ივანე ჯავახიშვილის სახელობის
თბილისის სახელმწიფო უნივერსიტეტი

ლექცია 5

Fourier Transform

f(t) – Function; F(ω) – Fourier harmonic

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)\exp(\mathrm{i}\omega t)\mathrm{d}t$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\exp(-\mathrm{i}\omega t)\mathrm{d}\omega$$

F = F(ω) : Spectral Distribution, Spectrum

Fourier Transform

Time / Frequency

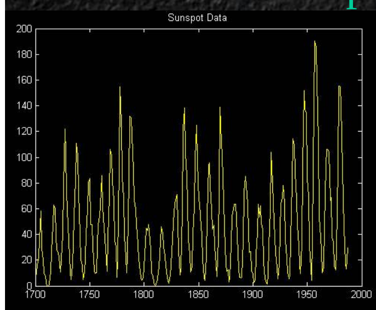
$$F(\omega) = \int_{-\infty}^{+\infty} f(t)\exp(\mathrm{i}\omega t)\mathrm{d}t$$

Co-ordinate / Wave-number

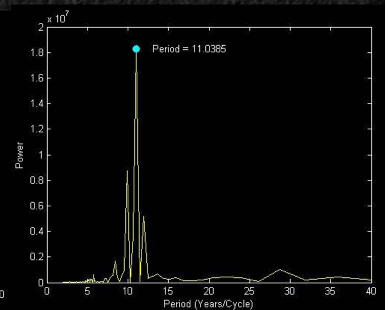
$$F(k) = \int_{-\infty}^{+\infty} f(x)\exp(\mathrm{i}kx)\mathrm{d}x$$

Temporal (Spatial) spectrum: **Spectral Analysis**

Sunspot Data



Time series



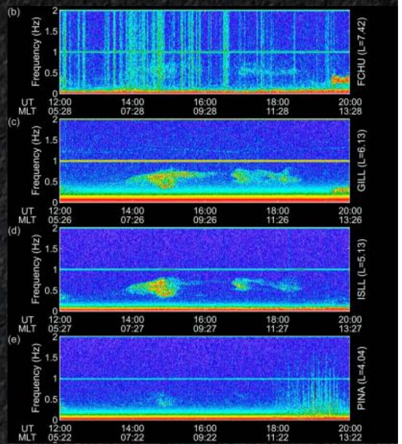
Spectrum

```
load sunspot.dat
Year = sunspot(:,1); Wolf = sunspot(:,2);
```

Spectrogram

Time evolution
of the temporal
spectrum (frequencies)

Solar wind pressure
and magnetic field
Fourier spectrograms
(CARISMA)



FT: Properties

Linear Superposition

$$f_1(t) + f_2(t) \rightarrow F_1(\omega) + F_2(\omega)$$

Correlation

$$\int_{-\infty}^{+\infty} f_1(\tau) f_2(t-\tau) \mathrm{d}\tau \rightarrow F_1(\omega) F_2(\omega)$$

Convolution

$$\int_{-\infty}^{+\infty} f_1(\tau) f_2(t+\tau) \mathrm{d}\tau \rightarrow F_1(\omega) F_2^*(\omega)$$

ასტროფიზიკის და პლაზმის ფიზიკის ამოცანების მოდელირება 2. ალ. თევზაძე (2016)

Discrete Fourier Transform

Continuous: ***X= (-∞ … ∞)***

Discrete: $\boldsymbol{X_k = (X_1 \ldots X_N)}$

$$X_k = \sum_{n=0}^{N-1} x_n \exp\left(-i2\pi \frac{k}{N} n\right)$$

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \exp\left(i2\pi \frac{k}{N} n\right)$$

k = 0 … (N-1) **Discrete Spectrum**

ასტროფიზიკის და პლაზმის ფიზიკის ამოცანების მოდელირება 2. ალ. თევზაძე (2016)

DFT constraints

Length-Scales:

Domain Size: $\boldsymbol{L = X_N - X_1}$

Step Size: $\boldsymbol{\Delta X = X_k - X_{k-1}}$, $\boldsymbol{\Delta X = L/(N-1)}$

Wave-Numbers:

Maximal wave-number: $\boldsymbol{K = 2\pi / \Delta X}$

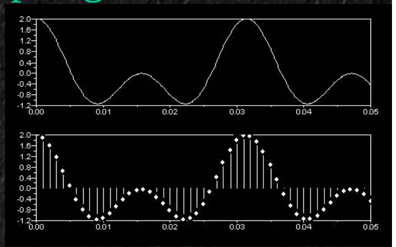
Minimal wave-number: $\boldsymbol{\Delta K = 2\pi / L}$

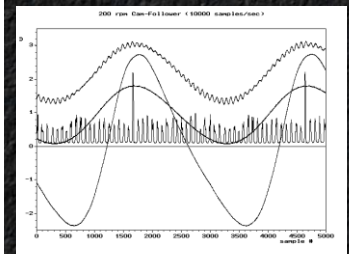
Number of Fourier Harmonics: **N**
(sampling rate, resolution)

ასტროფიზიკის და პლაზმის ფიზიკის ამოცანების მოდელირება 2. ალ. თევზაძე (2016)

Sampling

Discretization:
Sample continuous function



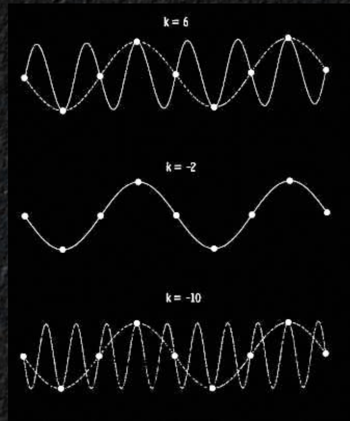


Nyquist critical frequency:

$\omega_c = 1 / 2\Delta$

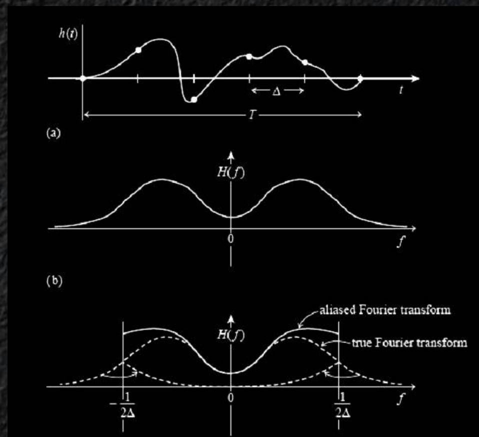
$\omega < \omega_c$

Sampling



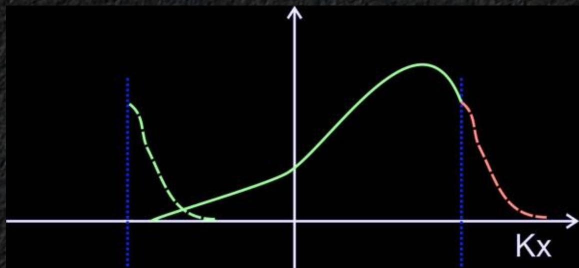
Aliasing

Aliasing



Aliasing

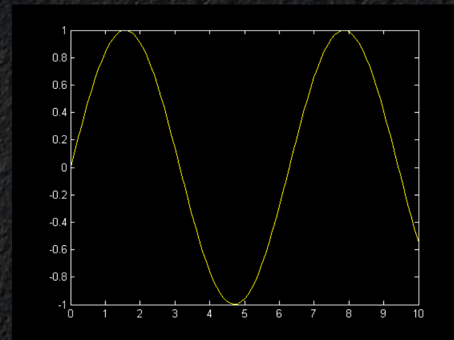
Aliasing defects to spectral power



$$k_x \to -k_x$$

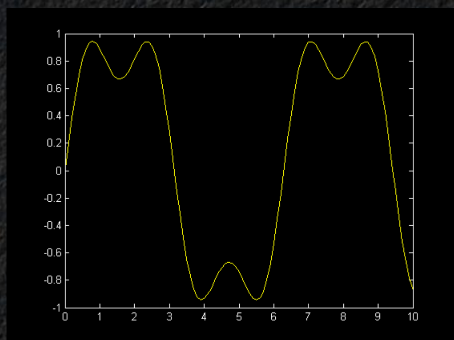
Number of Harmonics

```
t = (0:0.1:10);
y = sin(t);
plot(t,y);
```



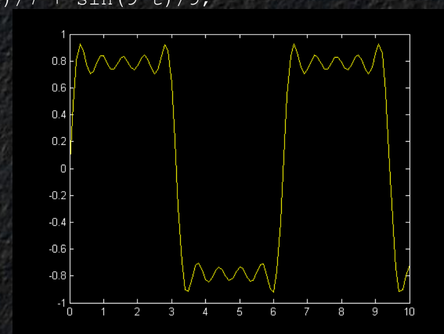
Number of Harmonics

```
t = (0:0.1:10);
y = sin(t) + sin(3*t)/3;
plot(t,y);
```



Number of Harmonics

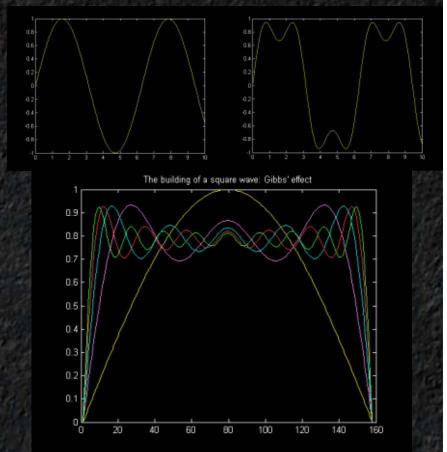
```
t = (0:0.1:10);
y = sin(t) + sin(3*t)/3 + ...
sin(5*t)/5+ sin(7*t)/7 + sin(9*t)/9;
plot(t,y);
```



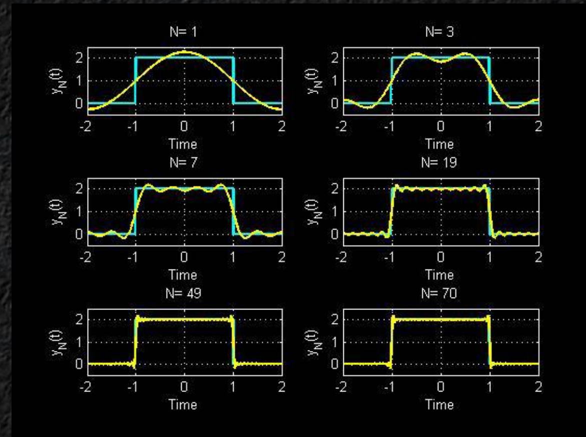
Gibbs Effect

N = 1…9

Density of the spectrum



Gibbs Effect



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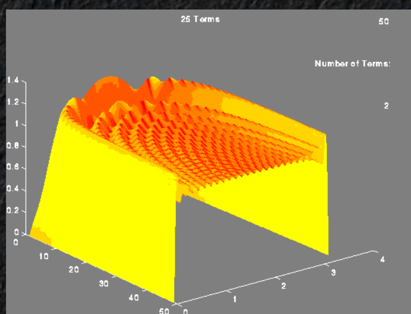
Gibbs in 2D

(X,Y) / (Kx, Ky)



ასტროფიზიკის და პლაზმის ფიზიკის ამოცანების მოდელირება 2. ალ. თევზაძე (2016)

Fast Fourier Transform

DFT:

***O*(N^2)** calculation process

FFT Algorithm: ($N = 2^m$)

***O*(N log N)** calculation process

(Cooley, Turkey 1960, … Gauss 1805)

$N = 10^6$ FFT : 30sec

FT : 2 week

ასტროფიზიკის და პლაზმის ფიზიკის ამოცანების მოდელირება 2. ალ. თევზაძე (2016)

PDE: Spectral Method

PDE:

$$\frac{\partial}{\partial t} A(x,t) = c^2 \frac{\partial^2}{\partial x^2} A(x,t)$$

Fourier decomposition in SPACE: $a(k,t) = \int_{-\infty}^{+\infty} A(x,t) \exp(ikx) dx$

$$\int \left\{ \frac{d}{dt} a(k,t) + c^2 k^2 a(k,t) \right\} \exp(ikx) dx = 0$$

ODE solver: $a = a(k,t)$

Inverse Fourier transform: $A = A(x,t)$

Spectral Simulations

Discrteize PDE: Sampling

DFT: mostly FFT

1. Transform PDE to spectral ODE
2. Solve ODE (e.g., R-K)
3. Inverse transform to reconstruct solutions

ასტროფიზიკის და პლაზმის ფიზიკის ამოცანების მოდელირება 2. ალ. თევზაძე (2016)

Spectral Method: Features

Initial Value Problem

- Calculate initial values in k-space

Boundary Value Problem

- Integrate boundaries into k-space

Spatial Inhomogeneities

- Introduce numerical variables to homogenize
- Integrate during reconstruction

ასტროფიზიკის და პლაზმის ფიზიკის ამოცანების მოდელირება 2. ალ. თევზაძე (2016)

Spectral Method: Problems

1. Shocks

Discontinuity: $\Delta \to 0$

$K_{cr} = 1/2\Delta \to \infty$

$K_{max} < K_{cr}$

2. **Complex Boundaries**

Ill-known numerical instabilities;

3. **Nonlinearities**

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Spectral Method: Variants

- Pseudo-spectral Method

Pseudo-spectral basis:
Legendre polynomials;
Chebishev polynomials;

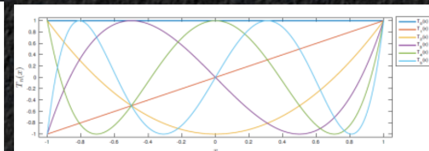
Expansion coefficients: colocation, Galerkin,…

Chebishev polynomials

$$\langle T_n, T_m \rangle \equiv \int_{-1}^{1} \frac{T_n(x) \cdot T_m(x)}{\sqrt{1 - x^2}} \, \mathrm{d}x = \begin{cases} 0, & m \neq n, \\ \pi, & m = n = 0, \\ \frac{\pi}{2}, & m = n > 0. \end{cases}$$

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ T_3(x) &= 4x^3 - 3x, \\ T_4(x) &= 8x^4 - 8x^2 + 1, \\ T_5(x) &= 16x^5 - 20x^3 + 5x. \end{aligned}$$

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x).$$



Example

$$\begin{aligned} u_t &= \nu\, u_{xx}, \\ u(x, 0) &= u_0(x). \end{aligned}$$

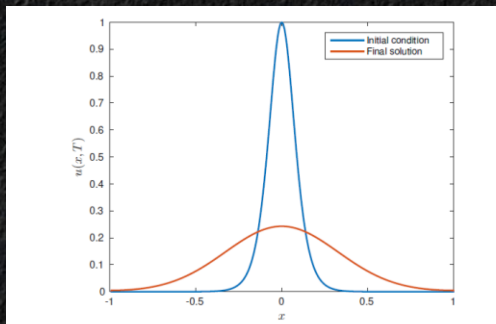


Figure 6. *Spectral solution to the linear heat equation (4.1) at $t = 5.0$, with $\nu = 10^{-2}$ and the initial condition is $u_0(x) = \mathrm{sech}^2(10\,x)$.*

Example

```
l = 1.0;          % half-length of the domain
N = 256;          % number of Fourier modes
dx = 2*l/N;       % distance between two collocation points
x = (1-N/2:N/2)*dx; % physical space discretization
nu = 0.01;        % diffusion parameter
T = 5.0;          % time where we compute the solution
dk = pi/l;        % discretization step in Fourier space
k = [0:N/2 1-N/2:-1]*dk; % vector of wavenumbers
k2 = k.^2;        % almost 2nd derivative in Fourier space

u0 = sech(10.0*x).^2; % initial condition
u0_hat = fft(u0); % Its Fourier transform

% and the solution at final time:
uT = real(ifft(exp(-nu*k2*T).*u0_hat));
```

Comparison

Spectral Method:
Linear combination of continuous functions;
Global approach;

Finite Difference, Flux conservation:
Array of piecewise functions;
Local approach;

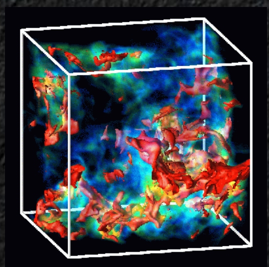
+ / -

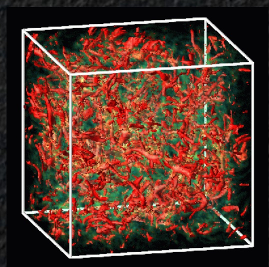
+ (very) fast for smooth solutions
+ Exponential convergence
+ Best for turbulent spectrum

- Shocks
- Inhomogeneities
- Complex Boundaries
- Need for serial reconstruction (integration)

Chaotic flows

Post-processing:
Partial Reconstruction at different length-scales





end

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