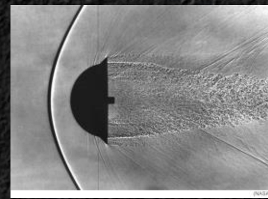
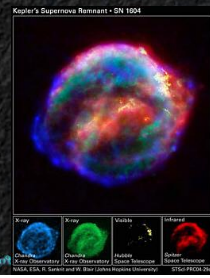
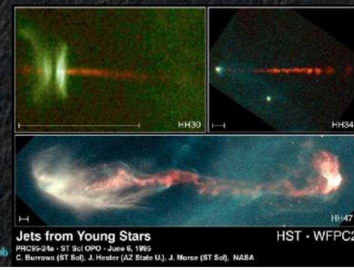
ივანე ჯავახიშვილის სახელობის
თბილისის სახელმწიფო უნივერსიტეტი

ლექცია 9

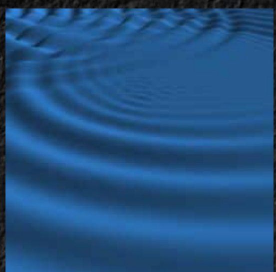
Shock Waves







Water Waves



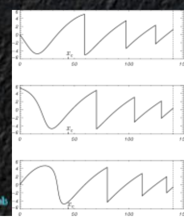
Small amplitude waves

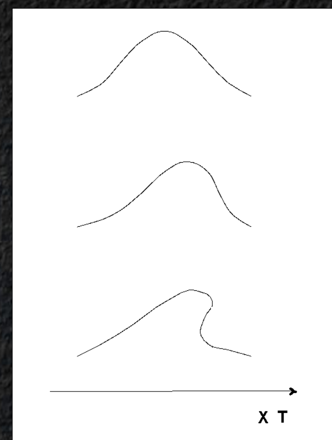
Large amplitude waves

Shock Development

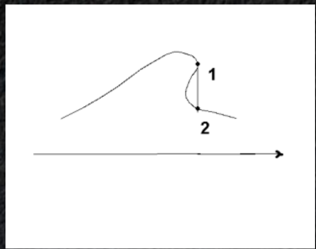
Wave front steepening

- Transverse waves
- Longitudinal waves





Jumps



Discontinuous solutions obeying conservation laws

Jumps in (P, Rho, V, T);

Continuous (E,M)

Rankine-Hugoniot Equation

1D Euler equations:

$$(1)\quad \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u)$$

$$(2)\quad \frac{\partial}{\partial t}(\rho u) = -\frac{\partial}{\partial x}(\rho u^2 + p)$$

$$(3)\quad \frac{\partial}{\partial t}(E^t) = -\frac{\partial}{\partial x}\left[u\left(E^t + p\right)\right],$$

$$E^t = \rho e + \rho \frac{1}{2} u^2,$$

$$(4)\quad p = (\gamma - 1)\,\rho e,$$

$$(5)\quad \frac{p}{\rho^\gamma} = \text{constant.}$$

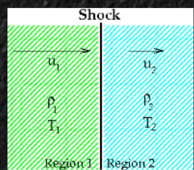
$$(6')\quad \frac{\partial w}{\partial t} + \frac{\partial}{\partial x} f(w) = 0$$

$$(6)\quad \frac{d}{dt}\int_{x_1}^{x_2} w\,dx = -f(w)\big|_{x_1}^{x_2}$$

$$(10)\quad u_s = \frac{f(w_1) - f(w_2)}{w_1 - w_2}.$$

Rankine-Hugoniot Equation

Jump conditions:



$$(12)\quad u_s(\rho_2 - \rho_1) = \rho_2 u_2 - \rho_1 u_1$$

$$(13)\quad u_s(\rho_2 u_2 - \rho_1 u_1) = (\rho_2 u_2^2 + p_2) - (\rho_1 u_1^2 + p_1)$$

$$(14)\quad u_s(E_2 - E_1) = \left[\rho_2 u_2\left(e_2 + \frac{1}{2}u_2^2 + p_2/\rho_2\right)\right] - \left[\rho_1 u_1\left(e_1 + \frac{1}{2}u_1^2 + p_1/\rho_1\right)\right].$$

$$c_1 = \sqrt{\gamma p_1/\rho_1}$$

$$(15)\quad u_s = u_1 + c_1\sqrt{1 + \frac{\gamma+1}{2\gamma}\left(\frac{p_2}{p_1} - 1\right)},$$

3D Shocks

3D Euler equation in the conservative form:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} + \frac{\partial \mathbf{H}}{\partial z} = 0$$

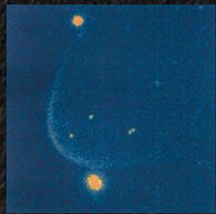
$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho e_t \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ (\rho e_t + p)u \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} \rho v \\ \rho v u \\ \rho v^2 + p \\ \rho v w \\ (\rho e_t + p)v \end{bmatrix} \quad \mathbf{H} = \begin{bmatrix} \rho w \\ \rho w u \\ \rho w v \\ \rho w^2 + p \\ (\rho e_t + p)w \end{bmatrix}$$

$$e_t = e + \frac{u^2 + v^2 + w^2}{2} + gz$$

Shock Consequences

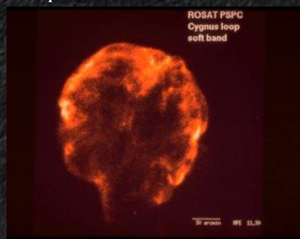
Viscous Heating

Bow shock produced by a neutron star



Supernova envelope

Supernova remnant



HD Linear Spectrum

Fourier Analysis:

$$(V, P, \rho) \sim exp(i\, k\, r - i\, \omega\, t)$$

Dispersion Equation:

$$\omega^2 (\omega^2 - Cs^2 k^2)=0$$

Solutions:

Vortices: $\omega^2 = 0$

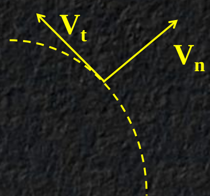
Sound waves: $\omega^2 = Cs^2 k^2$

HD Discontinuities

Sound waves –> Shocks

$(P_1,\rho_1,V_{n1}) \rightarrow (P_2,\rho_2,V_{n2})$;

$V_{t1}=V_{t2}$



Vortex –> Contact Discontinuity

$V_{t1} \rightarrow V_{t2}$;

$(P_1, \rho_1,V_{n1})= (P_2,\rho_2,V_{n2})$

MHD modes

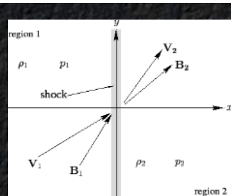
- Fast magnetosonic waves;
- Slow magnetosonic waves;
- Alfven waves;

- Fast shocks;
- Slow shocks;
- Contact shocks;

MHD shocks

$$\rho_1 v_{n1} = \rho_2 v_{n2},$$
$$B_{n1} = B_{n2},$$
$$\rho_1 v_{n1}^2 + p_1 + \frac{B_{t1}^2}{2\mu_0} = \rho_2 v_{n2}^2 + p_2 + \frac{B_{t2}^2}{2\mu_0},$$
$$\rho_1 v_{n1} \mathbf{v_{t1}} - \frac{\mathbf{B_{t1}} B_{n1}}{\mu_0} = \rho_2 v_{n2} \mathbf{v_{t2}} - \frac{\mathbf{B_{t2}} B_{n2}}{\mu_0},$$
$$\left(\frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} + \frac{v_1^2}{2}\right)\rho_1 v_{n1} + \frac{v_{n1} B_{t1}^2}{\mu_0} - \frac{B_{n1}(\mathbf{B_{t1}}\cdot\mathbf{v_{t1}})}{\mu_0} = \left(\frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} + \frac{v_2^2}{2}\right)\rho_2 v_{n2} + \frac{v_{n2} B_{t2}^2}{\mu_0} - \frac{B_{n2}(\mathbf{B_{t2}}\cdot\mathbf{v_{t2}})}{\mu_0}$$
$$(\mathbf{v}\times\mathbf{B})_{t1} = (\mathbf{v}\times\mathbf{B})_{t2},$$

$$a_{\rm slow}^2 = \frac{1}{2}\left[(c_s^2 + V_A^2) - \sqrt{(c_s^2 + V_A^2)^2 - 4c_s^2 V_A^2 \cos^2\theta_{Bn}}\right],$$
$$a_{\rm fast}^2 = \frac{1}{2}\left[(c_s^2 + V_A^2) + \sqrt{(c_s^2 + V_A^2)^2 - 4c_s^2 V_A^2 \cos^2\theta_{Bn}}\right],$$



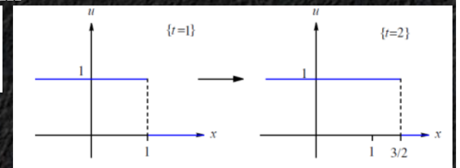
Riemann Shock Tube Problem

1D: Shock tube problem

$$\begin{cases} u_t + uu_x = 0, & t \ge 0 \\ u(x,0) = \phi(x), \end{cases}$$

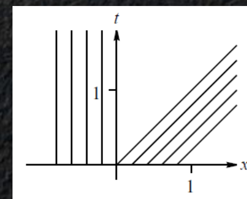
$$u_t + \left[\frac{u^2}{2}\right]_x = 0.$$

$$\phi(x) = \begin{cases} 1 & \text{for } x < 0 \\ 0 & \text{for } x > 0. \end{cases}$$



Method of characteristics:

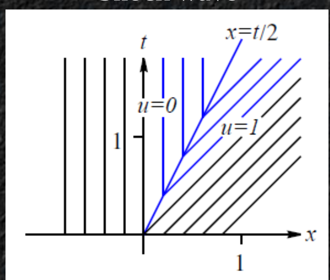
$$x = \phi(r)t + r.$$



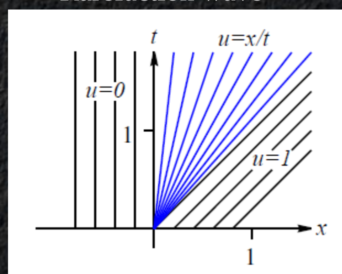
Riemann Shock Tube Problem

$$u_1(x,t) = \begin{cases} 0 & \text{for } x < \frac{t}{2} \\ 1 & \text{for } x > \frac{t}{2}. \end{cases}$$

Shock wave

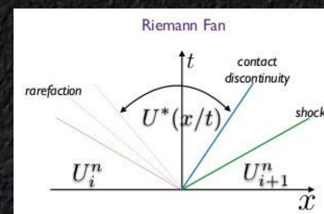


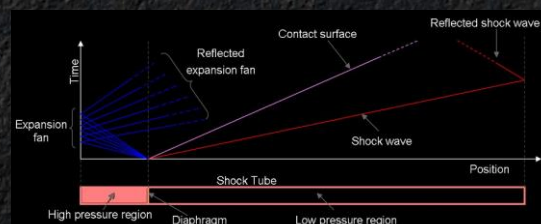
Rarefaction wave

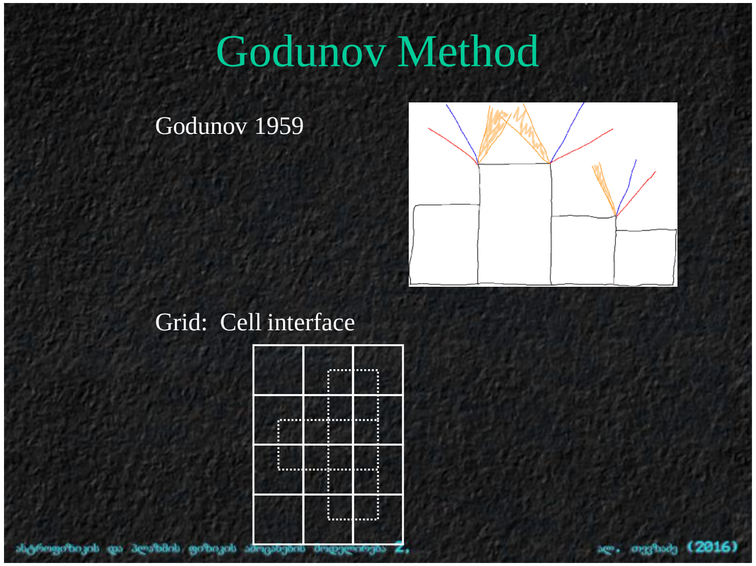


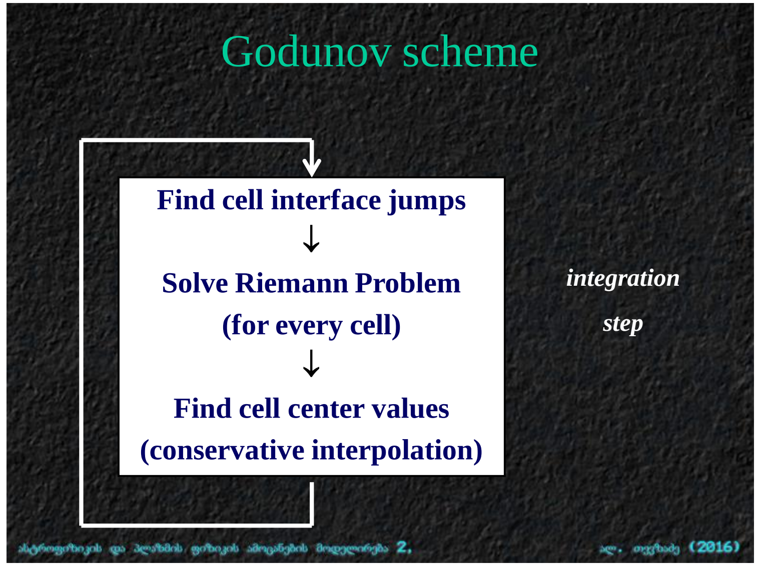
Riemann Problem

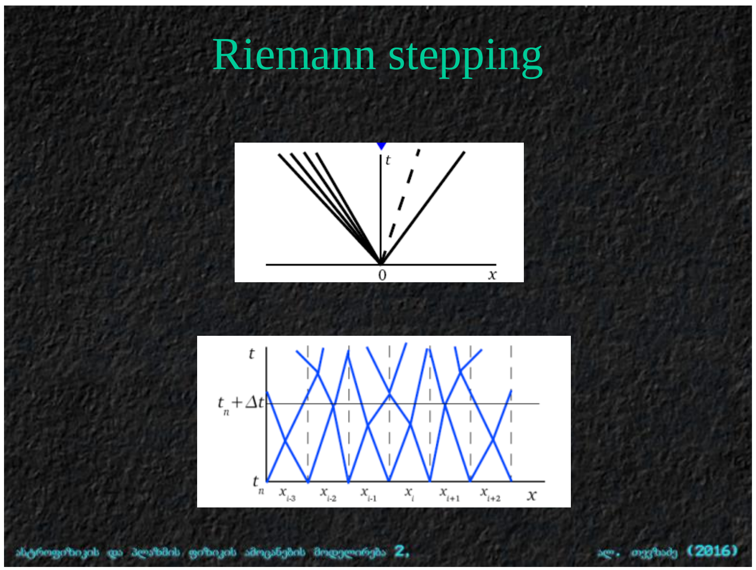
Shocks in Euler equations

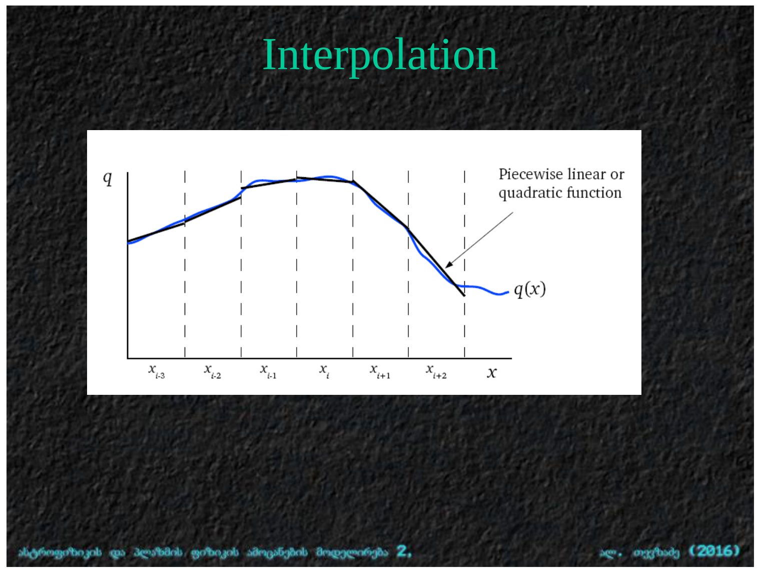




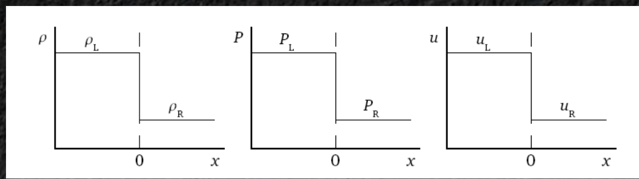

Godunov Method
Godunov 1959
Grid: Cell interface
(2016)


Godunov scheme
Find cell interface jumps
Solve Riemann Problem
(for every cell)
Find cell center values
(conservative interpolation)
integration
step
(2016)


Riemann stepping
t
0
x
t
$t_n + \Delta t$
t_n
x_{i-3} x_{i-2} x_{i-1} x_i x_{i+1} x_{i+2} x
(2016)


Interpolation
q
Piecewise linear or
quadratic function
q(x)
x_{i-3} x_{i-2} x_{i-1} x_i x_{i+1} x_{i+2} x
(2016)

Conservative Interpolation



Rho = Rho(P,S)

Rho<0

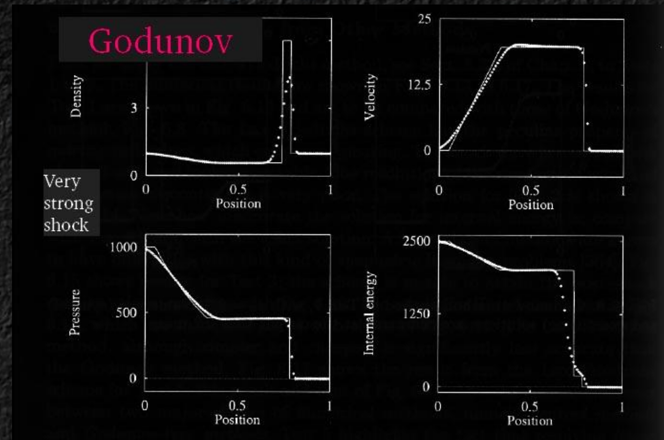


Fig. 6.10. Godunov's method applied to Test 3, with $x_0 = 0.5$. Numerical (symbol) and exact (line) solutions are compared at the output time 0.012 units

Approximations

Riemann Solver:

- Linear Riemann solver (ROE)
  Fast, medium accuracy
- Harten-Laxvan-Leer solver (HLL)
  Problems with contact discontinuities
- Two-Shock Rieman Solver
  Problems with entropy waves

Interpolation

- Linear (numerical stability)
- Parabolic (ppm)
- High order

Riemann Solvers

Linear Riemann solver (Roe, 1981)

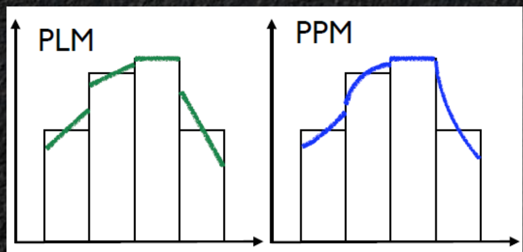
Formulate approximate linear problem

$d/dt\, U_t + A(U_L,U_R)\, d/dx\, U = 0$

$$\hat{u} = \frac{\sqrt{\rho_L} u_L + \sqrt{\rho_R} u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$
$$\hat{v} = \frac{\sqrt{\rho_L} v_L + \sqrt{\rho_R} v_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$
$$\hat{w} = \frac{\sqrt{\rho_L} w_L + \sqrt{\rho_R} w_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$

Interpolations

- Linear (numerical stability)
- Parabolic (ppm)
- High order



+/-

+ Best accuracy
+ Shock capturing
+ study of the Heating, viscosity, …

- Slow
- Turbulence
- Complicated

end

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