ივანე ჯავახიშვილის სახელობის
თბილისის სახელმწიფო უნივერსიტეტი

ლექცია 2

Finite Difference (FD) Methods

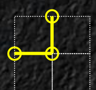
Conservation Law:

$$\frac{\partial U(t,x)}{\partial t} + \nabla \cdot J(t,x) = S(t,x) \qquad J(t,x) = a\,U(t,x)$$

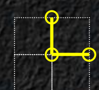
1D Linear Differential Equation:

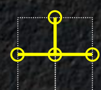
$$\frac{\partial U(t,x)}{\partial t} + a\,\frac{\partial U(t,x)}{\partial x} = 0$$

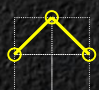
FD Methods

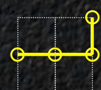
 One-sided backward

 Leapfrog

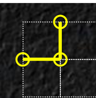
 One-sided forward

 Lax-Wendroff

 Lax-Friedrichs

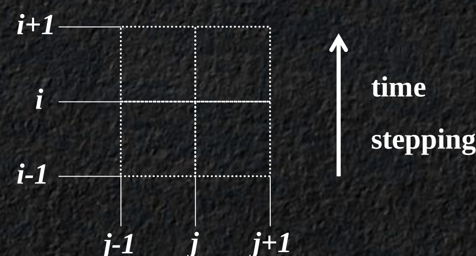
 Beam-Warming

One-sided backward



Difference equation:

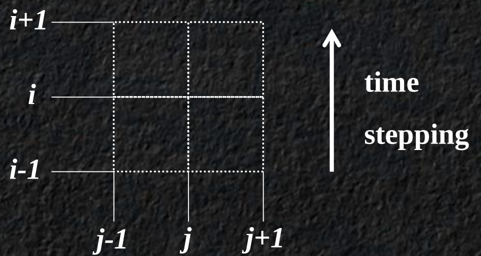
$$U(i+1,j) = U(i,j) - \frac{\Delta t}{\Delta x} A\left[U(i,j) - U(i,j-1)\right]$$



One-sided forward

Difference equation:

$$U(i+1,j) = U(i,j) - \frac{\Delta t}{\Delta x} A\left[U(i,j+1) - U(i,j)\right]$$



Lax-Friedrichs

Difference equation:

$$U(i+1,j) = \frac{1}{2}\left[U(i,j+1) + U(i,j-1)\right] - \frac{\Delta t}{2\Delta x} A\left[U(i,j+1) - U(i,j-1)\right]$$

Leapfrog

Difference equation:

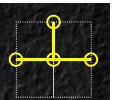
$$U(i+1,j) = U(i-1,j) - \frac{\Delta t}{\Delta x} A\left[U(i,j+1) - U(i,j-1)\right]$$

Lax-Wendroff

Difference equation:

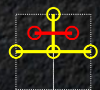
$$U(i+1,j) = U(i,j) - \frac{\Delta t}{2\Delta x} A\left[U(i,j+1) - U(i,j-1)\right] + \frac{\Delta t^2}{2\Delta x^2} A^2\left[U(i,j+1) - 2U(i,j) + U(i,j-1)\right]$$

Lax-Wendroff



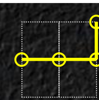
Derivation:

1. Lax-Friedrichs with half step;
2. Leapfrog half step;



derive

Beam-Warming



Difference equation:

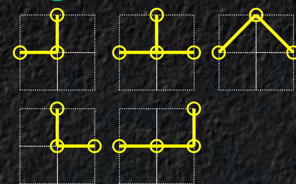
$$U(i+1,j) = U(i,j) - \frac{\Delta t}{2\Delta x} A[3U(i,j) - 4U(i,j-1) + U(i,j-2)] + \frac{\Delta t^2}{2\Delta x^2} A^2 [U(i,j) - 2U(i,j-1) + U(i,j-2)]$$

Comparison

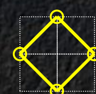
One-sided schemes:	$O(\Delta t, \Delta x)$
Lax-Friedrichs:	$O(\Delta t, \Delta x^2)$
Leapfrog:	$O(\Delta t^2, \Delta x^2)$
Lax-Wendroff:	$O(\Delta t^2, \Delta x^2)$
Beam-Warming:	$O(\Delta t^2, \Delta x^3)$

Time stepping

1st order methods in time:
U(i+1) = F{ U(i) }



2nd order methods in time:
U(i+1) = F{ U(i) , U(i-1) }



Starting method: A) *U(2) = F{ U(1) }* (*e.g.* One-Sided)
B) *U(3) = F{ U(2) , U(1) }* (Leapfrog)

Convergence

$u(t,x)$ - Analytical solution
$U(i,j)$ - Numerical solution

Error function: $E(i,j) = U(i,j) - u(t(i), x(j))$

Numerical convergence (**Norm** of the Error function):

$$\| E(i,j) \| \rightarrow 0, \qquad \Delta x \rightarrow 0$$

Norms

Norm for conservation laws: $\| u(x,y) \| \equiv \int_{-\infty}^{+\infty} | u(x,y) | \, dx$

$$\| E(i,j) \|_1 = \frac{1}{N} \sum_{j=1}^{N} | E(i,j) |$$

Other Norms (e.g. spectral problems)

$$\| E(i,j) \|_2 = \frac{1}{N} \sqrt{ \sum_{j=1}^{N} | E(i,j) |^2 }$$

Norms

P-Norm

$$\| E(i,j) \|_p = \frac{1}{N} \left(\sum_{j=1}^{N} | E(i,j) |^p \right)^{1/p}$$

Norm-2: Energy in numerical domain;
Numerical dissipation, boundary effects, etc.

Numerical Stability

Courant, Friedrichs, Levy (CFL, Courant number)

$$CFL = \max \left(\frac{a \Delta x}{\Delta t} \right)$$

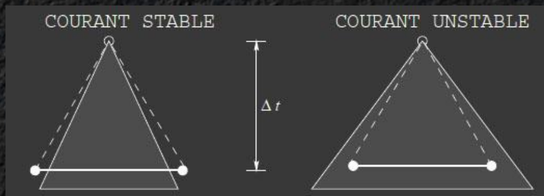
Upwind schemes for discontinity:

$a > 0$: One sided forward
$a < 0$: One sided backward

Numerical stability

CFL<1

Domain of dependence



Lax Equivalence Theorem

For a well-posed linear initial value problem, the method is convergent if and only if it is stable.

Given a properly posed initial-value problem and a finite difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence.

Necessary and sufficient condition for consistent linear method

Well posed: solution exists, continuous, unique;

- numerical stability (t)
- numerical convergence (xyz)

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Discontinuous solutions

Advection equation: $$\frac{\partial U(t,x)}{\partial t} + a\frac{\partial U(t,x)}{\partial x} = 0$$

Initial condition: $$U_0(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

Analytic solution: $$U(t,x) = U_0(x - at)$$

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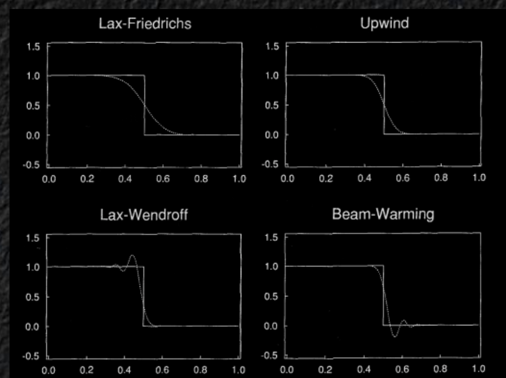
Analytic vs Numerical



Δx = 0.01

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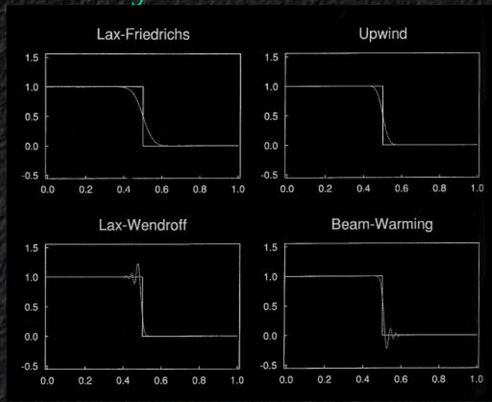
Analytic vs Numerical



$\Delta x = 0.001$

Nonlinear Equations

Linear conservation: $$\frac{\partial U(t,x)}{\partial t} + a\frac{\partial U(t,x)}{\partial x} = 0$$

Nonlinear conservation: $$\frac{\partial U(t,x)}{\partial t} + \frac{\partial}{\partial x}F(U) = 0$$

$$a\frac{\partial U(t,x)}{\partial x} \to \frac{\partial}{\partial x}F(U)$$

Nonlinear FD methods

Lax-Friedrichs

<u>*linear stencil:*</u> $$U(i+1,j) = \frac{1}{2}\left[U(i,j-1) + U(i,j+1)\right] - \frac{\Delta t}{2\Delta x}a\left[U(i,j+1)\text{-}U(i,j-1)\right]$$

<u>*nonlinear stencil:*</u> $$U(i+1,j) = \frac{1}{2}\left[U(i,j-1) + U(i,j+1)\right] - \frac{\Delta t}{2\Delta x}\left[F(U(i,j+1))\text{-}F(U(i,j-1))\right]$$

Nonlinear FD methods

Lax-Wendroff

$$U(i+1,j) = U(i,j) - \frac{\Delta t}{2\Delta x}A\left[U(i,j+1)\text{-}U(i,j-1)\right] + \frac{\Delta t^2}{2\Delta x^2}A^2\left[U(i,j+1)\text{-}2U(i,j) + U(i,j-1)\right]$$

MacCormack’s two step method:

$$U'(j) = U(i,j) - \frac{\Delta t}{\Delta x}\left[F(U(i,j+1))\text{-}F(U(i,j))\right] \quad ,$$

$$U(i+1,j) = \frac{1}{2}\left(U(i,j) + U'(j)\right) - \frac{\Delta t}{2\Delta x}\left[F(U'(j))\text{-}F(U'(j-1))\right]$$

FD Methods

Methods to supress numerical instabilities

Numerical diffusion (first order methods)

$$\frac{\partial U(t,x)}{\partial t} + a\frac{\partial U(t,x)}{\partial x} = D\frac{\partial^2 U(t,x)}{\partial x^2}$$

$$D = \frac{\Delta x^2}{2\Delta t}\left[I - \left(\frac{\Delta t}{\Delta x}a\right)^2\right]$$

Numerical dispersion

$$\frac{\partial U(t,x)}{\partial t} + a\frac{\partial U(t,x)}{\partial x} = \mu\frac{\partial^3 U(t,x)}{\partial x^3}$$

Lax-Wendroff (or second order methods)

$$\mu = \frac{\Delta x^2}{6}a\left[\frac{\Delta t^2}{\Delta x^2}a^2 - I\right]$$

Beam-Warming method

$$\mu = \frac{\Delta x^2}{6}a\left[2I - \frac{3\Delta t}{\Delta x}a + \frac{\Delta t^2}{\Delta x^2}a^2\right]$$

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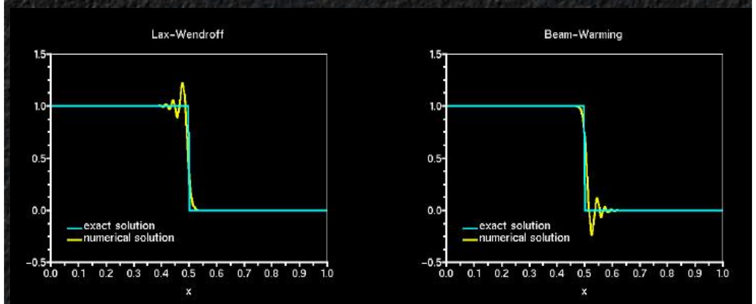
Numerical dispersion

(a > 0) (a < 0)



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Numerical dispersion

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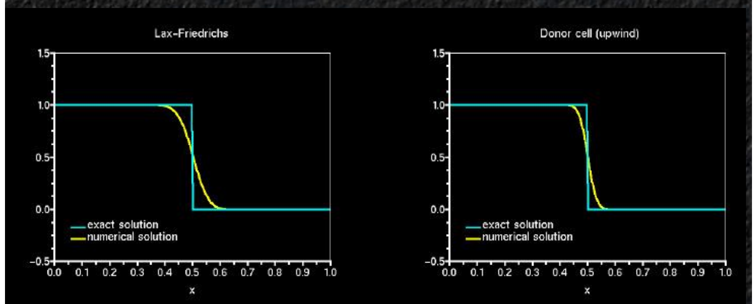

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Convergence

Lax-Friedrichs:

$$\left\| E(i,j) \right\| \approx C \sqrt{t \cdot \Delta x}$$

Convergence:

$$\left\| E(i,j) \right\| \to 0, \qquad \Delta x \to 0$$

FD Methods

+ / Primitive
+ / Fast

- / Accuracy
- / Numerical instabilities

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end

http://www.tevza.org/home/course/modelling-II_2016/

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