Chapter 2

General properties of shear flows

In this chapter we focus on simple shear flows aiming to describe the non-modal method as well as the basic properties of flows with inhomogeneous velocity fields. We start with the equations of magnetohydrodynamics and then study the linear perturbations of the non-magnetized equilibrium state. We illustrate the mathematical formalism of the non-modal method for a parallel flow with a constant linear velocity shear. In order to demonstrate the effect of the background shear flow on the perturbation modes individually we consider limiting cases separately. In particular, we analyze the effects of shear flow on the vortical and compressible perturbations when simple analytic solutions may be obtained. Finally, we estimate the possible values of the velocity shear that may occur in equilibrium flows and introduce the convention used further throughout the thesis.

2.1 Basic equations

The basic MHD equations are the conservation laws for mass, momentum and energy. Conservation of the mass is described by the continuity equation:

$$\left\{\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right\}\rho + \rho(\nabla\cdot\mathbf{V}) = 0. \tag{2.1}$$

Momentum conservation is written as the equation of motion:

$$\rho\left\{\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right\}\mathbf{V} = -\nabla P + \mathbf{F}_B + \sum_n \mathbf{F}_n, \tag{2.2}$$

where we have neglected all dissipative effects and assume that the flow is inviscid. The sum of external forces $\mathbf{F}_n$ includes the inertial Coriolis force in the case of rotating medium and the gravity force in the case of the flow affected by gravity. The effect of the magnetic field in the equation of motion may be described by the Lorenz force:

$$\mathbf{F}_B = \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}. \tag{2.3}$$

The complete description of the magnetized flow needs additional equations for the evaluation of the magnetic field. For this purpose we employ the ideal magnetohydrodynamic (MHD) approximation. It uses the magnetic induction equation for the description of the field dynamics:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}), \tag{2.4}$$

where the magnetic diffusivity is neglected. In this ideal case the magnetic flux is “frozen” into the fluid flow. The magnetic field follows the solenoidal condition which excludes the existence of the magnetic monopoles:

$$\nabla \cdot \mathbf{B} = 0. \tag{2.5}$$

The energy conservation law may be written for the entropy of the flow as:

$$\left\{\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla\right\} S = 0, \tag{2.6}$$

where we assume that the flow is isentropic. This is a well justified approximation when the effects of heat conductivity and emission or absorption of radiation are negligible.
For the description of the thermodynamic equilibrium state we can use the equation of entropy in the ideal gas approximation:

$$S = c_{\mathrm{v}} \ln \frac{P}{\rho^\gamma} + \mathrm{constant}, \tag{2.7}$$

where γ is the adiabatic index and c_{v} is the specific heat. In many cases it is useful to employ the dynamical equation for the hydrodynamic pressure which can be derived from Eqs. (2.1,6,7):

$$\left\{\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla\right\} P + \gamma P (\nabla \cdot \mathbf{V}) = 0. \tag{2.8}$$

Eqs. (2.1)-(2.7) are the set of equations that describe MHD flow influenced by the Coriolis and gravity forces. In this chapter we focus on non-rotating flows, we neglect gravity and magnetic field, and focus on hydrodynamics in order to describe the basic properties of flows with velocity shear.

2.2 Shear flow equilibrium and linear perturbations

Let us consider an unbounded inviscid hydrodynamic flow. The basic properties of shear flows may be studied for 2D flows. Thus we consider the simplest possible inhomogeneous flow, namely a 2D flow with constant linear shear of velocity and homogeneous pressure and density (plane Couette flow):

$$\mathrm{V}_0 = (Ay, 0), \quad P_0 = \text{constant}, \quad \rho_0 = \text{constant}. \tag{2.9}$$

A is the constant parameter of the velocity shear, which is assumed to be positive (see Fig. 2.1). This gives an exact time independent equilibrium solution of Eqs. (2.1), (2.2) and (2.8). However, realistic flows are much more complicated. For their analysis we may employ the perturbative method. We distinguish the equilibrium and perturbed parts in various physical quantities and study the perturbations on a known equilibrium flow. Formally:

$$\mathbf{V} = \mathbf{V}_0 + \mathbf{V}', \quad P = P_0 + P', \quad \rho = \rho_0 + \rho', \tag{2.10}$$

where $\mathbf{V}'$, P' and ρ' are perturbations of the velocity, pressure and density, respectively. When the perturbations are much smaller in amplitude than the corresponding equilibrium quantities we can neglect terms that are higher than the first order in the perturbations and analyze the linear problem.
Substitution of the variables (2.10) into the Eqs. (2.1), (2.2) and (2.8) and linearization leads to the following system of partial differential equations:

$$\left\{ \frac{\partial}{\partial t} + Ay\frac{\partial}{\partial x} \right\} V_x' = -AV_y' - \frac{1}{\rho_0}\frac{\partial P'}{\partial x}, \tag{2.11}$$

$$\left\{ \frac{\partial}{\partial t} + Ay\frac{\partial}{\partial x} \right\} V_y' = -\frac{1}{\rho_0}\frac{\partial P'}{\partial y}, \tag{2.12}$$

$$\left\{ \frac{\partial}{\partial t} + Ay\frac{\partial}{\partial x} \right\} \rho' + \rho_0 \left(\nabla \cdot \mathbf{V}'\right) = 0, \tag{2.13}$$

$$\left\{ \frac{\partial}{\partial t} + Ay\frac{\partial}{\partial x} \right\} P' + \gamma P_0 \left(\nabla \cdot \mathbf{V}'\right) = 0. \tag{2.14}$$

The effect of the background shear flow enters through the explicit coordinate dependence in the convective derivative.
The principal advantage of linear analysis is that it is possible to identify the perturbation spectrum. The linear character of the governing equations permits any solution to be expanded into a linear superposition of different modes.

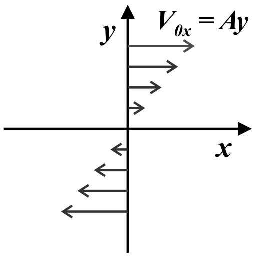

Figure 2.1: Shear flow with constant linear velocity profile $\mathbf{V}_0 = (Ay, 0)$. This is a simplest example of the incompressible flow with parallel streamlines and non-zero vorticity: curl$\mathbf{V}_0 = -A$.

It is straightforward to define the linear spectrum in the limiting case of a static background, when $\mathbf{V}_0 = 0$ ($A = 0$). In this case Eqs. (2.11) - (2.14) are homogeneous in time as well as in space. Hence, the corresponding solutions may be expressed in terms of harmonic functions of coordinate and time variables and may be found prescribing the same $(\mathbf{r}, t)$ dependence to all physical variables:

$$\mathbf{V}'(\mathbf{r}, t) \propto \exp(\mathrm{i}k_x x + \mathrm{i}k_y y - \mathrm{i}\omega t). \qquad (2.15)$$

Substitution of this into Eqs. (2.11-14) in the case of $A = 0$ leads to the well known dispersion equation:

$$\omega^2(\omega^2 - c_s^2 k^2) = 0, \qquad (2.16)$$

where $k^2 = k_x^2 + k_y^2$ and $c_s^2 = \gamma P_0/\rho_0$ is the sound speed. This dispersion describes two types of perturbations:

- Acoustic wave mode with $\omega^2 = c_s^2 k^2$, which corresponds to the oscillating compressible perturbations with purely potential velocity field: $P', \rho' \neq 0$, $\nabla \times \mathbf{V}' = 0$.
- Vortex mode with $\omega = 0$, which corresponds to the aperiodic incompressible perturbations with purely vortical velocity field: $P', \rho' = 0$, $\nabla \times \mathbf{V}' \neq 0$.

The different linear modes have clearly distinguishable eigenfrequencies and do not posses any similar feature in this shearless limit ($A = 0$). New effects are introduced when the background flow is sheared ($A \neq 0$) and the mean velocity has nonzero vorticity ($\nabla \times \mathbf{V}_0 \neq 0$).[1] In this case the two modes have a mixed character: the acoustic wave mode acquires vortical features, while the vortex mode become compressible.
To understand how a shearing (vortical) background flow affects a linear mode we calculate the contribution of the linear perturbations to the total velocity circulation of the flow. After straightforward manipulations

[1] Please, note that we use $\nabla \times \mathbf{F}$ notation for the rotation of a vector in the general case, not specifying the dimension of the vector $\mathbf{F}$ (3D or 2D). While we employ curl$\mathbf{F}$ notation only in the 2D case, stressing therewith that the product of this operator is a scalar quantity.

of Eqs. (2.11) - (2.14) we obtain:

$$\left\{\frac{\partial}{\partial t}+Ay\frac{\partial}{\partial x}\right\}\left[\left(\frac{\partial V_y'}{\partial x}-\frac{\partial V_y'}{\partial x}\right)-A\frac{\rho'}{\rho_0}\right]=0. \qquad (2.17)$$

We can somewhat generalize this equation in the case of a 2D flow with constant velocity shear: [1]

$$\left\{\frac{\partial}{\partial t}+(\mathbf{V}_0\cdot\nabla)\right\}\left[\frac{\mathrm{curl}\mathbf{V}'}{\rho_0}+\frac{\rho'\ \mathrm{curl}\mathbf{V}_0}{\rho_0^2}\right]=0, \qquad (2.18)$$

Hence, the linear perturbations of the potential vorticity $\mathrm{curl}\mathbf{V}/\rho$ are convected with the equilibrium flow.
This is the basic property of vorticity which helps us to identify the vortical modes in simple hydrodynamic situation. However, this conservation law has its limitations. For instance, the flow viscosity leads to the decrease of potential vorticity through the thermal dissipation. Potential vorticity also varies in baroclinic flows: perturbations of $\nabla P\times\nabla\rho$ act as the source or the sink of potential vorticity depending on the thermodynamic properties of the background flow. Most profound is the action of magnetic field, when Lorenz force leads to the oscillatory behavior of the potential vorticity. Neglecting all these factors in the present section allows us to consider potential vorticity conservation and analyze its consequences for the vortex and acoustic modes in shear flows.
The term in the square brackets of Eq. (2.18) remains constant in Lagrangian coordinates (moves together with fluid) and actually determines the contribution of perturbations to the total velocity circulation of the flow. Consequently, perturbations that do not change the circulation of the background flow velocity should obey the following condition:

$$\rho_0\mathrm{curl}\mathbf{V}'+\rho'\mathrm{curl}\mathbf{V}_0=0. \qquad (2.19)$$

Hence purely compressible perturbations ($\rho'\neq 0$, $\mathrm{curl}\mathbf{V}'=0$) may not be expressed as a superposition of only acoustic wave perturbations. Linear perturbations of the density (or pressure) contribute to the potential vorticity by the background vortical momentum $\rho'\mathrm{curl}\mathbf{V}_0$. This means that acoustic wave modes acquire a vortical nature in flows with inhomogeneous velocity profiles. Perturbations of acoustic vorticity will propagate with the sound wave frequency:

$$\mathrm{curl}\mathbf{V}'(t)=-\mathrm{curl}\mathbf{V}_0\frac{\rho'(t)}{\rho_0}.$$

[1] The same equation may be obtained by the direct calculation of the linear perturbation of velocity circulation in Eq. (2.18) having in mind that $\rho'/\rho_0=\int_{S'}\mathrm{d}f/\int_{S_0}\mathrm{d}f$, where S_0 and S' are mean and perturbed components of the area over which the vorticity is integrated. Note, that in this 2D case $\mathrm{d}f$ is a scalar quantity.

On the other hand vortex modes loose their purely vortical nature and acquire compressible nature: $\mathrm{curl}\mathbf{V}' = 0,\ \rho' \neq 0$ necessarily contains the vortex mode perturbations, since the resulting contribution into the total velocity circulation is non-zero. This vortex mode perturbations of density (and pressure) do not propagate and behave aperiodically[2]. In other words, vortical or compressible perturbations are not uniquely associated with vortex and acoustic wave modes, respectively.
For this simplified 2D hydrodynamic example we have seen that the vortical character of the equilibrium flow affects basic characteristics of the perturbation. The effect increases with the velocity shear parameter $A = -\mathrm{curl}\mathbf{V}_0$. As we will see later, this is the crucial factor that determines the necessary conditions for different modes to interact in the linear regime.
It is helpful to reduce the system of partial differential equations to a set of ordinary differential equations. The common approach to this problem has been modal analysis. It employs an expansion of the physical variables in Fourier modes with predefined temporal and x-spatial structure and study of the boundary value problem with respect to the y coordinate. For the reasons discussed in the introduction we do not use the modal analysis but employ an alternative method.

2.3 Non-modal formalism

For shortness of notation we introduce the generalized vector:

$$\Psi'(\mathbf{r},t) \equiv \begin{pmatrix} \mathbf{V}'(\mathbf{r},t) \\ \rho'(\mathbf{r},t) \\ P'(\mathbf{r},t) \end{pmatrix}. \tag{2.20}$$

In the non-modal method, we use a transformation of variables from the stationary to the co-moving frame, a so-called shearing sheet transformation:

$$x' \equiv x - Ayt, \quad y' \equiv y, \quad t' \equiv t. \tag{2.21}$$

This substitution of variables transforms the spatial inhomogeneity of the equations (2.8-12) into a inhomogeneity:

$$\frac{\partial}{\partial t} + Ay\frac{\partial}{\partial x} = \frac{\partial}{\partial t'}, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial x'}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'} - At'\frac{\partial}{\partial x'}. \tag{2.22}$$

Hence, the full *spatial* Fourier expansion is straightforward:

$$\Psi'(\mathbf{r}',t') = \int\int_{-\infty}^{+\infty} \psi(\mathbf{k}',t') \exp\left(\mathrm{i}k'_x x' + \mathrm{i}k'_y y'\right) \mathrm{d}k'_x \mathrm{d}k'_y, \tag{2.23}$$

[2]This density perturbations are sometimes referred as a pseudo-sound in hydrodynamic literature.

where

$$\psi(\mathbf{k},t) \equiv \left(\begin{array}{c} \mathbf{v}(\mathbf{k},t) \\ \varrho(\mathbf{k},t) \\ p(\mathbf{k},t) \end{array} \right). \tag{2.24}$$

Similarly, we may choose a particular form of the spatial harmonics and analyze perturbations in the laboratory frame. Substitution of (2.21) into the Fourier expansion (2.23) gives us an understanding of the intrinsic shape of the spatial harmonics (SFH) of the perturbed quantities in shear flows: $\exp(\mathrm{i}\mathbf{k'r'}) = \exp[\mathrm{i}k'_x x + \mathrm{i}(k'_y - Atk'_x)y]$. Hence we seek solutions of Eqs. (2.11)-(2.14) of the form:

$$\Psi'(\mathbf{r},t) \propto \psi(\mathbf{k}(t),t)\exp(\mathrm{i}\mathbf{k}(t),\mathbf{r}), \tag{2.25}$$

$$\mathbf{k}(t) = (k_x, k_y(t)), \quad k_y(t) = k_y(0) - Ak_x t. \tag{2.26}$$

The expansion in spatial harmonics with time dependent wave-numbers cancels the explicit coordinate dependence in the original equations. This leads to a system of ordinary differential equations for the perturbation SFH in time:

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} + Av_y + \mathrm{i}k_x\frac{p}{\rho_0} = 0, \tag{2.27}$$

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} + \mathrm{i}k_y(t)p = 0, \tag{2.28}$$

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} + \mathrm{i}\rho_0(k_x v_x + k_y(t)v_y) = 0, \tag{2.29}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} + \mathrm{i}\gamma P_0(k_x v_x + k_y(t)v_y) = 0, \tag{2.30}$$

Eqs. (2.27) - (2.30) pose an initial value problem. The solution of this initial value problem describes the temporal evolution of the SFH in shear flows. Note that the perturbations are described by the individual SFH obtained from the solution of the above initial value problem, and also by the fact that every harmonic linearly drifts in the wave-number space (**k**-space): $\mathbf{k} = \mathbf{k}(t)$. The behavior of the perturbations in the **r**-space is determined by the combined effect of the amplitude of SFHs and their drift in **k**-space. The linear drift of the SFH in a shear flow is easy to understand. It reveals the fact, that the shearing background flow stretches the wave-crests in the direction of the streamlines. Linear drift of SFH is an important property of all perturbations of shear flows. It reveals an inherent anisotropy of the linear process of energy redistribution in the **k**-space, the background trend of the energy transport between the different spatial scales. Temporal characteristics of perturbation modes are defined by their spatial characteristis – wave-numbers.

Therefore, the time dependence of the wave-numbers indicates the temporal variation of the effective frequencies of the perturbation SFH in shear flows:

$$\omega = \omega(\mathbf{k}), \quad k_y = k_y(t): \quad \omega = \omega(t)\ . \tag{2.31}$$

A more detailed description of this process will be given later for particular examples.
Let us consider two limiting cases where simple solutions for sound waves and vortical perturbations are obtained. This will help us to get a general insight into the influence of velocity shear on the dynamics.

2.4 Vortex mode perturbations

First we look at the limiting case of a 2D incompressible flow to study purely vortical perturbations. We neglect the perturbation of the fluid density $\varrho \equiv 0$ and adopt a divergence free velocity field:

$$\nabla \cdot \mathbf{V} = 0. \tag{2.32}$$

We can rewrite Eqs. (2.27)-(2.30) to the following:

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} + Av_y + \mathrm{i}k_x \frac{p}{\rho_0} = 0, \tag{2.33}$$

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} + \mathrm{i}k_y(t) \frac{p}{\rho_0} = 0, \tag{2.34}$$

$$k_x v_x + k_y(t) v_y = 0. \tag{2.35}$$

After straightforward manipulations we obtain the following solution of the system:

$$v_x(t) = -\frac{k_y(t)}{k_x k^2(t)} C, \tag{2.36}$$

$$v_y(t) = \frac{C}{k^2(t)}, \tag{2.37}$$

$$\frac{\mathrm{i}p(t)}{\rho_0} = -2A \frac{k_x}{k^4(t)} C, \tag{2.38}$$

where C is a constant of integration. This aperiodic mode is sometimes called the Kelvin mode, with reference to the pioneering paper by Lord Kelvin 1887. The solution describes a transient amplification of the vortical SFH in shear flow. The pressure SFH is complex due to the $\pi/2$ phase difference between the velocity and pressure perturbations. In the incompressible limit the total spectral energy density of perturbations is only due to kinetic terms and may be written as:

$$E_{\mathbf{k}}(t) = \frac{\rho_0}{2}(|v_x(t)|^2 + |v_y(t)|^2) = \frac{C^2 \rho_0}{2k_x^2 k^2(t)} \tag{2.39}$$

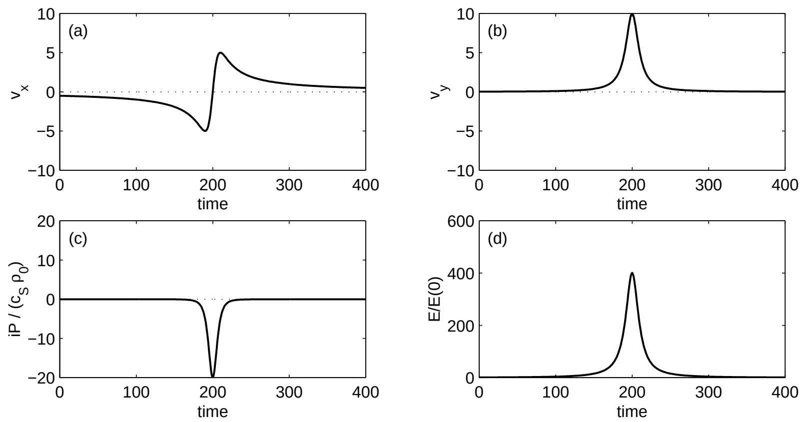

Figure 2.2: Transient amplification of the vortical perturbation SFH in shear flows. $k_x = 1$, $k_y = 20$, $C = 10$ and $A = 0.1$. The velocity $v_x(t)$, $v_y(t)$, pressure $\mathrm{i}p(t)/(c_s\rho_0)$, and normalized spectral energy $E_{\mathbf{k}}(t)/E_{\mathbf{k}}(0)$ of SFH are shown on the a,b,c and d graphs, respectively.

This solution is plotted on Fig. 2.2. It seems that the vortical perturbations are able to extract energy from the mean shear flow and are amplified by several orders of magnitude (by factor of 400 in the considered case). The maximal amplification is reached when the wave-number in the direction of the flow velocity shear is zero: in the present case – at $t = t^* = 200$, when $k_y(t^*) = k_y(0) - Ak_xt^* = 0$.

The character of the evolution of SFH is defined by its wave-number. Namely, SFH with $k_y/Ak_x > 0$ undergoes amplification and SFH with $k_y/Ak_x \leq 0$ looses energy to the background flow and decreases in amplitude.

This amplification phenomenon is a direct consequence of the eigenmode interference and has a transient character. As we shall see later, the transient amplification may be realized not only for aperiodic vortex modes. It can also occur for oscillating wave perturbations. The condition in the latter case is that the oscillating solution has a period much longer than the time scales on which the amplification occurs.

From a physical point of view, the growth of the energy of perturbations is due to the anisotropic character of the momentum exchange of the SFH with specific phases affected by the background velocity shear.

2.5 Acoustic wave mode perturbations

Combining Eqs. (2.13)-(2.14) we obtain the dynamical equation for the pressure and density perturbations:

$$\frac{\mathrm{d}P'}{\mathrm{d}t} = c_s^2 \frac{\mathrm{d}\rho'}{\mathrm{d}t} \ . \tag{2.40}$$

For the study of acoustic waves it is useful to assume that there are no stationary (constant) pressure and density perturbations in the flow. Stationary perturbations can always be removed by including them into the mean flow, e.g. by renormalization of the equilibrium flow. Hence, we may reduce the Eq. (2.39) to the following algebraic form

$$p = c_s^2 \varrho. \tag{2.41}$$

The perturbations are adiabatically compressible. Hence, the equations that govern the compressible SFH in a 2D unbounded shear flow are:

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} + Av_y + \mathrm{i}k_x c_s^2 \frac{\varrho}{\rho_0} = 0 \ , \tag{2.42}$$

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} + \mathrm{i}k_y(t) c_s^2 \frac{\varrho}{\rho_0} = 0 \ , \tag{2.43}$$

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} + \mathrm{i}\rho_0 \left(k_x v_x + k_y(t) v_y\right) = 0 \ . \tag{2.44}$$

These equations describe both, the acoustic (sound) waves as well as the vortex mode, which we have studied in the incompressible limit. In order to study the purely acoustic modes we should select the initial perturbations so that the vortex mode is excluded from the consideration.
Eqs. (2.42)-(2.44) allow for the following invariant form in time:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 0 : \quad I(\mathbf{k}) = k_x v_y - k_y(t) v_x + A\frac{\mathrm{i}\varrho}{\rho_0}, \tag{2.45}$$

Referring to the Eq. (2.18) we may deduce that I($\mathbf{k}$) is a spectral form of the potential vorticity of the SFH. In order to remove the vortex mode from the analysis we select initial conditions with zero potential vorticity:

$$I(\mathbf{k}) \equiv 0. \tag{2.46}$$

Using this initial condition in Eqs. (2.42)-(2.44) we derive the second order differential equation that governs the acoustic wave mode:

$$\frac{\mathrm{d}^2 v_x}{\mathrm{d}t^2} + c_s^2 k^2(t) v_x = 0. \tag{2.47}$$

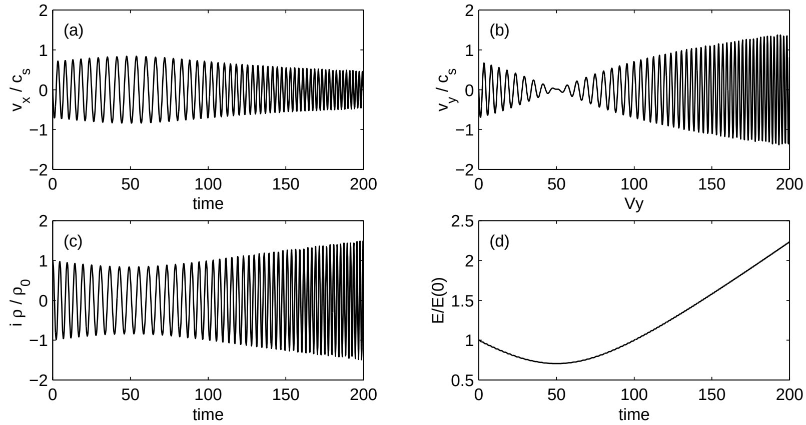

Figure 2.3: Evolution of the acoustic SFH of perturbations in shear flow. $k_x = 1$, $k_y = 1$, and $A = 0.02$. The velocity $v_x(t)$, $v_y(t)$, density $\mathrm{i}\varrho(t)/\rho_0$, and normalized spectral energy $E_{\mathbf{k}}(t)/E_{\mathbf{k}}(0)$ of the harmonic are shown on the a,b,c and d graphs, respectively. Oscillations of the velocity and density show the variation of the its frequency, which is apparently increased at longer times.

To get more insight into the qualitative behaviour of the perturbations we consider flows with low shear and solve Eq. (2.47) approximately, rather than trying to find the exact solution in the general case. When the time dependent parameters of the second order system are varying adiabatically, the system may be considered as an oscillatory system with a varying frequency

$$\omega^2(t) = c_s^2 k^2(t), \tag{2.48}$$

provided that the adiabatic condition of the slow variation is satisfied:

$$\left| \frac{\mathrm{d}\omega(t)}{\mathrm{d}t} \right| \ll \omega^2(t). \tag{2.49}$$

Hence, the velocity shear affects the acoustic wave frequency due to the wave-number variation. During the evolution in shear flows the frequency of the acoustic waves is increased or decreased, depending on the phase of the SFH. Conservation of the adiabatic invariant in such system reveals the characteristic behavior of the wave energy:

$$E_{\mathbf{k}}(t) \propto \omega(t) \propto |\mathbf{k}(t)|. \tag{2.50}$$

These effects are readily illustrated on Fig. 2.3 where we present the exact numerical solution of Eqs. (2.42) - (2.44) for specific initial conditions. On the other hand, the analytical solution of this system in the

WKB approximation is:

$$\mathrm{i}\varrho(t)/\rho_0 = A_\rho(t)\exp\left[\mathrm{i}\phi(t)\right] \ , \tag{2.51}$$

$$v_x(t) = A_{vx}(t)\exp\left[\mathrm{i}(\phi(t) + \phi_{vx}(t))\right] \ , \tag{2.52}$$

$$v_y(t) = A_{vy}(t)\exp\left[\mathrm{i}(\phi(t) + \phi_{vy}(t))\right] \ , \tag{2.53}$$

where

$$\phi(t) = \phi_0 + \int_0^t \omega(t)\mathrm{d}t \ . \tag{2.54}$$

$A_{vx}(t)$, $A_{vy}(t)$, $A_\rho(t)$ are amplitudes and $\phi_{vy}(t)$, $\phi_\rho(t)$ time dependent phase differences between the $v_x(t)$, $v_y(t)$ and $\varrho(t)$, respectively (We do not analyze the acoustic wave properties in detail here and refer the reader to Chagelishvili et al. 1994, 1997 for more details). We rather emphasize the characteristic properties of the acoustic waves that are inherent to waves influenced by a mean shear flow.

First of all, as we have already seen in the incompressible limit, the perturbations exchange energy with the background shear flow. The character of this process is largely defined by the velocity shear parameter and the character of perturbations. At low shear rates the equation describing the evolution of SFH are analogous to an oscillatory equation with a time dependent frequency. Therefore the shear induced temporal variation of the mode frequency is due to the time dependence of the SFH wave-numbers, the linear drift in **k**-space. $\omega(t) = \omega(\mathbf{k}(t))$. The adiabatic character of the system then defines the character of energy exchange between perturbations and the background flow: $E_{\mathbf{k}}(t) \propto \omega(\mathbf{k}(t))$.

Shear of the flow also affects energy transfer by waves. In the simple case of uniform flow the energy transport of wave package may be estimated by the group velocity. However, in the present case the definition of the group velocity becomes problematic.

Indeed, when we use the classical definition of group velocity (cf. Lighthill 1978) and use Eq. (2.54) we obtain:

$$\mathbf{U} = \frac{\partial\omega(\mathbf{k})}{\partial\mathbf{k}} = \frac{\partial^2\phi(\mathbf{k},t)}{\partial\mathbf{k}\,\partial t} \tag{2.55}$$

However, the phase difference between the SFH of the different physical quantities (see Eqs. 2.51 - 2.53) is time dependent: the density and velocity perturbations evolve with their corresponding phases. Therefore, the group velocity calculated for different physical quantities is different:

$$\mathbf{U}_\rho = \mathbf{U} \ , \tag{2.56}$$

$$\mathbf{U}_{vx} = \mathbf{U} + \frac{\partial^2\phi_{vx}(\mathbf{k},t)}{\partial\mathbf{k}\,\partial t} \ , \tag{2.57}$$

$$\mathbf{U}_{vy} = \mathbf{U} + \frac{\partial^2 \phi_{vy}(\mathbf{k}, t)}{\partial \mathbf{k}\, \partial t} \ . \tag{2.58}$$

So, the compressible energy is transported with a different group velocity than the kinetic one. Even the definition of the kinetic energy is problematic, since the harmonics of different polarization transport the energy at different speeds. Two otherwise identical waves with different velocity components projected along the x and y directions have different mean group velocities due to the difference between $\mathbf{U}_{vx}$ and $\mathbf{U}_{vy}$. Hence, even the definition of the average group velocity becomes a subject of approximation. This property is generally inherent to all waves in shear flows, but is most profound during the non-adiabatic stage of the wave evolution, when the phases of different physical quantities significantly diverge. For the considered sound waves non-adiabaticity is profound when $k_y(t)/(Ak_x) < 1$ (see Chagelishvili et al. 1997).

2.6 Velocity shear rate

The dynamics of sheared flows crucially depends on the type of the velocity inhomogeneity. A major concern in this respect is the stability of a given flow. In fact, there are several different factors that may lead to exponential destabilization of the laminar shear flow. Flow may be fundamentally unstable due to the large amplitude or the specific geometry of the velocity shear. For instance, a powerful instability may occur in a flow that has an inflection point in the velocity profile. In the present thesis we study flows with smooth shear of velocity, i.e., velocity profile without an infection point and sufficiently smooth to be approximated by a linear profile locally. Therefore, we do not consider effects related to the flow velocity profile, but simply estimate the largest possible velocity shear that may occur in a spectrally stable flow.

The influence of the velocity shear is different on perturbations of different characteristic length-scales. We have already seen that a sheared flow results in a temporal variation of the wave-numbers thereby stretching the initial pattern into the direction of the streamlines. Therefore, for an adequate description of the effect of the shearing background on the perturbations, it is useful to introduce the non-dimensional shear rate:

$$R \equiv A/c_s k_{\|}, \tag{2.59}$$

where $k_{\|}$ is the characteristic wave-number of the perturbation along the flow. The value of the velocity shear rate may be estimated from several different points of view. The first natural question in this respect: what is the value of the shear rate R which does not lead to kinematic shocks. The latter process may occur when the variation of the flow velocity over a dissipative length scales is of the same order as the sound speed of the medium c_s. The shear parameter A defines the velocity variation over a unit length scale. Thus the formation of extreme velocity gradients necessary for the formation of shock waves is possible when

$$Al_\nu / c_s \geq 1, \tag{2.60}$$

where l_ν is the characteristic dissipative length scale of the flow. Having in mind that $l_\| \sim 1/k_\|$ and using Eq. (2.59) we can rewrite Eq. (2.60) as:

$$R \geq l_\| / l_\nu . \tag{2.61}$$

The length scales of the dynamical perturbations are longer than the dissipative ones: $l_\| > l_\nu$. Therefore, the critical value of the velocity shear rate necessary for the formation of shock waves in a laminar flow should obey the following condition:

$$R > 1. \tag{2.62}$$

Note that the stability threshold may be much higher ($R \gg 1$) for large scale perturbations ($l_\| \gg l_\nu$). Our further analysis is restricted by the maximal value of the velocity shear rate for small scale perturbations: $l_\| \geq l_\nu$. Hence, we define the velocity shear rate to be low when $R \leq 0.1$, moderate when $0.1 < R < 1$, and high when $R \geq 1$.