Lecture 2

Finite Difference (FD) Methods

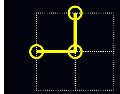
Conservation Law:

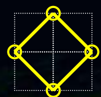
$$\frac{\partial \mathrm{U}(t,x)}{\partial t} + \nabla(\mathrm{J}(t,x)) = 0 \qquad \mathrm{J}(t,x) = a\mathrm{U}(t,x)$$

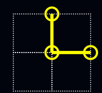
1D Linear Differential Equation:

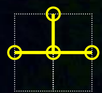
$$\frac{\partial \mathrm{U}(t,x)}{\partial t} + a\frac{\partial \mathrm{U}(t,x)}{\partial x} = 0$$

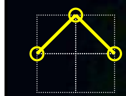
FD Methods

 One-sided backward

 Leapfrog

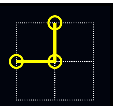
 One-sided forward

 Lax-Wendroff

 Lax-Friedrichs

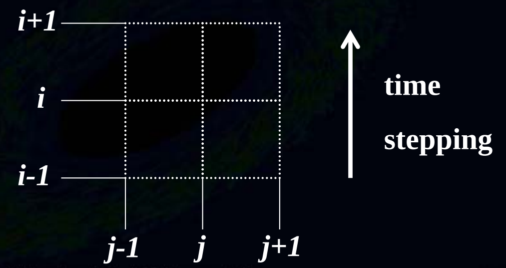
 Beam-Warming

One-sided backward

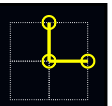


Difference equation:

$$\mathrm{U}(i+1,j) = \mathrm{U}(i,j) - \frac{\Delta t}{\Delta x} A\left[\mathrm{U}(i,j) - \mathrm{U}(i,j-1)\right]$$

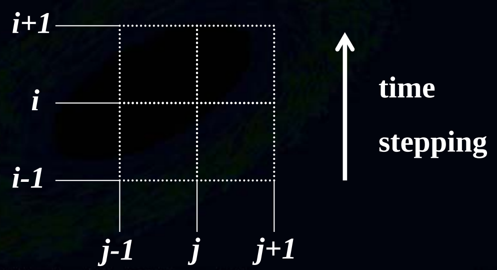


One-sided forward

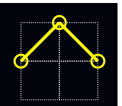


Difference equation:

$$U(i+1,j) = U(i,j) - \frac{\Delta t}{\Delta x} A\left[U(i,j+1) - U(i,j)\right]$$



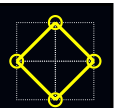
Lax-Friedrichs



Difference equation:

$$U(i+1,j) = \frac{1}{2}\left[U(i,j-1) + U(i,j+1)\right] - \frac{\Delta t}{2\Delta x} A\left[U(i,j+1) - U(i,j-1)\right]$$

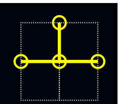
Leapfrog



Difference equation:

$$U(i+1,j) = U(i-1,j) - \frac{\Delta t}{2\Delta x} A\left[U(i,j+1) - U(i,j-1)\right]$$

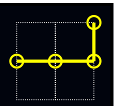
Lax-Wendroff



Difference equation:

$$U(i+1,j) = U(i,j) - \frac{\Delta t}{2\Delta x} A\left[U(i,j+1) - U(i,j-1)\right] + \frac{\Delta t^2}{2\Delta x^2} A^2\left[U(i,j+1) - 2U(i,j) + U(i,j-1)\right]$$

Beam-Warming



Difference equation:

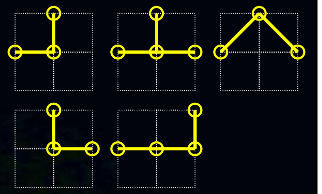
$$\mathrm{U}(i+1,j) = \mathrm{U}(i,j) -$$

$$-\frac{\Delta t}{2\Delta x} A\left[3\mathrm{U}(i,j)\text{-}4\mathrm{U}(i,j-1)+\mathrm{U}(i,j-2)\right]$$

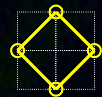
$$+\frac{\Delta t^2}{2\Delta x^2} A^2\left[\mathrm{U}(i,j)\text{-}2\mathrm{U}(i,j-1)+\mathrm{U}(i,j-2)\right]$$

Time stepping

1st order methods in time:
U(i+1) = F{ U(i) }



2nd order methods in time:
U(i+1) = F{ U(i) , U(i-1) }



Starting A) *U(2) = F{ U(1) }* (*e.g.* One-Sided)
method: B) *U(3) = F{ U(2) , U(1) }* (Leapfrog)

Convergence

$\mathrm{u}(t,x)$ - Analytical solution
$\mathrm{U}(i,j)$ - Numerical solution

Error function: $\mathrm{E}(i,j) = \mathrm{U}(i,j) - u(t(i), x(j))$

Numerical convergence (**Norm** of the Error function):

$$\left\| \mathrm{E}(i,j) \right\| \to 0, \qquad \Delta x \to 0$$

Norms

Norm for conservation laws: $\left\| u(x,y) \right\| \equiv \int_{-\infty}^{+\infty} |u(x,y)|\, dx$

$$\left\| E(i,j) \right\|_1 = \sum_{j=1}^{N} |E(i,j)|$$

Other Norms (e.g. spectral problems)

$$\left\| E(i,j) \right\|_2 = \sqrt{\sum_{j=1}^{N} |E(i,j)|^2}$$

Numerical Stability

Courant, Friedrichs, Levy (CFL, Courant number)

$$CFL = \max\left(\frac{a\Delta x}{\Delta t}\right)$$

Upwind schemes for discontinuity:

a > 0 : One sided forward
a < 0 : One sided backward

Lax Equivalence Theorem

For a <u>well-posed</u> linear initial value problem, the method is <u>convergent</u> if and only if it is <u>stable.</u>

Necessary and sufficient condition
for consistent linear method

Well posed: solution exists, continuous, unique;

- numerical stability (t)
- numerical convergence (xyz)

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Discontinuous solutions

Advection equation: $$\frac{\partial U(t,x)}{\partial t} + a\frac{\partial U(t,x)}{\partial x} = 0$$

Initial condition: $$U_0(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

Analytic solution: $$U(t,x) = U_0(x - at)$$

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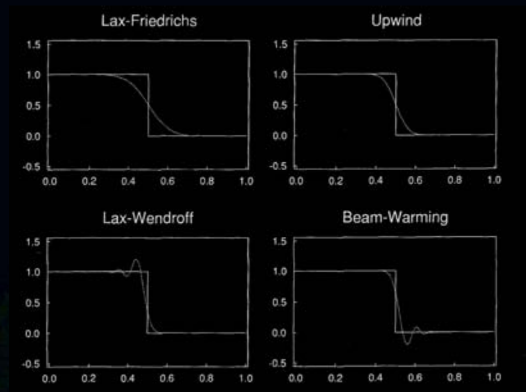
Analytic vs Numerical



Δx = 0.01

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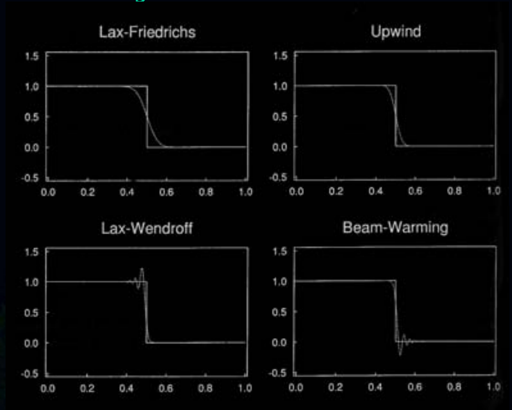
Analytic vs Numerical



Δx = 0.001

Nonlinear Equations

Linear conservation: $$\frac{\partial \mathrm{U}(t,x)}{\partial t} + a\frac{\partial \mathrm{U}(t,x)}{\partial x} = 0$$

Nonlinear conservation: $$\frac{\partial \mathrm{U}(t,x)}{\partial t} + \frac{\partial}{\partial x}F(\mathrm{U}) = 0$$

$$a\frac{\partial \mathrm{U}(t,x)}{\partial x} \rightarrow \frac{\partial}{\partial x}F(\mathrm{U})$$

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Nonlinear FD methods

Lax-Friedrichs

linear stencil:
$$\mathrm{U}(i+1,j) = \frac{1}{2}\left[\mathrm{U}(i,j-1) + \mathrm{U}(i,j+1)\right] - \frac{\Delta t}{2\Delta x}a\left[\mathrm{U}(i,j+1) - \mathrm{U}(i,j-1)\right]$$

nonlinear stencil:
$$\mathrm{U}(i+1,j) = \frac{1}{2}\left[\mathrm{U}(i,j-1) + \mathrm{U}(i,j+1)\right] - \frac{\Delta t}{2\Delta x}\left[F(\mathrm{U}(i,j+1)) - F(\mathrm{U}(i,j-1))\right]$$

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Nonlinear FD methods

Lax-Wendroff
$$\mathrm{U}(i+1,j) = \mathrm{U}(i,j) - \frac{\Delta t}{2\Delta x}A\left[\mathrm{U}(i,j+1) - \mathrm{U}(i,j-1)\right] + \frac{\Delta t^2}{2\Delta x^2}A^2\left[\mathrm{U}(i,j+1) - 2\mathrm{U}(i,j) + \mathrm{U}(i,j-1)\right]$$

MacCormack's two step method:

$$\mathrm{U}'(j) = \mathrm{U}(i,j) - \frac{\Delta t}{\Delta x}\left[F(\mathrm{U}(i,j+1)) - F(\mathrm{U}(i,j))\right] \quad ,$$

$$\mathrm{U}(i+1,j) = \frac{1}{2}\left(\mathrm{U}(i,j) + \mathrm{U}'(j)\right) - \frac{\Delta t}{2\Delta x}\left[F(\mathrm{U}'(j)) - F(\mathrm{U}'(j-1))\right]$$

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FD Methods

Methods to supress numerical instabilities

Numerical diffusion (first order methods)

$$\frac{\partial \mathrm{U}(t,x)}{\partial t} + a\frac{\partial \mathrm{U}(t,x)}{\partial x} = D\frac{\partial^2 \mathrm{U}(t,x)}{\partial x^2}$$

$$D = \frac{\Delta x^2}{2\Delta t}\left[I - \left(\frac{\Delta x}{\Delta t}a\right)^2\right]$$

Numerical dispersion

$$\frac{\partial \mathrm{U}(t,x)}{\partial t} + a\frac{\partial \mathrm{U}(t,x)}{\partial x} = \mu\frac{\partial^3 \mathrm{U}(t,x)}{\partial x^3}$$

Lax-Wendroff (or second order methods)

$$\mu = \frac{\Delta x^2}{6}a\left[\frac{\Delta t^2}{\Delta x^2}a^2 - I\right]$$

Beam-Warming method

$$\mu = \frac{\Delta x^2}{6}a\left[2I - \frac{3\Delta t}{\Delta x}a + \frac{\Delta t^2}{\Delta x^2}a^2\right]$$

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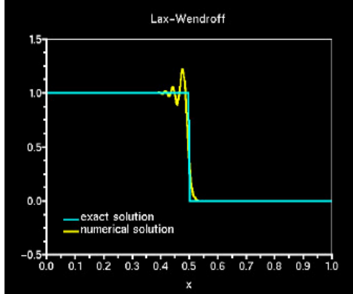
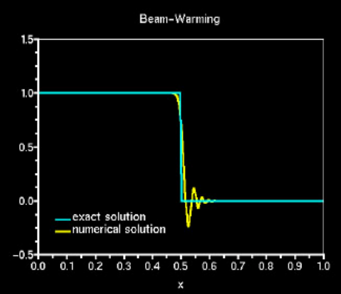
Numerical dispersion

(a > 0) (a < 0)





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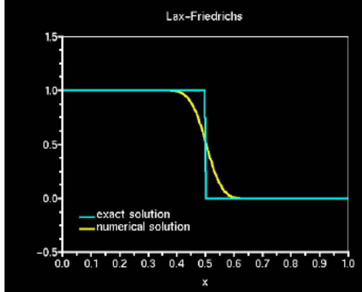

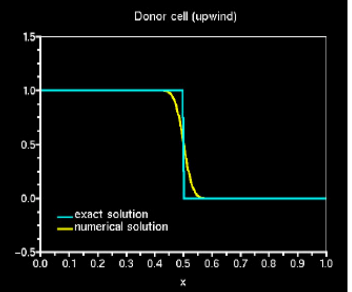

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Convergence

Lax-Friedrichs:

$$\left\| \mathrm{E}(i,j) \right\| \approx C\sqrt{t \cdot \Delta x}$$

Convergence:

$$\left\| \mathrm{E}(i,j) \right\| \to 0, \qquad \Delta x \to 0$$

FD Methods

+ / Primitive
+ / Fast

- / Accuracy
- / Numerical instabilities

end

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