Theoretical Physics

Analytical and Numerical Study of Soliton Collisions

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Abstract

Solitons are stable propagating solutions of nonlinear differential equations. We present the historical events and discoveries that have lead to the advent of soliton theory. Furthermore, we present three equations which have soliton solutions: the Korteweg-de Vries (KdV) equation, the sine-Gordon equation and the nonlinear Schrödinger equation. The inverse scattering transform (IST) is a method of solution which can be applied to a number of nonlinear partial differential equations which have soliton solutions. We present the steps involved in the IST for the case of the KdV equation.

Using the IST, we derive a two-soliton solution of the KdV equation. All three equations are solved numerically using the finite difference method. This confirms the existence of soliton solutions, as well as other theoretical results presented herein, including results concerning collisions between various kinds of solitons. In those cases in which a direct comparison can be made between analytical and numerical solutions the agreement is very good.

Sammanfattning

Solitoner är stabila propagerande lösningar till ickelinjära differentialekvationer. Vi presenterar de historiska händelser och upptäckter som har lett till solitonteorins uppkomst. Vidare presenterar vi tre ekvationer som har solitonlösningar: Korteweq-de-Vries-ekvationen (KdV-ekvationen), sine-Gordon-ekvationen och den ickelinjära schrödingerekvationen. Den inversa spridningstransformen (IST) är en lösningsmetod som kan tillämpas på ett antal ickelinjära partiella differentialekvationer som har solitonlösningar. Vi presenterar lösningsstegen i IST för fallet med KdV-ekvationen.

Med hjälp av IST härleder vi en två-solitonlösning till KdV-ekvationen. Alla tre ekvationer löses numeriskt med hjälp av finita differensmetoden. Därigenom bekräftas existensen av solitonlösningar, liksom andra häri presenterade teoretiska resultat, inklusive resultat rörande kollisioner mellan olika typer av solitoner. I de fall där en direkt jämförelse kan göras mellan analytiska och numeriska lösningar är överensstämmelsen väldigt god.

Contents

Chapter 1

Introduction

Part of the 2009 Nobel Prize in Physics was awarded to Charles Kao for his prediction that glass fibres could be used as a means of long-distance communication. The fact that glass fibres can be used to guide beams of light had long been known, although the efficiency was too low for long-distance communication. Kao suggested that imperfections in the glass were the reason for the low efficiency. Only a few years after his prediction Kao was proven right: in purified glass, light transmission is greatly improved. Admittedly, the signal is still somewhat attenuated even in this improved glass, but it can be reinforced with the help of optical amplifiers. Also, one can use a signal which does not change shape as it propagates. Such a signal is called a soliton. [11]

A soliton is a certain kind of wave which travels with permanent speed and form. This is however not the only special property; another characteristic is that solitons maintain their shape and speed after collisions with other solitons. Solitons can be found in a variety of materials and are used for many different purposes. In some substances, they may carry electrical charge. When charged solitons travel through certain polymer chains, the chains tend to curve. This property may one day be used in applications such as artificial muscles. Solitons are also used in communication, as previously mentioned, since they can transfer large amounts of information over large distances with no errors in the signal. [6]

The first observation of a wave with similar characteristics to those of a soliton was made in 1834 by John Scott Russell. This was the beginning of a whole new field of study to which scientists and mathematicians over the years have contributed extensively. In general, a soliton is obtained as the solution of a nonlinear partial differential equation (PDE). Today there are a number of equations known to have soliton solutions. The most famous of these is the Korteweg-de Vries equation (KdV equation), which is our main focal point.

We will begin this report with a brief preliminary discussion of a few basic concepts, after which we will give an overview of the historical events and discoveries which have lead to the advent of soliton theory. We will then describe the KdV equation more in detail, together with two other equations that have soliton solutions, namely the sine-Gordon equation (SG equation) and the nonlinear Schrödinger equation (NLS equation). Also, we shall describe a method of solution that can be applied to a number of nonlinear PDEs which have soliton solutions, including those mentioned. This method is known as the inverse scattering transform (IST), and will be presented using the example of the KdV equation.

In our investigation we will, using the IST, derive a solution of the KdV equation which consists of two solitons. We will then perform a number of simulations, whereby we shall study collisions between solitons which are obtained as solutions of the three aforementioned equations. Finally, we will compare our numerical results with some analytical results, including the two-soliton solution of the KdV equation which we will derive.

Chapter 2

Background Material

We begin this chapter by defining and discussing some basic concepts which will be needed in what follows. We then provide a fairly brief introduction to the historical events and discoveries which have led to the advent of soliton theory. Furthermore we present the three aforementioned PDEs which admit soliton solutions, and finally we describe the IST method for the case of the KdV equation.

2.1 Basic Concepts

2.1.1 Solitary Waves and Solitons

A *solitary wave* is a localised wave which propagates with unchanging shape and constant speed in one spatial direction only. A *soliton* is a special kind of solitary wave, which is obtained as the solution of a nonlinear PDE. The distinctive characteristic of a soliton is that it can collide with other solitons, after which both solitons re-emerge with their original form and speed. [10]

It should be pointed out that other definitions than those given above do exist. In particular, there are more formal mathematical definitions [2] of the concept of a soliton, as well as less strict definitions [10] frequently used in physics. However, the definition given above is sufficient for our purposes.

2.1.2 Nonlinearity and Dispersion

According to the well known superposition principle, it is the case that if y_1 and y_2 are solutions of a given linear differential equation, then $y_1 + y_2$ is also a solution. This is true for both ordinary differential equations and PDEs. We are primarily concerned with PDEs of two variables; one is interpreted as a time variable and the other as a space variable.

Consider a plane wave solution of a PDE of the aforementioned type. Such a solution has the form

$$u = Ce^{i(kx-\omega t)}, \tag{2.1}$$

where C is a constant, k is the wave number and ω is the angular frequency. In general there exists some relation $\omega = \omega(k)$ between the wave number and the angular frequency. This relation, which depends on the PDE of which Eq. (2.1) is a solution, is called the *dispersion relation*. The velocity c of the plane wave (2.1) is given by $c(k) = \omega(k)/k$.

Now consider a pulse-like wave, which can be written as a superposition of plane waves. According to the superposition principle, if Eq. (2.1) is a solution of a given linear PDE, then any sum of terms of the form of Eq. (2.1) is also a solution. Thus the aforementioned pulse-like wave is a solution. If the dispersion relation is linear (in which case the PDE is said to be *nondispersive*), then each of the plane wave components of the pulse move with the same velocity, and thus the shape of the pulse remains constant. If, on the other hand, the dispersion relation is not linear (in which case the PDE is said to be *dispersive*), then the velocity c has a non-trivial dependence on k. Hence the plane wave components travel with different velocities, and the pulse does not maintain a constant shape, but spreads out with time. Thus we conclude that a linear PDE can have solutions which represent a wave which travels with unchanging shape only if it is nondispersive.

Solutions of nonlinear nondispersive PDEs cannot represent a wave which travels with unchanging shape. Instead, such solutions gradually steepen until they break, i.e. become multi-valued. Only in the presence of dispersion can the solution of a nonlinear PDE be a travelling wave with constant shape. This is because the effects of dispersion, which tend to spread out a pulse, can balance the effects of nonlinearity, which tend to steepen a pulse. [8]

2.2 Historical Background

The field of soliton theory, and nonlinear studies in general, differs from many other fields in that it has clearly traceable origins, beginning with the work of a single individual [1]. We now describe how the work of him and others led to the discovery of solitons, and thus to a whole new field of study in mathematics and physics.

2.2.1 Russell's Discovery of Solitary Waves

The first observation of a solitary wave was made in 1834 by the Scottish scientist and engineer John Scott Russell. He first encountered this phenomenon whilst observing a boat on the Edinburgh-Glasgow canal, and his original account of this event is repeated below. [12]

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed.

Russell, who in fact coined the phrase "solitary wave", conducted extensive water tank experiments in order to investigate this new phenomenon. He generated solitary waves by dropping weights in one end of a tank and by releasing accumulations of water through the removal of partitions (Fig. 2.1). He found that depending on the amount of water displaced, one or several solitary waves can be formed. In either case, a residual wave train may or may not also be formed. Russell also discovered that solitary waves of elevation can be formed, but not waves of depression. Attempts at creating such a wave always result in the creation of an oscillatory wave train with gradually decreasing amplitude. [10, 12]

Furthermore, Russell found empirically that the speed of propagation c of a solitary water wave is given by

$$c^2 = g(h + a),$$

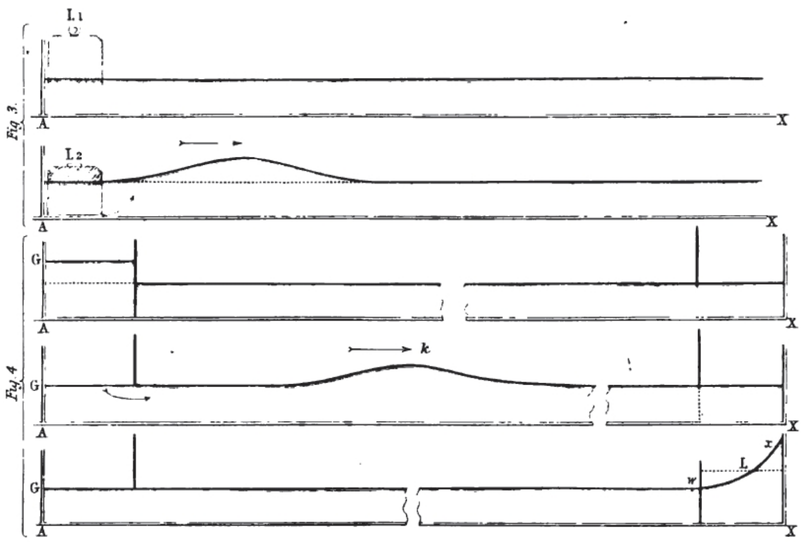

Figure 2.1: Illustration of water tank experiments carried out by Russell. Original image from Ref. [12].

where a is the amplitude of the wave, h the depth of water in equilibrium and g the acceleration of gravity. From this it follows that waves of greater amplitude travel faster. Hence, if a wave of greater amplitude is initially behind a wave of smaller amplitude, the former will eventually catch up and collide with the latter. After such a collision has taken place, both waves re-emerge with their original shape and velocity. This was determined experimentally by Russell. [10, 12]

At the time Russell made his observations public, they appeared to contradict the existing water wave theories, which predicted that a wave cannot propagate without change of form. The disagreement occurred because the then current theory neglected dispersion, which, as noted above, tends to prevent waves from steepening. However, it was shown by Joseph Boussinesq in 1871, and independently by Lord Rayleigh in 1876, that the effects of nonlinearity and dispersion can balance each other out in such a way that waves propagate without change in shape. [10]

2.2.2 Korteweg-de Vries Equation

In 1895 the Dutch scientists Diedrik Korteweg and Gustav de Vries derived an equation which describes the propagation of waves on the surface of a shallow channel. Subsequently, the equation has been named after its discoverers. If the depth of the channel is l, and η is the elevation of the surface above equilibrium level, then the wave motion is governed by

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{l}}\frac{\partial}{\partial x}\left(\frac{2}{3}\alpha\eta + \frac{1}{2}\eta^2 + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial x^2}\right), \tag{2.2}$$

where ρ is the density of the water, T is the surface tension, α is an arbitrary constant and $\sigma = l^3/3 - Tl/\rho g$ [1]. This equation can be transformed into a simpler form by letting

$$\eta = -\frac{2}{3}\alpha\,(6u+1)\,, \quad \tau = \sqrt{\frac{2\alpha^3 g}{\sigma l}}t \quad \text{and} \quad \xi = -\sqrt{\frac{2\alpha}{\sigma}}x.$$

With these transformations Eq. (2.2) becomes

$$u_\tau - 6uu_\xi + u_{\xi\xi\xi} = 0, \tag{2.3}$$

which is the form we use. The factor 6 in the second term is chosen for convenience, and can be rescaled through the transformation $u \to \beta u$.

Korteweg and de Vries found a localised, travelling-wave solution of the KdV equation which corresponds to the solitary waves discovered by Russell. In section 2.3.1 we derive this solution. We also demonstrate the existence of multi-soliton solutions, some of which correspond to the multiple solitary waves also found by Russell.

Almost 60 years after the discoveries of Korteweg and de Vries, Enrico Fermi, John Pasta and Stan Ulam (FPU) were studying the heat conductivity of solids. They modelled the solid as a one-dimensional chain of masses connected by weakly nonlinear springs. In the case of linear springs, the energy in each normal mode of the system remains constant. FPU expected that the introduction of nonlinearity would lead to an even distribution of the energy among the normal modes. However, the results of numerical experiments performed by FPU contradicted this hypothesis. [4]

The relevance of the above to us is that the unexpected results obtained by FPU lead to further study of nonlinear systems, which in turn lead to the development of soliton theory. Although the FPU-problem described above may appear to have nothing to do with solitons, there is in fact a connection. The equation of motion for a particle in the above chain model can namely, in the continuum limit and using certain approximations, be transformed into the KdV equation. [9]

2.2.3 Discovery of Solitons

Solitons were discovered in 1965 by the applied mathematicians Martin Zabusky and Norman Kruskal (ZK), who studied the KdV equation numerically. The word soliton stems from the greek "on", meaning particle. The name was coined by ZK due to the particle-like behaviour of the pulses they discovered.

ZK showed that the FPU-problem leads, in the continuum limit, to the KdV equation in the form

$$u_t + 6uu_x + \delta^2 u_{xxx} = 0,$$

where δ is a small parameter. ZK used the value $\delta = 0.022$ and imposed periodic boundary conditions, as well as a sinusoidal initial condition. They found that the part of the curve whose derivative was negative initially steepened. Also, small oscillations formed immediately to the left of the region of negative slope. Eventually these oscillations grew and separated from one another, each forming a pulse very similar to the solitary wave solution of the KdV equation. The pulses were of different amplitude, the larger ones being further to the right and travelling faster. Due to the periodic boundary conditions, the larger pulses eventually caught up with the smaller ones. [9, 14]

Then something remarkable occurred: the larger pulses collided with the smaller ones, after which both the larger and smaller pulses re-emerged with their original form and velocity. The only trace of the nonlinear interaction was a small phase shift through which the larger pulses were further ahead, and the smaller pulses further behind, than they would have been if they had undergone a linear interaction. When two pulses of approximately equal amplitude collided, the pulse which was ahead grew as it came into contact with the leading edge of the other pulse, which simultaneously shrunk. This continued until both pulses had assumed the original size and shape of the other, after which they separated. If there was a large difference between the amplitudes, the larger pulse simply rode over the smaller one. These findings are confirmed by our simulations. [9, 14]

2.3 Equations with Soliton Solutions

In this section we present three equations which have soliton solutions. We describe the contexts in which these equations arise, derive some simple solutions and discuss some properties of these. This has already been done in part for the KdV-equation, which is our main focal point. Although there are more equations with soliton solutions than those presented here, we limit our presentation to those equations which we study numerically and, to some extent, analytically. It should be pointed out that soliton equations are very special, as most nonlinear PDEs which have travelling wave solutions do not have soliton solutions [1].

Of particular interest to us are so called multi-soliton solutions, since these admit the study of soliton collisions. We will, however, not be able to deal with these very effectively until we reach the next section, in which we describe the IST method.

2.3.1 Korteweg-de Vries Equation

As previously mentioned, the KdV equation arises in the context of the study of shallow water waves. The form of the KdV equation which we use is

$$u_t - 6uu_x + u_{xxx} = 0, \tag{2.4}$$

which is identical to Eq. (2.3). Note that all the coefficients can be rescaled to arbitrary nonzero values through appropriate transformations. The choice of having a negative nonlinear term results in the soliton solutions of Eq. (2.4) having negative sign. In general, a solution of Eq. (2.4) consists of two parts: a number of rightward moving solitons, and a leftward moving dispersive wave train. Typically a solution incorporates both solitons and dispersive waves, but this is not necessary. If $u(x,0) > 0$ then $u(x,t)$ will for $t > 0$ consist of dispersive waves only. The solutions which we are most interested in are the pure soliton solutions, which contain no dispersive waves. [3]

We now derive the one-soliton solution of the KdV equation [2, 3]. Since we are interested in travelling wave solutions, we make the ansatz $u(x,t) = f(\xi)$, where $\xi = x - ct$ and c is a constant. By inserting this into Eq. (2.4) we get

$$-cf' - 6ff' + f''' = 0.$$

Integrating this equation gives

$$-cf - 3f^2 + f'' = A,$$

where A is a constant of integration. If we multiply by f' and integrate once again, this yields

$$\frac{1}{2}(f')^2 = f^3 + \frac{1}{2}cf^2 + Af + B,$$

where B is another constant of integration. Since we are interested in soliton solutions, which are localised, we require that $f, f', f'' \to 0$ as $\xi \to \pm\infty$. From this it follows that $A = B = 0$, and thus

$$(f')^2 = f^2(2f + c).$$

It is clear that $2f + c \geq 0$ must hold in order for a real solution to exist. We proceed by writing the above as

$$\int \frac{df}{f\sqrt{2f + c}} = \pm \int d\xi.$$

We now make the substitution

$$f = -\frac{c}{2\cosh^2\theta},$$

which yields

$$-2\int \frac{d\theta}{\sqrt{c}} = \pm \int d\xi.$$

This gives us the solution

$$f(\xi) = -\frac{c}{2\cosh^2 \frac{1}{2}\sqrt{c}(\xi - x_0)},$$

where x_0 is a constant of integration. If we recall that $u(x,t) = f(x - ct)$, we get the final solution

$$u(x,t) = -\frac{c}{2\cosh^2 \frac{1}{2}\sqrt{c}(x - ct - x_0)}. \tag{2.5}$$

The one-soliton solution (2.5) contains two arbitrary constants: c and x_0. The constant x_0 simply determines the position of the soliton at $t = 0$. Apart from being the speed of propagation, c determines the amplitude and breadth of the soliton: taller solitons are narrower and travel faster. Although the solution (2.5) exists for all $c \geq 0$, it is only up to a certain value of c that Eq. (2.5) is valid as an approximation of a water wave. Note that $u(x,t) \leq 0$ in Eq. (2.5); as mentioned above, this corresponds to the negative sign of the nonlinear term in Eq. (2.4).

Apart from the one-soliton solution (2.5) there exist solutions of Eq. (2.4) which contain an arbitrary number of solitons. An example of a multi-soliton solution is the two-soliton solution

$$u(x,t) = -12\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{[3\cosh(x - 28t) + \cosh(3x - 36t)]^2}. \tag{2.6}$$

A common feature of all multi-soliton solutions of the KdV equation is that they asymptotically approach a superposition of one-soliton solutions [of the form of Eq. (2.5)] as $t \to \pm\infty$. For the case of Eq. (2.6) we have that

$$u(x,t) \sim -\sum_{n=1}^{2} \frac{c_n}{2\cosh^2 \frac{1}{2}\sqrt{c_n}(x - c_n t - x_n^{\pm})} \quad \text{for} \quad t \to \pm\infty$$

with $c_1 = 16$ and $c_2 = 4$. This is the justification for describing Eq. (2.6) as a two-soliton solution. The constants $x_n^{\pm}$ have different values for $t \to +\infty$ and $t \to -\infty$; this is because the solitons are phase shifted as they collide. [2, 3]

2.3.2 Sine-Gordon Equation

The SG equation is normally written in one of two forms: the laboratory coordinates

$$u_{xx} - u_{tt} = \sin u, \tag{2.7}$$

or the original form

$$u_{\xi\eta} = \sin u. \tag{2.8}$$

Introducing the substitutions $\xi = \frac{1}{2}(x-t)$ and $\eta = \frac{1}{2}(x+t)$, Eq. (2.7) can be transformed into Eq. (2.8). [2]

The SG equation originates from differential geometry, in which it describes a certain kind of surface. The SG equation appears in a number of physical applications, including dislocations in crystals and the motion of a rigid pendulum attached to a stretched wire [2]. Also, the SG equation can be used to describe elementary particles and the propagation of ultra short optic pulses in lasers [3]. The name sine-Gordon is a pun on the Klein-Gordon equation $u_{xx} - u_{tt} = u$, the form of which is quite similar to that of Eq. (2.7) [2].

A one-soliton solution of the SG equation in the form of Eq. (2.7) is

$$u(x,t) = 4\arctan\left(\exp\frac{x-\lambda t}{\sqrt{1-\lambda^2}}\right), \tag{2.9}$$

where $-1 < \lambda < 1$. A solution of the form of Eq. (2.9), for which u increases monotonically from zero to 2π as x increases from $-\infty$ to ∞, is called a kink. There exists one other kind of soliton solution, called an antikink, for which u decreases from 2π to zero as x increases from $-\infty$ to ∞. Also, adding an integral multiple of 2π to u gives similar kinks and antikinks. In general, a solution of the SG equation can contain both kinks and antikinks. The SG equation has an infinite set of trivial solutions $u(x,t) = n\pi$, where n is an integer, and the soliton solutions interpolate between different trivial solutions. [2]

Finally, we present two multi-soliton solutions of the SG equation. A kink-kink solution, for which two kinks collide with each other, is

$$u(x,t) = 4\arctan\frac{\lambda\sinh\left(x/\sqrt{1-\lambda^2}\right)}{\cosh\left(\lambda t/\sqrt{1-\lambda^2}\right)}. \tag{2.10}$$

A kink-antikink solution, for which a kink and an antikink collide, is

$$u(x,t) = 4\arctan\frac{\lambda\cosh\left(x/\sqrt{1-\lambda^2}\right)}{\sinh\left(\lambda t/\sqrt{1-\lambda^2}\right)}. \tag{2.11}$$

As $t \to \pm\infty$, the kink-kink solution (2.10) approaches a superposition of two kinks, and the kink-antikink solution (2.11) approaches a superposition of a kink and an antikink. [2]

2.3.3 Nonlinear Schrödinger Equation

The NLS equation is given by

$$iu_t + \frac{1}{2}u_{xx} + \kappa|u|^2u = 0, \tag{2.12}$$

where u is a complex function and κ is an arbitrary nonzero real constant. The soliton solutions of the NLS equation are represented by $|u|$. Through scale transformations, the coefficients in Eq. (2.12) can be almost arbitrarily chosen, so that fundamentally there are two variants of the NLS equation. These are

$$iu_t + u_{xx} \pm |u|^2u = 0, \tag{2.13}$$

where the variant with a plus sign in front of the last term is the one we use. This variant has a common property with the KdV equation [10], namely that an initial profile evolves into a number of solitons and a dispersive tail. [5]

The NLS equation describes phenomena in nonlinear media with strong dispersion, e.g. the self-focusing of an optical pulse [13]. The NLS equation is important in nonlinear optics, where it is used, among other things, in the construction of soliton lasers. In a

$$
\begin{array}{ccc}
u(x,0) = u_0(x) & \xrightarrow{\text{Scattering}} & \kappa_n(0), c_n(0), b(k;0) \\
\text{KdV equation} \Big\downarrow & & \Big\downarrow \text{Time evolution} \\
u(x,t) & \xleftarrow{\text{Inverse scattering}} & \kappa_n(t), c_n(t), b(k;t)
\end{array}
$$

Figure 2.2: Diagram illustrating the steps involved in the IST method.

soliton laser, a laser amplifier sends solitons into an optical fibre. The output from the fibre is then sent back into the amplifier, where the pulses are reshaped and amplified. Since the NLS equation is scale-invariant, the equation remains the same for the pulses as they become shorter and shorter as the process is repeated. The idea is to produce a final result of narrow clean soliton pulses, which then are used in information technology and other areas of application. [5]

The one-soliton solutions of the NLS equation (2.12) are of the form

$$u(x,t) = \sqrt{\rho(x,t)}e^{i\sigma(x,t)}. \tag{2.14}$$

If $\kappa > 0$, the solution is called an envelope soliton or a bright soliton and is given by

$$\rho = \frac{\rho_0}{\cosh^2 \sqrt{\kappa\rho_0}\epsilon(x - vt)} \quad \text{and} \quad \sigma = \frac{\kappa}{2}\epsilon\rho_0 t.$$

If $\kappa < 0$, the solution is called a dark soliton or dark hole and is given by

$$\rho = \rho_0 \left[1 - a^2/\cosh^2 \sqrt{-\kappa\rho_0}a\epsilon(x - vt)\right]$$

and

$$\sigma = \sqrt{-\kappa}\left[\sqrt{\rho_0\left(1 - a^2\right)}\epsilon(x - vt) + \arctan\left(\frac{a}{\sqrt{1 - a^2}}\tanh\sqrt{-\kappa\rho_0}a\epsilon(x - vt)\right)\right] - \frac{\kappa}{2}\epsilon^2\rho_0\left(3 - a^2\right)t.$$

In the above, v is the velocity of the solitons, ρ_0 determines the amplitude and $0 \leq a \leq 1$. [10]

2.4 Inverse Scattering Transform

In this section we describe the IST, which, as mentioned above, is a method of solution for initial value problems for certain nonlinear PDEs. The IST is treated in all books on introductory soliton theory, including Ref. [1, 2, 9, 10]. Our presentation is based mostly on Ref. [2, 3], the latter of which contains a brief introduction to the topic.

The IST can be applied to a number of PDEs which have soliton solutions, including the KdV equation, the SG equation and the NLS equation. The basic idea of the IST is akin to that of solving PDEs by Fourier transformation, in that it consists of an initial transformation of the problem into an alternative form, followed by the solution of this alternative problem and finally by an inverse transformation, which yields the solution of the original problem. These steps are illustrated in Fig. 2.2 for the case of the KdV equation; for other equations the steps are analogous. The meaning of the components of Fig. 2.2 are explained below.

It can be shown that all PDEs which have soliton solutions are equivalent to the so called Lax equation

$$L_t = LB - BL, \tag{2.15}$$

where L and B are some linear differential operators. In general, the eigenvalue problem $L\psi = \lambda\psi$ forms part of the IST solution of the corresponding PDE. If we choose

$$L = -\frac{\partial^2}{\partial x^2} + u(x) \quad \text{and} \quad B = -4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3\frac{\partial u}{\partial x}$$

then Eq. (2.15) becomes equivalent to the KdV equation (2.4). In this case, the eigenvalue problem of interest is

$$\left(-\frac{d^2}{dx^2} + u(x,t)\right)\psi(x) = \lambda\psi(x). \tag{2.16}$$

For equations other than the KdV equation, the corresponding eigenvalue problem is more complicated than Eq. (2.16). Therefore we focus on the KdV equation. Hence our goal is to solve the initial value problem

$$u_t - 6uu_x + u_{xxx} = 0, \quad u(x,0) = u_0(x).$$

2.4.1 Scattering Problem

The analogue in the IST method of performing a Fourier transformation is the determination of the so called *scattering data*, which are defined below. This involves Eq. (2.16), which is the time-independent Schrödinger equation for a particle moving in one dimension under the influence of the potential $u(x,t)$. It is important to realise that t plays the role of a parameter and has nothing to do with the time of the time-dependent Schrödinger equation.

We assume that $u(x,t)$ decays rapidly as $|x| \to \infty$, so that

$$\int_{-\infty}^{\infty} (1+|x|)|u(x,t)|dx < \infty \tag{2.17}$$

holds for all t. The reasons for making this assumption, rather than just assuming (say) that $u(x,t)$ is absolutely integrable with respect to x, are quite technical and are not discussed here.

The first step in the IST method is to insert the initial condition $u_0(x)$ into Eq. (2.16) as the potential. The behaviour of the solution of Eq. (2.16) as $x \to \pm\infty$ for the case $u(x,t) \to 0$ as $x \to \pm\infty$ is well known, and depends on the sign of λ. (The case $\lambda = 0$ does not occur, except for the trivial potential $u_0(x) = 0$.) For $\lambda = k^2 > 0$ we have

$$\psi(x;k) \sim \begin{cases} e^{-ikx} + b(k)e^{ikx} & \text{for } x \to +\infty \\ a(k)e^{-ikx} & \text{for } x \to -\infty. \end{cases} \tag{2.18}$$

Note that this holds for all $k > 0$, and thus for all $\lambda > 0$. In quantum mechanics the first term in Eq. (2.18) is interpreted as the wave function of an incident ray of particles, the term with factor $b(k)$ is interpreted as the wave function of a reflected ray and the term with factor $a(k)$ is interpreted as the wave function of a transmitted ray. Thus $a(k)$ and $b(k)$ describe how particles incident from $x = +\infty$ are scattered by the potential in question. It can be shown that $a(k)$ and $b(k)$ must fulfil the relation

$$|a(k)|^2 + |b(k)|^2 = 1 \tag{2.19}$$

(this is equivalent to the requirement that the total number of particles is conserved). From Eq. (2.19) it follows that we only need one of $a(k)$ and $b(k)$ to describe the scattering. We choose $b(k)$, and this is (by definition) the scattering data for the case $\lambda > 0$, also known as the *continuous spectrum*.

We now consider the remaining case $\lambda = -\kappa^2 < 0$, for which

$$\psi(x) \sim ce^{-\kappa x} + de^{\kappa x} \quad \text{for} \quad x \to +\infty \tag{2.20}$$

since $u(x,t) \to 0$ as $x \to +\infty$. In quantum mechanics, $|\psi(x)|^2$ is interpreted as the probability density for finding the particle in question at x. This gives rise to the condition

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1. \tag{2.21}$$

From Eq. (2.21) it follows that $d = 0$ in Eq. (2.20). Furthermore, there are only a finite number of values of κ, $\kappa_1 < \kappa_2 < \cdots < \kappa_N$, which are consistent with Eq. (2.21). To these values of κ there corresponds values of c: $c_1, c_2, \ldots, c_N$. The set of acceptable values of κ, which constitutes the so called *discrete spectrum*, and the corresponding values of c constitute the scattering data for the case $\lambda < 0$.

In total, the scattering data for the potential $u = u_0(x)$ in Eq. (2.16) consists of

$$\{\kappa_n\}_{n=1}^N, \quad \{c_n\}_{n=1}^N \quad \text{and} \quad \{b(k)\}_{k\in\mathbb{R}}.$$

2.4.2 Time Evolution of the Scattering Data

Thus far we have determined the scattering data for the potential $u_0(x) = u(x,0)$ in Eq. (2.16). The next step, as illustrated in Fig. 2.2, is to determine how the scattering data evolves with time, assuming that $u(x,t)$ satisfies the KdV equation. It can be shown (see pp. 67-71 in Ref. [2]) that the time evolution is given by

$$\kappa_n(t) = \kappa_n(0), \quad c_n(t) = c_n(0)e^{4\kappa_n^3 t}, \quad b(k;t) = b(k;0)e^{8ik^3 t}. \tag{2.22}$$

Note that the discrete spectrum is constant in time.

2.4.3 Inverse Scattering Problem

It is clear that the scattering data are uniquely determined by the potential u in Eq. (2.16). In fact, the converse is also true, i.e. the potential u is determined by the scattering data. Finding u from the scattering data is what constitutes the inverse scattering problem. This is a difficult problem, the complete solution of which is not presented here.

The first step in the determination of u is to form a function $F(X;t)$ from the scattering data;

$$F(X;t) = \sum_{n=1}^{N} c_n^2(0) \exp\left(8\kappa_n^3 t - \kappa_n X\right) + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k;0) \exp\left(8ik^3 t + ikX\right) dk. \tag{2.23}$$

The next step is to insert $F(X;t)$ into the so called Marchenko equation

$$K(x,z;t) + F(x+z;t) + \int_x^{\infty} K(x,y;t)F(y+z;t)dy = 0. \tag{2.24}$$

This equation is solved for $K(x,z;t)$, after which u is given by

$$u(x,t) = -2\frac{\partial}{\partial x}\hat{K}(x,t), \tag{2.25}$$

where $\hat{K}(x,t) = K(x,x;t)$.

We have now, in principle, solved the initial value problem for the KdV equation. The method of solution we have presented contains two potentially difficult steps: the solution of the time independent Schrödinger equation (2.16), and the solution of the Marchenko equation (2.24). However, the Marchenko equation is a so called Fredholm integral equation, which can be solved using standard techniques. Finally, it is worth noting that the problem of solving a nonlinear PDE (the KdV equation) has been reduced to the problem of solving the two *linear ordinary* equations (2.16) and (2.24).

Chapter 3

Investigation

The main purpose of our investigation is to study soliton collisions. We do this for the KdV equation, the SG equation and the NLS equation. To this end we perform analytical calculations based on the IST method, as well as numerical simulations based on the finite difference method (FDM).

3.1 Analytical Calculations

In this section we solve the initial value problem for the KdV equation for a certain special case, using the IST. The initial condition we use constitutes a so called *reflectionless potential*. The defining characteristic of a reflectionless potential is that $b(k) = 0$ for all k. From Eq. (2.22), we see that if this is true for $t = 0$ it is also be true for all $t > 0$.

The reason for focusing on a reflectionless potential is that if $b(k) \neq 0$ then the Marchenko equation cannot be solved in closed form [2]. Also, reflectionless potentials correspond to pure soliton solutions, whereas solutions for which $b(k) \neq 0$ contain dispersive waves [3]. Naturally, the calculations we perform can be found in the literature, e.g. in Ref. [2].

In general, it can be shown that the initial condition

$$u(x,0) = -\frac{N(N+1)}{\cosh^2 x} \tag{3.1}$$

yields an N-soliton solution [2]. We choose $N = 2$, i.e. $u(x,0) = -6/\cosh^2 x$. With this choice the time independent Schrödinger equation (2.16) becomes

$$\psi''(x) + \left(\lambda + \frac{6}{\cosh^2 x}\right)\psi(x) = 0. \tag{3.2}$$

If we make the substitution $T = \tanh x$, we have that

$$\frac{d}{dx} = \frac{1}{\cosh^2 x}\frac{d}{dT} = \left(1 - T^2\right)\frac{d}{dT}.$$

Hence Eq. (3.2) becomes

$$\left(1 - T^2\right)\frac{d}{dT}\left(\left(1 - T^2\right)\frac{d\psi}{dT}\right) + \left[\lambda + 6\left(1 - T^2\right)\right]\psi = 0,$$

which can be written

$$\frac{d}{dT}\left(\left(1 - T^2\right)\frac{d\psi}{dT}\right) + \left(6 + \frac{\lambda}{1 - T^2}\right)\psi = 0.$$

This is an associated Legendre equation, the bounded solutions of which are the associated Legendre polynomials

$$P_2^1(T) = -3T\sqrt{1-T^2} = -3\frac{\tanh x}{\cosh x}$$

and

$$P_2^2(T) = 3\left(1-T^2\right) = \frac{3}{\cosh^2 x}.$$

By imposing the normalisation condition (2.21) we get the eigenfunctions

$$\psi_1(x) = \sqrt{\frac{3}{2}}\frac{\tanh x}{\cosh x} \quad \text{and} \quad \psi_2(x) = \frac{\sqrt{3}}{2}\frac{1}{\cosh^2 x},$$

with corresponding eigenvalues $\lambda_1 = -\kappa_1^2 = -1$ and $\lambda_2 = -\kappa_2^2 = -4$. In order to determine c_1 and c_2 we study the behaviour of ψ_1 and ψ_2 as $x \to +\infty$, which is given by

$$\psi_1(x) \sim \sqrt{6}e^{-x} \quad \text{and} \quad \psi_2(x) \sim 2\sqrt{3}e^{-2x}.$$

We see that $c_1(0) = \sqrt{6}$ and $c_2(0) = 2\sqrt{3}$, and thus $c_1(t) = \sqrt{6}e^{4t}$ and $c_2(t) = 2\sqrt{3}e^{32t}$.

As previously mentioned, the continuous spectrum is simply given by $b(k;t) = 0$ (see pp. 46-47 in Ref. [2]). Hence we now know the time evolution of the scattering data, and can construct the function $F(X;t)$ in Eq. (2.23), which becomes

$$F(X;t) = 6e^{8t-X} + 12e^{64t-2X}.$$

By inserting this into the Marchenko equation (2.24) we get

$$K(x,z;t) + 6e^{8t-(x+z)} + 12e^{64t-2(x+z)} + \int_x^\infty K(x,y;t)\left(6e^{8t-(y+z)} + 12e^{64t-2(y+z)}\right)dy = 0. \quad (3.3)$$

This can be rewritten as

$$K(x,z;t) + e^{-z}\left(6e^{8t-x} + 6\int_x^\infty K(x,y;t)e^{8t-y}dy\right) + e^{-2z}\left(12e^{64t-2x} + 12\int_x^\infty K(x,y;t)e^{64t-2y}dy\right) = 0,$$

whence it is clear that the solution for $K(x,z;t)$ must be of the following form:

$$K(x,z;t) = L_1(x,t)e^{-z} + L_2(x,t)e^{-2z}.$$

By inserting this into Eq. (3.3), and noting that terms with factors of e^{-z} and e^{-2z} independently must sum to zero, we get the following pair of equations:

$$L_1 + 6e^{8t-x} + 6e^{8t}\left(L_1\int_x^\infty e^{-2y}dy + L_2\int_x^\infty e^{-3y}dy\right) = 0,$$
$$L_2 + 12e^{64t-2x} + 12e^{64t}\left(L_1\int_x^\infty e^{-3y}dy + L_2\int_x^\infty e^{-4y}dy\right) = 0.$$

By evaluating the integrals we can write these equations as

$$L_1 + 6e^{8t-x} + 3L_1e^{8t-2x} + 2L_2e^{8t-3x} = 0,$$
$$L_2 + 12e^{64t-2x} + 4L_1e^{64t-3x} + 3L_2e^{64t-4x} = 0.$$

The solution of these equations can easily be found to be

$$L_1(x,t) = 6\frac{e^{72t-5x} - e^{8t-x}}{1 + 3e^{8t-2x} + 3e^{64t-4x} + e^{72t-6x}},$$
$$L_2(x,t) = -12\frac{e^{64t-2x} + e^{72t-4x}}{1 + 3e^{8t-2x} + 3e^{64t-4x} + e^{72t-6x}}.$$

According to Eq. (2.25), the solution of the KdV equation is given by

$$\begin{aligned} u(x,t) &= -2\frac{\partial}{\partial x}\left(L_1 e^{-x} + L_2 e^{-2x}\right) \\ &= 12\frac{\partial}{\partial x}\left(\frac{e^{8t-2x} + e^{72t-6x} - 2e^{64t-4x}}{1 + 3e^{8t-2x} + 3e^{64t-4x} + e^{72t-6x}}\right), \end{aligned}$$

which can be rewritten as

$$u(x,t) = -12\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{[3\cosh(x - 28t) + \cosh(3x - 36t)]^2}.$$

This is precisely the two-soliton solution (2.6), which is valid for all values of t, both positive and negative.

3.2 Numerical Analysis

In this section we present numerical methods of solution for the initial value problems for the KdV equation, the SG equation and the NLS equation. We begin by a fairly detailed discussion of the case of the KdV equation, which then carries over in a straight-forward way to the other equations. Descriptions of the FDM, which is used throughout, can be found in any introductory textbook on numerical methods, including Ref. [7].

3.2.1 Korteweg-de Vries Equation

The fundamental idea behind the FDM is to replace the continuous variables x and t by discrete sets $\{x_i\}_{i=0}^{n}$ and $\{t_j\}_{j=0}^{m}$. These sets are chosen so that the x_i and t_j range over the intervals of interest. Ideally these would be $-\infty < x < \infty$ and $t \geq 0$, but for the sake of the numerical implementation we must choose finite sets. For the simulations we perform we choose $-100 \leq x \leq 100$. Also, we let the differences between successive values of x and t be constant, i.e.

$$\Delta x_i = x_i - x_{i-1} = h \quad \text{and} \quad \Delta t_j = t_j - t_{j-1} = k.$$

For brevity we introduce the notation $u(x_i, t_j) = u_{i,j}$.

The replacement of the continuous variables x and t by discrete sets entails the replacement of the KdV equation by a corresponding difference equation. This equation is found by replacing the derivatives in the KdV equation by suitable difference approximations. For the first derivatives we use the central difference approximations

$$u_x(x_i, t_j) \approx \frac{u_{i+1,j} - u_{i-1,j}}{2h} \tag{3.4}$$

and

$$u_t(x_i, t_j) \approx \frac{u_{i,j+1} - u_{i,j-1}}{2k}. \tag{3.5}$$

In order to obtain a difference approximation for the third derivative we begin by writing

$$u_{xxx} \approx \frac{u_{xx}(x_{i+1}, t_j) - u_{xx}(x_{i-1}, t_j)}{2h}, \tag{3.6}$$

that is, we approximate the third derivative from the second derivative using a central difference approximation. A well known difference approximation for the second derivative is

$$u_{xx}(x_i, t_j) \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}. \tag{3.7}$$

By inserting Eq. (3.7) into Eq. (3.6) we get

$$u_{xxx}(x_i, t_j) \approx \frac{u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j}}{2h^3}, \tag{3.8}$$

which is the desired approximation of the third derivative.

We now combine the above approximations in order to obtain the desired difference equation. Insertion of Eqs. (3.4), (3.5) and (3.8) into the KdV equation yields

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} - 6u_{i,j}\frac{u_{i+1,j} - u_{i-1,j}}{2h} + \frac{u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j}}{2h^3} = 0,$$

which can be rewritten as

$$u_{i,j+1} = u_{i,j-1} + 2k\left(3u_{i,j}\frac{u_{i+1,j} - u_{i-1,j}}{h} - \frac{u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j}}{2h^3}\right). \tag{3.9}$$

We have now found an expression for the value of u in a given point in space and time, which contains the values of u for the two previous t-values and the five closest x-values (i.e. the x-value in question and two on either side). This expression can be used to compute $u_{i,j}$ for $2 \leq i \leq n-2$ and $2 \leq j \leq m$, provided that all values of u for $j-1$ and $j-2$ are given. As for the values of $u_{0,j}$, $u_{1,j}$, $u_{n-1,j}$ and $u_{n,j}$, which cannot be computed using Eq. (3.9), we simply set these equal to zero. The justification for this is that in the simulations we perform, the u-values in question are very close to zero anyway, and thus the error introduced by setting them to zero is small.

We are now very close to having obtained an algorithm for the numerical solution of the initial value problem for the KdV equation. Provided that all values of $u_{i,0}$ and $u_{i,1}$ are given, we can use Eq. (3.9) to compute all successive values of $u_{i,j}$. However, the initial condition $u(x,0) = u_0(x)$ only gives us the values of $u_{i,0}$. One way of dealing with this is to simply set $u_{i,1} = u_{i,0}$ for all i and then proceed from there. The other alternative is to compute the values of $u_{i,1}$ from those of $u_{i,0}$ in some more sophisticated way. This can be done by replacing the central difference approximation (3.5) with the forward difference approximation

$$u_t(x_i, t_j) \approx \frac{u_{i,j+1} - u_{i,j}}{k}. \tag{3.10}$$

By making this replacement we obtain

$$u_{i,j+1} = u_{i,j} + k\left(3u_{i,j}\frac{u_{i+1,j} - u_{i-1,j}}{h} - \frac{u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j}}{2h^3}\right). \tag{3.11}$$

This is an equation which is very similar to Eq. (3.9), but differs from Eq. (3.9) in that the right hand side contains only $u_{i-2,j} \ldots u_{i+2,j}$, not $u_{i,j-1}$. Hence Eq. (3.11) can be used to compute the values of $u_{i,1}$ from those of $u_{i,0}$.

3.2.2 Accuracy and Divergence

Out of the two aforementioned alternatives for the computation of the values of $u_{i,1}$ from those of $u_{i,0}$ we choose the latter, although numerical trials suggest that this choice is of little consequence. One might wonder why we do not dispense with Eq. (3.9) and simply use Eq. (3.11) instead. The answer rests on the fact that the central difference approximation (3.5) is a more precise approximation of the time derivative than the forward difference approximation (3.10). More precisely, the error introduced by the replacement of the time derivative by a difference approximation is approximately proportional to k for Eq. (3.10), and to k^2 for Eq. (3.5). A consequence of this is that much larger values of h and k can be used in conjunction with Eq. (3.9) than with Eq. (3.11). This means that much less computing power is required when using Eq. (3.9), since fewer computations need to be performed.

What, then, are the consequences of choosing too large values for h and k? One possibility is that the obtained result is, although qualitatively correct, not accurate enough for the purpose at hand. Another, more dramatic possibility is that divergence occurs, giving very large and nonsensical values for u. It is above all to avoid divergence that we choose Eq. (3.9) over Eq. (3.11). Apart from h and k being sufficiently small, to avoid divergence it is also necessary that k be much smaller (perhaps three orders of magnitude) than h.

It should be pointed out that smaller values of h and k do not necessarily give a more accurate result, at least not when a computer is used for the implementation (which obviously is necessary). This is because very small values of h and k lead to considerable round-off error. There are thus optimal values of h and k which give the highest possible accuracy.

It is not clear (to us) whether applications of Eq. (3.9) or Eq. (3.11) always lead to divergence, or whether suitable values of h and k may give a solution which is stable for arbitrarily large values of t. Although questions such as this no doubt can be investigated theoretically, the information which can be obtained through numerical experiments is sufficient for our purposes.

3.2.3 Sine-Gordon Equation

We begin by replacing the SG equation (2.7) by a suitable difference equation, which is obtained analogously to that for the KdV equation. By using the difference approximation (3.7) for the second derivative we get

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} - \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = \sin u_{i,j},$$

which can be rewritten as

$$u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + k^2 \left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} - \sin u_{i,j} \right). \tag{3.12}$$

This expression can be used to compute $u_{i,j}$ for $1 \le i \le n-1$ and $2 \le j \le m$, provided that all values of u for $j-1$ and $j-2$ are given.

We limit our investigation to kinks and antikinks which are not too close to the edges of the x-interval over which the simulation is performed. Provided we do this, we can set $u_{0,j}$ and $u_{n,j}$ equal to some suitable constants, which are integral multiples of π. The reason for this is of course that u approaches integral multiples of π as $x \to \pm\infty$.

In the case of the KdV equation, we had the problem of determining the values of $u_{i,1}$ from those of $u_{i,0}$. In the case of the SG equation, there is no sensible way to perform such a calculation. The reason for this is that while the KdV equation contains only the first derivative with respect to t, the SG equation contains the second derivative with respect to t. A consequence of this is that both $u(x,0)$ and $u_t(x,0)$ must be given in order for a unique solution of the SG equation to exist. In our numerical scheme this corresponds to the need for both $u_{i,0}$ and $u_{i,1}$ to be given as initial data.

3.2.4 Nonlinear Schrödinger Equation

In order to obtain the desired difference equation we use the approximations (3.5) and (3.7) for the first and second derivative. Insertion of Eqs. (3.5) and (3.7) into the NLS equation (2.13) (with a plus sign in front of the last term) yields

$$i\frac{u_{i,j+1}-u_{i,j-1}}{2k}+\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}+|u_{i,j}|^2u_{i,j}=0,$$

which can be rewritten as

$$u_{i,j+1}=u_{i,j-1}+2ki\left(\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}+|u_{i,j}|^2u_{i,j}\right). \tag{3.13}$$

This equation can be used to compute $\{u_{i,j}\}_{i=1}^{n-1}$ from $\{u_{i,j-1}\}_{i=0}^{n}$ and $\{u_{i,j-2}\}_{i=0}^{n}$. The values of $u_{0,j}$ and $u_{n,j}$ are set to zero; the motivation for this is the same as for the KdV equation. Provided that the values of $u_{i,1}$ and $u_{i,0}$ are given, we can compute all successive values of $u_{i,j}$. As with the KdV equation, we can compute the values of $u_{i,1}$ from those of $u_{i,0}$ using an equation very similar to Eq. (3.13), which is obtained by using the forward difference approximation (3.10) instead of the central difference approximation (3.5).

3.3 Results and Discussion

For the KdV equation, we perform simulations with initial profiles given by Eq. (3.1) with $N=1,2,3$. In each case, the profile evolves into N solitons with the largest, and thus fastest, soliton furthest to the right. As we have seen in section 3.1, the case $N=2$ corresponds to the two-soliton solution (2.6). We compare the exact solution (2.6) with the corresponding numerical solution by plotting them together in the same graph for $0 \le t \le 20$. The agreement is very good, at least provided that h and k are sufficiently small; we use $h=0.05$ and $k=0.00001$.

In order to study a collision between two solitons, we perform a simulation with an initial profile given by the two-soliton solution (2.6) with $t=-0.5$ (Fig. 3.1). (Note that the initial profile corresponds to $t=0$ in the simulation.) The initial profile consists of two well-separated solitons with the larger and faster to the left. As the two solitons collide they momentarily form a single pulse [which is given by Eq. (3.1) with $N=2$], after which they separate and resume their original identities. The formation of a single pulse is characteristic of collisions between solitons with sufficiently large difference in amplitude.

The interaction between solitons shown in Fig. 3.1 is nonlinear, which can be seen in two ways. Firstly, the amplitude of the single pulse formed by the two solitons is smaller than the amplitude of the larger soliton. This would not be the case if the interaction was linear, since in that case the solitons would obey the superposition principle. Secondly, the solitons undergo a phase shift, whereby the larger soliton is further ahead, and the smaller soliton further behind, than they would have been if the interaction had been linear.

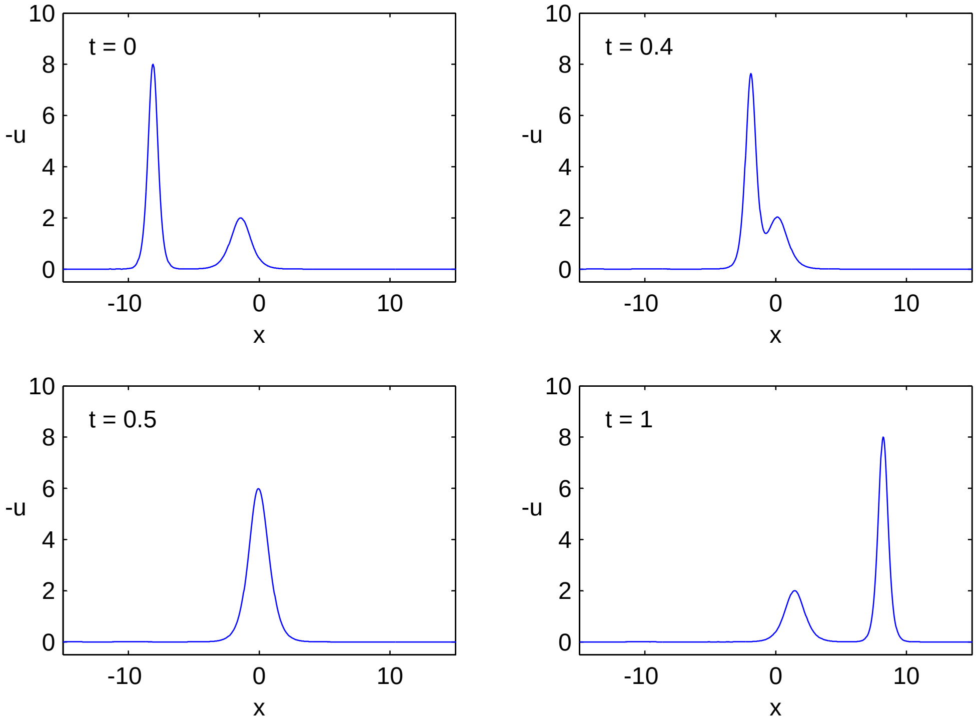

Figure 3.1: Result of simulation of the KdV equation with initial condition given by the two-soliton solution (2.6) with $t = -0.5$. Note that $-u$ is plotted against x.

Next we simulate a collision between solitons of similar amplitude (Fig. 3.2). The initial profile is given by a superposition of two solitons of the form of Eq. (2.5); one soliton has $c = 10$ and $x_0 = 5$, the other has $c = 7$ and $x_0 = 10$. When the leading edge of the larger soliton reaches the rear edge of the smaller soliton, the larger soliton begins to shrink and the smaller soliton begins to grow. This process continues until the solitons have swapped identities, after which they separate. During the entire interaction there are two local maxima; this is characteristic of collisions between solitons of similar amplitude.

Furthermore, we perform a number of simulations with initial profiles given by various superpositions of solitons of the form of Eq. (2.5). Provided that the solitons are initially well-separated, they eventually become ordered by size, with the largest solitons furthest to the right. During this process the solitons may undergo multiple collisions, each of which leads to a phase shift. We also perform simulations with initial profiles which are not given by superpositions of well-separated solitons. In these cases, the initial profile evolves into a number of rightward-moving solitons and a leftward-moving dispersive wave train. An example of such a profile is $u(x, 0) = -4/\cosh^2 x$, which evolves into two solitons and a dispersive wave train (Fig. 3.3). Also, we confirm numerically that initial profiles for which $u > 0$ evolve into a dispersive wave train only.

We perform two simulations for the SG equation: one which shows a collision between two kinks (Fig. 3.4), and one which shows a collision between a kink and an antikink (Fig. 3.5). The initial profile for the kink-kink collision is given by Eq. (2.10) with $t = -10$; the initial profile for the kink-antikink collision is given by Eq. (2.11) with $t = -10$ (in fact, this profile has been shifted upward by 2π so that $u \to 0$ as $x \to \pm\infty$). As seen in section 3.2.3, we must also give the profiles in the second instant in time ($j = 1$). Naturally, these are also given by Eqs. (2.10) and (2.11). Direct comparisons between the numerical solutions and the analytical solutions (2.10) and (2.11) show good

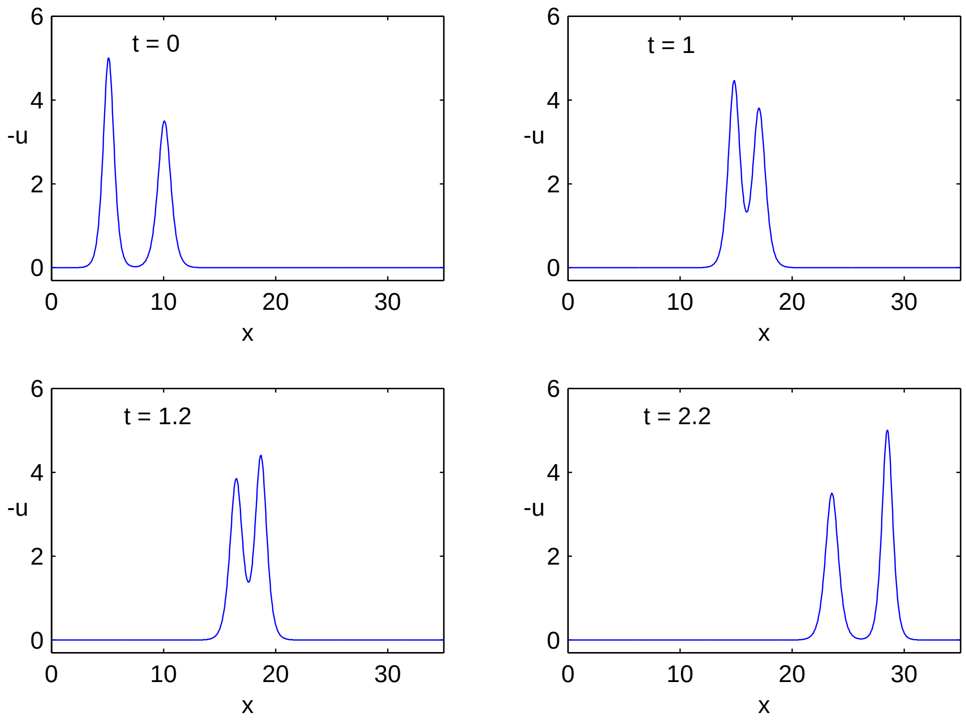

Figure 3.2: Result of simulation of the KdV equation with initial condition given by a superposition of two one-soliton solutions of the form of Eq. (2.5). One soliton has $c = 10$ and $x_0 = 5$, the other has $c = 7$ and $x_0 = 10$. Note that $-u$ is plotted against x.

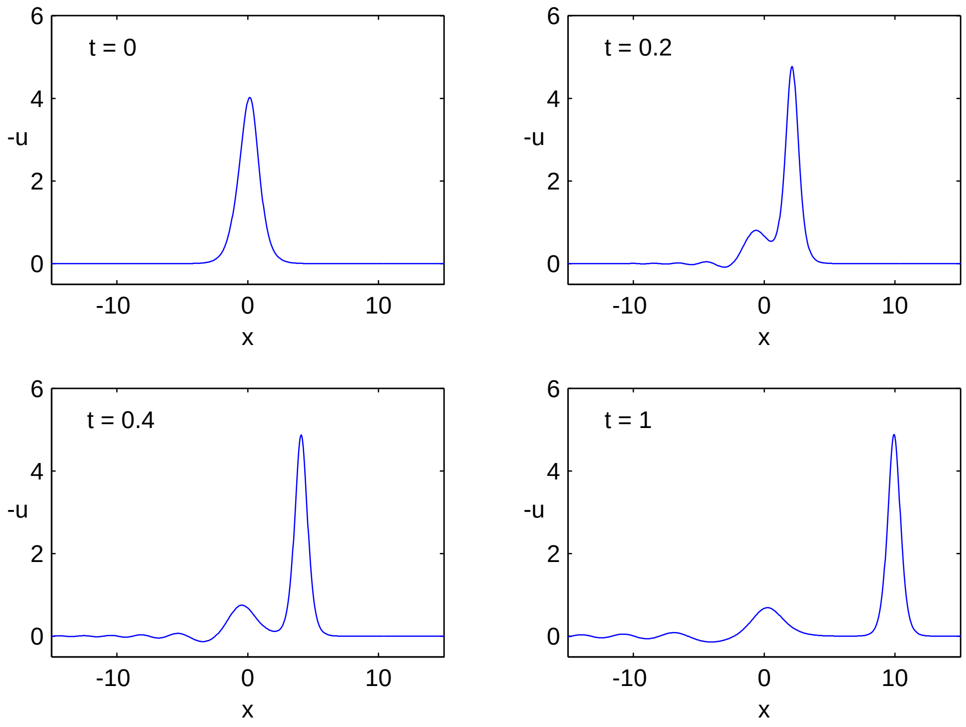

Figure 3.3: Result of simulation of the KdV equation with initial condition $u(x, 0) = -4/\cosh^2 x$. Note that $-u$ is plotted against x.

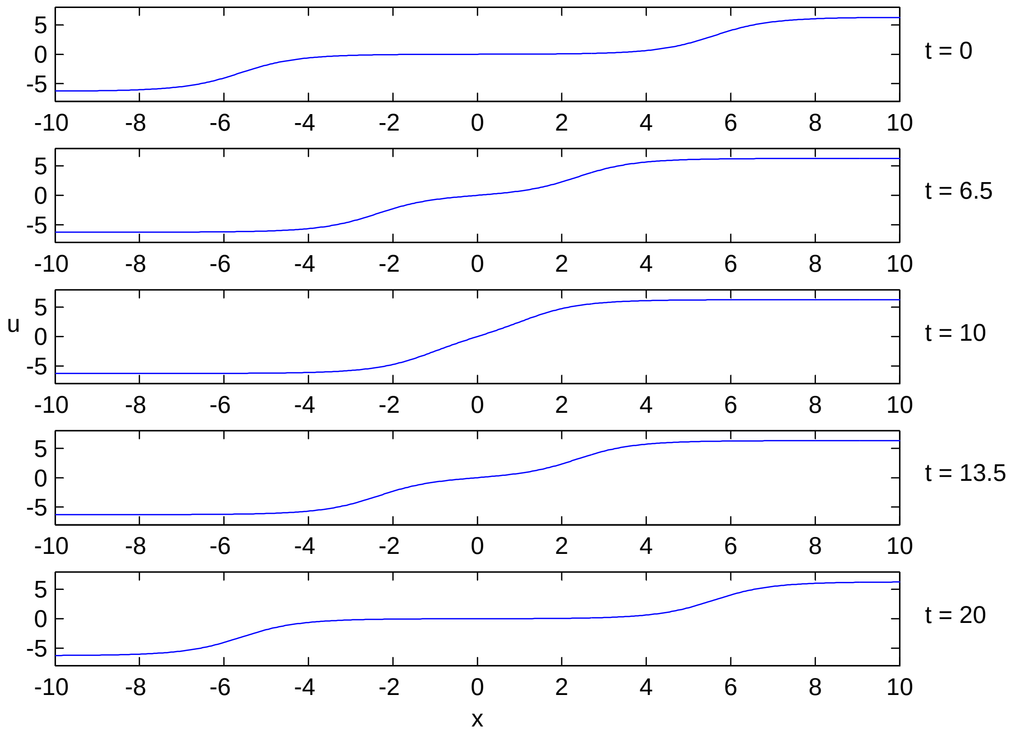

Figure 3.4: Result of simulation of the SG equation with initial condition given by the kink-kink solution (2.10) with $t = -10$.

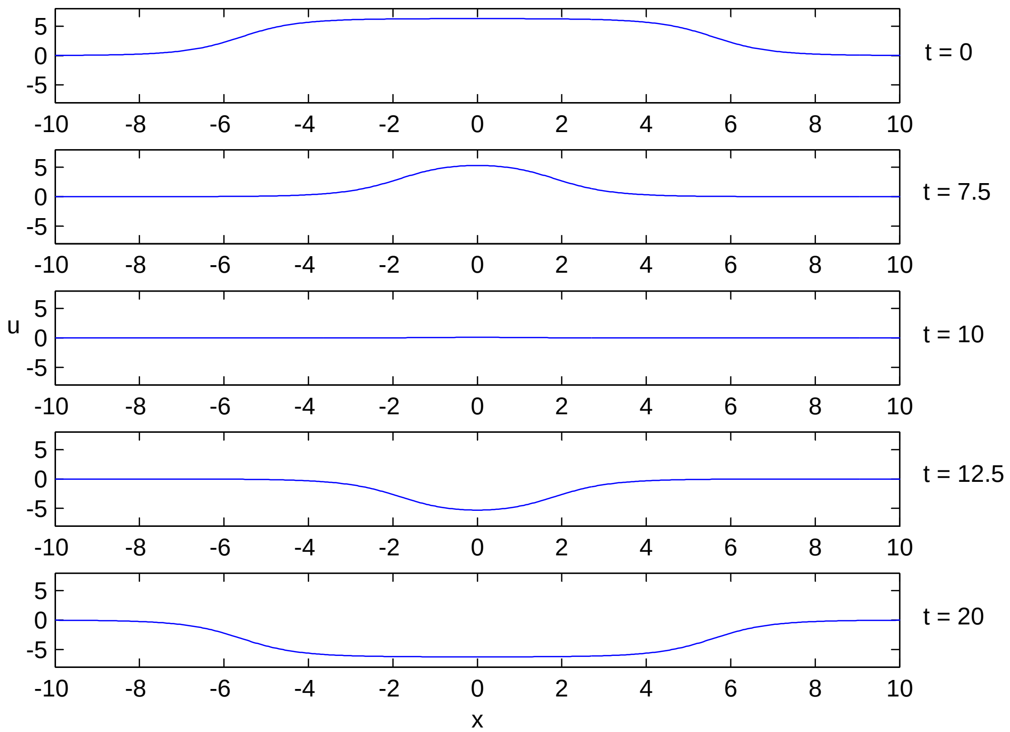

Figure 3.5: Result of simulation of the SG equation with initial condition given by the kink-antikink solution (2.11) with $t = -10$, shifted upward by 2π.

agreement.

In the kink-kink simulation, the kinks initially move inward towards the origin. Eventually they collide, whereby they are reflected and move away from one another. In the kink-antikink simulation, the kink and the antikink initially move inward towards the origin. As they collide, the profile instantaneously becomes equal to zero. Subsequently the kink and the antikink move away from one another in their original directions of motion. During the collision the kink and the antikink are both shifted downward, so that $u < 0$ after the collision, whereas initially $u > 0$.

Finally, we perform a simulation for the NLS equation which shows a collision between two bright solitons [corresponding to the choice of a positive last term in Eq. (2.13)]. The initial profile is

$$u(x,0) = \sqrt{2}\left(\frac{e^{i(x-5)/2}}{\cosh(x-5)} + \frac{e^{i(x-15)/2}}{\cosh(x-15)}\right), \tag{3.14}$$

which is a superposition of two one-soliton solutions. Initially, the two solitons move inward toward one another. As the two solitons collide, they undergo a fairly complicated interaction, whereby a tall spike is formed. The process is then reversed, whereby the solitons separate.

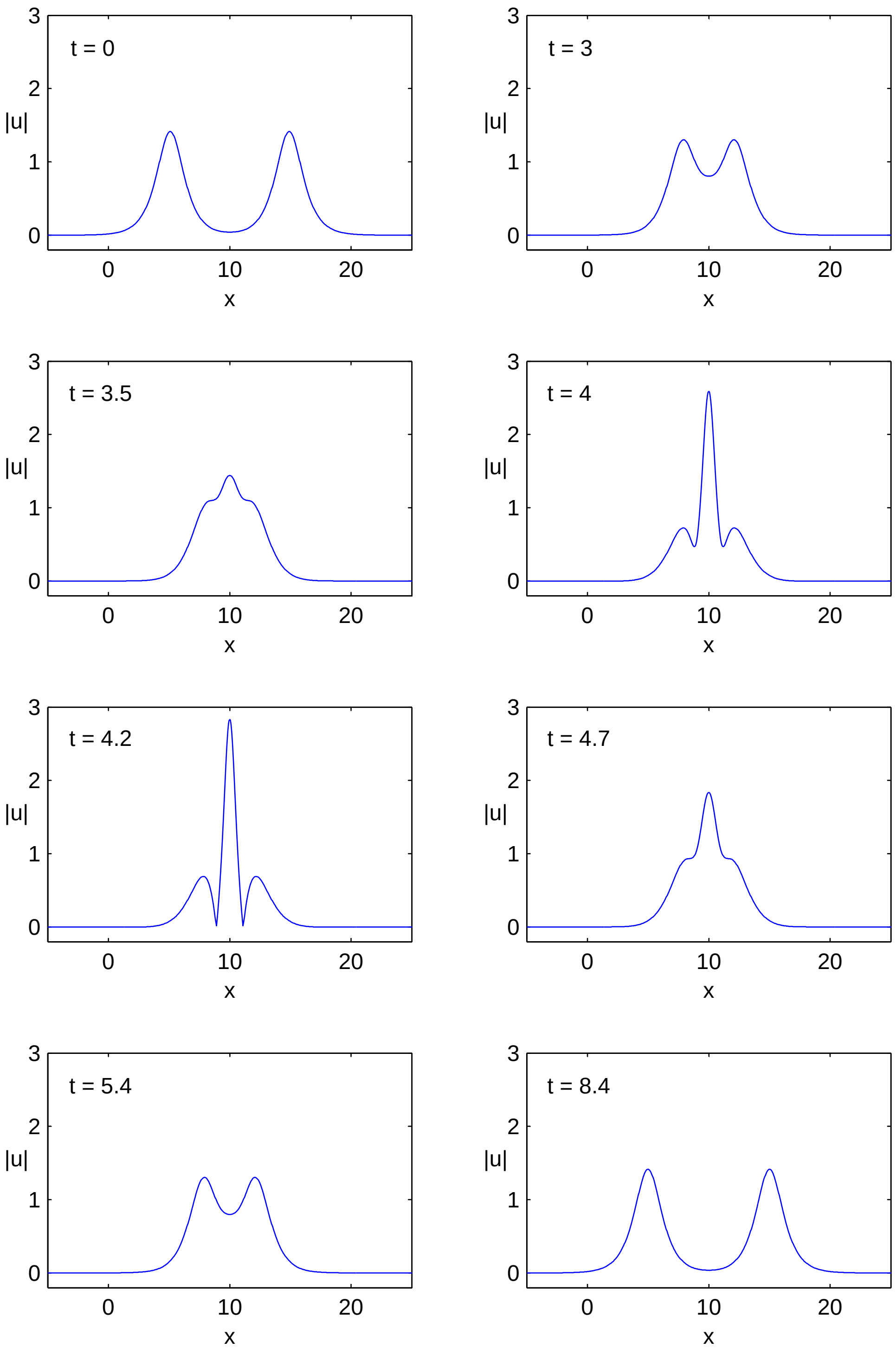

Figure 3.6: Result of simulation of the NLS equation with initial condition given by Eq. (3.14). Note that $|u|$ is plotted against x.

Chapter 4

Summary and Conclusions

In this report we have studied solitons, which are stable propagating solutions of nonlinear PDEs. We began by observing that in order for a nonlinear PDE to have soliton solutions, it is necessary that the PDE be dispersive. This is because the effects of dispersion, which cause a travelling wave to spread, can balance the effects of nonlinearity, which cause a travelling wave to steepen.

We continued by presenting the historical events and discoveries which have lead to the advent of soliton theory. The first of these is the discovery of the solitary wave by John Scott Russell in 1834. By conducting extensive water tank experiments, Russell determined several important properties of solitary water waves. Perhaps most interesting to us is the result that solitary water waves can collide with each other and subsequently reassume their original form and velocity. The next important discovery is that of the KdV equation by Korteweg and de Vries in 1895. The KdV equation describes the propagation of waves on the surface of a shallow channel. Korteweg and de Vries found a solution of this equation which corresponds to the solitary waves discovered by Russell. In 1965 Zabusky and Kruskal were inspired by the work of Fermi, Pasta and Ulam to study the KdV equation numerically. Zabusky and Kruskal found that an initial profile evolved into a number of solitons, which could interact strongly without loosing their identity.

Furthermore, we have presented a number of equations which have soliton solutions. These are the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0,$$

the SG equation

$$u_{xx} - u_{tt} = \sin u$$

and the NLS equation

$$iu_t + u_{xx} \pm |u|^2 u = 0.$$

The KdV equation has pulse-like rightward moving soliton solutions which have a profile of the form $1/\cosh^2 x$. In general, the solution of the initial value problem for the KdV equation consists of a number of such solitons and a leftward moving dispersive wave train. The soliton solutions of the SG equation are called kinks and antikinks, both of which approach different integer multiples of π as $x \to \pm\infty$. Finally, the NLS equation has two types of soliton solution, depending on the sign of the last term. If the sign is positive the solitons are so called bright solitons, whereas if the sign is negative they are so called dark solitons.

The IST is a method of solution for initial value problems for nonlinear PDEs, which can be applied to a number of equations which have soliton solutions, including those we have mentioned. The steps involved in the IST (Fig. 2.2) are the determination of the scattering data from the initial profile $u(x, 0)$, the determination of the time evolution of these scattering data and finally the solution of the inverse scattering problem, which yields the desired solution $u(x, t)$. These steps have been presented for the case of the KdV equation.

The main purpose of our investigation is to study soliton collisions, which we have done for the three equations mentioned above. Using the IST, we have derived a two-soliton solution of the KdV equation. Using the FDM, we have simulated solutions of all three equations. In doing so we have confirmed the existence of soliton solutions. We have also been able to study collisions between different kinds of solitons. For the KdV equation we have seen that the nature of a collision between two solitons depends on their relative amplitudes (Figs. 3.1 and 3.2), and that any initial profile will evolve into a number of solitons and (possibly) a dispersive wave train (Fig. 3.3). For the SG equation we have studied the two fundamental types of soliton collision: a kink-kink collision (Fig. 3.4) and a kink-antikink collision (Fig. 3.5). Finally, we have for the NLS equation studied the relatively complicated behaviour involved in a collision between two bright solitons (Fig. 3.6).

In conclusion, all our simulations agree, at least qualitatively, with the theoretical results which have been presented. A consequence of this is that we have confirmed the numerical results of Zabusky and Kruskal, along with a number of other results concerning the KdV equation, as well as the SG and NLS equations. Finally, in those cases in which we have been able to make a direct comparison between an exact and a numerical solution, the agreement has been very good.

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