# Linear Stability of Differentially Rotating Keplerian Flows with Granular Rheology

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## Abstract

The linear stability of Keplerian disks is studied taking into account rheological properties of incompressible flow that contains gas and dust orbiting central gravitating object. Interaction of solid particles is studied within the granular flow model, when local constitutive equation can be used. Granular viscosity of the flow depends on the pressure, as well as the velocity shear of the flow. In this model 2D equilibrium flow with Keplerian velocity profile and radially stratified pressure, surface density and granular viscosity parameter is derived.

Local shearing sheet analysis is used to neglect flow curvature to study the linear dynamics of perturbations under the influence of granular rheology. The linear spectrum of the problem reveals different factors leading to the growth and decay of perturbations in this limit. Solution shows viscous damping, velocity shear induced transient amplification and rheological effects that can lead to asymptotic growth in linear limit.

Found instability differs from viscous overstability of dense granular flows: it occurs at shorter scales and can potentially promote particle agglomeration and planetesimal formation at early stages in protoplanetary disks.

## Granular Keplerian Flows

Protoplanetary disks are flows of gas and dust rotating around central object.

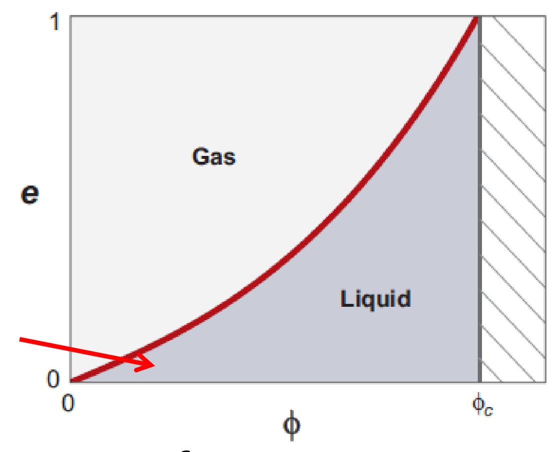
Keplerian rotation: $\Omega_K(r) = r^{-3/2}$

### Observations

- Mixture of gas and dust granules;
- Denser dust sub-disk at the central plane;

### Important Factors

- Low values of dust to gas ratio;
- Low restitution parameter;




*Flow can still exhibit properties of "granular fluid" (Forterre & Pouliquen 2008)*

In this limit we effectively neglect kinematic viscosity of gas component, gas-to-dust drug and assume that momentum dissipation is primarily due to dust/particle collisions and associated rheological behavior in the incompressible single fluid approximation.

## Viscous Disk Model

Navier-Stoke s equation: $$\rho\left\{\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial x_k}\right\}v_i=\frac{\partial P}{\partial x_k}+\frac{\partial \Phi}{\partial x_k}+\frac{\partial}{\partial x_k}\tau_{ik}$$

Viscous stress tensor: $$\tau_{ik}=\eta\dot{\gamma}_{ik},\qquad \dot{\gamma}_{ik}=\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}$$

Local rheological viscosity parameter: $$\eta=\eta(P,\mathrm{V})$$

### Equilibrium disk model in polar coordinates

Radial balance: $$r\Omega^2=\frac{1}{\bar\rho}\frac{\partial\bar P}{\partial r}-\frac{\partial\Phi}{\partial r}$$

Azimuthal balance: $$\left(r\frac{\partial^2\Omega}{\partial r^2}+3\frac{\partial\Omega}{\partial r}\right)\bar\eta+r\frac{\partial\Omega}{\partial r}\frac{\partial\bar\eta}{\partial r}=0$$

$$\bar P(r)=P_0\left(\frac{r}{r_0}\right)^{-\beta_P}$$

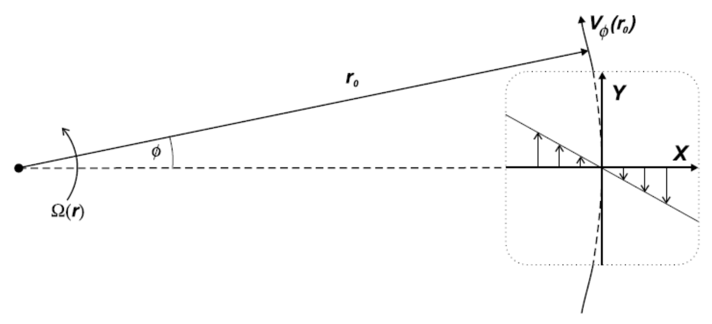
*Radially stratified rotationally supported disk model*

$$\bar\rho(r)=\rho_0\left(\frac{r}{r_0}\right)^{-\beta_\sigma}$$

$$\bar\eta(r)=\eta_0\left(\frac{r}{r_0}\right)^{-\beta_\eta}$$

- Viscosity ($q$=3/2) $$\frac{\partial\ln\bar\eta}{\partial\ln r}=q-2$$
- Isentropic flow: $$\frac{\partial}{\partial r}\left(\frac{\bar P(r)}{\bar\rho^\gamma(r)}\right)=0$$

## Linear Perturbations




*Analysis in local Cartesian co-rotating shear sheet frame: Non-normal modes.*

Local ODE equations:

*A*,*B*-Oort's constants.

$$\begin{aligned}\frac{du_x}{dt}&=2\Omega u_y-k_x(t)p-\nu\left(k^2+2ik_\Sigma k_x(t)\right)u_x+2iAk_y\hat\eta\\ \frac{du_y}{dt}&=2Bu_x-k_yp-\nu\left(k^2+2ik_\Sigma k_x(t)\right)u_y+2iAk_x(t)\hat\eta\\ \hat\eta&=-iG_pp+2iG_\gamma\left((k_x(t)+ik_\Sigma)u_y+k_yu_x\right)\end{aligned}$$

1. Pressure rheology: $$G_p=\left(\frac{\partial\eta}{\partial p}\right)_\gamma$$

2. Shear rheology: $$G_\gamma=\frac{2}{\rho_0}\left(\frac{\partial\eta}{\partial\gamma_{r\phi}}\right)_P$$

Linear spectrum: adiabatic approximation. $$\left|\frac{d}{dt}\omega(t)\right|\ll\omega^2(t)$$

## Viscous Stability

General form of adiabatic dispersion equation for local linear perturbations

$$\omega=i\sigma_\nu+i\sigma_A+i\sigma_P+i\sigma_\gamma$$

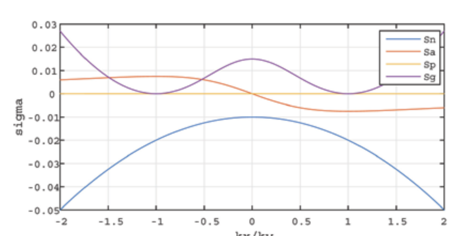
Assuming that viscous time-scales are longer then disk rotation period:

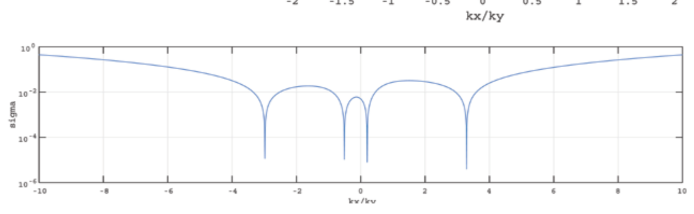
$$\Omega G_p=\Omega_\nu/\Omega\ll1\,,\quad A=-3\Omega/4\,,\quad B=-\Omega/4$$

- Viscous decay $$\sigma_\nu=-\nu k^2$$
- Transient amplification $$\sigma_A=-\frac{3\Omega}{2}\frac{k_xk_y}{k^2}$$
- Pressure rheology $$\sigma_P=-3\Omega G_P\left(\nu k_xk_y+\frac{\Omega k_x^2+Bk_y^2}{k^2}\right)$$
- Shear rheology $$\sigma_\gamma=3\Omega G_\gamma\frac{(k_x^2-k_y^2)^2}{k^2}$$

Linear small-scale asymptotic instability:

$$\frac{k_x^2}{k_y^2}>\frac{1+\alpha^2}{1-\alpha^2}\,,\quad\alpha^2\equiv\left|\frac{\eta}{3\Omega G_\gamma}\right|$$





## Summary

Viscous rheological model of the incompressible granular flow in gravitating astrophysical disks is developed.

Local linear exponential instability in Keplerian protoplanetary disks originating from the granular properties of dust particles is found.

Instability growth rate is calculated analytically is weakly stratified disks, where radial stratification scale is much larger then characteristic length-scales of perturbations: $|k/k_\sigma|>>1$.

Instability is due to viscosity shear rheology and is revealed for small scale alongated perturbations: $|k_x/k_y|>1$.

It differs in principle from viscous overstability originating from pressure (density) rheology for larger scale perturbations.

Instability can lead to the growth of gas density and acceleration of dust particle agglomeration process, thus leading to rapid formation of planetesimals.

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