ლექცია 3

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ჯინსის არამდგრადობა ბრუნავ გარემოში: ტუმრის კრიტერიუმი

Pringle, King –

Chapter 12
12.4.1 ბრუნავი თხელი დისკი

ჯინსის განტოლება
James Binney, Galactic Dynamics (2007)

An alternative form of the collisionless Boltzmann equation can be derived by extending to six dimensions the concept of the convective or Lagrangian derivative (see eq. F.8). We define

$$\frac{\mathrm{d}f}{\mathrm{d}t} \equiv \frac{\partial f}{\partial t} + \dot{\mathbf{w}} \cdot \frac{\partial f}{\partial \mathbf{w}}; \qquad (4.8)$$

$\mathrm{d}f/\mathrm{d}t$ represents the rate of change of the local probability density as seen by an observer who moves through phase space with a star. Comparison of equations (4.6) and (4.7) shows that $\dot{\mathbf{w}} \cdot (\partial f/\partial \mathbf{w}) = [f, H]$, so the convective derivative can also be written

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + [f, H], \qquad (4.9)$$

and the collisionless Boltzmann equation (4.6) is simply

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0. \qquad (4.10)$$

In terms of inertial Cartesian coordinates, in which $H = \frac{1}{2}v^2 + \Phi(\mathbf{x}, t)$ with Φ the gravitational potential, the collisionless Boltzmann equation reads

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0. \qquad (4.11)$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}}\cdot\frac{\partial f}{\partial \mathbf{v}} = B - D, \tag{4.15}$$

where $B(\mathbf{x},\mathbf{v},t)$ and $D(\mathbf{x},\mathbf{v},t)$ are the rates per unit phase-space volume at which stars are born and die. In the collisionless Boltzmann equation, $B-D$ is set to zero. This is a useful approximation to the truth if and only if $B-D$ is smaller in magnitude than terms on the left of equation (4.15). The term $\mathbf{v}\cdot\partial f/\partial\mathbf{x}$ is of order vf/R, where v and R are the characteristic speed and radius in the galaxy. The ratio R/v is simply the crossing time t_{cross} (§1.2). Similarly, $\partial\Phi/\partial\mathbf{x}$ is of order the characteristic acceleration a, so the term $(\partial\Phi/\partial\mathbf{x})\cdot(\partial f/\partial\mathbf{v})$ is of order af/v. Since $a \approx v/t_{\mathrm{cross}}$, the two last terms in the middle section of equation (4.15) are of order f/t_{cross}. Thus consider the ratio

$$\gamma = \left|\frac{B-D}{f/t_{\mathrm{cross}}}\right|. \tag{4.16}$$

4.8 The Jeans and virial equations

In §4.1.2 we saw that comparisons between theoretical models and observational data often center on velocity moments of the DF, such as $\overline{\mathbf{v}}$ and $\overline{v_i v_j}$. Calculating moments is easy if one knows the DF, but finding a DF that is compatible with a given probability density distribution $\nu(\mathbf{x})$ is not straightforward, and even if a DF can be found, it is often not unique. Therefore in this section we discuss techniques for inferring moments from stellar densities without actually recovering the DF. Dejonghe (1986) gives an extensive discussion of this problem.

Integrating equation (4.11) over all velocities, we obtain

$$\int \mathrm{d}^3\mathbf{v}\,\frac{\partial f}{\partial t} + \int \mathrm{d}^3\mathbf{v}\,v_i\frac{\partial f}{\partial x_i} - \frac{\partial\Phi}{\partial x_i}\int \mathrm{d}^3\mathbf{v}\,\frac{\partial f}{\partial v_i} = 0, \tag{4.203}$$

where we have employed the summation convention (page 772). The range of velocities over which we are integrating does not depend on time, so the partial derivative $\partial/\partial t$ in the first term of this equation may be taken outside the integral. Similarly, since v_i does not depend on x_i, the partial derivative $\partial/\partial x_i$ in the second term of the equation may be taken outside the integral sign. Furthermore, the last term on the left side of the equation vanishes on application of the divergence theorem (eq. B.46), given that $f(\mathbf{x}, \mathbf{v}, t) = 0$ for sufficiently large $|\mathbf{v}|$, i.e., there are no stars that move infinitely fast. Recalling the definitions of the density ν (eq. 4.20) and the mean velocity $\overline{\mathbf{v}}$ (eq. 4.24b), we have that

$$\frac{\partial \nu}{\partial t} + \frac{\partial(\nu \overline{v}_i)}{\partial x_i} = 0. \tag{4.204}$$

Equation (4.204) differs from the continuity equation (F.3) only in that it describes conservation of probability rather than that of mass, and replaces the fluid velocity by the mean stellar velocity.

We now multiply equation (4.11) by v_j and integrate over all velocities, and obtain

$$\frac{\partial}{\partial t}\int \mathrm{d}^3\mathbf{v}\, f v_j + \int \mathrm{d}^3\mathbf{v}\, v_i v_j \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\int \mathrm{d}^3\mathbf{v}\, v_j \frac{\partial f}{\partial v_i} = 0. \tag{4.205}$$

The last term on the left side can be transformed by applying the divergence theorem, using the fact that f vanishes for large $|\mathbf{v}|$:

$$\int \mathrm{d}^3\mathbf{v}\, v_j \frac{\partial f}{\partial v_i} = -\int \mathrm{d}^3\mathbf{v}\, \frac{\partial v_j}{\partial v_i} f = -\int \mathrm{d}^3\mathbf{v}\, \delta_{ij} f = -\delta_{ij}\nu. \tag{4.206}$$

Thus equation (4.205) may be rewritten

$$\frac{\partial(\nu\overline{v}_j)}{\partial t}+\frac{\partial(\nu\overline{v_i v_j})}{\partial x_i}+\nu\frac{\partial\Phi}{\partial x_j}=0. \tag{4.207}$$

This can be put into a more familiar form by subtracting from it $\overline{v}_j$ times the equation of continuity (4.204) to yield

$$\nu\frac{\partial\overline{v}_j}{\partial t}-\overline{v}_j\frac{\partial(\nu\overline{v}_i)}{\partial x_i}+\frac{\partial(\nu\overline{v_i v_j})}{\partial x_i}=-\nu\frac{\partial\Phi}{\partial x_j}, \tag{4.208}$$

and then using the definition (4.26) of the velocity-dispersion tensor to eliminate $\overline{v_i v_j}$. The result is an analog of Euler's equation (F.7) of fluid flow;

$$\nu\frac{\partial\overline{v}_j}{\partial t}+\nu\overline{v}_i\frac{\partial\overline{v}_j}{\partial x_i}=-\nu\frac{\partial\Phi}{\partial x_j}-\frac{\partial(\nu\sigma_{ij}^2)}{\partial x_i}. \tag{4.209}$$

The left side and the first term on the right side of equation (4.209) differ from terms in the ordinary Euler equation only by the replacement of the mass density by the probability density, and by the replacement of the fluid velocity by the mean stellar velocity. The last term on the right side of equation (4.209) represents something akin to the pressure force $-\boldsymbol{\nabla}p$. More exactly, $-\nu\sigma_{ij}^2$ is a **stress tensor** that describes an anisotropic pressure. Since equations (4.204) and (4.209) were first applied to stellar dynamics by Jeans (1919), we call them the **Jeans equations**.[16]

Equations (4.204) and (4.209) are valuable because they relate observationally accessible quantities, such as the streaming velocity, velocity dispersion, and so forth. However, this is an incomplete set of equations in the following sense. If we know the potential $\Phi(\mathbf{x}, t)$ and the density $\nu(\mathbf{x}, t)$, we have nine unknown functions—the three components of $\overline{\mathbf{v}}$ and the six independent components of the symmetric tensor $\boldsymbol{\sigma}^2$—but only four equations—the scalar continuity equation and the three components of Euler's equation. Thus we cannot solve for $\overline{\mathbf{v}}$ and $\boldsymbol{\sigma}^2$ without additional information. The reader may argue that if we multiply the collisionless Boltzmann equation (4.11) through by $v_i v_k$ and integrate over all velocities, we obtain a new set of differential equations for $\boldsymbol{\sigma}^2$ which might supply the missing information. Unfortunately, these equations involve quantities like $\overline{v_i v_j v_k}$ for which we would require still further equations. Thus these additional equations are of no use unless we can in some way truncate or *close* this regression to ever higher moments of the velocity distribution. We shall find that closure is possible only in special circumstances, for example when the system is spherical and we know that its DF is ergodic, $f(H)$ (Box 4.3), or when the system is axisymmetric and its DF is of the form $f(H, L_z)$. The equations can also be closed for any Stäckel potential (van de Ven et al. 2003).

The line-of-sight velocity dispersion is determined both by the variation in the mean velocity $\overline{v}_{\parallel}(\mathbf{x})$ along the line of sight, and the spread in stellar velocities at each point in the galaxy around $\overline{\mathbf{v}}(\mathbf{x})$. This spread is characterized by the **velocity-dispersion tensor**

$$\begin{aligned}\sigma^2_{ij}(\mathbf{x}) &\equiv \frac{1}{\nu(\mathbf{x})}\int \mathrm{d}^3\mathbf{v}\,(v_i-\overline{v}_i)(v_j-\overline{v}_j)f(\mathbf{x},\mathbf{v})\\ &= \overline{v_iv_j}-\overline{v}_i\overline{v}_j.\end{aligned} \tag{4.26}$$

The velocity-dispersion tensor is manifestly symmetric, so we know from matrix algebra that at any point $\mathbf{x}$ we may choose a set of orthogonal axes $\hat{\mathrm{e}}_i(\mathbf{x})$ in which $\boldsymbol{\sigma}^2$ is diagonal, that is, $\sigma^2_{ij}=\sigma^2_{ii}\delta_{ij}$ (no summation over i, and $\delta_{ij}=1$ for $i=j$ and zero otherwise). The ellipsoid that has the

4.8.1 Jeans equations for spherical systems

To obtain the Jeans equations in spherical coordinates, we start from the collisionless Boltzmann equation in the form (4.14), which involves the canonical momenta

$$p_r=\dot{r}=v_r \quad ; \quad p_\theta=r^2\dot{\theta}=rv_\theta \quad ; \quad p_\phi=r^2\sin^2\theta\dot{\phi}=r\sin\theta\,v_\phi. \tag{4.210}$$

We have

$$\int \mathrm{d}p_r\mathrm{d}p_\theta\mathrm{d}p_\phi\,f=r^2\sin\theta\int \mathrm{d}v_r\mathrm{d}v_\theta\mathrm{d}v_\phi\,f=r^2\sin\theta\,\nu. \tag{4.211}$$

We assume that the system is spherical and time-independent, so we can drop $\partial\Phi/\partial\theta$, $\partial\Phi/\partial\phi$, $\partial f/\partial t$ and $\partial f/\partial\phi$ from (4.14); we retain $\partial f/\partial\theta$ because any dependence of f on v_ϕ is likely to introduce θ-dependence through the last of equations (4.210) when v_ϕ is expressed in terms of p_ϕ. After simplification, equation (4.14) becomes

$$p_r \frac{\partial f}{\partial r} + \frac{p_\theta}{r^2}\frac{\partial f}{\partial \theta} - \left(\frac{\mathrm{d}\Phi}{\mathrm{d}r} - \frac{p_\theta^2}{r^3} - \frac{p_\phi^2}{r^3 \sin^2\theta}\right)\frac{\partial f}{\partial p_r} + \frac{p_\phi^2 \cos\theta}{r^2 \sin^3\theta}\frac{\partial f}{\partial p_\theta} = 0. \quad (4.212)$$

We now multiply by $p_r\,\mathrm{d}p_r\mathrm{d}p_\theta\mathrm{d}p_\phi$ and integrate over all momenta. With equation (4.211) and similar results, and using the divergence theorem to eliminate derivatives with respect to the momenta, we find

$$\frac{\partial}{\partial r}(r^2 \sin\theta\, \nu \overline{p_r^2}) + \frac{\partial}{\partial \theta}(\sin\theta\, \nu \overline{p_r p_\theta}) + r^2 \sin\theta\, \nu \left(\frac{\mathrm{d}\Phi}{\mathrm{d}r} - \frac{\overline{p_\theta^2}}{r^3} - \frac{\overline{p_\phi^2}}{r^3 \sin^2\theta}\right) = 0. \quad (4.213)$$

Additional Jeans equations can be obtained by multiplying (4.212) by p_θ or p_ϕ, but these are not useful.

If the line-of-sight velocity dispersion has been measured as a function of radius, equation (4.215) can be used to constrain the radial dependence of β. The most direct approach is to assume a functional form for $\beta(r)$ and treat (4.215) as a first-order linear differential equation for $\nu\overline{v_r^2}$. The integrating factor is $\exp(2\int \mathrm{d}r\, \beta/r)$, so the solution can be written in closed form. Different choices of $\beta(r)$ yield different predictions for the line-of-sight velocity dispersion as a function of radius (see Problem 4.28), so β can be constrained by optimizing the fit between predictions obtained from (4.215) and the observed velocity-dispersion profile.

The case of constant non-zero β is particularly simple. Then the solution of (4.215) that satisfies the boundary condition $\lim_{r\to\infty}\overline{v_r^2} = 0$ is

$$\overline{v_r^2}(r) = \frac{1}{r^{2\beta}\nu(r)} \int_r^\infty \mathrm{d}r'\, r'^{2\beta}\nu(r')\frac{\mathrm{d}\Phi}{\mathrm{d}r'}. \qquad (4.216)$$

In any static spherical system, $\overline{p_r p_\theta} = r\overline{v_r v_\theta}$ must vanish because the DF is of the form $f(H, \mathbf{L})$, and is therefore an even function of v_r. Finally, dividing through by $r^2 \sin\theta$ and using equations (4.210) we obtain

$$\frac{\mathrm{d}(\nu\overline{v_r^2})}{\mathrm{d}r} + \nu\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r} + \frac{2\overline{v_r^2} - \overline{v_\theta^2} - \overline{v_\phi^2}}{r}\right) = 0. \qquad (4.214)$$

In terms of the anisotropy parameter of equation (4.61), equation (4.214) reads

$$\frac{\mathrm{d}(\nu\overline{v_r^2})}{\mathrm{d}r} + 2\frac{\beta}{r}\nu\overline{v_r^2} = -\nu\frac{\mathrm{d}\Phi}{\mathrm{d}r}. \qquad (4.215)$$