ლექცია 2

Finite Difference (FD) Methods

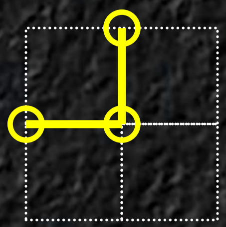
Conservation Law:

$$\frac{\partial U(t,x)}{\partial t} + \nabla \cdot J(t,x) = S(t,x) \qquad J(t,x) = aU(t,x)$$

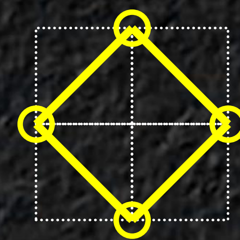
1D Linear Differential Equation:

$$\frac{\partial U(t,x)}{\partial t} + a\frac{\partial U(t,x)}{\partial x} = 0$$

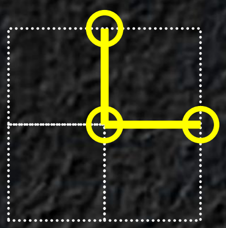
FD Methods



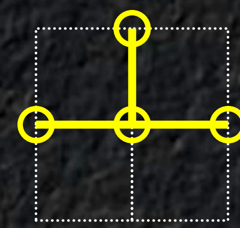
One-sided backward



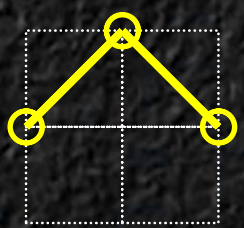
Leapfrog



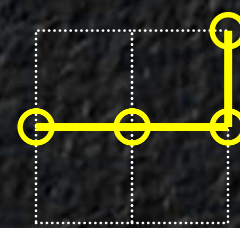
One-sided forward



Lax-Wendroff

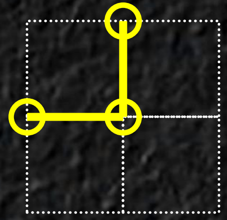


Lax-Friedrichs



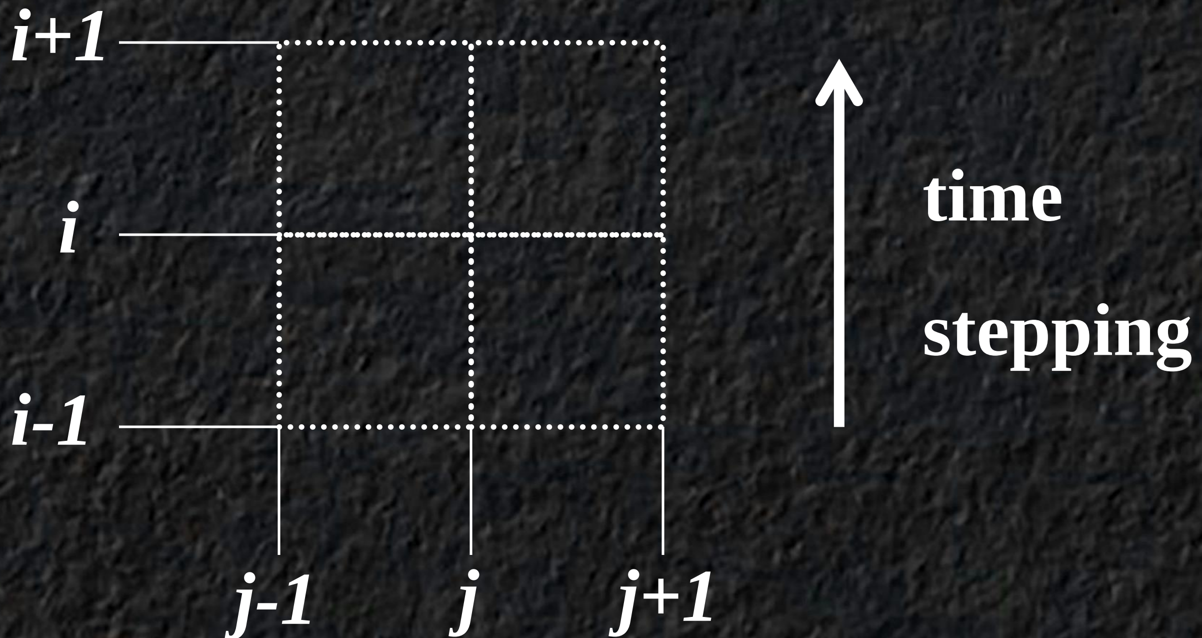
Beam-Warming

One-sided backward

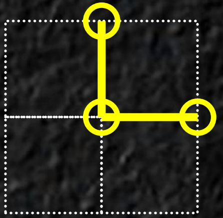


Difference equation:

$$U(i+1,j) = U(i,j) - \frac{\Delta t}{\Delta x} A\left[U(i,j)-U(i,j-1)\right]$$

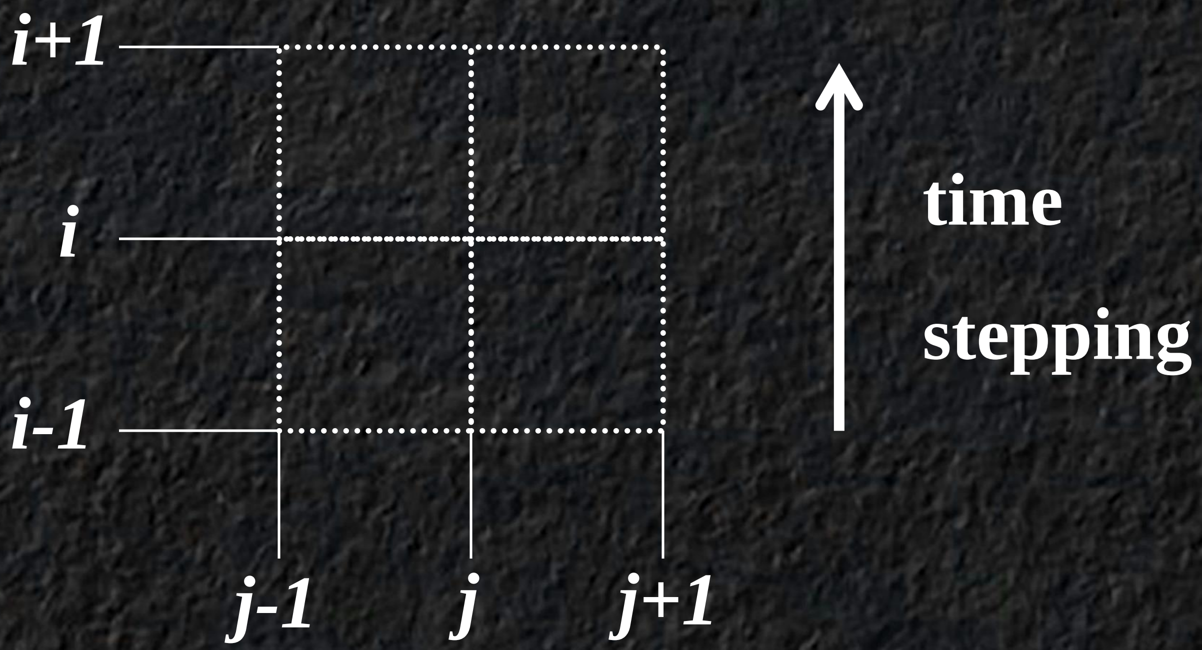


One-sided forward

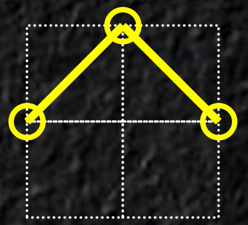


Difference equation:

$$U(i+1,j) = U(i,j) - \frac{\Delta t}{\Delta x} A\left[U(i,j+1) - U(i,j)\right]$$



Lax-Friedrichs



Difference equation:

$$U(i+1,j) = \frac{1}{2}\left[U(i,j+1) + U(i,j-1)\right] - $$

$$- \frac{\Delta t}{2\Delta x} A\left[U(i,j+1) - U(i,j-1)\right]$$

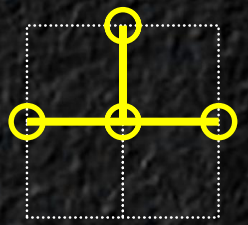
Leapfrog

Difference equation:

$$\mathrm{U}(i+1,j)=\mathrm{U}(i-1,j)-$$

$$-\frac{\Delta t}{\Delta x}A\left[\mathrm{U}(i,j+1)\text{-}\mathrm{U}(i,j-1)\right]$$

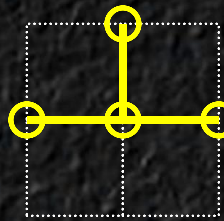
Lax-Wendroff



Difference equation:

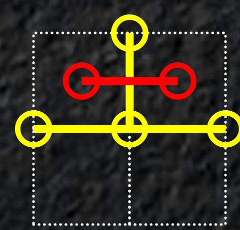
$$
\begin{aligned}
U(i+1,j) = U(i,j) &- \\
&- \frac{\Delta t}{2\Delta x} A\left[U(i,j+1) - U(i,j-1)\right] \\
&+ \frac{\Delta t^2}{2\Delta x^2} A^2\left[U(i,j+1) - 2U(i,j) + U(i,j-1)\right]
\end{aligned}
$$

Lax-Wendroff



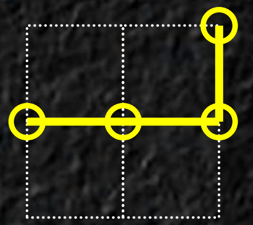
Derivation:

1. Lax-Friedrichs with half step;
2. Leapfrog half step;



derive

Beam-Warming



Difference equation:

$$U(i+1,j) = U(i,j) -$$

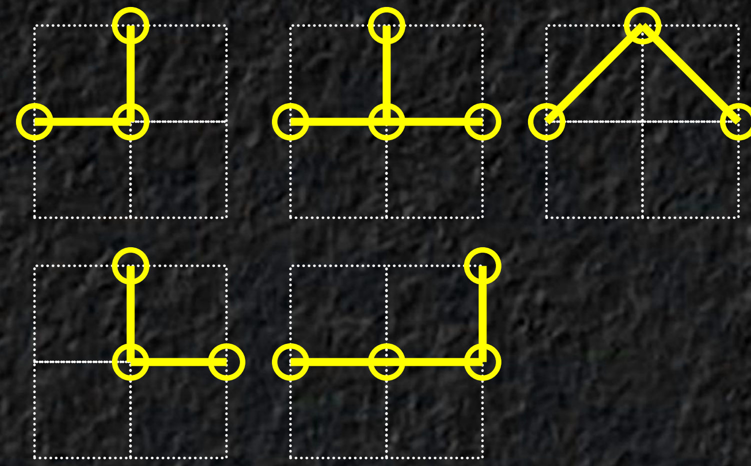
$$-\frac{\Delta t}{2\Delta x} A\left[3U(i,j) - 4U(i,j-1) + U(i,j-2)\right]$$

$$+\frac{\Delta t^2}{2\Delta x^2} A^2\left[U(i,j) - 2U(i,j-1) + U(i,j-2)\right]$$

Comparison

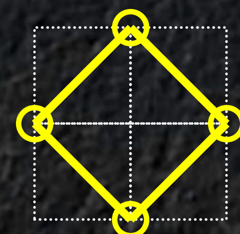
One-sided schemes:	$O(\Delta t, \Delta x)$
Lax-Friedrichs:	$O(\Delta t, \Delta x^2)$
Leapfrog:	$O(\Delta t^2, \Delta x^2)$
Lax-Wendroff:	$O(\Delta t^2, \Delta x^2)$
Beam-Warming:	$O(\Delta t^2, \Delta x^3)$

Time stepping



1st order methods in time:

$U(i+1) = F\{ U(i) \}$



2nd order methods in time:

$U(i+1) = F\{ U(i) , U(i-1) \}$

Starting method: A) $U(2) = F\{ U(1) \}$ (*e.g.* One-Sided)

B) $U(3) = F\{ U(2) , U(1) \}$ (Leapfrog)

Convergence

$u(t,x)$ - Analytical solution

$U(i,j)$ - Numerical solution

Error function: $E(i,j) = U(i,j) - u(t(i), x(j))$

Numerical convergence (**Norm** of the Error function):

$$\|E(i,j)\| \to 0, \qquad \Delta x \to 0$$

Norms

Norm for conservation laws: $\| u(x,y) \| \equiv \int_{-\infty}^{+\infty} | u(x,y) | \, dx$

$$\| E(i,j) \|_1 = \frac{1}{N} \sum_{j=1}^{N} |E(i,j)|$$

Other Norms (e.g. spectral problems)

$$\| E(i,j) \|_2 = \frac{1}{N} \sqrt{\sum_{j=1}^{N} |E(i,j)|^2}$$

Norms

P-Norm

$$\left\| E(i,j) \right\|_p = \frac{1}{N}\left(\sum_{j=1}^{N} \left| E(i,j) \right|^p \right)^{1/p}$$

Norm-2: Energy in numerical domain;

Numerical dissipation, boundary effects, etc.

Numerical Stability

Courant, Friedrichs, Levy (CFL, Courant number)

$$CFL = \max\left(\frac{a\Delta x}{\Delta t}\right)$$

Upwind schemes for discontinity:

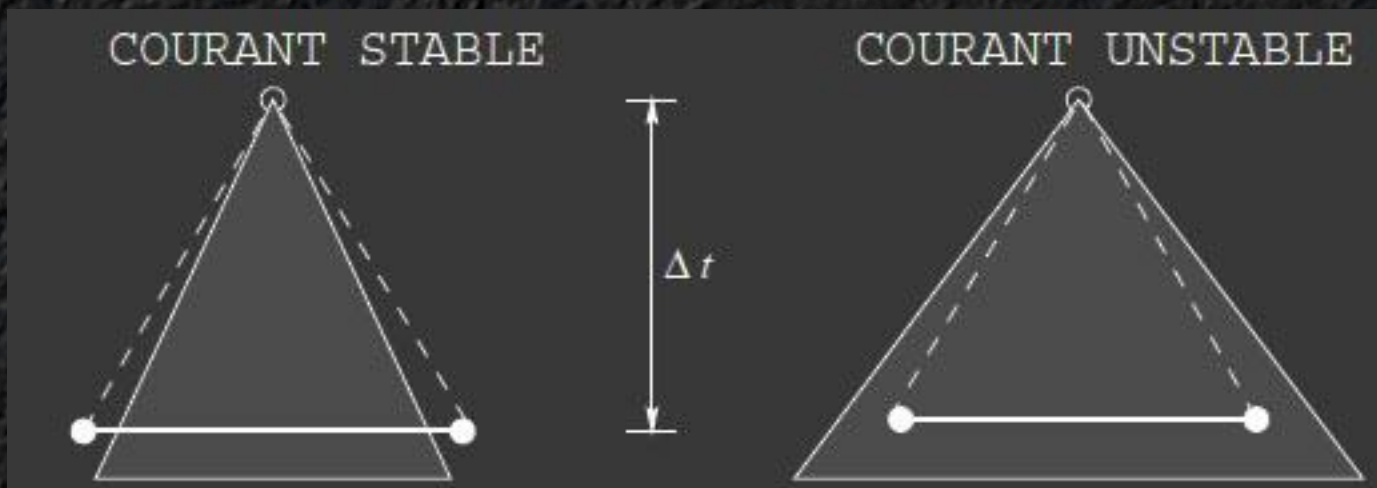
a > 0 : One sided forward

a < 0 : One sided backward

Numerical stability

CFL<1

Domain of dependence



Lax Equivalence Theorem

For a <u>well-posed</u> linear initial value problem, the method is <u>convergent</u> if and only if it is <u>stable.</u>

Given a properly posed initial-value problem and a finite difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence.

Necessary and sufficient condition
for consistent linear method

Well posed: solution exists, continuous, unique;

- numerical stability (t)
- numerical convergence (xyz)

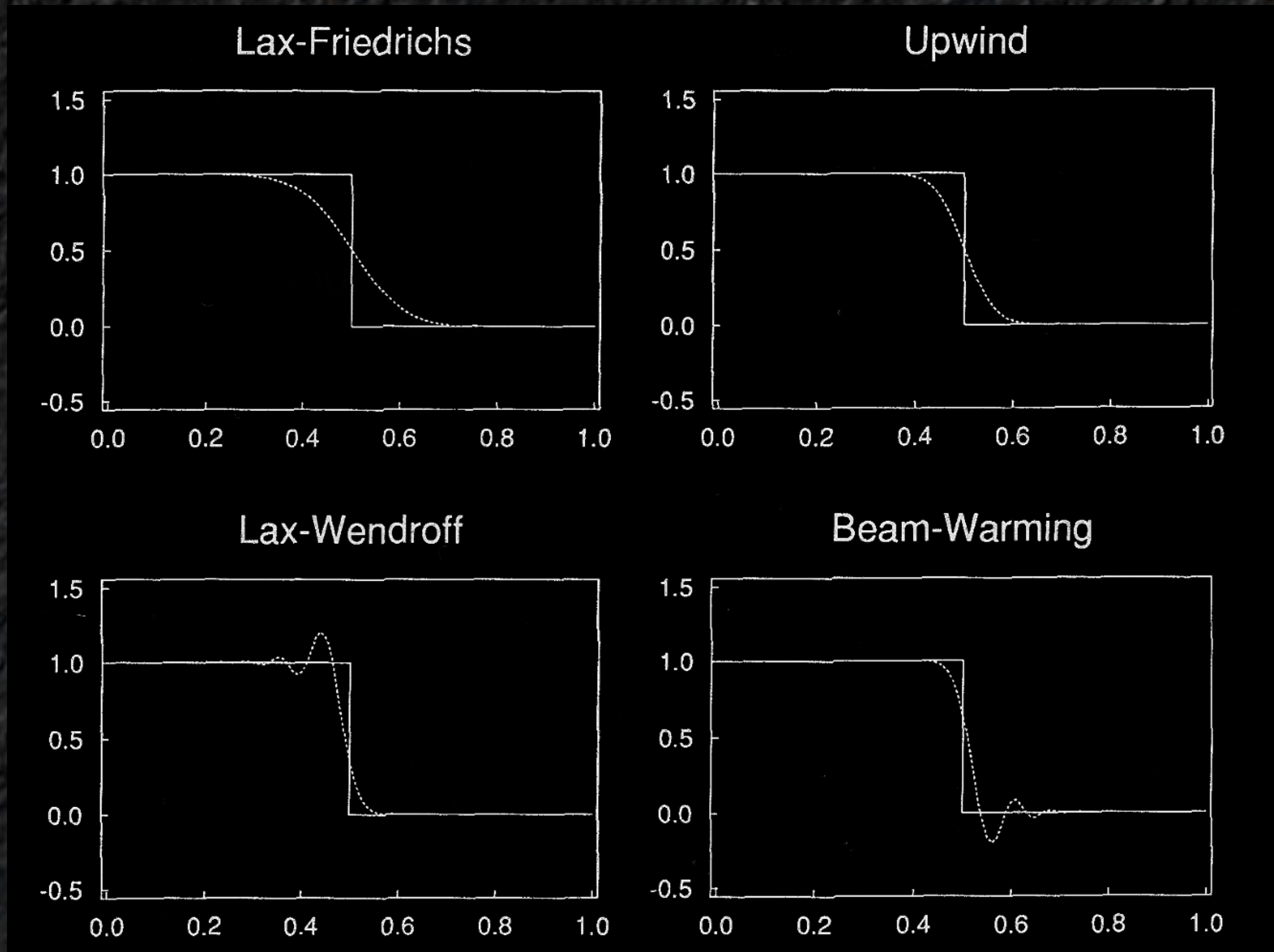
Discontinuous solutions

Advection equation: $$\frac{\partial U(t,x)}{\partial t} + a\,\frac{\partial U(t,x)}{\partial x} = 0$$

Initial condition: $$U_0(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

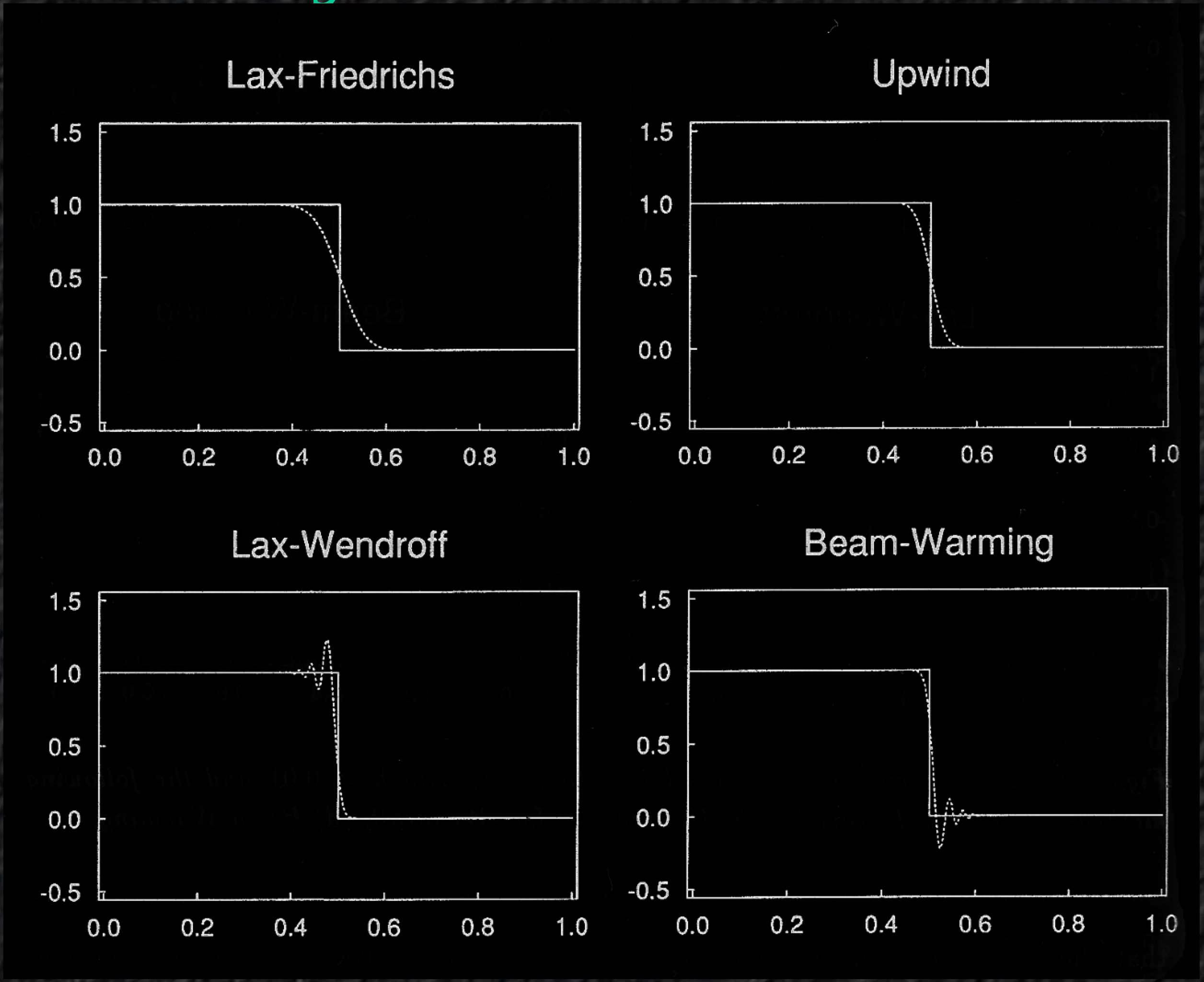
Analytic solution: $$U(t,x) = U_0(x - at)$$

Analytic vs Numerical



$\Delta x = 0.01$

Analytic vs Numerical



$\Delta x = 0.001$

Nonlinear Equations

Linear conservation: $$\frac{\partial U(t,x)}{\partial t} + a\,\frac{\partial U(t,x)}{\partial x} = 0$$

Nonlinear conservation: $$\frac{\partial U(t,x)}{\partial t} + \frac{\partial}{\partial x}F(U) = 0$$

$$a\,\frac{\partial U(t,x)}{\partial x} \rightarrow \frac{\partial}{\partial x}F(U)$$

Nonlinear FD methods

Lax-Friedrichs

linear stencil:

$$U(i+1,j)=\frac{1}{2}\left[U(i,j-1)+U(i,j+1)\right]-$$
$$-\frac{\Delta t}{2\Delta x}a\left[U(i,j+1)\text{-}U(i,j-1)\right]$$

nonlinear stencil:

$$U(i+1,j)=\frac{1}{2}\left[U(i,j-1)+U(i,j+1)\right]-$$
$$-\frac{\Delta t}{2\Delta x}\left[F(U(i,j+1))\text{-}F(U(i,j-1))\right]$$

Nonlinear FD methods

Lax-Wendroff

$$U(i+1,j) = U(i,j) - \frac{\Delta t}{2\Delta x} A\left[U(i,j+1) - U(i,j-1)\right] + \frac{\Delta t^2}{2\Delta x^2} A^2\left[U(i,j+1) - 2U(i,j) + U(i,j-1)\right]$$

MacCormack’s two step method:

$$U'(j) = U(i,j) - \frac{\Delta t}{\Delta x}\left[F(U(i,j+1)) - F(U(i,j))\right] \quad ,$$

$$U(i+1,j) = \frac{1}{2}\left(U(i,j) + U'(j)\right) - \frac{\Delta t}{2\Delta x}\left[F(U'(j)) - F(U'(j-1))\right]$$

FD Methods

Methods to supress numerical instabilities

Numerical diffusion (first order methods)

$$\frac{\partial U(t,x)}{\partial t} + a\frac{\partial U(t,x)}{\partial x} = D\frac{\partial^2 U(t,x)}{\partial x^2}$$

$$D = \frac{\Delta x^2}{2\Delta t}\left[I - \left(\frac{\Delta t}{\Delta x}a\right)^2\right]$$

Numerical dispersion

$$\frac{\partial \mathrm{U}(t,x)}{\partial t} + a\frac{\partial \mathrm{U}(t,x)}{\partial x} = \mu \frac{\partial^3 \mathrm{U}(t,x)}{\partial x^3}$$

Lax-Wendroff (or second order methods)

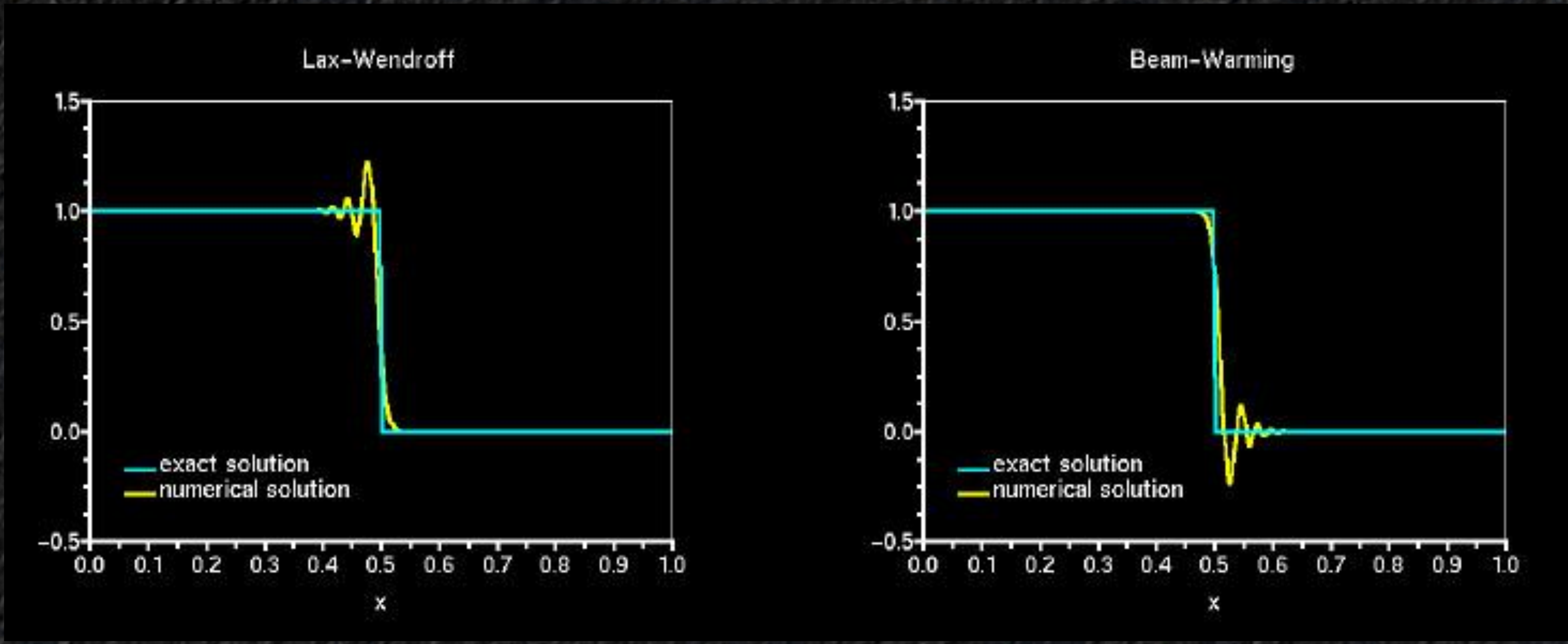
$$\mu = \frac{\Delta x^2}{6} a \left[\frac{\Delta t^2}{\Delta x^2} a^2 - I \right]$$

Beam-Warming method

$$\mu = \frac{\Delta x^2}{6} a \left[2I - \frac{3\Delta t}{\Delta x} a + \frac{\Delta t^2}{\Delta x^2} a^2 \right]$$

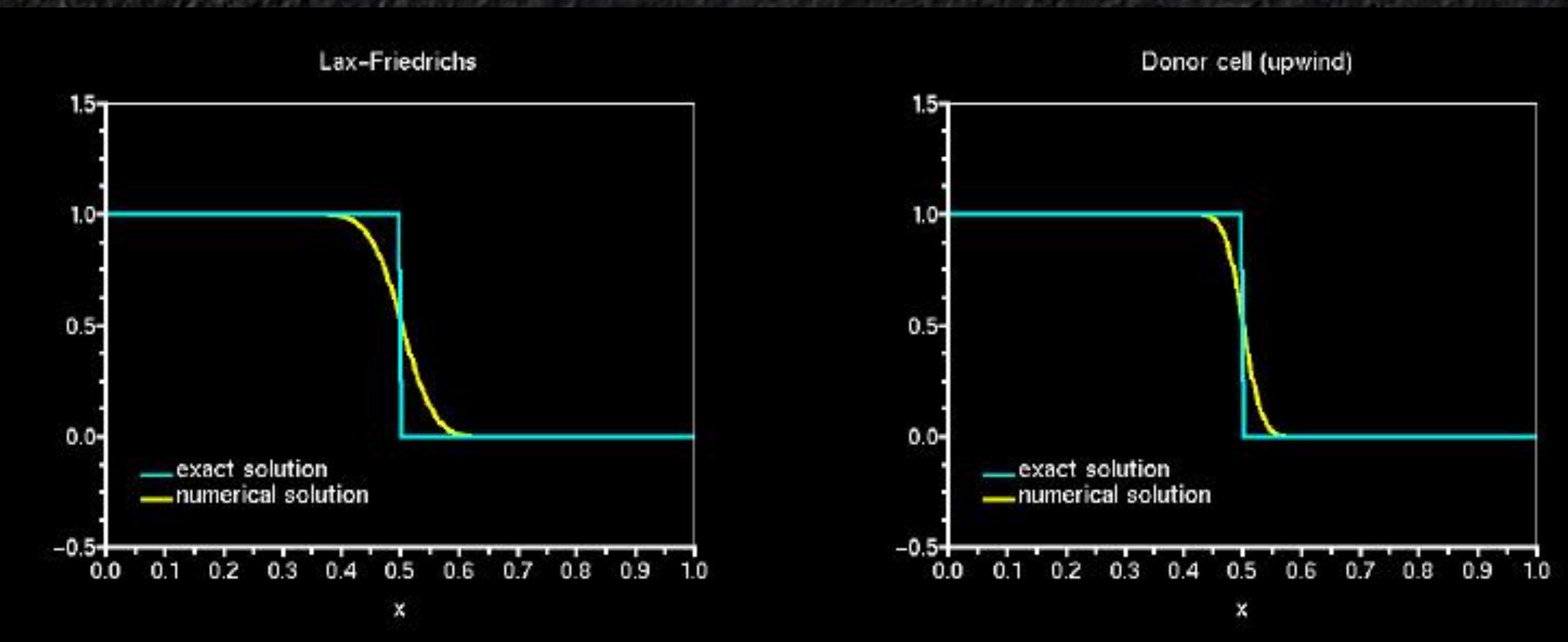
Numerical dispersion

(a > 0) (a < 0)



Numerical dispersion

(a > 0)　　　　　　　　　(a < 0)



Convergence

Lax-Friedrichs:

$$\|E(i,j)\| \approx C\sqrt{t\cdot\Delta x}$$

Convergence:

$$\|E(i,j)\| \to 0, \qquad \Delta x \to 0$$

FD Methods

+ / Primitive
+ / Fast

- / Accuracy
- / Numerical instabilities

end

http://www.tevza.org/home/course/modelling-II_2016/