ივანე ჯავახიშვილის სახელობის
თბილისის სახელმწიფო უნივერსიტეტი

ლექცია 4

Finite Volume

Conservation equation in the integral form

$$\frac{\partial}{\partial t}\int_{\Omega} \mathrm{U}(t,x)\, d\Omega + \int_{\Omega} \nabla \cdot \mathrm{F}(t,x)\, d\Omega = 0$$

$$\frac{\partial}{\partial t}\int_{\Omega} \mathrm{U}(t,x)\, d\Omega + \oint_{\mathrm{S}} \mathrm{F}(t,x) \cdot \mathrm{n}\, ds = 0$$

Finite Volume

Evaluating flux

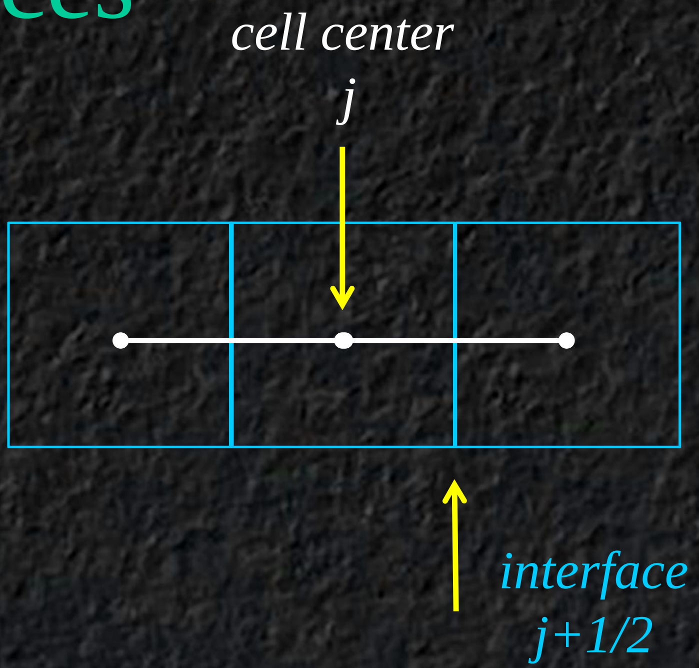
$$\oint_S \mathrm{F}(t,x)\cdot \mathrm{n}ds \approx \sum_{faces} F_k n_k$$

Numerical flux calculation problem

Cells and Faces

Grid cell centers:

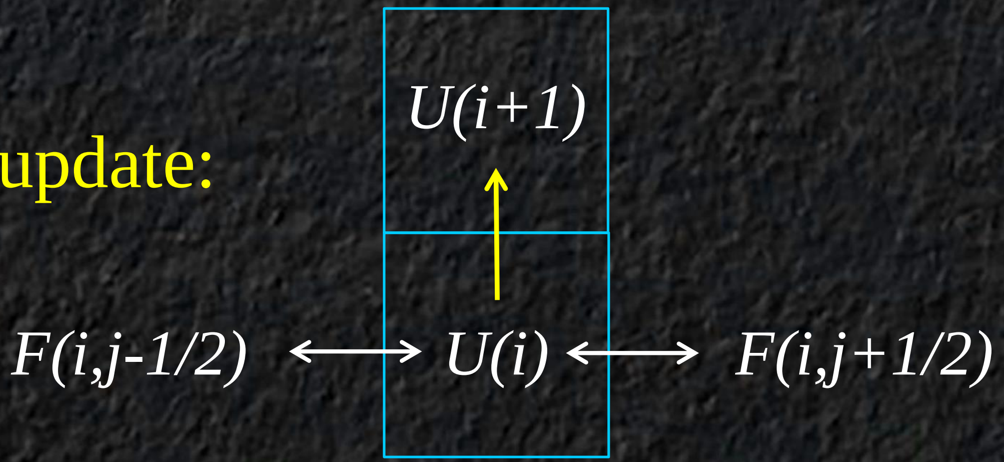
$j=1, 2, 3, \ldots, n.$



Cell interfaces:

$j=1/2, 3/2, \ldots, n-1/2.$

Finite Volume update:



Finite Volume

Time stepping algorithm

$$U(i+1, j) = U(i, j) - \frac{\Delta t}{\Delta x_j}\left(F^*(i, j+1/2) - F^*(i, j-1/2)\right)$$

flux on the cell faces

Finite Volume

Multidimensional problem:

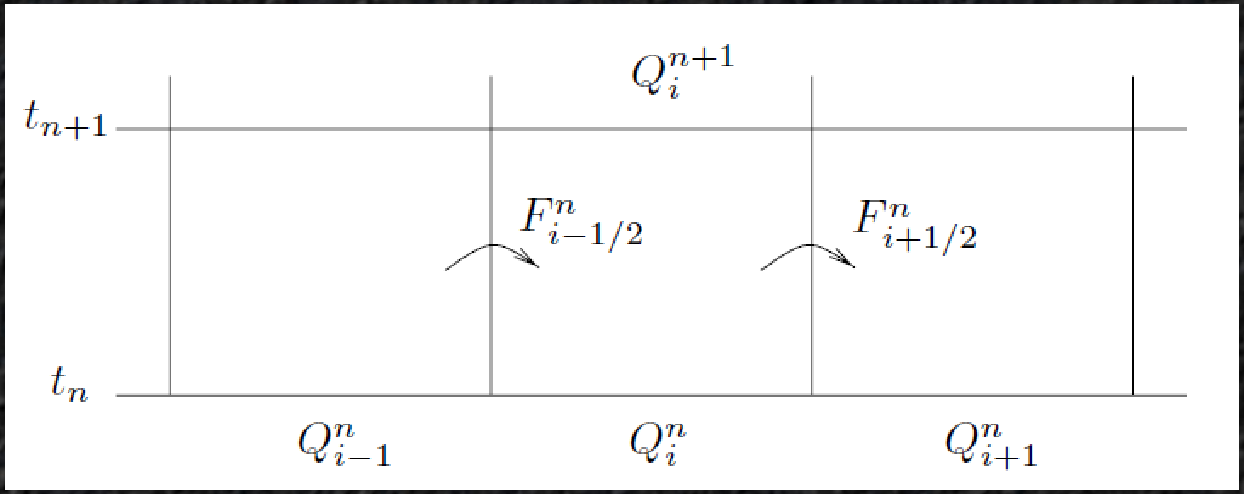
Unsplit integration

$$U(i+1,j,k)=U(i,j,k)-\frac{\Delta t}{\Delta x_j}\left(F^*(i,j+1/2,k)-F^*(i,j-1/2,k)\right)-$$
$$\frac{\Delta t}{\Delta y_j}\left(G^*(i,j,k+1/2)-G^*(i,j,k-1/2)\right)$$

Finite Volume

Leveque, Sec. 4

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right),$$



Boundary Conditions

Numerical flux calculation over the domain:

$$\sum_{volume} \sum_{faces} F_k n_k = \sum_{boundary} F_k n_k$$

Numerical Flux Function

Estimating flux on the faces

$$U(i+1, j) = U(i, j) - \frac{\Delta t}{\Delta x_j}\left(F(i, j+1/2) - F(i, j-1/2)\right) =$$

$$= \frac{\Delta t}{\Delta x_j}\left(F(U(i, j+1/2)) - F(U(i, j-1/2)\right)$$

$U(i,j+1/2)$, $U(i,j-1/2)$ - ?

Numerical Flux Function

Estimating numerical flux:

Upwind – Zeroth approximation

Central difference – Averaging conservation variable

Central difference – Averaging fluxes

Lux-Wendroff – Finite difference analog calculation of the numerical fluxes

Upwind

$$F^*_{i+1/2} = \begin{cases} F(U_i) & \text{if } \lambda_{i+1/2} \geq 0 \\ F(U_{i+1}) & \text{if } \lambda_{i+1/2} < 0 \end{cases}$$

Finding flux direction and projecting cell centered flux value on the cell face.

"wave velocity" $\lambda = \partial F / \partial U$

Central Difference

1. Averaging U(i,j)
2. Averaging F(i,j)

$$F^*_{i+1/2} = \frac{1}{2}(F(U_i) + F(U_{i+1})) \quad \text{or} \quad F^*_{i+1/2} = F\left(\frac{U_i + U_{i+1}}{2}\right)$$

Finite Volume

Lax-Wendroff flux estimation

$$F^*_{i+1/2} = \frac{1}{2}(F(U_i) + F(U_{i+1})) - \frac{\lambda_{i+1/2}\Delta t}{2\Delta x_{i+1/2}}(F(U_{i+1}) - F(U_i))$$

Calculating numerical flux using finite difference methods on cell center to face transition.

Finite Volume Method

Discrete formulation of the conservation equation at every cell/interface

$$\frac{\partial}{\partial t}\int_{\Omega} \mathrm{U}(t,x)d\Omega + \int_{\Omega} \nabla \cdot \mathrm{F}(t,x)d\Omega =$$

$$= \frac{\partial}{\partial t}U_{cell.average}(t,x_j) + F_R(t,x_j) - F_L(t,x_j) = 0$$

Diffusion Equations

Implicit (Crank-Nicolson)

$$q_t = \beta q_{xx},$$

$$f(q_x, x) = -\beta(x) q_x.$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right),$$

$$\frac{d}{dt}\int_{\mathcal{C}_i} q(x,t)\,dx = f\left(q\left(x_{i-1/2},t\right)\right) - f\left(q\left(x_{i+1/2},t\right)\right).$$

$$Q_i^{n+1} = Q_i^n + \frac{\Delta t}{2\,\Delta x^2}\Big[\beta_{i+1/2}\left(Q_{i+1}^n - Q_i^n\right) - \beta_{i-1/2}\left(Q_i^n - Q_{i-1}^n\right) + \beta_{i+1/2}\left(Q_{i+1}^{n+1} - Q_i^{n+1}\right) - \beta_{i-1/2}\left(Q_i^{n+1} - Q_{i-1}^{n+1}\right)\Big].$$

Finite Volume

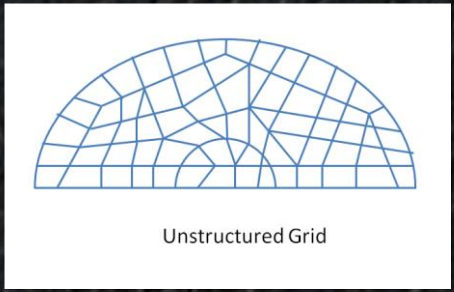
Finite Difference

- Problems with conservation
- Irregular geometries

Finite Volume

+ Explicit (hyperbolic)
+ easily adaptable
+ unstructured grids

- problems towards the edges



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www.tevza.org/home/course/modelling-II_2016/

Leveque – *Finite difference Methods for Hyperbolic equations*,

Sec. 4.1-4.9, pages 64-76