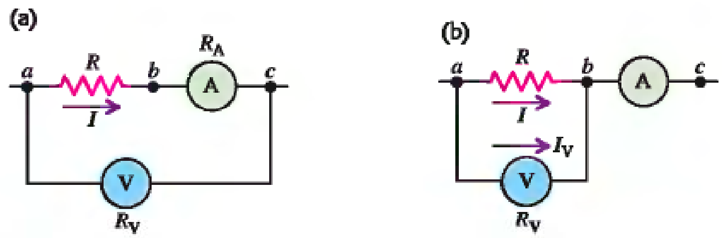
$$R_s = \frac{V_V}{I_{fs}} - R_c = \frac{10.0\ \text{V}}{0.00100\ \text{A}} - 20.0\ \Omega = 9980\ \Omega$$

EVALUATE: At full-scale deflection, $V_{ab} = 10.0$ V, the voltage across the meter is 0.0200 V, the voltage across R_s is 9.98 V, and the current through the voltmeter is 0.00100 A. In this case most of the voltage appears across the series resistor. The equivalent meter resistance is $R_{eq} = 20.0\ \Omega + 9980\ \Omega = 10{,}000\ \Omega$. Such a meter is described as a "1000 ohms-per-volt meter," referring to the ratio of resistance to full-scale deflection. In normal operation the current through the circuit element being measured (I in Fig. 26.15b) is much greater than 0.00100 A, and the resistance between points a and b in the circuit is much less than 10,000 Ω. So the voltmeter draws off only a small fraction of the current and disturbs only slightly the circuit being measured.

Ammeters and Voltmeters in Combination

A voltmeter and an ammeter can be used together to measure *resistance* and *power*. The resistance R of a resistor equals the potential difference V_{ab} between its terminals divided by the current I; that is, $R = V_{ab}/I$. The power input P to any circuit element is the product of the potential difference across it and the current through it: $P = V_{ab}I$. In principle, the most straightforward way to measure R or P is to measure V_{ab} and I simultaneously.

With practical ammeters and voltmeters this isn't quite as simple as it seems. In Fig. 26.16a, ammeter A reads the current I in the resistor R. Voltmeter V, however, reads the *sum* of the potential difference V_{ab} across the resistor and the potential difference V_{bc} across the ammeter. If we transfer the voltmeter terminal from c to b, as in Fig. 26.16b, then the voltmeter reads the potential difference V_{ab} correctly, but the ammeter now reads the *sum* of the current I in the resistor and the current I_V in the voltmeter. Either way, we have to correct the reading of one instrument or the other unless the corrections are small enough to be negligible.



26.16 Ammeter–voltmeter method for measuring resistance.

Example 26.10 Measuring resistance I

Suppose we want to measure an unknown resistance R using the circuit of Fig. 26.16a. The meter resistances are $R_V = 10{,}000\ \Omega$ (for the voltmeter) and $R_A = 2.00\ \Omega$ (for the ammeter). If the voltmeter reads 12.0 V and the ammeter reads 0.100 A, what are the resistance R and the power dissipated in the resistor?

SOLUTION

IDENTIFY: The ammeter reads the current $I = 0.100$ A through the resistor, and the voltmeter reads the potential difference between a and c. If the ammeter were *ideal* (that is, if $R_A = 0$), there would be zero potential difference between b and c, the voltmeter reading $V = 12.0$ V would be equal to the potential difference V_{ab} across the resistor, and the resistance would simply be equal to $R = V/I = (12.0\ \text{V})/(0.100\ \text{A}) = 120\ \Omega$. The ammeter is *not* ideal, however (its resistance is $R_A = 2.00\ \Omega$), so the voltmeter reading V is actually the sum of the potential differences V_{bc} (across the ammeter) and V_{ab} (across the resistor).

SET UP: We use Ohm's law to find the voltage V_{bc} across the ammeter from its known current and resistance. Then we solve for V_{ab} and the resistance R. Given these, we are able to calculate the power P into the resistor.

EXECUTE: From Ohm's law, $V_{bc} = IR_A = (0.100\ \text{A})(2.00\ \Omega) = 0.200\ \text{V}$ and $V_{ab} = IR$. The sum of these is $V = 12.0\ \text{V}$, so the potential difference across the resistor is $V_{ab} = V - V_{bc} = (12.0\ \text{V}) - (0.200\ \text{V}) = 11.8\ \text{V}$. Hence the resistance is

$$R = \frac{V_{ab}}{I} = \frac{11.8\ \text{V}}{0.100\ \text{A}} = 118\ \Omega$$

The power dissipated in this resistor is

$$P = V_{ab}I = (11.8\ \text{V})(0.100\ \text{A}) = 1.18\ \text{W}$$

EVALUATE: You can confirm this result for the power by using the alternative formula $P = I^2R$. Do you get the same answer?

Example 26.11 Measuring resistance II

Suppose the meters of Example 26.10 are connected to a different resistor in the circuit shown in Fig. 26.16b, and the readings obtained on the meters are the same as in Example 26.10. What is the value of this new resistance R, and what is the power dissipated in the resistor?

SOLUTION

IDENTIFY: In Example 26.10 the ammeter read the actual current through the resistor, but the voltmeter reading was not the same as the potential difference across the resistor. Now the situation is reversed: The voltmeter reading $V = 12.0$ V shows the actual potential difference V_{ab} across the resistor, but the ammeter reading $I_A = 0.100$ A is *not* equal to the current I through the resistor.

SET UP: Applying the junction rule at b in Fig. 26.16b shows that $I_A = I + I_V$, where I_V is the current through the voltmeter. We find I_V from the given values of V and the voltmeter resistance R_V, and we use this value to find the resistor current I. We then determine the resistance R from I and the voltmeter reading, and calculate the power as in Example 26.10.

EXECUTE: We have $I_V = V/R_V = (12.0\ \text{V})/(10{,}000\ \Omega) = 1.20\ \text{mA}$. The actual current I in the resistor is $I = I_A - I_V = 0.100\ \text{A} - 0.0012\ \text{A} = 0.0988\ \text{A}$, and the resistance is

$$R = \frac{V_{ab}}{I} = \frac{12.0\ \text{V}}{0.0988\ \text{A}} = 121\ \Omega$$

The power dissipated in the resistor is

$$P = V_{ab}I = (12.0\ \text{V})(0.0988\ \text{A}) = 1.19\ \text{W}$$

EVALUATE: Our results for R and P are not too different than the results of Example 26.10, in which the meters are connected in a different way. That's because the ammeter and voltmeter are nearly ideal: Compared to the resistance R under test, the ammeter resistance R_A is very small and the voltmeter resistance R_V is very large. Nonetheless, the results of the two examples *are* different, which shows that you must account for how ammeters and voltmeters are used when interpreting their readings.